Graphz

The Graphites

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Our goal is to completely understand the cardinality of k-vertex-critical $(P_4 + P_1,H)$-free graphs when each is order 4 or higher

- K_4 - unknown
- *diamond* - finite [1]
- *paw* - finite [1]
- $C4$ - finite [3]
- *claw* - [4]
- P_4 - finite since perfect
- $\overline{K_4}$ - finite ramseys theorem
- $P_2 + 2P_1$ - [2]
- $P_3 + P_1$ - [2]
- $2K_2$ - unknown
- $K_3 + P_1$ - unknown

Theorem 0.1. *hello*

Lemma 0.2. *there*

Proof. woop woop $\square$

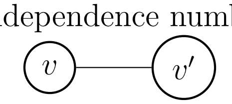
Let G be a graph. Let S be the maximal independent set of G. The subgraph $V(G) - S$ has order $|V(G) - S|$, thus it is at most $|V(G) - S|$-colourable. The remaining vertices S are 1-colourable, since they are an independent set and are not adjacent in G. So then our graph is at most $|V(G) - S| + 1$colourable

Let G be a k-vertex-critical $(P_4+P_1, 2P_2)$-free graph. Let S be a maximum independent set in G. Let the vertices outside S, $V(G) - S$, be partitioned by A and B, where $A = \{v \in V(G) - S : |N(v) \cap S = 1\}$. Then let $B = V(G) - S - A$. Let $\forall v \in S$, $v_A = N(v) \cap A$
Claim 1: $\forall v, v' \in S$, v_A is complete to v'_A
Proof: If $a \in v_A$ and $a' \in v'_A$ such that $a \not\sim a'$, then $\{v, v', a, a'\}$ induces a $2P_2$— a contradiction.
Claim 2: $|S_A| \leq 1$
Proof: If $|S_A| \geq 2$, then let $v, v' \in S_A$ with $a \in v_A anda' \in v'_A$. From claim 1 we know that $a \sim a'$, so $\{v, v', a, a', x\}$ induces a $P_4 + P_1$.

For this graph to be k-vertex critical we can also assert that it has no comparable vertices. Which is to say, $\forall u, v \in S$, $N(u) \not\subseteq N(v)$, or every vertex in S has at least one unique neighbour compared to another vertex.
Let v_1, v_2 be two vertices in $V(G) - S$ s.t. $v_1 \sim S_A$ and $v_2 \sim$some stuff in S_B. We can now assert that any element in B has no neighbours in S_A. And that S_A has no neighbours in B.

Let us find what happens when we look for a $(P_4 + P_1, K_3 + P_1, 2P_2)$-free graph. Let S be the maximum independent set. Let $A = \{v \in V(G) - S : |N(V) \cap S| = 1\}$ Let $B = V(G) - S - A$. Let $S_A = \{v \in S : N(V) \cap A \neq \emptyset\}$
Claim 3: $\forall v, v' \in S$, v_A is complete to v'_a. Proof: Assume $v_A \not\sim v'_A$. Then, $\{v, v', v_A, v'_A\}$ induces a $2P_2$, a contradiction.

Claim 4: $|S_A| \leq 1$ Proof: Assume $|S_A| \geq 2$. Then let $u, u' \in S_A$ with $a \in v_A$ and $a' \in v'_A$. From claim 1 we know that $a \sim a'$, so $\{v, v'a, a', x\}$ induces a $P_4 + P_1$ for any $x \in S - \{v, v'\}$. Note that x exists, other wise the independence number of this graph is 2.



Claim 5: $|A| \leq 1$ First lets make a new notation. Since $|S_A| \leq 1$, let us name that potential vertex v_S. Proof: Assume $|A| \geq 2$. Then we have any two vertices in A v, v'. From claim 4 we know that $|S_A| \leq 1$ and from claim 3 that v is complete to v'. Since both v, v' are adjacent to v_S, then $\{v, v', v_S, x\}$ induces a $K_3 + P_1$, where $x \in S_B$

Once again, another claim:

There are three cases we have to work with. For any given $v \in B$, v is either adjacent to v_S, a, or both. Let's look at when $v \sim v_S$

If $v \sim v_S$, v is complete to S_B. Proof: Assume that v is not complete to S_B, then there exists some vertex $u \in S_B$ s.t $u \not\sim v$. we also know there is a vertexss $u' \in S_B$ that is adjacent to v, since v is adjacent to at least 2 vertices in S. Then $\{v_S, a, v, u, u'\}$ induces a $P_4 + P_1$.

Claim: let there be a vertex v' in B s.t $v' \not\sim a$ and $v' \not\sim v_S$.

Claim: let there be a vertex v' in B s.t $v' \not\sim v_S$. v' is not adjacent to anything in S and other things. Proof: Assume at v' is adjacent to some vertex u in S_B. Then, $\{v', u, v_S, a\}$ induces a $2P_2$.

Some random other claim: Let $v' \not\sim v$ and $v \sim a$, then v' is complete to S. Proof: Assume that v' is not complete to S_B, at least. Then, let u be some vertex $\in S_B$ such that $u \sim v$ and $u \not\sim v'$. Then, $\{v_S, a, v, u, v'\}$ induces a $P_4 + P_1$. Further, if $v' \not\sim v_S$, then this leassves us with comparable vertices and thus a non-k-critical graph. Therefore, $v' \sim v_S$ and v' is complete to S.

Let $s = S - S_B$ and $a = V(G) - B$ Let us further split B into three partitions. Let $B_1 = \{v \in B : v \sim s \text{ and } v \not\sim a\}$ and $B_2 = \{v \in B : v \sim a \text{ and } v \not\sim s\}$ and finally $B_3 = \{v \in B : v \sim s \text{ and } v \sim a\}$. Claim: $B_1 \cup B_3$ is complete to S and B_2 is complete to $S - \{s\}$ $\forall s_1, s_2 \in S - \{s\}$, $N(s_1) = B_1 \cup B_2 \cup B_3 = N(s_2)$, this makes s_1 and s_2 comparable, contradicting the criticality of the graph. Now we must test the case for when $|A| = 0$. In this, we have S as the maximum independent set. For this to be critical, each vertex in S must have a unique neighbour such that for two vertices $s_1, s_2 \in S$, $N(s_1) \not\subset N(s_2)$ and $N(s_2) \not\subset N(s_1)$. Let us call the set of unique neighbours U and the common neighbours C. Common neighbours are defined $\{v_1, v_2 \in V - S : N(v_1) \cap S = N(v_2) \cap S\}$. Then $U = V - C$.

Look at this claw-free graph.
Everytime I do it makes me laugh.
How did the nodes get so red
And where's the independent set?

This is what I proofed up
I think the logic is good enough
I don't how we ever went without
The comparable nodes lemma

Every memory of looking for K_3
I sorted all of them by criticality

It’s hard to say it
Time to say it
It’s finite, it’s finite

Here’s another direction for 6-critical graphs that are $P5$ free: Any 6-critical graph that is P_5 free is also $P_4 + P_1$free.

References

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