Stochastic process

1 Progressive process

Definition 1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ be a filtration. A stochastic process $X : [0,\infty) \times \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-progressively measurable if, for any T, the restriction $X|_{[0,T]}$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-measurable.

1.1 Relation with integration

Let us define

$$I_t(\omega) := \int_0^t X_u(\omega)\, du,$$

assuming for simplicity that X is bounded, so the integral is well-defined.

If $s \in [0,t]$, then I_s is $\mathcal{F}_s$-measurable by Fubini's theorem, because the mapping $(u,\omega) \mapsto X_u(\omega)$ is $\mathcal{B}([0,s]) \otimes \mathcal{F}_s$-measurable. That is, the process $(I_s)_{s\geq 0}$ is adapted to the filtration $(\mathcal{F}_s)$. Moreover, the map $s \mapsto I_s(\omega)$ is continuous (for fixed ω) by standard analysis.

It follows that the process (I_s) is even *predictable*, and in particular, *progressively measurable.*

2 Predictable process

Definition 2.1 (Discrete-Time Process). Given a filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P}),$$

a stochastic process $(X_n)_{n\in\mathbb{N}}$ is said to be *predictable* if, for each $n \in \mathbb{N}$, the random variable X_{n+1} is $\mathcal{F}_n$-measurable.

Definition 2.2 (Continuous-Time Process). Given a filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}),$$

a continuous-time stochastic process $(X_t)_{t\geq 0}$ is said to be *predictable* if the mapping

$$X : \Omega \times \mathbb{R}_+ \to \mathbb{R}$$

is measurable with respect to the *predictable σ-algebra*, which is the σ-algebra generated by all left-continuous, adapted processes.

2.1 How to understand the predictable process?

Fact 1. The predictable sigma-algebra is generated by continuous and adapted processes.

Fact 2. The predictable sigma-algebra is generated by the sets of the form

$$\{(s,t] \times A : 0 \leq s < t, A \in \mathcal{F}_s\} \cup \{\{0\} \times A : A \in \mathcal{F}_0\} \tag{2.1}$$

3 Stoppimg time

> **Manager:** 야, 알바 오면 제빙기 멈춰.
>
> **Staff:** 지금 11시인데, 알바 왔음?

> **CCTV:** 11시까지 돌린 CCTV로는 알바 왔는지 판단 못함. 12시 CCTV 봐야 알 수 있음.
> **Manager:** *WTF????*

그래서 Stopping time은 $\{\tau \le t\} \in \mathcal{F}_t$ 조건을 요구한다.

3.1 Stopping sigma field

Let $(\Omega, \Sigma, \langle \mathcal{F}_n \rangle_{n\ge 0}, \mathbb{P})$ be a filtered probability space.

Let T be a stopping time with respect to $\langle \mathcal{F}_n \rangle_{n\ge 0}$.

Define the stopped σ-algebra associated with T as follows:

$$\mathcal{F}_T := \{A \in \Sigma : A \cap \{\omega \in \Omega : T(\omega) \le t\} \in \mathcal{F}_t \text{ for all } t \in \mathbb{Z}_{\ge 0}\}.$$

Intuition:
The stopped σ-algebra $\mathcal{F}_T$ represents the information available *up to the (random) time* T, without looking into the future.
For an event A to be in $\mathcal{F}_T$, we require that *on the part of the sample space where* $T \le t$, the intersection $A \cap \{T \le t\}$ must be observable using the information available at time t (i.e., be in $\mathcal{F}_t$), for all t. This ensures that A is determined *at the time the process is stopped*, based only on the information up to that time.
$\Rightarrow$ 즉, 알바가 언제 왔는지 알고 나서, 그 시점까지만 보고 판단 가능한 정보만 $\mathcal{F}_T$에 들어감.

Properties

1. Difference of stopping times is not a stopping time in general. (Link)
2. Regulairty and measurability are well preserved in the stoppped process X^τ. (Link)
3. 조금 미래 정보가 섞여 있는 시스템에서도, 지금 정보만 보고 '그 일이 이미 일어났는지'를 판단할 수 있으면, 그건 여전히 정직한 stopping time이다. (Link)

3.2 Debut Theorem

Theorem 3.1 (Debut Theorem). Let X be an adapted, right-continuous stochastic process defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$.

Let K be any real number, and define the random time $\tau : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\tau(\omega) := \inf\{t \in \mathbb{R}_+ \ : \ X_t(\omega) \ge K\}.$$

Then τ is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$.

4 Local martingale

4.1 Quadratic variation

Fact. If X is a continuous local martingale, then $[X]_t < \infty$ a.s. for every $t \ge 0$, where $[X]$ denote the quadratic variation of the process X.

4.2 Stochastic integral

Proposition 4.1. For any continuous L^2-martingale M where $M_0 = 0$, and any predictable step process V where $|V| \le 1$, the process $(V \cdot M)$ is an L^2-martingale with $\mathbb{E}(V \cdot M)_t^2 \le \mathbb{E} M_t^2$.

5 Doob's h transform

Set

$$h(x) = \underset{x}{\mathbb{P}}(\tau_A < \tau_B) \tag{5.1}$$

Then, $h(x)$ is the probability, starting from x to hit A before hitting B. Then h is positive on $\mathcal{X} \setminus (A \cup B)$. Furthermore, for $x \notin A \cup B$

$$\hat{P}(x, y) = \underset{x}{\mathbb{P}}[X_1 = y | \tau_A < \tau_B] \tag{5.2}$$

Finally, $h(x) = \mathbb{P}_x(\tau_A < \tau_B)$ satisfies both

1. $h(x) = 1$ for $x \in A$ and $h(z) = 0$ for $z \in B$
2. h is harmonic at x for every $x \notin A \cup B$

and $h(\cdot)$ is the unique solution of linear system given by 1 and 2 above. (Good source)

6 Girsanov Theorem

Girsanov 정리는 확률 공간 위에서 drift를 바꾸는 과정을 정당화해주는 "확률 세계의 물리 법칙 바꾸기"이다.

Setting: 기본 확률 공간 $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ 위에서, W_t는 $\mathbb{P}$ 하의 표준 Brownian motion 이라고 하자.

We can think of following stochastic differential equation (SDE):

$$dX_t = \mu dt + \sigma dW_t.$$

Now, we want to change μ to $\tilde{\mu}$ (i.e., new drift). Now, we need **risk-neutral measure** $\mathbb{Q}$ for the adjustment.

Theorem 6.1.

$$\theta := \frac{\tilde{\mu} - \mu}{\sigma} \quad \text{에 대해,} \quad Z_t := \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t\right)$$

를 $\mathbb{P}$에 대한 Radon-Nikodym 도함수로 하면, 다음이 성립한다:

$$\tilde{W}_t := W_t + \theta t$$

는 $\mathbb{Q}$ 하에서 표준 Brownian motion이 된다. 즉, 새로운 drift $\tilde{\mu}$ 하의 dynamics는 $\mathbb{Q}$에서 해석 가능하다:

$$dX_t = \tilde{\mu} dt + \sigma d\tilde{W}_t.$$

직관: 기존 확률 세계 $\mathbb{P}$에서 샘플링한 Brownian path W_t를 그대로 두고, 그 위에 새로운 drift를 덧붙인 $\tilde{W}_t$를 만들어 $\mathbb{Q}$에서 해석하는 것이다.

즉, simulation은 한 번만 하고, 해석을 다르게 가져가는 마법같은 정리이다.

실전 용도:

- **Option pricing**: risk-neutral measure $\mathbb{Q}$ 하에서 기대값 계산

- **Importance sampling**: 희귀 사건을 더 자주 발생시키고 Girsanov weight로 보정
- **Stochastic control**: 실제 환경은 $\mathbb{P}$지만, 정책에 따른 drift를 $\mathbb{Q}$로 바꿔 실험

정리 요약:

$$\mathbb{E}^{\mathbb{P}}[f(X)] = \mathbb{E}^{\mathbb{Q}}[f(X) \cdot Z_T] \quad \text{where } \frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$$

한 줄 요약: *"확률 measure는 바뀌었지만, path는 그대로 둔 채로 drift를 바꿔 해석하는 수학적 마법."*

7 Brownian Motion

Proposition 7.1. Let B be a pre-Brownian motion. Then,

1. $-B$ is also a pre-Brownian motion (symmetry property).
2. for every $\lambda > 0$, the process $B_t^{\lambda} = \frac{1}{\lambda} B_{\lambda^2 t}$ is also a pre-Brownian motion (invaraince under scaling).
3. for every $s \geq 0$, the process $B_t^{(s)} = B_{t+s} - B_s$ is a pre-Brownian motion and is independent of $\sigma(B_r, r \leq s)$ (simple markov property).