Introduction to Commutative Algebra

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Solution to exercises

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Chapter 1 — Rings and Ideals

Exercise 1. Let x be a nilpotent element of the ring A, and $n > 0$ an integer such that $x^n = 0$. Then,

$$(1+x)\sum_{i\geq 0}^{n-1}(-1)^i x^i = 1.$$

Therefore, $1+x$ is a unit. The sum of a nilpotent x and a unit u can be written in the form $u(1+u^{-1}x)$, which makes it again a unit since $u^{-1}x$ is nilpotent.

Exercise 2.

i) If f is a unit in $A[x]$ then there is $g \in A[x]$ such that $fg = 1$ and in turn $f(0)g(0) = 1$, whence $f(0)$ is a unit in A. If $f = a_0 + a_1 x + \cdots + a_n x^n$ and $g = b_0 + b_1 x + \cdots + b_m x^m$ is the inverse of f in $A[x]$, then we shall prove by induction on r that $a_n^{r+1} b_{m-r} = 0$. Indeed, this is true for $r = 0$ (last coefficient of fg), and if it is true for $0 \leq r < n+m-1$, then taking the coefficient of degree $n+m-r-1$ in $fg = 1$ and multiplying by a_n^r yields

$$\sum_{i+j=n+m-r} a_n^r a_i b_j = 0$$

and using the inductive hypothesis we can drop all summands with $j > m-r$, leaving only $a_n^{r+1} b_{m-r} = 0$.

We get $a_n^{m+1} b_0 = 0$ but b_0 is a unit so a^n is nilpotent. Thus, $f - a_n x^n$ is a unit as per the previous exercise, which implies by induction that all the nonconstant coefficients are nilpotent. The reciprocal property is immediate, since a sum of nilpotents is nilpotent and a unit plus a nilpotent is a unit.

ii) If f is nilpotent, $1+f$ is a unit (exercise 1) and as per the previous point, $a_1, \cdots, a_n$ are nilpotent, and $a_0 = f(0)$ is clearly nilpotent. The converse is clear as well (sum of nilpotents is nilpotent per the binomial formula).

iii) Let f be a zero divisor in $A[x]$ and g be a least degree polynomial $b_0 + \cdots + b_m x^m$ such that $fg = 0$. Then, $a_n b_m = 0$ hence $a_n g = 0$ since $f a_n g = 0$ and $a_n g$ has degree $< m = \deg(g)$. Suppose $a_{n-r} g = 0$ for $r < n$. Then writing explicitely the coefficients of $fg = 0$, one finds

$$fg = (a_0 + \cdots + a_{n-r-1} x^{n-r-1})g = 0$$

thus $a_{n-r-1} b_m = 0$ and $a_{n-r-1} g = 0$. We deduce $b_m f = 0$ (all the coefficients cancel).

iv) Let $\mathfrak{a} = (a_0, \cdots, a_n)$, $\mathfrak{b} = (b_0, \cdots, b_n)$ and $\mathfrak{c} = (a_0 b_0, a_1 b_0 + a_0 b_1, \cdots, a_n b_n)$. Clearly, $\mathfrak{c} \subset \mathfrak{a} \cap \mathfrak{b}$, thus if $\mathfrak{c} = (1)$ then $\mathfrak{a} = \mathfrak{b} = (1)$. Assume now that $\mathfrak{a} = \mathfrak{b} = (1)$. If $\mathfrak{c} \subsetneq (1)$, then there is a maximal ideal $\mathfrak{m}$ containing $\mathfrak{c}$. The image $\overline{fg}$ of fg in $A/\mathfrak{m}[x]$ is 0, thus f and g are zero divisors (they are both nonzero) in the integral domain $A/\mathfrak{m}[x]$, contradiction.

Exercise 4. The nilradical is contained in every prime ideal, thus in every maximal ideal: $\mathbf{N} \subset \mathbf{R}$. Then, let f be an element of the Jacobson radical $\mathbf{R}$. Then, $1 - fx$ is a unit, therefore all the coefficients from fx besides the first one are nilpotent, that is, all the coefficients from f are nilpotent and therefore f is nilpotent too.

Exercise 5.

i) Let $f = \sum_{n \geq 0} a_n x^n$ be an element of $A[[x]]$ with a_0 a unit in A. Take

$$g = a_0^{-1}\Big(1 - (a_0^{-1}f - 1) + (a_0^{-1}f - 1)^2 - (a_0^{-1}f - 1)^3 + \cdots\Big).$$

This is a well-defined element of $A[[x]]$ since $f - a_0$ has only nonconstant monomial terms thus each coefficient of g is defined only by a finite amount of terms in the infinite sum. We know from the theory of infinite series that $fg = 1$, which makes f a unit. The converse is immediate.

ii) Let f be a nilpotent formal power series. For every prime ideal $\mathfrak{p}$ of A, the image $f_{\mathfrak{p}}$ of f in $A/\mathfrak{p}[[x]]$ is null since that ring is integral and f is nilpotent. Therefore, the coefficients of f are in $\mathfrak{p}$ for every prime $\mathfrak{p}$ of A. Since the nilradical is the intersection of all prime ideals, all the coefficients are nilpotent.

iii) f is in the Jacobson if and only if $1 - fg$ is a unit for all $g \in A[[x]]$ if and only if $1 - a_0 c$ is a unit for all $c \in A$ if and only if a_0 is in the Jacobson of A.

iv) $A[[x]]/(x) = A$ whence the ideals of A correspond (in an order-preserving way) to ideals of $A[[x]]$ containing x. Moreover, for all $f \in A[[x]]$, $1 - xf$ is a unit (its constant coefficient is 1), therefore x is in the Jacobson radical and thus in every maximal ideal of $A[[x]]$. Thus, there is a bijection between maximal ideals $\mathfrak{m}$ of $A[[x]]$ and maximal ideals of A, given by $\mathfrak{m} \mapsto \mathfrak{m}^c$. Moreover, the extension $\mathfrak{m} \mapsto \mathfrak{m}^e$ is clearly $\mathfrak{m} \mapsto (\mathfrak{m}, x)$ (given by the canonical inclusion $A \longrightarrow A[[x]]$).

v) Let $\mathfrak{p}$ be a prime ideal in A, and denote by π the canonical quotient map $A[[x]] \longrightarrow A[[x]]/(\mathfrak{p}, x)$. Elements of $A[[x]]/(\mathfrak{p}, x)$ are of the form $\pi(a)$ for some $a \in A$. Denote by π' the canonical map $A \longrightarrow A/\mathfrak{p}$. The map $\pi(a) \in A[[x]]/(\mathfrak{p}, x) \longmapsto \pi'(a) \in A/\mathfrak{p}$ is well defined and defines an isomorphism. As such, $A[[x]]/(\mathfrak{p}, x)$ is an integral domain and $(\mathfrak{p}, x)$ is a prime ideal, which concludes.

Exercise 6. Suppose that the Jacobson is not contained in the nilradical. Then there is a non-zero idempotent e contained in the Jacobson but not in the nilradical. Therefore, $e(1 - e) = 0$ and $1 - e$ is a unit (because e is in the Jacobson), so $e = 0$ which is absurd.

Exercise 7. Let $\mathfrak{p}$ be a prime ideal of A, and let x be an element outside $\mathfrak{p}$. There is $n > 1$ such that $x(1 - x^{n-1}) = 0$, and since $A/\mathfrak{p}$ is an integral domain, this relation yields $\overline{x}^{n-1} = 1$ in $A/\mathfrak{p}$. Therefore, $A/\mathfrak{p}$ is a field (all non-zero elements have a multiplicative order and thus are units), and $\mathfrak{p}$ is maximal.

Exercise 8. This is a straightforward application of Zorn's lemma. To build a lower bound for each descending chain, take the intersection of the primes in that chain, which is still prime thanks to the inclusion relation between primes in the chain.

Exercise 9. If $\mathfrak{a} = r(\mathfrak{a})$ then we already know that $\mathfrak{a}$ is the intersection of all prime ideals containing $\mathfrak{a}$ (Prop 1.14). If $\mathfrak{a} = \bigcap_i \mathfrak{p}_i$ and $x^n \in \mathfrak{a}$, then $x^n \in \mathfrak{p}_i$ for all i, thus $x \in \mathfrak{p}_i$ for all i and $x \in \mathfrak{a}$, whence $\mathfrak{a} = r(\mathfrak{a})$.

Exercise 10.

- i) $\Rightarrow$ iii) The nilradical of A is the intersection of all of its prime ideals. Thus, $\mathfrak{N}$ is the sole prime ideal, making it maximal, and thus $A/\mathfrak{N}$ is a field.
- iii) $\Rightarrow$ ii) Let $a \in A$ be a non-nilpotent element, so that $\overline{a} \neq 0$ in $A/\mathfrak{N}$ (which is a field). Take $b \in A$ to be in the class of inverses of $\overline{a}$: $\overline{ab} = 1$ in $A/\mathfrak{N}$. Then $ab = 1 + x$ for some nilpotent x, but the sum of a nilpotent and a unit is again a unit (Exercise 1.1), thus ab is a unit and a is a unit.
- ii) $\Rightarrow$ i) Assume $\mathfrak{p}$ is a prime ideal distinct from the nilradical $\mathfrak{N}$. Then $\mathfrak{p}$ contains an element x which is not nilpotent, and that makes it a unit by hypothesis. This is a contradiction, since a prime ideal can not contain units. Therefore, the nilradical is the only possible prime ideal, and one easily checks that it is.

Exercise 11.

i) $(1+x)^2 = 1 + x = 1 + 2x + x^2 = 1 + 2x + x \Longrightarrow 2x = 0$.

ii) Every prime is maximal (Exercise 1.7), therefore $A/\mathfrak{p}$ is a field. If $x \in A/\mathfrak{p}$ is non-zero then $x = x^2 x^{-1} = x x^{-1} = 1$.

iii) Let a, b be elements of A. We have $a(a + b + ab) = a + 2ab = a$ and $b(a + b + ab) = b + 2ab = b$ so $(a, b) = (a + b + 2ab)$. By induction, this shows that A is a PID.

Exercise 12. Say A is local with maximal ideal $\mathfrak{m}$ and e is an idempotent different from $0, 1$. We have that $e(e-1) = 0$ whence e is not a unit, meaning $e \in \mathfrak{m}$. The maximal ideal is also the Jacobson radical, therefore $1 - e$ is a unit, which contradicts $e(e-1) = 0$.

Exercise 14. Apply Zorn's lemma to show existence of maximal elements (take the union of each term as the maximum of a chain). Let S be maximal in Σ, $x, y \notin S$. If $xy \in S$ then both x and y are zero divisors, meaning $(x) + S$ and $(y) + S$ are in Σ which contradicts maximality of Σ. Thus, $xy \notin S$.

Exercise 15.

i) If a prime ideal $\mathfrak{p}$ contains $\mathfrak{a}$, then it also contains $r(\mathfrak{a})$ since $r(\mathfrak{a})$ is an intersection of some primes, $\mathfrak{p}$ included. The rest is clear.

ii) Immediate.

iii) If $\mathfrak{p}$ contains all E_i then $\mathfrak{p} \in \bigcap_i V(E_i)$, and vice-versa.

iv) $r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$

Exercise 16.

- $\mathrm{Spec}(\mathbf{Z}) = \{(0)\} \cup \{(p), p \ \text{ prime}\}$
- $\mathrm{Spec}(\mathbf{R}) = \{(0)\}$
- $\mathrm{Spec}(\mathbf{C}[x]) = \{(0)\} \cup \{(x - a), a \in \mathbf{C}\}$

Exercise 17. We first show that the principal opens form a basis for the Zariski topology. If $U = X \setminus V(\mathfrak{a})$ is an open subset then for any $f \in \mathfrak{a}$, $X_f \subset U$. Then,

$$\bigcup_{f \in \mathfrak{a}} X_f = X \setminus \left(\bigcap_{f \in \mathfrak{a}} V(f) \right) = X \setminus \left(V\left(\bigcup_{f \in \mathfrak{a}} (f) \right) \right) = X \setminus V(\mathfrak{a}) = U.$$

which shows that the $\left(X_f\right)_{f \in \mathfrak{a}}$ form a basis for the Zariski topology.

i) $X_f \cap X_g = X \setminus (V(f) \cup V(g)) = X \setminus V((fg)) = X_{fg}$

ii) $X_f = \emptyset \Longleftrightarrow V(f) = X \Longleftrightarrow f \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p} = \mathfrak{N}(A)$

iii) $X_f = X \iff V(f) = 0 \iff f$ is a unit (otherwise (f) is a proper ideal contained in a prime maximal ideal).
iv) $X_f = X_g \iff V(f) = V(g) \iff r((f)) = r((g))$
v) Every open covering of X can be reduced to an open covering by basic open sets $X_f, f \in I \subset A$ (cover each open with basic open sets). We get $X = \bigcup_{f \in I} X_f$ whence

$$\emptyset = \bigcap_{f \in I} V(f) = V\left(\bigcup_{f \in I} (f)\right) = V(I)$$

thus $r(I) = (1)$ and $I = (1)$. Thus, there is a finite relationship

$$1 = \sum_{i=1}^{n} g_i f_i$$

with $f_i \in I$. Thus, $X = \bigcup_{i=1}^{n} X_{f_i}$ which concludes.
vi) $X_f = \bigcup_{f' \in I} X_{f'}$ yields $V(f) = \bigcap_{f' \in I} V(f') = V(I)$ whence $r((f)) = r(I)$ and $f^n = \sum_{f' \in J} g_{f'} f'$ for J a finite subset of I. The rest is the same as before since $f^n \in \mathfrak{p} \iff f \in \mathfrak{p}$ for any prime ideal $\mathfrak{p}$.
vii) If U is quasi compact then since U has an open cover of basic open sets, then it is a finite union of X_f. Conversely, if it is a finite union of X_f, $f \in I$, and $\{U_j\}_{j \in J}$ is another open cover, then each $\{U_j \cap X_f\}_{j \in J}$ is an open cover of X_f which is quasi compact. Exctract the indices for a finite covering to yield a finite open covering of U from the $\{U_j\}$.

Exercise 18.
i) If $\mathfrak{p}_x$ is maximal, then indeed $\{x\} = V(\mathfrak{p}_x)$ which is closed. Conversely, if $\{x\}$ is closed, then there is no ideal $\mathfrak{a}$ such that $\mathfrak{p}_x \subset \mathfrak{a}$, meaning $\mathfrak{p}_x$ is maximal.
ii)
$$\overline{\{x\}} = \bigcap_{\substack{Y \text{ closed} \\ x \in Y}} Y = \bigcap_{\substack{f \in A \\ \mathfrak{p}_x \subset r(f)}} V(f) = V\left(\bigcup_{\substack{f \in A \\ \mathfrak{p}_x \in r(f)}} (f)\right) = V(\mathfrak{p}_x)$$

iii) $y \in \overline{\{x\}} \iff y \in V(\mathfrak{p}_x) \iff \mathfrak{p}_x \subset \mathfrak{p}_y$
iv) From previous point, either $X \setminus \overline{\{x\}}$ or $X \setminus \overline{\{y\}}$ works.

Exercise 19. Assume $\mathfrak{N}(A)$ is not prime, i.e. there exists $a, b \in A \setminus \mathfrak{N}(A)$ such that ab is nilpotent. Then $X_a \cap X_b = X \setminus (V(a) \cup V(b)) = X \setminus V(ab)$, whence $X_a \cap X_b = \emptyset$ since $X = V(\mathfrak{N}(A)) \subset V(ab)$ (the nilradical is contained in every prime ideal, and $ab \in \mathfrak{N}(A)$). Note also that neither X_a nor X_b is empty, since a and b are not nilpotent. Thus, $\operatorname{Spec}(A)$ is not irreducible.

Assume now that the nilradical is prime, and that X_f, X_g are two basic open sets with empty intersection: $X_f \cap X_g = \emptyset$. Thus, $V(f) \cup V(g) = V(fg) = X$. In particular, $fg \in \mathfrak{N}(A)$ and since that ideal is prime, either f or g is nilpotent, which implies that one of X_f and X_g is empty. Therefore, $\operatorname{Spec}(A)$ is indeed irreducible.

Exercise 20.
i) Open subsets of $\overline{Y}$ are also open in Y, thus dense in Y, thus dense in $\overline{Y}$.
ii) Apply Zorn's lemma. To find a maximal element of a chain, take the closure of the union of the terms.
iii) From i) the maximal irreducible subspaces are necessarily closed. Then, every point of X is contained in the irreducible subspace $\overline{\{x\}}$ and therefore in a maximal irreducible subspace. This shows that maximal irreducible subspaces cover X. In a Hausdorff space, since any two points can be separated by neighborhoods, the irreducible components are the singletons.

iv) In $X = \mathrm{Spec}(A)$, candidates are closed thus of the form $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Since $V(-)$ is inclusion-reversing, it will be sufficient to show that whenever $\mathfrak{p}$ is prime, $V(\mathfrak{p})$ is irreducible (maximality will automatically ensue for minimal primes). Let $\mathfrak{p}$ be such a prime and assume $V(\mathfrak{p})$ is not irreducible, that is, there are nonempty open subspaces U, V of $V(\mathfrak{p})$ with empty intersection: $U \cap V = \emptyset$. We can write $U = V(\mathfrak{p}) \setminus V(\mathfrak{a})$ and $V = V(\mathfrak{p}) \setminus V(\mathfrak{b})$ for some ideals $\mathfrak{a}, \mathfrak{b}$ containing $\mathfrak{p}$ and we get $U \cap V = \emptyset = V(\mathfrak{p}) \setminus (V(\mathfrak{a}) \cup V(\mathfrak{b})) = V(\mathfrak{p}) \setminus V(\mathfrak{a}\mathfrak{b})$, whence $V(\mathfrak{p}) \subset V(\mathfrak{a}\mathfrak{b})$ and $r(\mathfrak{a}\mathfrak{b}) \subset \mathfrak{p}$. Since $\mathfrak{p} \subset \mathfrak{a}, \mathfrak{b}$, we get $\mathfrak{p} \subset r(\mathfrak{a}\mathfrak{b}) \subset \mathfrak{p}$ thus $\mathfrak{p} = r(\mathfrak{a}\mathfrak{b}) \supseteq \mathfrak{a}\mathfrak{b}$, which implies $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$ since $\mathfrak{p}$ is prime. Since we started with ideals containing $\mathfrak{p}$, this means either $\mathfrak{a} = \mathfrak{p}$ or $\mathfrak{b} = \mathfrak{p}$, which contradicts the nonemptyness of U and V.

Note that this shows two things: irreducible components are of the form $V(\mathfrak{p})$ for $\mathfrak{p}$ a minimal prime, and $V(\mathfrak{p})$ is always irreducible regardless of minimality, provided $\mathfrak{p}$ is prime.

Exercise 21. $\phi : A \longrightarrow B$ a ring homomorphism, $\mathfrak{q} \subset Y$ a prime ideal. Assume $ab \in \phi^{-1}(\mathfrak{q})$, then $\phi(a)\phi(b) \in \mathfrak{q}$ so $\phi(a) \in \mathfrak{q}$ or $\phi(b) \in \mathfrak{q}$ and thus a or b is in $\phi^{-1}(\mathfrak{q})$, thus $\phi^{-1}(\mathfrak{q})$ is prime. Define $\phi^* : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ as $\mathfrak{q} \mapsto \phi^{-1}(\mathfrak{q})$. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$.

i) If $f \in A$ then

$$\begin{aligned}\phi^{*-1}(X_f) &= \{y \in Y \mid \phi^*(\mathfrak{p}_y) \in X_f\} \\ &= \{y \in Y \mid f \notin \phi^{-1}(\mathfrak{p}_y)\} \\ &= \{y \in Y \mid \phi(f) \notin \mathfrak{p}_y\} = X_{\phi(f)}\end{aligned}$$

Preimages of open subsets are open subsets, making ϕ^* continuous.

ii) If $\mathfrak{a}$ is an ideal of A then

$$\begin{aligned}\phi^{*-1}(V(\mathfrak{a})) &= \{y \in Y \mid \mathfrak{a} \subset \phi^{-1}(\mathfrak{p}_y)\} \\ &= \{y \in Y \mid \phi(\mathfrak{a}) \subset \mathfrak{p}_y\} \mid \\ &= \{y \in Y \mid B\phi(\mathfrak{a}) \subset \mathfrak{p}_y\} \\ &= \{y \in Y \mid \mathfrak{a}^e \subset \mathfrak{p}_y\} = V(\mathfrak{a}^e)\end{aligned}$$

iii) Let $\mathfrak{b}$ be an ideal of B.

$$\phi^*(V(\mathfrak{b})) = \{x \in X \mid \mathfrak{b} \subset \phi(\mathfrak{p}_x)\} \subseteq \{x \in X \mid \phi^{-1}(\mathfrak{b}) \subset \mathfrak{p}_x\} = V(\mathfrak{b}^c).$$

By closedness of $V(\mathfrak{b}^c)$, $\overline{\phi^*(V(\mathfrak{b}))} \subset V(\mathfrak{b}^c)$. There is an ideal $\mathfrak{a}$ of A such that $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{a})$ and

$$V(\mathfrak{a}^e) = \phi^{*-1}(V(\mathfrak{a})) = \phi^{*-1}\left(\overline{\phi^*(V(\mathfrak{b}))}\right) \supseteq V(\mathfrak{b})$$

so $\mathfrak{a}^e \subset r(\mathfrak{b})$ and $\mathfrak{a} \subset r(\mathfrak{b}^c)$ whence $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{a}) \supseteq V(\mathfrak{b})$.

iv) If ϕ is surjective, then it factors into an isomorphism $\tilde{\phi} : A/\ker(\phi) \longrightarrow B$ which has an inverse $\tilde{\psi} : B \longrightarrow A/\ker(\phi)$. Then clearly, $\tilde{\phi}^*$ is a homeomorphism (of continuous inverse $\tilde{\psi}^*$) of Y onto $\mathrm{Spec}(A/\ker(\phi))$.

There is a one-to-one correspondance between ideals of $A/\ker(\phi)$ and ideals of A containing $\ker(\phi)$. Let π be the quotient map, then π^* is a continuous bijection $\mathrm{Spec}(A/\ker(\phi)) \longrightarrow V(\ker(\phi))$ with inverse

$$\begin{aligned}\pi' : V(\ker(\phi)) &\longrightarrow \mathrm{Spec}(A/\ker(\phi)) \\ \mathfrak{p} &\longmapsto \pi(\mathfrak{p}).\end{aligned}$$

It only remains to show that this map is continuous too. Since it is bijective, we only need to show that it is closed. Let $\mathfrak{a}$ be an ideal in A containing $\ker(\phi)$, i.e. $V(\mathfrak{a})$ is a closed subspace of $V(\ker(\phi))$. Then

$$\pi'(V(\mathfrak{a})) = \{\pi(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(A), \mathfrak{a} \subset \mathfrak{p}\} = \{\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec}(A/\ker(\phi)), \mathfrak{a} \subset \pi^{-1}(\mathfrak{p})\} = V(\pi(\mathfrak{a}))$$

which shows that the map is closed, thus a homeomorphism.

v) $\phi^*(Y)$ is dense $\iff V(0^c) = V(0) \iff V(\ker(\phi)) = V(0) \iff \ker(\phi) \subset \mathfrak{N}(A)$

vi) Apply the definitions

vii) Note that $\operatorname{Spec}(B) = \{(0) \times (1), (1) \times (0)\}$ (the only two other ideals are the zero ideal and B itself, both of which are not prime) and $\operatorname{Spec}(A) = \{0, p\}$. We have $\phi^{-1}((0) \times (1)) = \mathfrak{p}$ and $\phi^{-1}((1) \times (0)) = 0$, so ϕ^* is bijective. However, it is not closed since $\phi^*(V((1) \times (0))) = \{0\}$ and $\{0\}$ is not a closed point of $\operatorname{Spec}(A)$ (it is a generic point).

Exercise 22. $A = \prod_{i=1}^n A_i$, p_i the projections on each A_i and $\mathfrak{p}$ a prime ideal of A. Naturally, for all i, $p_i(\mathfrak{p})$ is either prime in A_i or is A_i itself (the primality condition is verified, but the ideal may not be proper). Assume there is $i < j$ such that both $p_{i(\mathfrak{p})}$ and $p_{j(\mathfrak{p})}$ are primes, which we denote by $\mathfrak{p}_i$ and $\mathfrak{p}_j$ respectively. Without loss of generality, assume $i = 1, j = 2$. Then for $a \in \mathfrak{p}_1$, $b \in \mathfrak{p}_2$, we have $(1, b, 1, \cdots) \cdot (a, 1, 1, \cdots) = (a, b, 1, \cdots) \in \mathfrak{p}$ but $(a, 1, 1, \cdots)$ and $(1, b, 1, \cdots)$ are both not in $\mathfrak{p}$, which contradicts the primality of $\mathfrak{p}$. Therefore, $\mathfrak{p}$ is of the form

$$\mathfrak{p} = (1) \times \cdots \times (1) \times \mathfrak{p}_i \times (1) \times \cdots \times (1),$$

and one verifies easily that this is indeed a prime ideal. This shows that $\operatorname{Spec}(A) = \coprod_{i=1}^n X_i$ where

$$X_i = A_1 \times \cdots \times A_{i-1} \times \operatorname{Spec}(A_i) \times A_{i+1} \times \cdots \times A_n.$$

The X_i are evidently canonically homeomorphic to $\operatorname{Spec}(A_i)$ via p_i (continuous bijective and closed).

Now let A be any ring.

- i) $\implies$ iii) Assume $X = \operatorname{Spec}(A)$ is disconnected, i.e. there are two nonempty disjoint open subsets U, V covering X. Mechanically, U and V are also closed and of the form $V(\mathfrak{a})$, $V(\mathfrak{b})$ for some ideals $\mathfrak{a}, \mathfrak{b}$ of A. We have $V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b}) = \emptyset$ so $\mathfrak{a} + \mathfrak{b} = (1)$, and $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = X$ so $r(\mathfrak{a}\mathfrak{b}) = \mathfrak{N}(A)$ and $\mathfrak{a}\mathfrak{b} \subset \mathfrak{N}(A)$. Consider $a \in \mathfrak{a}, b \in \mathfrak{b}$ such that $a + b = 1$. Then ab is nilpotent of cancelling order $n > 0$. We have

$$1 = (a+b)^{2n} = \underbrace{\sum_{k=1}^{n-1} \binom{2n}{k} a^k b^{2n-k}}_{s_1} + \binom{2n}{n} \underbrace{a^n b^n}_{=0} + \underbrace{\sum_{k=n+1}^{2n} \binom{2n}{k} a^k b^{2n-k}}_{s_2}.$$

  We found two elements s_1, s_2 such that: $s_1 + s_2 = 1$ and $s_1 s_2 = 0$ (all the terms have $a^n b^n = 0$ as a factor). Therefore, they are roots to $X^2 - X$ and at least one of them is nonzero (since $s_1 + s_2 = 1$) and not 1 since it is in either $\mathfrak{a}$ or $\mathfrak{b}$ which are proper ideals.

- iii) $\implies$ ii) Let e be a nontrivial idempotent. We shall show that the canonical map

$$\begin{aligned} \varphi : A &\longrightarrow A/eA \times A/(1-e)A \\ x &\longmapsto (x \bmod eA, x \bmod (1-e)A) \end{aligned}$$

  is an isomorphism. If $\varphi(x) = 0$ then $x = es = (1-e)t$ so $ex = e^2 s = es = x = e(1-e)t = 0$ whence $x = 0$ and φ is injective. Then, if $\left(\overline{a}, \overline{b}\right)$ is in the product above, then take $x = (1-e)a + eb$ so that $\varphi(x) = \left(\overline{(1-e)a}, \overline{eb}\right) = \left(\overline{a}, \overline{b}\right)$. Therefore, φ is an isomorphism.

- ii) $\implies$ i) This was done above at the start of the exercise.

Exercise 23.

i) For each f, X_f is open. Let $g = 1 - f$. We have $V(g) \cap V(f) = \emptyset$ since $s + f = 1$, and $V(g) \cup V(f) = V(gf) = V(0) = X$, whence $V(g)$ and $V(f)$ are complements, and $X_f = V(g)$ is closed.

ii) $X_{f_1} \cup \cdots \cup X_{f_n} = X \setminus V((f_1, \cdots, f_n)) = X \setminus V(f)$ since every finite type ideal is principal in A.

iii) $Y \subseteq X$ clopen, $Y = \cup_f X_f$, Y is closed in X which is quasi-compact, so Y is quasi compact and Y a finite union of X_f, and that union is again of the form X_f as per the previous point.

iv) We only need to show that X is Hausdorff. Let $x, y \in X$ be distinct points, wlog $\mathfrak{p}_x \not\subset \mathfrak{p}_y$ and there is $f \in \mathfrak{p}_y \setminus \mathfrak{p}_x$ so that $V(f)$ and X_f are both opens and they separate x and y.

Exercise 26. There is no problem *per se*. The book shows that if X is compact Hausdorff, then $X \simeq \mathrm{Max}(C(X))$ where $C(X)$ is the ring of continuous functions $X \longrightarrow \mathbf{R}$.

Exercise 27. Once again, there is no problem *per se*. Let k be algebraically closed and let I be an ideal of $k[t_1, \cdots, t_n]$. The set of points $x \in k^n$ such that $f(x) = 0$ for all $f \in I$ is called *algebraic affine variety*, which we denote by X. Let $I(X)$ be the ideal of $k[t_1, \cdots, t_n]$ consisting of zero everywhere polynomials (the kernel of the map $k[t_1, \cdots, t_n] \longrightarrow \{X \longrightarrow k\}$). The quotient ring $P(X) = k[t_1, \cdots, t_n]/I(X)$ is called the ring of polynomial functions on X.

The image ξ_i of t_i in $P(X)$ is called the i-th coordinate function, and together they generate $P(X)$ as a k-algebra, hence why $P(X)$ is also called the coordinate ring of X.

For $x \in X$, the ideal $\mathfrak{m}_x$ of all functions $f \in P(X)$ such that $f(x) = 0$ is a maximal ideal of $P(X)$, so that if $\tilde{X} = \mathrm{Max}(P(X))$, there is a canonical map $\mu : X \longrightarrow \tilde{X}, x \mapsto \mathfrak{m}_x$. This map is bijective! Showing this property yields Hilbert's Nullstellensatz: there is a one-to-one correspondance between maximal ideals of $P(X)$ and solutions to $\{f(x) = 0, f \in I(X)\}$

Exercise 28. Let φ be the map

$$\begin{aligned} \varphi : \{X \longrightarrow Y \text{ regular}\} &\longrightarrow \{P(Y) \longrightarrow P(X) \ k\text{-algebra morphism}\} \\ \phi &\longmapsto (\eta \mapsto \eta \circ \phi) \end{aligned}$$

- *Injectivity.* Let ϕ, ϕ' be regular mappings $X \longrightarrow Y$ such that $\varphi(\phi) = \varphi(\phi')$, that is, for all $\eta \in P(Y)$, $\eta \circ \phi = \eta \circ \phi'$. In particular, taking $\eta = \xi_i$ for each $1 \leq i \leq m$ shows that ϕ and ϕ' share all of their coordinates on X, whence $\phi = \phi'$.
- *Surjectivity.* Let $f : P(Y) \longrightarrow P(X)$ be a k-algebra morphism. Define

$$\phi = (f(\xi_1), \cdots, f(\xi_m))$$

where the ξ_i are the coordinate functions in $P(Y)$. For each $1 \leq i \leq m$, we have

$$\xi_i \circ \phi = f(\xi_i),$$

which implies that for all $\eta \in P(Y)$, $\eta \circ \phi = f(\eta)$ (since $P(Y)$ is generated as a k-algebra by the ξ_i, and f is a k-algebra morphism) and thus $\varphi(\phi) = f$. The only remaining thing is to show that ϕ is regular, which directly comes from the fact that each coordinate ϕ_i of ϕ is an element of $P(X)$ which is the ring of polynomial functions. To realise ϕ as a restriction of a polynomial mapping, one only has to choose an element of $\pi^{-1}(\phi_i) = \lambda + I(X)$ where $\pi : k[t_1, \cdots, t_n] \longrightarrow P(X)$ is the canonical quotient map and λ is any polynomial in $\pi^{-1}(\phi_i)$ (nonempty by surjectivity of π).

Chapter 2 — Modules

Exercise 1. There are u, v such that $un + vm = 1$. Let's compute simple tensors:

$$x \otimes y = (un + vm)(x \otimes y) = x \otimes uny + vmx \otimes y = x \otimes 0 + 0 \otimes y = 0.$$

Exercise 2. We have a short exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0.$$

The tensor product is right-exact, so we get the exact sequence

$$\mathfrak{a} \otimes_A M \longrightarrow A \otimes_A M \longrightarrow A/a \otimes_A M \longrightarrow 0$$

which is equivalent to

$$\mathfrak{a}M \longrightarrow M \longrightarrow A/\mathfrak{a} \otimes_A M \longrightarrow 0.$$

We immediately get an isomorphism $M/\mathfrak{a}M \simeq A/A \otimes_A M$.

Exercise 3. A a local ring, M and N finitely generated A-modules, $M \otimes_A N = 0$. Let $\mathfrak{m}$ be the maximal ideal of A and $k = A/\mathfrak{m}$ the residue field. By Exercise 2, we have $M_k = k \otimes M \simeq M/\mathfrak{m}M$ and by Nakayama's lemma, if $M_k = 0$ then $M = 0$. We have

$$M \otimes_A N = 0 \Longrightarrow (M \otimes_A N)_k = 0 \Longrightarrow M_k \otimes_k N_k = 0 \Longrightarrow M_k = 0 \quad \text{or} \quad N_k = 0$$

since M_k and N_k are k-vector spaces.

Note that $(M \otimes_A N)_k = (M \otimes_A k) \otimes_A (N \otimes_A k) = M_k \otimes_A N_k$ and that $M_k \otimes_a N_k \simeq M_k \otimes_k N_k$ (canonically with the obvious map).

Exercise 4. Let $M = \bigoplus_{i \in I} M_i$ where $M_i, i \in I$ is a family of A-modules. If each M_i is flat, there is M_i' such that $M_i \oplus M_i'$ is free, and $M \oplus M'$ is free with $M' = \bigoplus_i M_i'$, making M flat. Assume now that M is flat, and consider a short exact sequence

$$0 \longrightarrow B \xrightarrow{f} C \longrightarrow D \longrightarrow 0.$$

Since M is flat, we know that $f \otimes_A \mathrm{id}_M : B \otimes_A M \longrightarrow C \otimes_A M$ is injective. Write

$$B \otimes_A M = \bigoplus_i (B \otimes_A M_i)$$

and assume there is $i \in I$ such that $B \otimes_A M_i \longrightarrow C \otimes_A M_i$ is not injective, i.e. there is a nonzero tensor $t \in B \otimes_A M_i$ such that $\left(f \otimes_A \mathrm{id}_{M_i}\right)(t) = 0$. Denote by $\iota_i : M_i \longrightarrow M$ the inclusion map. We get $(f \otimes_A \mathrm{id}_M)((\mathrm{id}_B \otimes_A \iota_i)(t)) = 0$ (this is because $\mathrm{id}_M = \bigoplus_j \iota_j$ whence $f \otimes_A \mathrm{id}_M = \bigoplus_j f \otimes_A \iota_i$). Evidently, $(\mathrm{id}_B \otimes_A \iota_i)(t) \neq 0$ which contradicts the injectivity of $f \otimes_A \mathrm{id}_M$. Thus, every map $B \otimes_A M_i \longrightarrow C \otimes_A M_i$ is injective and each M_i is flat.

Exercise 5. Write $A[x] = \oplus_{n \geq 0} x^n A$ and $\forall n \geq 0, x^n A \simeq A$ as an A-module. Exercise 4 concludes.

Exercise 6. $M[x]$ is an $A[x]$-module (verify each axiom). Write $A_i = x^i A$ and $M_i = A_i \otimes_A M = x^i M$ so that $A[x] = \bigoplus_i A_i$ and $M = \bigoplus_i M_i$. We get

$$A[x] \otimes_A M = \bigoplus_i (A_i \otimes_A M) = \bigoplus_i M_i = M.$$

Exercise 7. Let $\mathfrak{p}$ be a prime ideal in A, and let $f(x), g(x)$ be elements of $A[x]$ such that

$$
\begin{aligned}
f(x) &= a_0 + a_1 x + \cdots + a_n x^n \\
g(x) &= b_0 + b_1 x + \cdots + b_m x^m.
\end{aligned}
$$

Assume neither $f(x)$ nor $g(x)$ belongs to $\mathfrak{p}[x]$. Let $i \leq n$ be the minimal index such that $a_i \notin \mathfrak{p}$ and $j \leq m$ be the minimal index such that $b_j \notin \mathfrak{p}$. The coefficient of $f(x)g(x)$ in degree $i+j$ is

$$
\sum_{k=0}^{i+j} a_k b_{i+j-k} = \sum_{k=0}^{i-1} \underbrace{a_k}_{\in \mathfrak{p}} b_{i+j-k} + \sum_{k=0}^{j-1} a_{i+j-k} \underbrace{b_k}_{\in \mathfrak{p}} + a_i b_j \notin \mathfrak{p},
$$

thus $f(x)g(x) \notin \mathfrak{p}[x]$, and $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. Let $\mathfrak{m}$ be a maximal ideal of $A[x]$. The ideal $\mathfrak{m}[x] + (x)$ is a bigger proper ideal, whence $\mathfrak{m}[x]$ is not maximal.

Exercise 8.

i) $B \longrightarrow C$ injective, $M \otimes B \longrightarrow M \otimes C$ injective, $M \otimes N \otimes B \longrightarrow M \otimes N \otimes C$ injective.

ii) Let $j : M \longrightarrow M'$ be an injective morphism of A-modules. The map $j \otimes_A \mathrm{id}_B$ is an injective map of A-modules between B-modules (via extension of scalars), and $(j \otimes_A \mathrm{id}_B) \otimes_B \mathrm{id}_N$ is an injective morphism of B-modules. Since

$$
(M \otimes_A B) \otimes_B N = M \otimes_A (B \otimes_B N) = M \otimes_A N
$$

per Exercise 2.15 (in the notes), and the same goes for M', we found that the map $j \otimes_A N$ is injective, thus N is flat.

Exercise 9. Consider the short exact sequence of A-modules

$$
0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.
$$

Assume M' and M'' are finitely generated, that is, there are epimorphisms $\alpha : A^n \longrightarrow M'$ and $\beta : A^m \longrightarrow M''$. We get the following diagram, with exact rows and columns

$$
\begin{tikzcd}
0 \arrow[r] & A^n \arrow[r, "{(\mathrm{id},0)}"] \arrow[d, "\alpha"'] & A^n \times A^m \arrow[r, "{(0,\mathrm{id})}"] \arrow[d, dashed, "\exists ?\, \gamma"'] & A^m \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"'] \arrow[d] & M \arrow[r, "g"'] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

To build γ, we'll define it on A^n and on A^m. Choose $\gamma|_{A^n} = f \circ \alpha$. Choose y_i an element of $g^{-1}(\beta(e_i))$ for each $1 \leq i \leq m$, where $(e_i)_i$ is the canonical basis of A^m. Then, set $\gamma|_{A^m} : e_i \mapsto y_i$.

It only remains to show that this map is indeed surjective. Write $M = \bigcup_{y \in M''} g^{-1}(\{y\})$ and if $x \in g^{-1}(\{y\})$ then $g^{-1}(\{y\}) = x + \ker(g) = x + \mathrm{im}(f) = x + \mathrm{im}(\gamma|_{A^n})$ and for all $x' \in g^{-1}(\{y\})$ we have $x' - x \in \mathrm{im}(\gamma|_{A^n})$ so there is $a \in A^n$ such that $x' - x = \gamma|_{A^n}(a)$ and there is $b \in A^m$ such that $\beta(b) = g(x)$ so that $x = \gamma|_{A^m}(b)$. Finally, we get $\gamma(a,b) = \gamma(a,0) + \gamma(0,b) = x' - x + x = x'$. Done!

Exercise 10. Let A be a ring and $\mathfrak{a} \subset \mathfrak{R}$ an ideal; M an A-module and N a finitely generated A-module. Let $u : M \longrightarrow N$ be a morphism such that the induced map $M/\mathfrak{a}M \longrightarrow N/\mathfrak{a}N$ is surjective.

The map $M \longrightarrow N/\mathfrak{a}N$ sends m to $\overline{u(m)} \in N/\mathfrak{a}N$ and is surjective (composition of surjective maps), thus $\overline{u(M)} = N/\mathfrak{a}N$ and $u(M) + \mathfrak{a}N = N$. By Nakayama's lemma (Corollary 2.7) we have $u(M) = N$, hence the surjectivity of u.

Exercise 11. Let A be a nontrivial ring and let m, n be integers such that $A^m \overset{\phi}{\simeq} A^n$. Let $\mathfrak{m}$ be a maximal ideal of A. Then $\mathrm{id}_{A/\mathfrak{m}} \otimes_A \phi$ is an isomorphism between $A/\mathfrak{m}$-vector spaces, and equality of dimension yields $m = n$. To see that this map is indeed an isomorphism, apply the right-exact tensor to the exact sequence

$$0 \longrightarrow A^m \overset{\phi}{\longrightarrow} A^n \longrightarrow 0.$$

- Suppose now that $\phi : A^m \longrightarrow A^n$ is only surjective. Using the short exact sequence

  $$0 \longrightarrow A^m/\ker(\phi) \longrightarrow A^m \longrightarrow A^n \longrightarrow 0,$$

  and using the right-exactness of $(A/\mathfrak{m}) \otimes_A -$, we get surjectivity of the k-vector space morphism $\mathrm{id}_k \otimes_A \phi$ where $k = A/\mathfrak{m}$.
- Now assume that the map $\phi : A^m \longrightarrow A^n$ is only injective. If $m > n$ the morphism ϕ can be seen as an injective endomorphism $\phi : A^m \longrightarrow A^m = A^n \oplus A^{m-n}$. Since A^m is finitely generated, we have a relationship

  $$\phi^r + a_1\phi^{r-1} + \cdots + a_r = 0$$

  for some $r > 0$. Since ϕ is injective, it is left-regular, thus if r is taken to be minimal we may assume $a_r \neq 0$ (otherwise LHS is of the form $\phi \circ P(\phi) = \phi \circ 0 \Longrightarrow P(\phi) = 0$ with $\deg P < r$, contradiction). We have $\mathrm{im}(\phi) = A^n \subsetneq A^n \oplus A^{m-n}$ and thus $\forall k > 0, \mathrm{im}(\phi^k) \subset A^n$. Take $x = (0, \cdots, 0, 1) \in A^n \oplus A^{m-n}$, $x \neq 0$. We have

  $$\phi^r(x) + \cdots + a_{r-1}\phi(x) + a_r x = 0$$

  and projecting this relation the last coordinate of A^m yields $a_r = 0$, contradiciton. Thus, $m \leq n$.

Exercise 12. Let M be a finitely generated A-module and $\phi : M \longrightarrow A^n$ be a surjective morphism. Let $e_1, \cdots, e_n$ be the canonical basis of A^n and choose $u_i \in \phi^{-1}(e_i)$ for each $1 \leq i \leq n$.

Define $\psi : A^n \longrightarrow M$ by $e_i \mapsto u_i$. Set $m \in M$. We have

$$\phi(m) = a_1e_1 + \cdots + a_ne_n$$

so $\psi(\phi(m)) = a_1u_1 + \cdots + a_nu_n$. Clearly, since $\phi \circ \psi = \mathrm{id}_{A^n}$, then $m - \psi(\phi(m)) \in \ker(\phi)$. The decomposition $m = (m - \psi(\phi(m))) + \psi(\phi(m))$ shows that $M = N + \ker(\phi)$ where N is the submodule generated by $u_1, \cdots, u_n$.

To show that the sum is direct, we merely need to show $\ker(\phi) \cap N = 0$, which is true since $m \in \ker(\phi) \cap N$ implies $m = a_1'u_1 + \cdots + a_n'u_n$ and $\phi(m) = 0 = a_1'e_1 + \cdots + a_n'e_n$ thus $a_i' = 0$ for all $1 \leq i \leq n$ since $e_1, \cdots, e_n$ is a basis, and $m = 0$.

To conclude, note that since M is finitely generated, there is a surjective morphism $A^m \longrightarrow M = N \oplus \ker(\phi)$ thus there is also a surjective mohpsism $A^m \longrightarrow \ker(\phi) = M/N$ which shows $\ker(\phi)$ is finitely generated.

Exercise 13. Let $f : A \longrightarrow B$ be a ring homomorphism and let N be a B-module seen as an A -module through restriction of scalars. Let $N_B = B \otimes_A N$, which is a B-module. Let $g : N \longrightarrow N_B$ be the morphism $y \mapsto 1 \otimes y$.

Define $p : N_B \longrightarrow N$ by $b \otimes y \mapsto by$ so that $p \circ g = \mathrm{id}_N$. This implies g is injective. Then, write for $y \in N_B$, $y = (y - g(p(y))) + g(p(y))$, so that the first term $y - g(p(y)) \in \ker(p)$. Thus, we have $N_B = \mathrm{im}(g) + \ker(p)$. To see that this sum is direct, take $y \in \ker(p) \cap \mathrm{im}(g)$ so that $y = g(x)$ and $p(y) = p(g(x)) = x = 0$ thus $y = 0$ and $\ker(p) \cap \mathrm{im}(g) = 0$.

Exercise 14. Nothing to do here. Let's break down the construction. $\mathbf{M} = \left(M_i, \mu_{ij}\right)$ a direct system over the directed set I. Let

$$C = \bigoplus_{i \in I} M_i$$

and D be the submodule generated by the $x - \mu_{ij}(x)$ when $i \leq j$. Taking the quotient C/D essentially comes down to taking the direct sum of the M_i but modulo the extra relation $x = \mu_{ij}(x)$ whenever $x \in M_i$. Note that when $x \in M_i$, $\mu_{ij}(x)$ is an element of M_j (where $j \geq i$). We are therefore glueing M_i and M_j together along $\mu_{ij}(M_i)$ with the map μ_{ij}. We define $M = C/D$ and we let $\mu : C \longrightarrow M$ be the canonical quotient map, with restrictions $\mu_i : M_i \longrightarrow M$.

Let's have a quick example. Let $I = \mathbf{N} \setminus \{0\}$ be the direct set ordered by divisibility ($n \leq m \iff n \mid m$, the LCM makes this a directed set). Let us consider the direct limit $M = \varinjlim_{n>0} \frac{1}{n}\mathbf{Z}$. The maps for the direct system are the inclusions $\frac{1}{n}\mathbf{Z} \hookrightarrow \frac{1}{m}\mathbf{Z}$ for $n \mid m$.

An element m of M is the class of an element in the direct sum of all the $\frac{1}{n}\mathbf{Z}$, meaning it is the class of a fraction of the form

$$x = \sum_{i=1}^{r} \frac{a_i}{i} = \frac{1}{\mathrm{lcm}(1, ..., r)} \sum_{i=1}^{r} a_i \underbrace{\frac{\mathrm{lcm}(1, ..., r)}{i}}_{\in \mathbf{Z}}$$

for some $r > 0$, whence $x \in \frac{1}{\mathrm{lcm}(1,...,r)}\mathbf{Z}$. In fact,

$$\varinjlim_{n>0} \frac{1}{n}\mathbf{Z} = \bigcup_{n>0} \frac{1}{n}\mathbf{Z} = \mathbf{Q}.$$

Exercise 15.

- Let x be an element of M, and choose $y \in \mu^{-1}(x)$. Since y lives in the direct sum $\bigoplus_{i \in I} M_i$ it can be written

  $$y = \sum_{j \in J} m_j$$

  for some finite subset $J \subset I$ and $m_j \in M_j$. Since I is direct, J has a maximal element i_J in I (quick induction), from which we get

  $$x = \mu(y) = \sum_{j \in J} \mu_j(m_j) = \sum_{j \in J} \mu_{i_J} \circ \mu_{ji_J}(m_j) = \mu_{i_J}\left(\sum_{j \in J} \mu_{ji_J}(m_j)\right).$$
- Now choose $i \in I$ and $x_i \in M_i$ such that $\mu_i(x_i) = 0$. Then x_i is in the class of $0 \in M_j$ for all $j \geq i$, meaning there is one $j \geq i$ such that $0 - \mu_{ij}(x_i) = 0$ whence the result.

Exercise 16. Let us first show that $M = \varinjlim_{i \in I} M_i$ does verify the property. Let N be an A -module and $\alpha_i : M_i \longrightarrow N$ be a collection of A-modules morphisms such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then naturally there exists a morphism $\tilde{\alpha} : \bigoplus_{i \in I} M_i \longrightarrow N$ which is the direct sum of all the α_i. Consider $i \leq j$ and $x \in M_i$. Then $\alpha\left(x - \mu_{ij}(x)\right) = \alpha_i(x) - \alpha_j\left(\mu_{ij}(x)\right) = \alpha_i(x) - \alpha_i(x)$. Therefore, the glueing submodule (which we referred to as D in previous exercises) is in the kernel of $\tilde{\alpha}$, which therefore induces a map $\alpha : M \longrightarrow N$. The identity $\alpha_i =$

$\alpha \circ \mu_i$ is immediate (remember: α is a factor of the direct sum of the α_i, and the other factor is $\bigoplus_i M_i \longrightarrow M$ which is the same factor as for μ_i). For uniqueness, note that we can take the direct sum on the source in the previous relationship to obtain $\left(\bigoplus_i \alpha_i\right) = \alpha \circ \mu$ and μ is a surjection, therefore it is right-invertible which leaves α to be uniquely defined.

To show universality we now take (M', μ_i') to be a module satisfying the property. We shall show $M \simeq M'$. Pick $N = M$ and $\alpha_i = \mu_i : M_i \longrightarrow M$. We have $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$ so we can apply the property, which yields a morphism $\alpha : M' \longrightarrow M$ such that $\alpha_i = \alpha \circ \mu_i'$. In other words, $\mu_i = \alpha \circ \mu_i'$, for all $i \in I$. Since every $x \in M$ can be written $x = \mu_i(x_i)$ (Exercise 15), then it can also be written $x = \alpha(\mu_i'(x_i))$ meaning α is surjective.

To conclude, we need injectivity of the morphism α we constructed. Note that the (yet-to-be) universal property claims unicity of such a morphism. This implies that $\bigoplus_i \mu_i'$ is surjective, as otherwise one could pick $x \in M' \setminus \left(\sum_i \operatorname{im}(\mu_i)\right)$ and define another morphism $\alpha' : M' \longrightarrow M$ satisfying the same relation but with $\alpha'(x) \neq \alpha(x)$. Therefore, every $x \in M'$ can be written as a (finite) sum of $\mu_j'(x_j)$ which, as we have seen in Exercise 15, with the relation $\mu_i' = \mu_j' \circ \mu_{ij}'$ for $j \geq i$, implies $x = \mu_i'(x_i)$ for some $i \in I$. Thus, if $x' \in \ker(\alpha)$, then write $x' = \mu_i'(x_i)$ so that $\alpha(x') = 0 = \alpha(\mu_i'(x_i)) = \alpha_i(x_i) = \mu_i(x_i)$. Then for some $j \geq i$, we have $\mu_{ij}(x_i) = 0$ (Exercise 15) and $\mu_i'(x_i) = \mu_j' \circ \mu_{ij}(x_i) = 0$, thus $x' = 0$ and the map α is an isomorphism.

Exercise 17. I is a directed set and $\left(M_i, \mu_{ij}\right)$ is a direct system over I. First, note that

$$\sum M_i = \bigcup M_i$$

in virtue of the fact that I is a directed system (if $x \in \sum M_i$ then $x = \sum_{j \in J} x_j$ for some finite subset J which has a maximal element i_J in I thus $x \in M_{i_J}$). It only remains to show that this union is the direct limit.

Let $\alpha_i : M_i \longrightarrow N$ be a collection of morphisms such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Since the μ_{ij} are inclusion maps, we get $\alpha_i = \alpha_j|_{M_i}$ whenever $M_i \subseteq M_j$. As such, we can define $\alpha : \varinjlim M_i \longrightarrow N$ as $x \mapsto \alpha_i(x_i)$ whenever $x = \mu_i(x_i)$ (which is always the case for some $i \in I$). This definition makes sense because if $x = \mu_i(x_i) = \mu_j(x_j)$, then there is k such that $i, j \leq k$ and in M_k (which contains both x_i and x_j), we have $\mu_k\left(x_i - x_j\right) = 0$ thus there is $k' \geq k$ such that $\mu_{kk'}\left(x_i - x_j\right) = 0 = x_i - x_j \in M_{k'}$, meaning $x_i = x_j$ (in the "big" containing module) and $\alpha_i(x_i) = \alpha_j\left(\mu_{ij}(x_i)\right) = \alpha_j(x_i) = \alpha_j\left(x_j\right)$. Moreover, surjectivity of $\bigoplus_i \mu_i : \bigoplus_i M_i \longrightarrow \varinjlim M_i$ once again proves that such a morphism with the relations $\alpha_i = \alpha \circ \mu_i$ is unique.

Therefore, the union $\bigcup M_i$ satisfies the universal property of $\varinjlim M_i$. For actual equality, notice that the isomorphism that arises from satisfying the universal property is the identity.

Exercise 18. Consider the maps $\nu_i \circ \phi_i : M_i \longrightarrow N$. They satisfy the hypothesis for the universal property, therefore they yield $\phi = \varinjlim \phi_i : M \longrightarrow N$ as requested.

$$\nu_i \circ \phi_i = \nu_j \circ \nu_{ij} \circ \phi_i = \nu_j \circ \phi_j \circ \mu_{ij}$$

Exercise 19. Let $\mathbf{M} \xrightarrow{f} \mathbf{N} \xrightarrow{g} \mathbf{P}$ be an exact sequence of direct systems. We have $f \circ \mu_i^{(M)} = \mu_i^{(N)} \circ f_i$ and $g \circ \mu_i^{(N)} = \mu_i^{(P)} \circ g_i$ thus for all $i \in I$,

$$g \circ f \circ \mu_i^{(M)} = g \circ \mu_i^{(N)} \circ f_i = \mu_i^{(P)} \circ g_i \circ f_i = \mu_i^{(P)} \circ 0 = 0,$$

which shows that $M \longrightarrow N \longrightarrow P$ is a sequence (i.e. $\operatorname{im} f \subset \ker g$). Now let x be an element of $\ker(g) \subset N$. Then x can be written $\mu_i^{(N)}(x_i)$ for some $i \in I$, $x_i \in N_i$. We have

$$g \circ \mu_i^{(N)}(x_i) = \mu_i^{(P)} \circ g_i(x_i) = 0$$

thus there is $j \geq i$ such that $\mu_{ij}^{(P)} \circ g_i(x_i) = 0$, i.e. $g_j \circ \mu_{ij}^{(N)}(x_i) = 0$, whence $\mu_{ij}^{(N)}(x_i) \in \ker(g_j)$. This means that x can be written

$$\mu_i^{(N)}(x_i) = \mu_j^{(N)} \circ \mu_{ij}^{(N)}(x_i) = \mu_j^{(N)}(x_j)$$

for $x_j = \mu_{ij}^{(N)}(x_i) \in \ker(g_j)$. However, we know $\ker(g_j) = \operatorname{im}(f_j)$, so $x_j = f_j(y_j)$ for some $y_j \in M_i$. We conclude with

$$x = \mu_j^{(N)}(x_j) = \mu_j^{(N)} \circ f_j(y_j) = f \circ \mu_j^{(M)}(y_j) = f(y)$$

for $y = \mu_j^{(M)}(y_j)$, which shows that $x \in \operatorname{im}(f)$ and therefore $\ker(g) \subset \operatorname{im}(f)$, which proves exactness of the sequence

$$M \longrightarrow N \longrightarrow P.$$

Exercise 20. Let $P = \varinjlim(M_i \otimes N)$ be the direct limit of $(M_i \otimes N, \mu_{ij} \otimes 1)$ and denote by $\mu_i^{(P)} : M_i \otimes N \longrightarrow P$ the maps associated to the direct limit.

- Let $g_i : M_i \times N \longrightarrow M_i \otimes N$ be the canonical mapping associated to the tensor product $M_i \otimes N$. Passing to the limit, we get a mapping $g : \varinjlim(M_i \times N) \longrightarrow P$. Canonically, $\varinjlim(M_i \times N) = M \times N$ (via the maps $(\mu_i, \mathrm{id}_N) : M_i \times N \longrightarrow M \times N$), so we get a map $g : M \times N \longrightarrow P$.
- Let's show that g is bilinear. Let $(m, n), (m', n) \in M \times N$ be two elements and $\lambda \in A$ be a scalar. There is $i \in I$ such that $m = \mu_i(m_i)$ for some $m_i \in M_i$ and $m' = \mu_i(m_i')$ for some $m_i' \in M_i$ (we can take the same i for both because I is directed). We have

  $$g((m + m', n)) = g((\mu_i, \mathrm{id}_N)(m_i + m_i', n)) = \mu_i^{(P)} \circ g_i((m + m', n))$$

  and both $\mu_i^{(P)}$ and g_i are linear, whence we get linearity in the first coordinate. The same steps show linearity for the second coordinate.
- By the universal property of the tensor product, g induces $\phi : M \otimes N \longrightarrow P$ such that $\phi(m \otimes n) = g(m, n)$ for all $(m, n) \in M \times N$.
- Let's compute $\phi \circ \psi$. Choose $p \in P$, we have $p = \mu_i^{(P)}(x_i)$ for some $i \in I$ and $x_i \in M_i \otimes P$. Thus,

  $$\phi \circ \psi(p) = \phi \circ \psi \circ \mu_i^{(P)}(x_i) = \phi \circ (\mu_i \otimes 1)(x_i).$$

  Since x_i is a finite sum of simple tensors and the relationship above is linear, proving that $\phi \circ \psi(p) = p$ when p comes from a simple tensor is enough. As such, we assume $x_i = m_i \otimes n \in M_i \otimes N$ (with $m_i \in M, n \in N$). We get

  $$\phi \circ \psi(p) = \phi(\mu_i(m_i) \otimes n) = g(\mu_i(m_i), n) = \mu_i^{(P)} \circ g_i(m_i, n) = \mu_i^{(P)}(m_i \otimes n) = p.$$

  We conclude $\phi \circ \psi = \mathrm{id}_P$.
- The same trick shows $\psi \circ \phi = \mathrm{id}_{M \otimes N}$.

This shows

$$\varinjlim(M_i \otimes N) \simeq \left(\varinjlim M_i\right) \otimes N.$$

Exercise 21. The maps $A_i \times A_i \longrightarrow A, (a, a') \mapsto \alpha_i(aa')$ induce a bilinear map $\varinjlim(A_i \times A_i) \longrightarrow A$, and canonically $\varinjlim(A_i \times A_i) = A \times A$. We get a product on A, we can easily check that it make A a ring. It also makes $\alpha_i : A_i \longrightarrow A$ into ring homomorphisms (verify directly with $a_i, a_i' \in A_i$ and the relation satisfied by the product map above that comes from passing to the limit).

If $A = 0$, then for any $i \in I$ we have $\alpha_i(1) = 0$ whence $\alpha_{ij}(1) = 0$ for some $j \geq i$, and since α_{ij} is a ring homomorphism, $A_j = 0$ (since $1 = 0$ in A_j).

Exercise 22. Note first that $\alpha_{ij}(\mathfrak{N}_i) \subset \mathfrak{N}_j$, whence indeed $\varinjlim \mathfrak{N}_i$ is well defined and $\mathfrak{N} = \varinjlim \mathfrak{N}_i \subset \varinjlim A_i = A$. If $x \in \mathfrak{N}_i$ then $x^n = 0$ for some $n > 0$ and $\mu_i^{(\mathfrak{N})}(x^n) = \mu_i^{(\mathfrak{N})}(x)^n = 0$ so all elements of $\mathfrak{N}$ are indeed nilpotent in A. Now let a be nilpotent in A. It can be written $a = \mu_i^{(A)}(x_i)$ for some $x_i \in A_i$. There is $n > 0$ such that $a^n = 0$ thus $\mu_i^{(A)}(x_i^n) = 0$ and there is $j \geq i$ such that $\mu_{ij}(x_i) \in \mathfrak{N}_j$ whence the nilradical of A is contained in $\varinjlim \mathfrak{N}_i$.

If $A = \varinjlim A_i$ is not integral, i.e. there are nonzero a, b such that $ab = 0$, then there is $i \in I$ such that $a = \alpha_i(a_i), b = \alpha_i(b_i)$ and $a_i b_i = 0$, and a_i, b_i are both nonzero since a, b are nonzero.

Exercise 23. The canonical maps are $\bigotimes_{j\in J} b_j \mapsto \bigotimes_{j\in J} b_j \otimes \bigotimes_{j\in J'\setminus J} 1$ (pick an ordering of J' to make sense of the notation).

Chapter 3 — Rings and Modules of Fractions

Exercise 1. Let $m_1, \cdots, m_n$ be generators of M as an A-module, S a multiplicatively closed subset of A.

- If $sM = 0$ for some $s \in S$ then obviously all fractions $m/s = 0$ are zero.
- If $S^{-1}M = 0$ then in particular there exist $s_1, \cdots, s_n \in S$ such that $s_i m_i = 0$ for all $1 \leq i \leq n$. Denote by s the product $s_1 \cdots s_n$ so that for all i, we have $sm_i = 0$ which in turn implies $sM = 0$.

Exercise 2. Let a/s be an element of $S^{-1}\mathfrak{a}$, and x/s' be an element of $S^{-1}A$. Then

$$1 - (a/s)(x/s') = (ss' - ax)/(ss')$$

and $ss' - ax \in S$ since $ss' \in 1 + \mathfrak{a}$, thus $ss' - ax$ is a unit in $S^{-1}A$. This shows $1 - (a/s)(x/s')$ is a unit for all x/s', whence a/s is in the Jacobson radical of $S^{-1}A$.

Assume now that M is finitely generated and $M = \mathfrak{a}M$ for some ideal $\mathfrak{a}$ of A. Then $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$ with $S^{-1}\mathfrak{a}$ is a subset of the Jacobson radical. By Nakayama's lemma, $S^{-1}M = 0$. By Exercise 1, there is $s \in S$ such that $sM = 0$, and we have $s \equiv 1 (\operatorname{mod} \mathfrak{a})$.

Exercise 3. The composite $\phi : A \longrightarrow S^{-1}A \longrightarrow U^{-1}(S^{-1}A)$ is such that:

- It sends every element of ST to a unit
- If $\phi(a) = 0$ then $ua = 0$ in $S^{-1}A$ for some $u \in U$, i.e. $(ta)/1 = 0$ for some $t \in T$ and $sta = 0$ in A for some $s \in S$, whence $ra = 0$ for some $r \in ST$.
- Elements of $U^{-1}(S^{-1}A)$ are of the form x/u for $u \in U$, write $u = t/1$ for some $t \in T$ and $x = a/s$ for some $s \in S$ to get $x/u = \phi(a)\phi(st)^{-1}$.

Corollary 3.2 shows that ϕ induces an isomorphism

$$(ST)^{-1}A \simeq U^{-1}(S^{-1}A).$$

Exercise 4. $S^{-1}B \longrightarrow T^{-1}B, x/s \mapsto x/f(s)$ is a well defined, a morphism, injective, surjective.

Exercise 5.

- Denote by $\mathfrak{N}$ the nilradical of A. For each prime ideal $\mathfrak{p}$, the nilradical of $A_\mathfrak{p}$ is $\mathfrak{N}_\mathfrak{p}$ (3.14) which is zero. By (3.8), $\mathfrak{N} = 0$.

- $A = \mathbf{Q} \times \mathbf{Q}$. The prime ideals of A are $\mathfrak{p} = 0 \times \mathbf{Q}$ and $\mathfrak{q} = \mathbf{Q} \times 0$. Note that as A-modules, $A = \mathfrak{p} \oplus \mathfrak{q}$. Let $S = A \setminus \mathfrak{p} = \mathbf{Q}^\times \times \mathbf{Q}$. We have $A_\mathfrak{p} = S^{-1}A = (S^{-1}\mathfrak{p}) \oplus (S^{-1}\mathfrak{q})$. Let's compute these two modules. We have $S^{-1}\mathfrak{q} = (\mathbf{Q}^\times)^{-1}\mathbf{Q} = \mathbf{Q}$ and $S^{-1}\mathfrak{p} = \mathbf{Q}^{-1}\mathbf{Q} = 0$. Thus, $A_\mathfrak{p} \simeq \mathbf{Q}$. Similarly, $A_\mathfrak{q} \simeq \mathbf{Q}$. This shows that being integral is not a local property (since $A_\mathfrak{p}$ is integral for each $\mathfrak{p} \in \mathrm{Spec}(A)$ but A is not integral).

Exercise 6. Apply Zorn's lemma to show that Σ has maximal elements (use the union as a maximal element for chains).

Let S be in Σ, $x, y \notin A \setminus S$, then $xy \in S$ therefore $xy \notin A \setminus S$, whence $A \setminus S$ is prime *if it is an ideal.* Assume now that S is maximal, i.e. for all $x \notin S$, we have $0 \in \{sx^n, s \in S, n \in \mathbf{N}\}$, i.e. $sx^n = 0$ for some $s \in S, n > 0$ and conversely if $sx^n = 0$ for some $s \in S, n > 0$ then $x \notin S$. Now take $a, b \in A \setminus S$. We have $sa^r = tb^s = 0$ for some $s, t \in S, r, s > 0$. We have $st(a+b)^{r+s} = 0$ thus $a + b \in A \setminus S$. Similarly, if $x \in A, y \in A \setminus S$, we have $sy^n = 0$ for some s, n, thus $s(xy)^n = 0$ and $xy \in A \setminus S$. If $\mathfrak{p} \subset A \setminus S$ is another prime ideal, then clearly $A \setminus \mathfrak{p}$ is a superset of S that belongs to Σ, whence $A \setminus \mathfrak{p} = S$ i.e. $\mathfrak{p} = A \setminus S$, and $A \setminus S$ is minimal.

One checks easily that any prime ideal $\mathfrak{p}$ yields an element $A \setminus \mathfrak{p}$ of Σ and since this correspondance is inclusion reversing, minimal primes are sent to maximal multiplicatively closed subsets (without 0).

Exercise 7.

i) If $A - S$ is a union of prime ideals $A \setminus S = \bigcup_i \mathfrak{p}_i$ then $S = \bigcap_i (A \setminus \mathfrak{p}_i)$. It is a multiplicatively closed subset (check). Now take $x, y \in A$ such that $xy \in S$. Then

$$\forall i \in I, \quad xy \in A \setminus \mathfrak{p}_i.$$

In particular, for all i, we have $xy \notin \mathfrak{p}_i$ i.e. $x \notin \mathfrak{p}_i$ and $y \notin \mathfrak{p}_i$, whence $x \in S$ and $y \in S$.

Conversely, assume S is saturated. Notice that the only saturated set containing 0 is A ($0a = 0 \in S \Rightarrow a \in S, \forall a \in A$). We shall show that $A \setminus S$ is the union of the primes $\mathfrak{p} \in \mathrm{Spec}(A)$ that do not meet S. It's obvious that for any such prime $\mathfrak{p}$, $\mathfrak{p} \subset A \setminus S$ and therefore

$$\bigcup_{\mathfrak{p} \cap S = \emptyset} \mathfrak{p} \subset A \setminus S.$$

Suppose now that $x \in A \setminus S$. Then the ideal (x) does not meet S since it is saturated ($xy \in S \Rightarrow x \in S$ which is absurd). It is therefore contained in an ideal, maximal for inclusion among the ideals that don't meet S. Let $\mathfrak{a}$ be that ideal. We claim that $\mathfrak{a}$ is prime, and this shall conclude the proof. Assume $x, y \notin \mathfrak{a}$. Maximality of $\mathfrak{a}$ ensures that there exist $s \in S \cap ((x) + \mathfrak{a})$ and $t \in S \cap ((y) + \mathfrak{a})$, whence $st \in (xy) + \mathfrak{a}$ and $xy \notin \mathfrak{a}$.

ii) Let $S = 1 + \mathfrak{a}$. Let $x \in A$ be such that there exists $y \in A$ such that $1 - xy \in \mathfrak{a}$. Then $xy \in S$ thus $x \in \overline{S}$. This shows that

$$\pi^{-1}((A/a)^\times) \subset \overline{S},$$

where π is the canonical map $A \longrightarrow A/\mathfrak{a}$. Now take $x \notin \pi^{-1}((A/\mathfrak{a})^\times)$, i.e. $x \pmod{\mathfrak{a}}$ is not a unit, let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{a} + (x)$ (this latter ideal is proper since x is not a unit in $A/\mathfrak{a}$). For all $s \in S$, $\pi(s) = 1$ whence $s \notin \mathfrak{m}$. Therefore, $\mathfrak{m} \cap S = \emptyset$, which implies $x \notin \overline{S}$. Theferore,

$$\overline{S} = \pi^{-1}((A/\mathfrak{a})^\times).$$

Exercise 8.

- i) $\implies$ ii) $(t/1)\phi^{-1}(1/t) = \phi^{-1}(t/t) = \phi^{-1}(1) = 1$
- ii) $\implies$ iii) In $S^{-1}A$, $(t/1)(x/s) = 1$ implies $(xs')t = ss' \in S$ for some $s' \in S$.
- iii) $\implies$ iv) $xt \in S \subset \overline{S} \implies t \in \overline{S}$.
- iv) $\implies$ v) $\mathfrak{p} \cap S = \emptyset \implies \mathfrak{p} \subset A \setminus \overline{S} \subset A \setminus T \implies \mathfrak{p} \cap T = \emptyset$
- v) $\implies$ iii) Choose $t \in T$ and suppose $t/1$ is not a unit in $S^{-1}A$. It is contained in a maximal ideal $\mathfrak{m}' \subset S^{-1}A$, which comes from a prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, with $\mathfrak{m} \cap S = \emptyset$, but $t \in \mathfrak{m}$ so $\mathfrak{m} \cap T \neq \emptyset$, contradiction.
- iii) $\implies$ i) Let a/s be such that $\phi(x/s) = 0$. Then there is $t \in T$ such that $tx = 0$ in A, and since $t \in S$, we have $x/s = 0$ in $S^{-1}A$, whence ϕ is injective.

  We shall now show that it is surjective. Choose a/t in $T^{-1}A$ for some $a \in A, t \in T$. Since $t/1$ is a unit in $S^{-1}A$, write $(t/1)^{-1} = b/s$ with $b \in A, s \in S$. Since

  $$a/t = (a/1)(t/1)^{-1} = (a/1)(b/s) = (ab)/s,$$

  we have $\phi((ab)/s) = a/t$, whence ϕ is surjective.

Exercise 9. Let $\mathfrak{p}$ be a minimal prime ideal, then (Exercise 6) $S = A \setminus \mathfrak{p}$ is maximal among the multiplicatively closed subsets of A not containing 0. S_0S is a multiplicatively closed subset that does not contain 0 and that contains S, thus $S = S_0S$ and $S_0 \subset S_0S = S$ which shows that $\mathfrak{p} \subset D$.

i) Let S ba a multiplicatively closed subset of A containing a zero divisor $x \in S$. Let $y \in S$ be such that $xy = 0$ and $y \neq 0$ (it exists since x is a zero divisor). Automatically, $y/1 = 0$ in $S^{-1}A$ (since $sy = 0$ for $s = x \in S$), thus $A \longrightarrow S^{-1}A$ is not injective.

ii) Let a/s be an element of $S_0^{-1}A$. Then either $a \in D$ in which case there is a nonzero b such that $ab = 0$, and $b/1 \neq 0$ (since elements of S_0 are non-zero-divisors) whence $(a/s)(b/1) = 0$, or $a \in S_0$ and a/s is automatically a unit.

iii) Use Exercise 8 with $S = \{1\}$ and $T = S_0$, and notice that in this case $t/1$ is a unit in $S^{-1}A = A$ for each $t \in S_0 = A^\times$. Use iii) $\implies$ i) to conclude.

Exercise 10.

i) Let $I' \subset S^{-1}A$ be a finitely generated ideal. It is the extended ideal (3.11) of a finitely generated ideal (write out the generators of I' and extract a family in A) thus there is an ideal $I \subset A$ such that $I' = S^{-1}I$. Since A is absolutely flat, there is an ideal J such that $I \oplus J = A$ thus

$$S^{-1}A = S^{-1}(I \oplus J) = S^{-1}I \oplus S^{-1}J = I' \oplus S^{-1}J,$$

thus I is a direct summand of $S^{-1}A$, which makes it an absolutely flat ring.

ii) ($\implies$) The rings $A_\mathfrak{m}$ are local and absolutely flat (as localisations of an absolutely flat ring) therefore they are fields (Exercise 2.28).

($\impliedby$) M an A-module. For each maximal $\mathfrak{m}$, $M_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module (vector space), thus it is flat. Since flatness is local, M is local.

Exercise 11.

- i) $\implies$ ii) Assume $A/\mathfrak{N}$ is absolutely flat. Note that $\operatorname{Spec}(A/\mathfrak{N}) \simeq \operatorname{Spec}(A)$ as topological spaces with a canonical isomorphism coming from the canonical quotient map $\pi : A \longrightarrow A/\mathfrak{N}$. Suppose $\mathfrak{p} \in \operatorname{Spec}(A)$ is contained in a maximal ideal $\mathfrak{m} \in \operatorname{Spec}(A)$, these ideals correspond to ideals $\overline{\mathfrak{p}} \subset \overline{\mathfrak{m}} \subset A/\mathfrak{N}$ with $\overline{\mathfrak{m}}$ maximal as well. This implies $\overline{\mathfrak{p}}_{\overline{\mathfrak{m}}}$ and $\overline{\mathfrak{m}}_{\overline{\mathfrak{m}}}$ are also ideals, respectively prime and maximal, of $(A/\mathfrak{N})_{\overline{\mathfrak{m}}}$ (3.13). By Exercise 10, $(A/\mathfrak{N})_{\overline{\mathfrak{m}}}$ is a field, whence $(0) = \overline{\mathfrak{m}}_{\overline{\mathfrak{m}}} = \overline{\mathfrak{p}}_{\overline{\mathfrak{m}}}$. This implies that $\overline{\mathfrak{p}} = \overline{\mathfrak{m}}$ since the correspondance is one-to-one, and this in turn implies $\mathfrak{p} = \mathfrak{m}$ from the homeomorphism above. Thus, all prime ideals are maximal.

- ii) $\Longrightarrow$ i) All prime ideals of $A/\mathfrak{N}$ are maximal (via the usual correspondance) thus for any maximal $\mathfrak{m} \in \operatorname{Spec}(A/\mathfrak{N})$, $(A/\mathfrak{N})_\mathfrak{m}$ is local and has a unique prime ideal, which is $\mathfrak{m}_\mathfrak{m}$. Therefore, $\mathfrak{m}_\mathfrak{m}$ is the nilradical of the localisation, which is zero since $A/\mathfrak{N}$ has no nilpotents (and the nilradical of the localisation is the localisation of the nilradical). Thus, $(A/\mathfrak{N})_\mathfrak{m}$ is a field, for all maximal $\mathfrak{m}$. Exercise 10 concludes.
- i/ii) $\Longrightarrow$ iv) Let $\mathfrak{p}_x \neq \mathfrak{p}_y$ be two prime (maximal) ideals of $A/\mathfrak{N}$. There are $f \in \mathfrak{p}_x, g \in \mathfrak{p}_y$ such that $f + g = 1$. The principal opens X_f and X_g are neighborhoods of y and x respectively. Since $A/\mathfrak{N}$ is absolutely flat, there are idempotents e, e' such that $(f) = (e)$ and $(g) = (e')$. Take $e'' = e(1-e')$, it is still idempotent and in (f). We have $e'' \in (f) \subset \mathfrak{p}_x$, $e' \in (g) \subset \mathfrak{p}_y$, thus if $\mathfrak{p}_z \in X_f \cap X_g$, then $e'', e' \notin \mathfrak{p}_z$. However, $e''e' = e(1-e')e' = 0 \in \mathfrak{p}_z$, which contradicts primality of $\mathfrak{p}_z$. Thus $X_f \cap X_g = \emptyset$ and $\operatorname{Spec}(A) \simeq \operatorname{Spec}(A/\mathfrak{N})$ is Hausdorff.
- iv) $\Longrightarrow$ iii) Hausdorff spaces are T_1.
- iii) $\Longrightarrow$ ii) See Exercise 1.18 for details. If $\mathfrak{p}_x \in \operatorname{Spec}(A)$ then $\mathfrak{p}_x \subset \mathfrak{p}_y$ implies $y \in \overline{\{x\}} = \{x\}$ i.e. $\mathfrak{p}_y = \mathfrak{p}_x$, thus $\mathfrak{p}_x$ is maximal.

Exercise 12. Clearly, $T(M)$ is a submodule of M.

i) Choose $\overline{m} \in T(M/T(M))$ for some $m \in M$ and $a \in \operatorname{Ann}(\overline{m}) \setminus \{0\}$, then $a\overline{m} = 0$ thus $am \in T(M)$ and since A is integral, $m \in T(M)$ thus $m = 0$ and $\overline{m} = 0$, whence $M/T(M)$ is torsion-free.

ii) $af(m) = f(am) = 0$ for $m \in T(M)$.

iii) Point ii) shows that the sequence makes sense. $T(M') \longrightarrow T(M)$ is injective as a restriction of $f : M' \longrightarrow M$. Choose $x \in \ker(g : T(M) \longrightarrow T(M''))$, then $x \in \ker(M \longrightarrow M'') = \operatorname{im}(M' \longrightarrow M)$ i.e. $x = f(m')$ for some $m' \in M'$, for $a \in \operatorname{Ann}(x) \neq 0$ we have $f(am) = ax = 0$ and by injectivity, $am = 0$ whence $m \in T(M')$ which shows exactness at $T(M)$.

iv) $a/b \otimes m \mapsto am/b$ is an isomorphism $K \otimes_A M \simeq \operatorname{Frac}(A)$, thus $1 \otimes m = 0$ if and only if $m/1 = 0$ in $\operatorname{Frac}(A)$ if and only if $\exists a \in A \setminus \{0\}$ s.t. $am = 0$.

Exercise 13. First, $T(S^{-1}M)$ and $S^{-1}(TM)$ are both submodules of $S^{-1}M$. Is $0 \in S$ then the result is obvious (all the modules are trivial). From now on we assume $0 \notin S$. For $m \in M, s \in S$,

$$\begin{aligned}
m/s \in T(S^{-1}M) &\Longleftrightarrow \exists a \neq 0, a(m/s) = 0 \\
&\Longleftrightarrow \exists a \neq 0, \exists t \in S, tam = 0 \\
&\Longleftrightarrow \exists a \neq 0, am = 0 \\
&\Longleftrightarrow m \in T(M) \\
&\Longleftrightarrow m/s \in S^{-1}(TM)
\end{aligned}$$

Now to show the equivalence, notice that *i)* is equivalent to injectivity of $M \longrightarrow \operatorname{Frac}(A) \otimes_A M$, and of course ii) and iii) are equivalent to injectivity of the corresponding localised map. Since injectivity is local, we get the equivalence for free.

Exercise 14. M an A-module and $\mathfrak{a}$ an ideal of A. Then $M/\mathfrak{a}M$ is an $A/\mathfrak{a}$-module. Maximal ideals of $A/\mathfrak{a}$ come from maximal ideals of A containing $\mathfrak{a}$. For each maximal ideal $\overline{\mathfrak{m}}$ of $A/\mathfrak{a}$ coming from $\mathfrak{m} \in \operatorname{Spec}(A)$ (maximal containing $\mathfrak{a}$), we have

$$(M/\mathfrak{a}M)_{\overline{\mathfrak{m}}} = (M/\mathfrak{a}M)_\mathfrak{m} = M_\mathfrak{m}/(\mathfrak{a}\mathbf{M})_\mathfrak{m} = 0,$$

whence $M/\mathfrak{a}M = 0$ and $M = \mathfrak{a}M$.

Exercise 15. The problem is essentially solved but let's go through the argument. Let A be a ring and let F be the A-module A^n. Let $x_1, \cdots, x_n$ be a set of generators of F and let $e_1, \cdots, e_n$ be the canonical basis of F. The application

$$\begin{aligned} \phi : F &\longrightarrow F \\ e_i &\longmapsto x_i \end{aligned}$$

This map is well defined by linearity, and surjective since the x_i generate F. We want to show that this map is injective, and since injectivity is local, we may assume A to be a local ring with maximal ideal $\mathfrak{m}$. Set $N = \ker\phi$ and $k = A/\mathfrak{m}$. Since F is a free A-module, it is also a flat A-module, therefore the short exact sequence $0 \longrightarrow N \longrightarrow F \longrightarrow F \longrightarrow 0$ yields the short exact sequence

$$0 \longrightarrow k \otimes_A N \longrightarrow k \otimes_A F \xrightarrow{1\otimes\phi} k \otimes_A F \longrightarrow 0.$$

We have $k \otimes_A F = k^n$ which is an n-dimensional vector space over k, and $1 \otimes_A \phi$ is surjective, thus surjective (injectivity and surjectivity are equivalent for vector space endomorphism in finite dimension). Thus, $k \otimes_A N = 0 = N/\mathfrak{m}N$ so $N = \mathfrak{m}N$. The ideal $\mathfrak{m}$ is contained in the Jacobson of A (it is the Jacobson), thus $N = 0$ by Nakayama's lemma, and ϕ is an isomorphism.

Suppose now that $x_1, \cdots, x_r$ is a generating family, with $r < n$. Then we can add any element to the family to get the generating family $x_1, \cdots, x_n$, which is a basis. Since it is a basis, x_n is not a linear combination of $x_1, \cdots, x_{n-1}$, which shows $x_1, \cdots, x_r$ does not generate F, contradiction.

Exercise 16. Let f be the (ring) map $A \longrightarrow B$ making B a flat A-algebra.

- i) $\Longrightarrow$ ii) $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is surjective if and only if $\forall \mathfrak{p} \in \mathrm{Spec}(A), \exists \mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{p} = f^{-1}(\mathfrak{q})$. By (3.16), this is verified if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$, which is true here by i).
- ii) $\Longrightarrow$ iii) Let $\mathfrak{m}$ be a maximal ideal of A, then there is $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{m} = \mathfrak{q}^c$, i.e. $\mathfrak{m}^e = \mathfrak{q}^{ce} \subseteq \mathfrak{q}$.
- iii) $\Longrightarrow$ iv) Let M be a nonzero A-module, it has a nonzero submodule $M' = Ax$ for some nonzero $x \in M$, which yields an exact sequence

  $$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0.$$

  Tensor it with the flat algebra B to get

  $$0 \longrightarrow M'_B \longrightarrow M_B \longrightarrow M_B/M'_B \longrightarrow 0.$$

  We have $M' \simeq A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ contained in some maximal ideal whose extension is not B, whence $M'_B \simeq B/\mathfrak{a}^e \neq 0$.
- iv) $\Longrightarrow$ v) Let M' be the kernel of $M \longrightarrow M_B$. Since B is flat, we have an exact sequence

  $$0 \longrightarrow M'_B \longrightarrow M_B \longrightarrow (M_B)_B,$$

  and the last map is injective per Exercise 2.13, whence $M'_B = 0$.
- v) $\Longrightarrow$ i) Take $M = A/\mathfrak{a}$.

Exercise 17. $A \xrightarrow{f} B \xrightarrow{g} C$, let $\phi : N \longrightarrow M$ be an A-module homomorphism. There is a commutative diagram

$$\begin{tikzcd} N_B \arrow[r, "\phi_B"] \arrow[d] & M_B \arrow[d] \\ N_C \arrow[r, hook, "\phi_C"'] & M_C \end{tikzcd}$$

The bottom map here is injective since $g \circ f$ is flat. Since g is faithfully flat, the map

$$N_B \longrightarrow (N_B)_C = (N \otimes_A B) \otimes_B C = N \otimes_B C = N_C$$

is injective. By commutativity, ϕ_B is injective.

Exercise 18. $f : A \longrightarrow B$ a flat ring morphism, $\mathfrak{q} \in \operatorname{Spec}(B)$, and $\mathfrak{p} = \mathfrak{q}^c$. The ring $B_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ by (3.10) (flatness is local). We have $B_\mathfrak{q} = (B \setminus \mathfrak{q})^{-1}B$, and since $f(A \setminus \mathfrak{p}) \subseteq B \setminus \mathfrak{q}$, we have a map $A_\mathfrak{p} \longrightarrow B_\mathfrak{q}$. Exercise 3 yields an isomorphism $B_\mathfrak{q} \simeq S^{-1}B_\mathfrak{p}$ with $S = \{s/1 \in B_\mathfrak{p}, s \in B \setminus \mathfrak{q}\}$. As a localisation, $B_\mathfrak{q}$ is flat over $B_\mathfrak{p}$, and we have a natural composite map

$$g : A_\mathfrak{p} \longrightarrow B_\mathfrak{p} \longrightarrow S^{-1}B_\mathfrak{p} \simeq B_\mathfrak{q}.$$

Both arrows are flat, hence $B_\mathfrak{q}$ is flat over $A_\mathfrak{p}$. Now notice that $A_\mathfrak{p}$ is local with maximal ideal $\mathfrak{p}A_\mathfrak{p}$. We have $g(\mathfrak{p}A_\mathfrak{p}) \subseteq \mathfrak{q}B_\mathfrak{q}$ whence $(\mathfrak{p}A_\mathfrak{p})^e \neq B_\mathfrak{p}$ and $A_\mathfrak{p} \longrightarrow B_\mathfrak{q}$ is faithfully flat per Exercise 16 iii). Condition ii) from that same exercise shows surjectivity of the required map.

Exercise 19.

i) Per (3.8), $M = 0 \iff \operatorname{Supp}(M) = 0$.

ii)

$$\begin{aligned}
\mathfrak{p} \in V(\mathfrak{a}) \iff \mathfrak{a} \subseteq \mathfrak{p} &\iff \mathfrak{a} \cap (A \setminus \mathfrak{p}) = \emptyset \\
&\iff \forall s \in A \setminus \mathfrak{p}, \forall a \in \mathfrak{a}, \quad s \neq a \\
&\iff \forall s, t \in A \setminus \mathfrak{p}, \forall a \in \mathfrak{a}, \quad t(s-a) \neq 0 \\
&\iff \forall a/s \in \mathfrak{a}_\mathfrak{p}, \quad a/s \neq 1 \\
&\iff \mathfrak{a}_\mathfrak{p} \subsetneq A_\mathfrak{p} \\
&\iff A_\mathfrak{p}/a_\mathfrak{p} = (A/\mathfrak{a})_\mathfrak{p} \neq 0 \\
&\iff \mathfrak{p} \in \operatorname{Supp}(A/\mathfrak{a})
\end{aligned}$$

iii)