Stochastic process

III Itô Formula and the Martingale Representation Theorem

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The 1-dimensional Itô formula

The 1-dimensional Itô formula

Definition 1 (1-dimensional Itô processes)

Let B_t be 1-dimensional Brownian motion $(\Omega, \mathcal{F}, \mathbb{P})$. A Itô process (or stochastic integral) is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$X_t = X_0 + \int_0^t u(s, w)\, \mathrm{d}s + \int_0^t v(s, w)\, \mathrm{d}B_s \tag{1.1}$$

where $v \in \mathcal{W}_{\mathcal{H}}$, so that

$$\mathbb{P}\left(\int_0^t v(s, w)^2\, \mathrm{d}s < \infty \text{ for all } t \geq 0\right) = 1 \tag{1.2}$$

We also assume that u is $\mathcal{H}_t$-adapted and

$$\mathbb{P}\left(\int_0^t |u(s, w)|\, \mathrm{d}s < \infty \text{ for all } t \geq 0\right) = 1 \tag{1.3}$$

If X_t is an Itô process of the form eq. (1.1), the eq. (1.1) sometimes written in the shorter differential form

$$\mathrm{d}X_t = u\, \mathrm{d}t + v\, \mathrm{d}B_t \tag{1.4}$$

The 1-dimensional Itô formula

We are now ready to state the first main result in this chapter.

Theorem 1 (The 1-dimensional Itô formula)

Let X_t be an Itô process given by

$$\mathrm{d}X_t = u\,\mathrm{d}t + v\,\mathrm{d}B_t$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$, then $Y_t = g(t, X_t)$ is again an Itô process, and

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}(t, X_t)\,\mathrm{d}t + \frac{\partial g}{\partial x}(t, X_t)\,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}(t, X_t) \cdot (\,\mathrm{d}X_t)^2 \tag{1.5}$$

where $(\,\mathrm{d}X_t)^2$ is computed according to the rules

$$\mathrm{d}t \cdot \mathrm{d}t = \mathrm{d}t \cdot \mathrm{d}B_t = \mathrm{d}B_t \cdot \mathrm{d}t = 0, \quad \mathrm{d}B_t \cdot \mathrm{d}B_t = \mathrm{d}t \tag{1.6}$$

Example We know that

$$\int_0^t B_s\,\mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

Apply the Itô formula to find the value of the integral with $X_t = B_t$ and $g(t,x) = \frac{1}{2}x^2$

Cont.

Then, $Y_t = g(t, B_t) = \frac{1}{2}B_t^2$ and by the Itô formula

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}\,\mathrm{d}t \frac{\partial g}{\partial x}\,\mathrm{d}B_t + \frac{1}{2} + \frac{\partial^2 g}{\partial x^2}(\,\mathrm{d}B_t)^2 = B_t\,\mathrm{d}B_t + \frac{1}{2}(\,\mathrm{d}B_t)^2$$

$\implies \mathrm{d}Y_t = B_t\,\mathrm{d}B_t + \frac{1}{2}\,\mathrm{d}t$. In other words,

$$\frac{1}{2}B_t^2 = \int_0^t B_s\,\mathrm{d}B_s + \frac{1}{2}t$$

Itô Integral의 정의를 사용하지 않고도 값을 구했다.

Example Find the value of $\int_0^t s\,\mathrm{d}B_s$

뭔가 적분했을 때 tB_t term이 나올 것 같으니 Itô formula를 함수를 $g(t, x) = tx$ 로 설정하는 것이 좋을 듯 하다. By the Itô formula,

$$\mathrm{d}Y_t = B_t\,\mathrm{d}t + t\,\mathrm{d}B_t + 0$$

$\implies tB_t = \int_0^t B_s\,\mathrm{d}s + \int_0^t s\,\mathrm{d}B_s, \quad \int_0^t s\,\mathrm{d}B_s = tB_t - \int_0^t B_s\,\mathrm{d}s$

The 1-dimensional Itô formula

Theorem 2 (Integral by parts)

Suppose $f(s,w)=f(s)$ only depends on s and that f is continuous and of bounded variation in $[0,t]$. Then

$$\int_0^t f(s)\,\mathrm{d}B_s = f(t)B_t - \int_0^t B_s\,\mathrm{d}f_s \tag{1.7}$$

Remark. f should not depend on w.

Multi-dimensional Itô Formula

Multi-dimensional Itô Formula

- Now we consider m-dimensional Brownian motion $B(t,w) := (B_1(t,w), \ldots, B_m(t,w))$. If each of the processes $u_i(t,w)$ and $v_{ij}(t,w)$ satisfies the conditions given in Definition 1 for $(1 \le 1 \le n, 1 \le j \le m)$, then we can form the following n Itô processes

$$\mathrm{d}X(t) = u\,\mathrm{d}t + v\,\mathrm{d}B(t), \tag{2.1}$$

where

$$X(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}, \; u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \; v = \begin{pmatrix} v_{11} & \ldots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \ldots & v_{nm} \end{pmatrix}, \; \mathrm{d}B(t) = \begin{pmatrix} \mathrm{d}B_t(t) \\ \vdots \\ \mathrm{d}B_m(t) \end{pmatrix} \tag{2.2}$$

Such a process $X(t)$ is called an n-dimensional Itô process.

Question: What is the result of Itô formula for $g(t, X_t)$ with smooth function g?

Multi-dimensional Itô Formula

Theorem 3

Let $\mathrm{d}X_t = u\,\mathrm{d}t + v\,\mathrm{d}B_t$ be an n-dimensional Itô process as above. Let $g(t,x) = (g_1(t,x),\ldots,g_p(t,x))$ be a C^2 map from $[0,\infty)\times\mathbb{R}^n \to \mathbb{R}^p$. Then the process

$$Y(t,w) = g(t,X(t))$$

is again an Itô process, whose Y_k $(k = 1,\ldots,p)$ is given by

$$\mathrm{d}Y_k = \frac{\partial g_k}{\partial t}(t,X)\,\mathrm{d}t + \sum_i \frac{\partial g_k}{\partial x_i}(t,X)\,\mathrm{d}X_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)\,\mathrm{d}X_i\,\mathrm{d}X_j$$

where $\mathrm{d}B_i\,\mathrm{d}B_j = \delta_{ij}\,\mathrm{d}t,\ \mathrm{d}B_i\,\mathrm{d}t = \mathrm{d}t\,\mathrm{d}B_i = 0$

Example. Let $B = (B_1,\ldots,B_n)$ be Brownian motion in $\mathbb{R}^n, n \geq 2$, and consider[1]

$$R(t,w) = |B(t,w)| = (B_1^2(t,w) + \ldots B_n^2(t,w))^{\frac{1}{2}}$$

Then, by the Itô formula

$$\mathrm{d}R = \sum_{i=1}^n \frac{B_i\,\mathrm{d}B_i}{R} + \frac{n-1}{2R}\,\mathrm{d}t$$

[1] Although $g(t,x) = |x|$ is not C^2 at the origin, B_t never hits the origin, a.s. when $n \geq 2$.

Multi-dimensional Itô Formula

Proof. We deal with the case $p = 1$. Fix $t > 0$ and consider an increasing sequence $0 = t_0^{(n)} < \cdots < t_{p_n}^{(n)} = t$ of subdivisions of $[0, t]$ whose mesh tends to 0. Then for every n,

$$g(X_t) = g(X_0) + \sum_{t=0}^{p_n - 1} (g(X_{t_{i+1}^{(n)}}) - g(X_{t_i^{(n)}}))$$

Now, for every $i \in \{0, 1, \ldots, p_n - 1\}$, we apply the Taylor-Lagrange formula to the function $[0, 1] \ni \theta \mapsto g(X_{t_i^{(n)}} + \theta(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}))$, between $\theta = 0$ and $\theta = 1$, and we get

$$g(X_{t_{i+1}^{(n)}}) - g(X_{t_i^{(n)}}) = g'(X_{t_i^{(n)}})(X_{t_{i+1}^{(n)}}) - X_{t_i^{(n)}}) + \frac{1}{2} g_{n,i}(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}})^2,$$

where $g_{n,i} = g''(X_{t_i^{(n)}} + c(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}))$ for some $c \in [0, 1]$. We also have

$$\lim_{n \to \infty} \sum_{i=0}^{p_n - 1} g'(X_{t_i^{(n)}})(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}) = \int_0^t g'(X_s)\, \mathrm{d}X_s$$

in probability. To complete the proof $p = 1$ case, it is enough to show that

$$\lim_{n \to \infty} \sum_{i=0}^{p_n - 1} g_{n,i}(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}})^2 = \int_0^t g''(X_s) \cdot (\,\mathrm{d}X_t)^2$$

in probability.

Cont.

We observe that

$$\sup_{0\le i\le p_n-1} |g_{n,i} - g''(X_{t_i^{(n)}})| \le \sup_{0\le i\le p_n-1}\left(\sup_{x\in[X_{t_i^{(n)}}\wedge X_{t_{i+1}^{(n)}},X_{t_i^{(n)}}\vee X_{t_{i+1}^{(n)}}]} |g''(x) - g''(X_{t_i^{(n)}})|\right)$$

The right-hand side tends to 0 a.s. $n\to\infty$ by the uniform continuity of g'' over a compact interval. Since $\sum_{i=0}^{p_n-1}(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}})^2$ converges in probability to 0, it follows that

$$\lim_{n\to\infty}\left|\sum_{i=0}^{p_n-1} g_{n,i}(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}})^2 - \sum_{i=0}^{p_n-1} g''(X_{t_i^{(n)}})(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}})^2\right| = 0$$

in probability. Therefore, it remains to show that

$$\lim_{n\to\infty}\sum_{i=0}^{p_n-1} g''(X_{t_i^{(n)}})(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}})^2 = \int_0^t g''(X_s)(\,\mathrm{d}X_s)^2$$

Cont (2).

To this end, we note that

$$\sum_{i=0}^{p_n-1} g''(X_{t_i^{(n)}})(X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}})^2 = \int_{[0,t]} g''(X_s)\mu_n(ds)$$

where μ_n is the random measure on $[0,t]$ defined by

$$\mu_n(\mathrm{d}r) := \sum_{i=0}^{p_n-1} (X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}})^2 \delta_{t_i^{(n)}}(\mathrm{d}r)$$

This random measure converges a.s. to the measure $\mathbf{1}_{[0,t]}(\mathrm{d}X_r)^2 = \mathbf{1}_{[0,t]}\,\mathrm{d}r$ [2]. This completes the proof of the case $p = 1$.

[2] 이건 지금은 그냥 받아들이자..

The Martingale Representation Theorem

The Martingale Representation Theorem

- In chapter 3, we proved that if $v \in \nu^n$ then the Itô integral

$$X_t = x_0 + \int_0^t v(s, w)\, \mathrm{d}B_s \quad t \geq 0$$

is always a $\mathcal{F}_t^{(n)}$-martingale (Corollary 3.2.6). In this section we will prove that the converse is also true: Any $\mathcal{F}_t^{(n)}$-martingale can be represented as an Itô integral.

The Martingale Representation Theorem

Lemma 1 (Dense subset in L^2 (I))

Fix $T > 0$. The set of random variables

$$\{\phi(B_{t_1}, \ldots, B_{t_n}) \,|\, t_i \in [0,T], \phi \in C_0^\infty(\mathbb{R}^n), n = 1, 2 \ldots\} \tag{3.1}$$

is dense in $L^2(\mathcal{F}_T, \mathbb{P})$

Lemma 2 (Dense subset in L^2 (II))

The linear span of random variables of the type

$$\exp\left(\int_0^T h(t)\,\mathrm{d}B_t(w) - \frac{1}{2}\int_0^T h^2(t)\,\mathrm{d}t\right); \quad h \in L^2[0,T] \tag{3.2}$$

is dense in $L^2(\mathcal{F}_T, \mathbb{P})$.

The Martingale Representation Theorem

- Suppose $B(t) = (B_1(t), \ldots, B_n(t))$ is n-dimensional. If $v(s,w) \in \nu^n(0,T)$ then the random variable

$$V(w) := \int_0^T v(t,w)\,\mathrm{d}B(t) \tag{3.3}$$

is $\mathcal{F}_T^{(n)}$-measurable and by the Itô isometry

$$\mathbb{E}\,V^2 = \int_0^T \mathbb{E}\,v^2(t,\cdot)\,\mathrm{d}t < \infty, \implies V \in L^2(\mathcal{F}_T^{(n)}, \mathbb{P})$$

The next lemma shows that any $F \in L^2(\mathcal{F}_T^{(n)}, \mathbb{P})$ can be represented this way:

Lemma 3 (The Itô representation theorem)

Let $F \in L^2(\mathcal{F}_T, \mathbb{P})$. Then there exists a unique stochastic process $f(t,w) \in \nu(0,T)$ such that

$$F(w) = \mathbb{E}\,F + \int_0^T f(t,w)\,\mathrm{d}B(t). \tag{3.4}$$

The Martingale Representation Theorem

Theorem 4 (The martingale representation theorem)

Let $B(t) = (B_1(t), \ldots, B_n(t))$ be n-dimensional. Suppose M_t is an $\mathcal{F}_t^{(n)}$-martingale and that $M_t \in L^2(P)$ for all $t \geq 0$. Then there exists a unique stochastic process $g(s, w)$ such that $g \in \nu^{(n)}(0, t)$ for all $t \geq 0$ and

$$M_t(w) = \mathbb{E}\, M_0 + \int_0^t g(s, w)\, \mathrm{d}B(s) \quad \text{a.s, for all } t \geq 0. \tag{3.5}$$

The Martingale Representation Theorem

Theorem 4 (The martingale representation theorem)

Let $B(t) = (B_1(t), \ldots, B_n(t))$ be n-dimensional. Suppose M_t is an $\mathcal{F}_t^{(n)}$-martingale and that $M_t \in L^2(P)$ for all $t \geq 0$. Then there exists a unique stochastic process $g(s, w)$ such that $g \in \nu^{(n)}(0, t)$ for all $t \geq 0$ and

$$M_t(w) = \mathbb{E}\, M_0 + \int_0^t g(s, w)\, \mathrm{d}B(s) \quad \text{a.s, for all } t \geq 0. \tag{3.5}$$

Proof. $(n = 1)$. By Lemma 3 with $T = t, F = M_t$, we have that for all t there exists a unique $f^{(t)}(s, w) \in L^2(\mathcal{F}_t, \mathbb{P})$ such that

$$M_t(w) = \mathbb{E}\, M_t + \int_0^t f^{(t)}(s, w)\, \mathrm{d}B_s(w) = \mathbb{E}\, M_0 + \int_0^t f^{(t)}(s, w)\, \mathrm{d}B_s(w)$$

Now fix $0 \leq t_1 < t_2$. Then,

$$M_{t_1} = \mathbb{E}\, M_{t_2} | \mathcal{F}_{t_1} = \mathbb{E}\, M_0 + \mathbb{E} \int_0^{t_2} f^{(t_2)}(s, w)\, \mathrm{d}B_s(w) | \mathcal{F}_{t_1} \tag{3.6}$$

$$= \mathbb{E}\, M_0 + \int_0^{t_1} f^{(t_2)}(s, w)\, \mathrm{d}B_s(w) \tag{3.7}$$

Cont.

But we also have

$$M_{t_1} = \mathbb{E}\, M_0 + \int_0^t f^{(t_1)}(s,w)\,\mathrm{d}B_s(w). \tag{3.8}$$

By comparing eq. (3.7) and eq. (3.8), we get

$$0 = \mathbb{E}\left(\int_0^{t_1} (f^{(t_2)} - f^{(t_1)})\,\mathrm{d}B\right)^2 = \int_0^{t_1} \mathbb{E}\,(f^{(t_2)} - f^{(t_1)})^2\,\mathrm{d}s \tag{3.9}$$

$\implies f^{(t_1)}(s,w) = f^{(t_2)}(s,w)$ for a.a. $(s,w) \in [0,t_1] \times \Omega$.

So we can define $f(s,w)$ for a.a. $s \in [0,t) \times \Omega$ by setting

$$f(s,w) = f^{(t)}(s,w) \quad \text{if } s \in [0,t) \tag{3.10}$$

then we get

$$M_t = \mathbb{E}\, M_0 + \int_0^t f^{(t)}(s,w)\,\mathrm{d}B_s(w) = \mathbb{E}\, M_0 + \int_0^t f(s,w)\,\mathrm{d}B_s(w) \quad \text{for all } t \geq 0 \tag{3.11}$$

The Martingale Representation Theorem

Problem 4.3. Let X_t and Y_t be Itô process in $\mathbb{R}$. Prove that

$$\mathrm{d}X_t Y_t = X_t \, \mathrm{d}Y_t + Y_t \, \mathrm{d}X_t + \mathrm{d}X_t \cdot \mathrm{d}Y_t$$

Deduce the following general integration by parts formula

$$\int_0^t X_s \, \mathrm{d}Y_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s \, \mathrm{d}Y_s - \int_0^t \mathrm{d}Y_s \cdot \mathrm{d}Y_s \tag{3.12}$$

The Martingale Representation Theorem

Problem 4.4. Suppose $\theta(t,w) = (\theta_1(t,w), \ldots, \theta_n(t,w)) \in \mathbb{R}^n$ with $\theta_k(t,w) \in \nu[0,T]$ for $k = 1, \ldots, n$, where $T \leq \infty$. Define

$$Z_t = \exp\left(\int_0^t \theta(s,w)\,\mathrm{d}B(s) - \frac{1}{2}\int_0^t \theta^2(s,w)\,\mathrm{d}s\right); \quad 0 \leq t \leq T \tag{3.13}$$

where $B(s) \in \mathbb{R}^n$ and $\theta^2 = \theta \cdot \theta$ (dot product).

a) Use Itô formula to prove that

$$\mathrm{d}Z_t = Z_t\theta(t,w)\,\mathrm{d}B(t).$$

b) Deduce that Z_t is a martingale for $t \leq T$, provided that

$$Z_t\theta_k(t,w) \in \nu[0,T] \quad \text{for } 1 \leq k \leq n$$

Remark. A sufficient condition that Z_t be a martingale is the Kazamaki condition

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^t \theta(s,w)\,\mathrm{d}B(s)\right) < \infty \quad \text{for all } t \leq T \tag{3.14}$$

This is implied by the following Novikov condition

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^T \theta^2(s,w)\,\mathrm{d}s\right) < \infty \tag{3.15}$$

Novikov condition $\Longrightarrow$ Kazamaki condition