Stochastic process

II Itô Integrals

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SNU MILAB

Construction of the Itô Integral

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- We will prove the existence, in a certain sense, of
$$\int_0^t f(s,w)\,\mathrm{d}B_s(w)$$
where $B_t(w)$ is 1-dimensional Brownian motion starting at the origin, for a wide class of functions $f : [0,\infty] \times \Omega \to$ r. Let us first assume that f has the form
$$\phi(t,w) = \sum_{j\geq 0} e_j(w) \cdot \chi_{[j\cdot 2^{-n},(j+1)2^{-n})}(t),$$
where χ denotes the characteristic (indicator) function and n is a natural number. For such functions, it is reasonable to define
$$\int_S^T \phi(t,w)\,\mathrm{d}B_t(w) = \sum_{j\geq 0} e_j(w)[B_{t_{j+1}} - B_j](w)$$
However, without any further assumptions on the function $e_j(w)$ this leads to difficulties, as the next example shows.

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However, without any further assumptions on the function $e_j(w)$ this leads to difficulties, as the next example shows.

Example. Choose
$$\phi_1(t,w) = \sum_{j\geq 0} B_{j\cdot 2^{-n}}(w)\chi_{[j\cdot 2^{-n},(j+1)2^{-n})}(t)$$
$$\phi_2(t,w) = \sum_{j\geq 0} B_{(j+1)2^{-n}}(w) \cdot \chi_{[j\cdot 2^{-n},(j+1)\cdot 2^{-n})}(t)$$

Construction of the Itô Integral

Example. Choose

$$\phi_1(t,w) = \sum_{j\geq 0} B_{j\cdot 2^{-n}}(w)\chi_{[j\cdot 2^{-n},(j+1)2^{-n})}(t)$$

$$\phi_2(t,w) = \sum_{j\geq 0} B_{(j+1)2^{-n}}(w)\cdot \chi_{[j\cdot 2^{-n},(j+1)\cdot 2^{-n})}(t)$$

Then

$$\mathbb{E}\int_0^T \phi_1(t,w)\,\mathrm{d}B_t(w) = \sum_{j\geq 0} \mathbb{E}\,B_{t_j}(B_{t_{j+1}} - B_{t_j}) = 0$$

since $\{B_t\}$ has independent increments. But

$$\begin{aligned}\mathbb{E}\int_0^T \phi_2(t,w)\,\mathrm{d}B_t(w) &= \sum_{j\geq 0} \mathbb{E}\,B_{t_{j+1}}\cdot(B_{t_{j+1}} - B_{t_j}) \\ &= \sum_{j\geq 0} \mathbb{E}\,(B_{t_{j+1}} - B_{t_j})^2 = T\end{aligned}$$

In spite of the fact that both ϕ_1 and ϕ_2 appear to be very reasonable approximates to $B_t(w)$, the integrals do not match each other.

Construction of the Itô Integral

In general, it is natural to approximate a given function $f(t,w)$ by

$$\sum_j f(t_j^*, w) \cdot \chi_{[t_j, t_{j+1})}(t)$$

where the points t_j^* belongs to the intervals $[t_j, t_{j+1}]$. Unlike the Riemann-Stieltjes integral, it does make a difference here what points t_j^* we choose. The following two choices have turned out to be the most useful ones:

1. $t_j^* = t_j$ (the left end point), which leads to the Itô integral, from now on denoted by
$$\int_S^T f(t,w)\,\mathrm{d}B_t(w),$$
and
2. $t_j^* = (t_j + t_{j+1})/2$ (the mid point), which leads to the Stratonovich integral, denoted by
$$\int_S^T f(t,w) \circ \mathrm{d}B_t(w).$$

Itô vs Staratonovich

- **Itô Integral**

  1. Does not follow classic the chain rule.
  2. Itô integral is more popular in mathematics and finance, where the interpretation as the limit of a discrete game is somewhat appealing, and (more importantly) the martingale property is convenient.

- **Stratonovich Integral**

  1. The advantage of this integral is that if f is smooth enough, you keep the standard chain rule for derivation for $f(X_t)$:
  $$\mathrm{d}f(X_t) = f'(X_t)dX_t$$

Construction of the Itô Integral

- The approximation procedure indicated above will work out successfully given that f has the property that each of the functions $w \to f(t_j, w)$ only depends on the behaviour of $B_s(w)$ up to time t_j. This leads to the following important concepts:

Definition 1 (Filtration)

Let $B_t(w)$ be n-dimensional Brownian motion. Then we define $\mathcal{F}_t = \mathcal{F}_t^{(n)}$ to be the σ-algebra generated bt the random variables $B_s(\cdot)$; $s \leq t$. In other words, $\mathcal{F}_t$ is the smallest σ-algebra containing all sets of the form

$$\{w \,|\, B_{t_1}(w) \in F_1, \cdots, B_{t_k}(w) \in F_k\},$$

where $t_j \leq t$ and $F_j \subset \mathbb{R}^n$ are Borel sets, $j \leq k = 1, 2, \cdots$.

- Intuitively, that $h(w)$ is $\mathcal{F}_t$-measurable means that the value of $h(w)$ can be decided from the values of $B_s(W)$ for $s \leq t$. For example, $h_1(w) = B_{t/2}(w)$ is $\mathcal{F}_t$-measurable, while $B_{2t}(w)$ is not.

- A function $h(w)$ will be $\mathcal{F}_t$-measurable if and only if h can be written as the pointwise a.e. limit of sums of functions of the form $g_1(B_{t_1})g_2(B_{t_2}) \ldots g_k(B_{t_k})$, where $g_1, \ldots, g_k$ are bounded continuous functions and $t_j \leq t$ for $j \leq k = 1, 2, \ldots$.

Construction of the Itô Integral

- We now describe our class of functions for which the Itô integral will be defined:

Definition 2

Let $\nu = \nu(S,T)$ be the class of functions $f(t,w) : [0,+\infty) \times \Omega \to \mathbb{R}$ such that

- $(t,w) \mapsto f(t,w)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$.
- $f(t,w)$ is $\mathcal{F}_t$-adapted (i.e. for each $t \geq 0$, $w \mapsto f(t,w)$ is $\mathcal{F}_t$-measurable).
- $\mathbb{E} \int_S^T f(t,w)^2 \, \mathrm{d}t < \infty$.

Construction of the Itô Integral

For functions $f \in \nu$ we will now show how to define the Itô integral

$$\mathcal{I}[f](w) = \int_S^T f(t, w) \, \mathrm{d}B_t(w),$$

where B_t is 1-dimensional Brownian motion. A function $\phi \in \nu$ is called elementary if it has the form

$$\phi(t, w) = \sum_j e_j(w) \cdot \chi_{[t_j, t_{j+1})}(t)$$

where $e_j(w)$ is $\mathcal{F}_{t_j}$-measurable. For elementary functions $\phi(t, w)$ we define the integral by

$$\int_S^T \phi(t, w) \, \mathrm{d}B_t(w) = \sum_{j \geq 0} e_j(w)[B_{t_{j+1}} - B_{t_j}](w).$$

Now we make the following important observation:

Lemma 1 (The Itô isometry)

If $\phi(t, w)$ is bounded and elementary then

$$\mathbb{E}\left(\int_S^T \phi(t, w) \, \mathrm{d}B_t(w)\right)^2 = \mathbb{E} \int_S^T \phi(t, w)^2 \, \mathrm{d}t. \tag{1.1}$$

Construction of the Itô Integral

Proof. Put $\Delta B_j = B_{t_{j+1}} - B_{t_j}$. Then

$$\mathbb{E}\, e_i e_j \Delta B_i \Delta B_j = \begin{cases} 0 & \text{if } i \neq j \\ \mathbb{E}\, e_j^2 \cdot (t_{j+1} - t_j) & \text{if } i = j \end{cases}$$

using that $e_i e_j \Delta B_i$ and ΔB_j are independent if $i < j$

$$\begin{aligned} \mathbb{E}\left(\int_S^T \phi \,\mathrm{d}B\right)^2 &= \sum_{i,j} \mathbb{E}\, e_i e_j \Delta B_i \Delta B_j = \sum_j \mathbb{E}\, e_j^2 (t_{j+1} - t_j) \\ &= \mathbb{E} \int_S^T \phi^2 \,\mathrm{d}t. \end{aligned}$$

Construction of the Itô Integral

- The idea is now to use the isometry to extend the definition from elementary functions to functions in ν. We do this in several steps:

Step 1. Let $g \in \nu$ be bounded and $g(\cdot, w)$ continuous for each w. Then there exist elementary functions ϕ_n such that

$$\mathbb{E} \int_S^T (g - \phi_n)^2 \, \mathrm{d}t \to 0 \quad \text{as } n \to \infty.$$

Step 2. Let $h \in \nu$ be bounded. Then there exist bounded functions $g_n \in \nu$ such that $g_n(\cdot, w)$ is continuous for all w and n, and

$$\mathbb{E} \int_S^T (h - g_n)^2 \, \mathrm{d}t \to 0.$$

Step 3. Let $f \in \nu$. Then there exists a sequence $\{h_n\} \in \nu$ such that h_n is bounded for each n and

$$\mathbb{E} \int_S^T (f - h_n)^2 \, \mathrm{d}t \to \text{ as } n \to \infty.$$

Construction of the Itô Integral

Step 1. Let $g \in \nu$ be bounded and $g(\cdot, w)$ continuous for each w. Then there exist elementary functions ϕ_n such that

$$\mathbb{E} \int_S^T (g - \phi_n)^2 \, \mathrm{d}t \to 0 \quad \text{as } n \to \infty.$$

Proof. Define $\phi_n(t, w) = \sum_j g(t_j, w) \cdot \chi_{[t_j, t_{j+1})}(t)$. Then ϕ_n is elementary since $g \in \nu$, and

$$\int_S^T (g - \phi_n)^2 \, \mathrm{d}t \to 0 \quad \text{as } n \to \infty, \text{ for each } w$$

since $g(\cdot, w)$ is continuous for each w. Then, by the bounded converge theorem $\lim_{n \to \infty} \mathbb{E} \int_S^T (g - \phi_n)^2 \, \mathrm{d}t \to 0$.

Construction of the Itô Integral

Step 2. Let $h \in \nu$ be bounded. Then there exist bounded functions $g_n \in \nu$ such that $g_n(\cdot, w)$ is continuous for all w and n, and

$$\mathbb{E} \int_S^T (h - g_n)^2 \, \mathrm{d}t \to 0.$$

Proof. Suppose $|h(t, w)| \leq M$ for all (t, w). For each n let ψ_n be non-negative, continuous function on $\mathbb{R}$ such that

1. $\psi_n(x) = 0$ for $x \leq -\frac{1}{n}$ and $x \geq 0$
2. $\int_{-\infty}^{\infty} \psi_n(x) \, \mathrm{d}x = 1$

Define

$$g_n(t, w) = \int_0^t \psi_n(s - t) h(s, w) \, \mathrm{d}s.$$

Then $g_n(\cdot, w)$ is continuous for each w and $g_n(t, w) \geq M$ and, $g_n(t, \cdot)$ is $\mathcal{F}_t$-measurable for all t. Moreover,

$$\int_S^T (g_n(s, w) - h(s, w))^2 \, \mathrm{d}s \to 0 \text{ as } n \to \infty, \text{ for each } w.$$

So by bounded convergence theorem

$$\lim_{n \to \infty} \mathbb{E} \int_S^T (h(t, w) - g_n(t, w))^2 \, \mathrm{d}t = 0$$

Construction of the Itô Integral

Step 3. Let $f \in \nu$. Then there exists a sequence $\{h_n\} \in \nu$ such that h_n is bounded for each n and

$$\mathbb{E} \int_S^T (f - h_n)^2 \, \mathrm{d}t \to \text{ as } n \to \infty.$$

Proof. Let

$$h_n(t, w) = \begin{cases} -n & \text{if } f(t, w) < -n \\ f(t, w) & \text{if } -n \le f(t, w) \le n \\ n & \text{if } f(t, w) > n \end{cases}$$

Then, $\int_S^T (f(t, w) - h_n(t, w))^2 \, \mathrm{d}t$ converges to 0 for each w. The conclusion follows by dominated convergence theorem.

- Conclusion: Define the integral for general f by

$$\mathcal{I}[f](w) := \int_S^T f(t, w) \, \mathrm{d}B_t(w) := \lim_{n \to \infty} \int_S^T \phi_n(t, w) \, \mathrm{d}B_t(w).$$

The limit exists since $\left\{ \int_S^T \phi_n(t, w) \, \mathrm{d}B_t(w) \right\}$ forms a Cauchy sequence in $L^2(P)$, by the Itô isometry.

Construction of the Itô Integral

Definition 3 (The Itô integral)

Let $f \in \nu(S,T)$. Then the Itô integral of f (from S to T) is defined by

$$\int_S^T f(t,w)\,\mathrm{d}B_t(w) = \lim_{n\to\infty} \int_S^T \phi_n(t,w)\,\mathrm{d}B_t(w) \quad \text{limit in } L^2(P) \tag{1.2}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$\mathbb{E}\int_S^t (f(t,w)-\phi_n(t,w))^2\,\mathrm{d}t \to 0 \quad \text{as } n\to\infty. \tag{1.3}$$

Note that such a sequence $\{\phi_n\}$ satisfying equation 1.3 exists and the limit does not depend on the actual choice of $\{\phi_n\}$.

Corollary 1 (The Itô isometry)

$$\mathbb{E}\left(\int_S^T f(t,w)\,\mathrm{d}B_t\right)^2 = \mathbb{E}\int_S^T f^2(t,w)\,\mathrm{d}t \;\text{ for all } f\in\nu(S,T). \tag{1.4}$$

Construction of the Itô Integral

Corollary 2

If $f(t,w) \in \nu(S,T)$ and $f_n(t,w) \in \nu(S,T)$ for $n = 1,2,\ldots$ and $\mathbb{E}\int_S^T (f_n(t,w) - f(t,w))^2 \,\mathrm{d}t \to 0$ as $n \to \infty$, then

$$\int_S^T f_n(t,w)\,\mathrm{d}B_t(w) \to \int_S^T f(t,w)\,\mathrm{d}B_t(w) \quad \text{in } L^2(P) \text{ as } n \to \infty.$$

Example. Assume $B_0 = 0$. Then

$$\int_0^t B_s\,\mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

The extra them $-\frac{1}{2}t$ shows that the Itô stochastic integral does not behave like ordinary integrals.

Construction of the Itô Integral

$$\int_0^t B_s \,\mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

Proof. Put $\phi_n(s,w) = \sum B_j(w) \cdot \chi_{[t_j,t_{j+1})}(s)$, where $B_j = B_{t_j}$. Then

$$\mathbb{E}\int_S^T (\phi_n - B_s)^2 \,\mathrm{d}s = \mathbb{E}\sum_j \int_{t_j}^{t_{j+1}} (B_j - B_s)^2 \,\mathrm{d}s$$

$$= \sum_j \int_{t_j}^{t_{j+1}} (s - t_j) \,\mathrm{d}s = \sum_j \frac{1}{2}(t_{j+1} - t_j)^2 \to 0 \text{ as } \Delta t_j \to 0.$$

So by Corollary 2

$$\int_0^t B_s \,\mathrm{d}B_s = \lim_{\Delta t_j \to 0} \int_0^t \phi_n \,\mathrm{d}B_s = \lim_{\Delta t_j \to 0} \sum_j B_j \Delta B_j$$

Note that for $\Delta(B_j^2)$

$$\Delta(B_j^2) = B_{j+1}^2 - B_j^2 = (\Delta B_j)^2 + 2B_j\Delta B_j,$$

Cont.

Therefore, since $B_0 = 0$

$$B_t^2 = \sum_j \Delta(B_j^2) = \sum_j (\Delta B_j)^2 + 2\sum_j B_j \Delta B_j$$

$$\implies \sum_j B_j \Delta B_j = \frac{1}{2} B_t^2 - \frac{1}{2}\sum_j (\Delta B_j)^2.$$

Since $\sum_j (\Delta B_j)^2 \to t$ in $L^2(P)$ as $\delta t_j \to 0$, the results follows.

Some properties of the Itô integral

Some properties of the Itô integral

First we observe the following:

Theorem 1

Let $f, g \in \nu(0,T)$ and let $0 \leq S < U < T$. Then

1. $\int_S^T f \, \mathrm{d}B_t = \int_S^U f \, \mathrm{d}B_t + \int_U^T f \, \mathrm{d}B_t$ for a.e. w
2. $\int_S^T (cf_+ g) \, \mathrm{d}B_t = c \int_S^T f \, \mathrm{d}B_t + \int_S^T g \, \mathrm{d}B_t$ for a.e. w.
3. $\mathbb{E} \int_S^T f \, \mathrm{d}B_T = 0$
4. $\int_S^T f \, \mathrm{d}B_t$ is $\mathcal{F}_T$-measurable.

Some properties of the Itô integral

Definition 4 (Martingale)

An n-dimensional stochastic process $\{M_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a $\mathcal{F}_t$-martingale if

1. M_t is $\mathcal{F}_t$-measurable for all t
2. $\mathbb{E}\,|M_t| < \infty$ for all t
3. $\mathbb{E}\,M_s|M_t = M_t$ for all $s \geq t$.

Example. Brownian motion B_t in $\mathbb{R}^n$ is a martingale w.r.t the σ-algebras generated by $\{B_s; s \leq t\}$.

Some properties of the Itô integral

For continuous martingales, we have the following important inequality due to Doob

Theorem 2 (Doob's martingale inequality)

If M_t is a martingale such that $t \to M_t(w)$ is continuous a.s., then for all $p \geq 1, T \geq 0$ and all $\lambda > 0$

$$\mathbb{P}[\sup_{0 \leq t \leq T} |M_t| \geq \lambda] \leq \frac{1}{\lambda^p} \mathbb{E} |M_T|^p.$$

We now use this inequality to prove that the Itô integral $\int_0^t f(s,w) \, \mathrm{d}B_s$ can be chosen to depend continuously on t.

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We now use this inequality to prove that the Itô integral $\int_0^t f(s, w)\,\mathrm{d}B_s$ can be chosen to depend continuously on t.

Theorem 3

Let $f \in \nu(0, T)$. Then there exists a t-continuous version of

$$\int_0^t f(s, w)\,\mathrm{d}B_s(w); \quad 0 \leq t \leq,$$

i.e. there exist a t-continuous stochastic process J_t on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{P}[J_t = \int_0^t f\,\mathrm{d}B] = 1 \quad \text{for all } 0 \leq t \leq T. \tag{2.1}$$

Extension of the Itô integral

Extension of the Itô integral

This allows us to define the multi-dimensional Itô integral as follows:

Definition 5

Let $B = (B_1, B_2, \ldots, B_n)$ be n-diemnsional Brownian motion. Then $\nu_{\mathcal{H}^{n\times n}}$ denotes the set of $m \times n$ matrices $v = [v_{ij}(t,w)]$ with respect some filtration $\mathcal{H} = \{\mathcal{H}_t\}_{t\geq 0}$. If $v \in \nu_{\mathcal{H}}^{m\times n}$ we define, using matrix notation

$$\int_S^T v\,\mathrm{d}B = \int_S^T \begin{pmatrix} v_{11} & \ldots & v_{1n} \\ \vdots & & \vdots \\ v_{m1} & \ldots & v_{mn} \end{pmatrix} \begin{pmatrix} \mathrm{d}B_1 \\ \vdots \\ \mathrm{d}B_n \end{pmatrix}$$

to be the $m \times 1$ matrix (column vector) whose i'th component is the following sum of 1-dimensional Ito integral:

$$\sum_{j=1}^{n} \int_S^T v_{ij}(s,w)\,\mathrm{d}B_j(s,w)$$