Explanation of the Gravitational Wave Detector Response Code

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1 Overview

This document explains the original NorthStar Algorithm (northStar1.py) file that was coded before 2025. This algorithm computes the response of a gravitational wave detector (such as those in the LIGO network) to incoming gravitational waves. To the best of my understanding, the code is organized as follows:

- **Constants and Detector Angles:** Physical constants, sampling rates, and detector-specific angles are defined.
- **Tensor Transformation:** A general function to transform tensors between different bases is provided.
- **Source Vector and Basis Transformations:** Functions compute the source direction in Cartesian coordinates and construct the change-of-basis matrices from the gravitational wave (GW) frame or the detector frame to the Earth-centered (EC) frame.
- **Gravitational Wave Tensor and Detector Response:** The gravitational wave tensor in the transverse-traceless (TT) gauge is constructed and then transformed to the Earth-centered frame. Finally, the detector response is computed by contracting the transformed GW tensor with the detector tensor.

2 Constants and Detector Angles

At the beginning, the code imports necessary modules and defines constants such as the number of detectors, the sampling rate, Earth's radius, and the speed of light. It also defines the detector angles for the Hanford and Livingston detectors. These angles are given in degrees and converted into radians. The detector angles typically include:

- A geographic coordinate (e.g., latitude or declination).

- A longitudinal coordinate (e.g., longitude or right ascension).
- An orientation angle (possibly shifted by $\pi/2$) which describes the detector's arm orientation.

3 Tensor Transformation

The function `transform_tensor` transforms a tensor from one coordinate system to another. In tensor analysis, the transformation laws depend on the variance (contravariant or covariant) of the tensor's indices. The function supports three types:

1. `'2_0'` **(Two contravariant indices):**

   $$T'^{ij} = \Lambda^{i}{}_{k}\Lambda^{j}{}_{l}\,T^{kl},$$

   where Λ is the change-of-basis matrix.

2. `'1_1'` **(Mixed tensor):**

   $$T'^{i}_{j} = \Lambda^{i}{}_{k}\,T^{k}_{l}\,(\Lambda^{-1})^{l}{}_{j}.$$

3. `'0_2'` **(Two covariant indices):**

   $$T'_{ij} = (\Lambda^{-1})^{k}_{i}(\Lambda^{-1})^{l}{}_{j}\,T_{kl}.$$

The code uses `np.einsum` to perform these index contractions in a concise manner.

4 Source Vector and Basis Transformations

Source Vector

The function `source_vector_from_angles` computes a unit vector in Cartesian coordinates from the angular coordinates (declination and right ascension) of the source:

$$\mathbf{s} = \begin{pmatrix} \cos(\delta)\cos(\alpha) \\ \cos(\delta)\sin(\alpha) \\ \sin(\delta) \end{pmatrix},$$

where δ is the declination and α the right ascension.

Change-of-Basis: Gravitational Wave Frame to Earth-Centered Frame

The function `change_basis_gw_to_ec` constructs a change-of-basis matrix from the gravitational wave frame to the Earth-centered (EC) frame. It does so by:

- Computing the source vector.
- Defining an orthonormal basis using:
  - $\mathbf{z}$ (set to $-$ the source vector),
  - A vector $\mathbf{y}$ that is orthogonal to the source vector,
  - A vector $\mathbf{x}$ defined as the cross product $\mathbf{x} = \mathbf{z} \times \mathbf{y}$.
- Combining these vectors into a matrix.
- Applying an additional rotation by the polarization angle.

The inverse of the product of the rotation matrix and the basis matrix gives the required change-of-basis matrix.

Change-of-Basis: Detector Frame to Earth-Centered Frame

Similarly, the function `change_basis_detector_to_ec` creates the change-of-basis matrix for a detector. It uses the detector's geographical coordinates and its arm orientation to define the basis vectors in the EC frame. After constructing the local x, y, and z vectors, an orientation rotation is applied.

5 Gravitational Wave Tensor and Detector Response

Gravitational Wave Tensor

In the transverse-traceless (TT) gauge, the gravitational wave is described by the tensor:

$$h^{\mathrm{TT}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where h_+ and $h_\times$ are the two polarization amplitudes. The function `gravitational_wave_ec_frame` computes this tensor and then transforms it to the Earth-centered frame using the previously defined change-of-basis matrix (with a covariant transformation, i.e., type `'0_2'`).

Detector Response

The detector tensor models the detector's sensitivity to the gravitational wave strain. It is given by:

$$D = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the function `detector_response`, the following steps are performed:

1. The detector tensor is transformed from the detector frame to the Earth-centered frame using a transformation of type `'2_0'`.

2. The gravitational wave tensor is computed in the Earth-centered frame.

3. A tensor contraction (using `np.tensordot`) is performed:

$$s = h_{ij}^{\mathrm{EC}} \, D^{ij},$$

   yielding the scalar detector response.

6 Summary

This code provides a modular framework for:

- Transforming tensors between different coordinate systems using appropriate transformation laws.
- Constructing the gravitational wave tensor in an Earth-centered frame.
- Evaluating the detector response by contracting the gravitational wave tensor with the detector tensor.

We will now make an attempt to make necessary improvements to the code.