# Equation Derivations for $\tau_\nu$

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## 1 Viscous Time

The non-dimensional equations for convection using $\tau_\nu$ are:

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P + \frac{\mathrm{R_F}}{\mathrm{Pr}} T\hat{\mathbf{z}} + \nabla^2 \mathbf{u} - \mathrm{Ta}^{\frac{1}{2}} \boldsymbol{\Omega} \times \mathbf{u}, \tag{2}$$

$$\partial_t T + (\mathbf{u} \cdot \nabla)T = \frac{1}{\mathrm{Pr}} \left[\nabla^2 T + \mathcal{H}\right], \tag{3}$$

## 2 Applying perturbation theory

From here we can decompose to an equilibrium state and a small order perturbation:

$$\mathbf{u} = \mathbf{u}_{\mathrm{eq}} + \epsilon \mathbf{u}', \quad T = T_{\mathrm{eq}} + \epsilon T', \quad P = P_{\mathrm{eq}} + \epsilon P', \tag{4}$$

where $\epsilon$ is a small parameter. We can substitute these into the non-dimensionalised equations, gather terms of the same order in epsilon, ignore terms of order $\epsilon^2$ and higher, and remember that all $\epsilon^0$ terms are a steady-state equilibrium, so $\partial_t$ terms = 0 and $\mathbf{u}_{\mathrm{eq}} = 0$. With this, we get:

**$\epsilon^0$ (equilibrium equations):**

$$\nabla \cdot \mathbf{u}_{\mathrm{eq}} = 0, \tag{5}$$

$$\nabla P_{\mathrm{eq}} = \frac{\mathrm{R_F}}{\mathrm{Pr}} T_{\mathrm{eq}} \hat{\mathbf{z}}, \tag{6}$$

$$-\nabla^2 T_{\mathrm{eq}} = \mathcal{H}. \tag{7}$$

**$\epsilon^1$ (perturbation equations):**

$$\nabla \cdot \mathbf{u}' = 0, \tag{8}$$

$$\partial_t \mathbf{u}' + \nabla P' - \frac{\mathrm{R_F}}{\mathrm{Pr}} T'\hat{\mathbf{z}} - \nabla^2 \mathbf{u}' + \mathrm{Ta}^{\frac{1}{2}} \boldsymbol{\Omega} \times \mathbf{u}' = 0, \tag{9}$$

$$\partial_t T' + (\mathbf{u}' \cdot \nabla) T_{\mathrm{eq}} - \frac{1}{\mathrm{Pr}} \nabla^2 T' = 0. \tag{10}$$

In the case of the Kazemi+22 heating function, we can solve the third $\epsilon^0$ equation to calculate $T_{\text{eq}}$. Since

$$\mathcal{H} = ae^{\frac{-z}{\ell}} - \beta, \tag{11}$$

we can show that

$$\partial_z T_{\text{eq}} = a\ell e^{\frac{-z}{\ell}} + \beta z + C, \tag{12}$$

and we can use the boundary condition that $\partial_z T_{\text{eq}}|_{z=0} = 0$ to show that $C = -a\ell$. We can then find

$$T_{\text{eq}} = -a\ell^2 e^{\frac{-z}{\ell}} + \frac{\beta z^2}{2} - a\ell z + C, \tag{13}$$

where $C$ is an arbitrary constant. It can be shown that choosing $C = 1 + a\ell^2$ will set $T_{\text{eq}}|_{z=0} = 1$, so we can deduce our equilibrium temperature profile as

$$T_{\text{eq}} = -a\ell^2 e^{\frac{-z}{\ell}} + \frac{\beta z^2}{2} - a\ell z + 1 + a\ell^2. \tag{14}$$

# 3 Boundary Conditions

The boundary conditions for the perturbation equations are:
Impermeable top and bottom:

$$w'(z=0) = w'(z=L) = 0, \tag{15}$$

Free-Slip top and bottom:

$$\partial_z u'(z=0) = \partial_z u'(z=L) = 0, \tag{16}$$

$$\partial_z v'(z=0) = \partial_z v'(z=L) = 0, \tag{17}$$

Insulating top and bottom:

$$\partial_z T'(z=0) = \partial_z T'(z=L) = 0. \tag{18}$$

The perturbation equations and boundary conditions are inputted into `eigenvalue.py` and solved using `eigentools`, which should recover the critical Rayleigh number and critical wave number for a given Taylor number.