MODULE *ChessRoyalMurder*

EXTENDS *Integers*

The set of all valid squares on the board that the knights can move to. The squares are represented as 2-tuples $\langle File, Rank \rangle$: ($x =$ valid square)

Rank
8 o o o x o o o o
7 o o o o o o o o
6 o x x x x x x x
5 x x x x x x x x
4 x x x x x x x x
3 x x x x x x x x
2 o o o o o o o o
1 o x o o o o x o
  1 2 3 4 5 6 7 8 File

This set includes their starting squares, $b1$ and $g1$, the black Queen's square, $d8$, and any square not occupied by another piece excluding $a6$, which the black knight can move to. There are other squares that the knights could visit, namely $a1$, $h1$, $a8$, $b8$, $g8$, and $h8$, but they aren't necessary to include to find a solution.

CONSTANTS *Squares*

These variables are the board location of each knight starting on files B/G and the number of moves each knight has made.

VARIABLES *knightBpos*, *knightGpos*, *knightBmove*, *knightGmove*

The knights must always be on valid squares. The number of moves must always be a natural number.

$$
\begin{aligned}
& \mathit{TypeOK} \triangleq \\
& \quad \land \ \mathit{knightBpos} \in \mathit{Squares} \\
& \quad \land \ \mathit{knightGpos} \in \mathit{Squares} \\
& \quad \land \ \mathit{knightBmove} \in \mathit{Nat} \\
& \quad \land \ \mathit{knightGmove} \in \mathit{Nat}
\end{aligned}
$$

The initial state of the board. The knights are on their starting squares and they haven't moved.

$$
\begin{aligned}
& \mathit{Init} \triangleq \\
& \quad \land \mathit{knightBpos} = \langle 2, 1 \rangle \\
& \quad \land \mathit{knightGpos} = \langle 7, 1 \rangle \\
& \quad \land \mathit{knightBmove} = 0 \\
& \quad \land \mathit{knightGmove} = 0
\end{aligned}
$$

An action describing all possible knight moves.

$$
\begin{aligned}
& \mathit{KnightMove}(\mathit{knightpos}) \triangleq \\
& \quad \land \mathit{knightpos}' \in \mathit{Squares} \\
& \quad \land \lor \land \mathit{knightpos}[1]' = \mathit{knightpos}[1] - 2 \\
& \qquad\quad \land \lor \mathit{knightpos}[2]' = \mathit{knightpos}[2] + 1 \\
& \qquad\qquad \lor \mathit{knightpos}[2]' = \mathit{knightpos}[2] - 1 \\
& \qquad \lor \land \mathit{knightpos}[1]' = \mathit{knightpos}[1] + 2 \\
& \qquad\quad \land \lor \mathit{knightpos}[2]' = \mathit{knightpos}[2] + 1
\end{aligned}
$$

$$
\begin{array}{l}
\qquad\quad \lor\ knightpos[2]' = knightpos[2] - 1 \\
\lor\ \land\ knightpos[1]' = knightpos[1] - 1 \\
\quad\ \land\ \lor\ knightpos[2]' = knightpos[2] + 2 \\
\qquad\quad \lor\ knightpos[2]' = knightpos[2] - 2 \\
\lor\ \land\ knightpos[1]' = knightpos[1] + 1 \\
\quad\ \land\ \lor\ knightpos[2]' = knightpos[2] + 2 \\
\qquad\quad \lor\ knightpos[2]' = knightpos[2] - 2
\end{array}
$$

This is what it means to move the knight starting on $b1$:

It is moved, its move count increases, and the other knight doesn't move.

$$
\begin{array}{l}
KnightBMove \triangleq \\
\quad \land\ KnightMove(knightBpos) \\
\quad \land\ knightBmove' = knightBmove + 1 \\
\quad \land\ \text{UNCHANGED}\ \langle knightGpos,\ knightGmove \rangle
\end{array}
$$

This is what it means to move the knight starting on $g1$:

It is moved, its move count increases, and the other knight doesn't move.

$$
\begin{array}{l}
KnightGMove \triangleq \\
\quad \land\ KnightMove(knightGpos) \\
\quad \land\ knightGmove' = knightGmove + 1 \\
\quad \land\ \text{UNCHANGED}\ \langle knightBpos,\ knightBmove \rangle
\end{array}
$$

This describes each possible move for white. Either knight is moved, but they can't end up on the same square.

$$
\begin{array}{l}
Next \triangleq \\
\quad \land\ (KnightBMove \lor KnightGMove) \\
\quad \land\ knightBpos' \neq knightGpos'
\end{array}
$$

By telling the model checker that a solution to the puzzle does not exist as described by this invariant, it will produce a counterexample when it finds one, which is a solution.

$$
\begin{array}{l}
Solution \triangleq \\
\quad \neg(\ \lor\ (knightBpos = \langle 4,\ 8 \rangle \land knightGpos = \langle 7,\ 1 \rangle \land knightBmove = 8) \\
\qquad\ \ \lor\ (knightBpos = \langle 7,\ 1 \rangle \land knightGpos = \langle 4,\ 8 \rangle \land knightGmove = 8))
\end{array}
$$