Equation Derivations for τ_{ff}

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1 Non-Dimensionalisation

Free-Fall time is the crossing time if buoyancy acts unimpeded, so the advection term will balance with the buoyancy term.

$$(\mathbf{u} \cdot \nabla)\mathbf{u} \sim \alpha g_0 T \tag{1}$$

$$\therefore U^2/L \sim \alpha g T \tag{2}$$

where U, L, T are the velocity, length and temperature scales respectively.

From this, we can see that the free-fall velocity,

$$U_{ff} \sim \sqrt{\alpha g T L}, \tag{3}$$

and so the free-fall time,

$$\tau_{ff} \sim \frac{L}{U_{ff}} \sim \frac{L}{\sqrt{\alpha g T L}} \sim \sqrt{\frac{L}{\alpha g T}}. \tag{4}$$

BUT: In my case the temperature scale is more complicated than the traditional $T \sim \Delta T$, due to the heating function $\mathcal{H}$. Following Kazemi+22, as $\mathcal{H} \sim Q$ with units temperature / time, the temperature scale becomes $T \sim \frac{L^2\mathcal{H}}{\kappa}$, which means

$$u_{ff} \sim \sqrt{\frac{\alpha g L^3 Q}{\kappa}}, \tag{5}$$

and

$$\tau_{ff} \sim \sqrt{\frac{\kappa}{\alpha g L Q}}. \tag{6}$$

From the dimensional equations:

$$\nabla \cdot \mathbf{u} = 0, \tag{7}$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla P + \alpha g T \hat{\mathbf{z}} + \nu \nabla^2 \mathbf{u} - 2\boldsymbol{\Omega} \times \mathbf{u}, \tag{8}$$

$$\partial_t T + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T + \mathcal{H}. \tag{9}$$

Using the scalings (where $\hat{}$ represents a non-dimensional variable):
$\nabla \rightarrow \frac{1}{L}\hat{\nabla}$; $\mathbf{u} \rightarrow \sqrt{\frac{\alpha g L^3 Q}{\kappa}}\hat{\mathbf{u}}$; $\partial_t \rightarrow \sqrt{\frac{\alpha g L Q}{\kappa}}\partial_{\hat{t}}$; $P \rightarrow \frac{\rho_o \kappa^2}{L^2}\hat{P}$; $T \rightarrow \frac{L^2 Q}{\kappa}\hat{T}$; $\mathbf{\Omega} \rightarrow \Omega_0 \hat{\mathbf{\Omega}}$; $\mathcal{H} \rightarrow Q\hat{\mathcal{H}}$,
and re-arranging, the equations become:

$$\hat{\nabla} \cdot \hat{\mathbf{u}} = 0, \tag{10}$$

$$\partial_{\hat{t}}\hat{\mathbf{u}} + (\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} = \frac{\kappa^3}{\alpha g L^5 Q}\hat{\nabla}\hat{P} + \hat{T}\hat{\mathbf{z}} + \sqrt{\frac{\nu^2 \kappa}{\alpha g L^5 Q}}\hat{\nabla}^2\hat{\mathbf{u}} - \sqrt{\frac{4\Omega_0^2 \kappa}{\alpha g L Q}}\hat{\mathbf{\Omega}} \times \hat{\mathbf{u}}, \tag{11}$$

$$\partial_{\hat{t}}\hat{T} + (\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{T} = \sqrt{\frac{\kappa^3}{\alpha g L^5 Q}}\left[\hat{\nabla}^2\hat{T} + \hat{\mathcal{H}}\right], \tag{12}$$

and since $\mathrm{R_F} = \frac{\alpha g L^5 Q}{\nu \kappa^2}$, $\mathrm{Pr} = \frac{\nu}{\kappa}$, $\mathrm{Ta} = \frac{4\Omega_0^2 L^4}{\nu^2}$, the equations become (dropping the $\hat{}$ notation):

$$\nabla \cdot \mathbf{u} = 0, \tag{13}$$

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{-1}{\mathrm{R_F Pr}}\nabla P + T\hat{\mathbf{z}} + \sqrt{\frac{\mathrm{Pr}}{\mathrm{R_F}}}\nabla^2\mathbf{u} - \sqrt{\frac{\mathrm{TaPr}}{\mathrm{R_F}}}\mathbf{\Omega} \times \mathbf{u}, \tag{14}$$

$$\partial_t T + (\mathbf{u} \cdot \nabla)T = \frac{1}{\sqrt{\mathrm{R_F Pr}}}\left[\nabla^2 T + \mathcal{H}\right]. \tag{15}$$

2 Applying perturbation theory

From here we can decompose to an equilibrium state and a small order perturbation:

$$\mathbf{u} = \mathbf{u}_{\mathrm{eq}} + \epsilon \mathbf{u}', \tag{16}$$

$$T = T_{\mathrm{eq}} + \epsilon T', \tag{17}$$

$$P = P_{\mathrm{eq}} + \epsilon P', \tag{18}$$

where ϵ is a small parameter. We can substitute these into the non-dimensionalised equations, gather terms of the same order in epsilon, ignore terms of order ϵ^2 and higher, and remember that all ϵ^0 terms are a steady-state equilibrium, so ∂_t terms = 0 and $\mathbf{u}_{\mathrm{eq}} = 0$. With this, we get:
ϵ^0 (equilibrium equations):

$$\nabla \cdot \mathbf{u}_{\mathrm{eq}} = 0, \tag{19}$$

$$\frac{1}{\mathrm{R_F Pr}}\nabla P_{\mathrm{eq}} = T_{\mathrm{eq}}\hat{\mathbf{z}}, \tag{20}$$

$$-\nabla^2 T_{\mathrm{eq}} = \mathcal{H}. \tag{21}$$

ϵ^1 **(perturbation equations):**

$$\nabla \cdot \mathbf{u}' = 0, \tag{22}$$

$$\partial_t \mathbf{u}' + \frac{1}{\mathrm{R_F Pr}}\nabla P' - T'\hat{\mathbf{z}} - \sqrt{\frac{\mathrm{Pr}}{\mathrm{R_F}}}\nabla^2 \mathbf{u}' + \sqrt{\frac{\mathrm{TaPr}}{\mathrm{R_F}}}\mathbf{\Omega} \times \mathbf{u}' = 0, \tag{23}$$

$$\partial_t T' + (\mathbf{u}'\nabla)T_{\mathrm{eq}} - \frac{1}{\sqrt{\mathrm{R_F Pr}}}(\nabla^2 T') = 0 \tag{24}$$

In the case of the Kazemi+22 heating function, we can solve the third ϵ^0 equation to calculate T_{eq}. Since

$$\mathcal{H} = ae^{\frac{-z}{\ell}} - \beta, \tag{25}$$

we can show that

$$\partial_z T_{\mathrm{eq}} = a\ell e^{\frac{-z}{\ell}} + \beta z + C, \tag{26}$$

and we can use the boundary condition that $\partial_z T_{\mathrm{eq}}|_{z=0} = 0$ to show that $C = -a\ell$. We can then find

$$T_{\mathrm{eq}} = -a\ell^2 e^{\frac{-z}{\ell}} + \frac{\beta z^2}{2} - a\ell z + C, \tag{27}$$

where C is an arbitrary constant. It can be shown that choosing $C = 1 + a\ell^2$ will set $T_{\mathrm{eq}}|_{z=0} = 1$, so we can deduce our equilibrium temperature profile as

$$T_{\mathrm{eq}} = -a\ell^2 e^{\frac{-z}{\ell}} + \frac{\beta z^2}{2} - a\ell z + 1 + a\ell^2. \tag{28}$$

3 Boundary Conditions

The boundary conditions for the perturbation equations are:
Impermeable top and bottom:

$$w'(z = 0) = w'(z = L) = 0, \tag{29}$$

Free-slip top and bottom:

$$\partial_z u'(z = 0) = \partial_z u'(z = L) = 0, \tag{30}$$

$$\partial_z v'(z = 0) = \partial_z v'(z = L) = 0, \tag{31}$$

Insulating top and bottom:

$$\partial_z T'(z = 0) = \partial_z T'(z = L) = 0. \tag{32}$$

The perturbation equations and boundary conditions are inputted into eigenvalue.py and solved eigentools, which should recover the critical Rayleigh number and critical wave number for a given Taylor number.