Stochastic process

VI Other Topics in Diffusion Theory

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INRIA

Kolmogorov’s Backward Equation. The Resolvent

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$$\frac{\partial u}{\partial t} = \mathbb{E}^x\left[Af(X_t)\right] \tag{1.1}$$

It turns out that the RHS of eq. (1.1) can be expressed in terms of u also:

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It turns out that the RHS of eq. (1.1) can be expressed in terms of u also:

Theorem 1 (Kolmogorov's backward equation)

Let $f \in C_0^2(\mathbb{R}^n)$.
a) Define $u(t,x) = \mathbb{E}^x\left[f(X_t)\right]$. Then, $u(t,\cdot) \in \mathcal{D}_A$ for each t and

$$\frac{\partial u}{\partial t} = Au_t, \quad t > 0, x \in \mathbb{R}^n \tag{1.2}$$

$$u(0,x) = f(x); \quad x \in \mathbb{R}^n \tag{1.3}$$

where u_t denotes $x \mapsto u(t,x)$

b) Moreover, if $w(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ is a bounded function satisfying eq. (1.2), eq. (1.3) then $w(t,x) = u(t,x)$ given by $u(t,x) = \mathbb{E}^x\left[f(X_t)\right]$.

Proof.

Let $g(x) = u(t, x)$. Then since $t \mapsto u(t, x)$ is differentiable we have

$$\begin{aligned} \frac{\mathbb{E}^x\left[g(X_r)\right] - g(x)}{r} &= \frac{1}{r} \cdot \mathbb{E}^x\left[\mathbb{E}^{X_r}\left[f(X_t)\right] - \mathbb{E}^x\left[f(X_t)\right]\right] \\ &= \frac{1}{r} \cdot \mathbb{E}^x\left[\mathbb{E}^x\left[f(X_{t+r})|\mathcal{F}_r\right] - \mathbb{E}^x\left[f(X_t)|\mathcal{F}_r\right]\right] \\ &= \frac{1}{r} \cdot \mathbb{E}^x\left[f(X_{t+r}) - f(X_t)\right] \\ &= \frac{u(t+r, x) - u(t, x)}{r} \to \frac{\partial u}{\partial t} \quad \text{as } r \downarrow 0 \end{aligned}$$

Hence

$$Au = \lim r \downarrow 0 \frac{\mathbb{E}^x\left[g(X_r)\right] - g(x)}{r} \quad \text{exists and } \frac{\partial u}{\partial t} = Au, \text{ as asserted .} \tag{1.4}$$

To prove the uniqueness, let $w(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ satisfies eqs. (1.2) and (1.3).

$$\tilde{A}w = -\frac{\partial w}{\partial t} + Aw = 0 \quad \text{for } t > 0, x \in \mathbb{R}^n \tag{1.5}$$

and $w(0.x) = f(x)$ $x \in \mathbb{R}^n$ hold.

Proof.

Fix $(s,x) \in \mathbb{R} \times \mathbb{R}^n$. Define the process $Y_t \in \mathbb{R}^{n+1}$ by $Y_t = (s-t, X_t^{0,x}), t \geq 0$. Then Y_t has generator $\tilde{A}$ and so by eq. (1.5) and Dynkin's formula we have, for all $t \geq 0$,

$$\mathbb{E}^{s,x}\left[w(Y_{t\wedge\tau_R}\right] = w(s,x) + \mathbb{E}^{s,x}\left[\int_0^{t\wedge\tau_R} \tilde{A}w(Y_r)\,\mathrm{d}r\right] = w(s,x), \tag{1.6}$$

where $\tau_R = \inf\{t > 0 : |X_t| \geq R\}$. Letting $R \to \infty$ we get

$$w(s,x) = \mathbb{E}^{s,x}\left[w(Y_t)\right]; \quad \forall t \geq 0. \tag{1.7}$$

In particular, by choosing $t = s$ we get

$$w(s,x) = \mathbb{E}^{s,x}\left[w(Y_s)\right] = \mathbb{E}\, w(0, X_s^{0,x} = \mathbb{E}\, f(X_s^{0,x}) = \mathbb{E}^x\left[f(X_s)\right]. \tag{1.8}$$

Remark. If we introduce the operator $Q_t : f \mapsto \mathbb{E}^{\cdot}\left[f(X_t)\right]$ then we have $u(t,x) = (Q_t f)(x)$ and we may rewrite as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(Q_t f) = Q_t(Af) \tag{1.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(Q_t f) = A(Q_t f); \quad f \in C_0^2(\mathbb{R}^n) \tag{1.10}$$

Kolmogorov's Backward Equation. The Resolvent

It is an important fact that the operator A always has an inverse, at least if a positive multiple of the identity is subtracted from A. This inverse can be expressed explicitly in terms of the diffusion X.

Definition 1 (Resolvent R_α)

For $\alpha > 0$ and $g \in C_b(\mathbb{R}^n)$ we define the resolvent operator R_α by

$$R_\alpha g(x) = \mathbb{E}^x \left[\int_0^\infty e^{-\alpha t} g(X_t)\,\mathrm{d}t \right]. \tag{1.11}$$

Next theorem states that R_α and $\alpha - A$ are inverse operators:

Theorem 2

a) If $f \in C_0^2(\mathbb{R}^n)$ then $R_\alpha(\alpha - A)f = f$ for all $\alpha > 0$.

b) If $g \in C_b(\mathbb{R}^n)$ then $R_\alpha g = \mathcal{D}_A$ and $(\alpha - A)R_\alpha g = g$ for all $\alpha > 0$.

Proof

a) If $f \in C_0^2(\mathbb{R}^n)$ then by Dynkin's formula

$$
\begin{aligned}
R_\alpha(\alpha - A)f(x) &= (\alpha R_\alpha f - R_\alpha A f)(x) \\
&= \alpha \int_0^\infty e^{-\alpha t}\mathbb{E}^x\left[f(X_t)\right]\,\mathrm{d}t - \int_0^\infty e^{-\alpha t}\mathbb{E}^x\left[Af(X_t)\right]\,\mathrm{d}t \\
&= -e^{\alpha t}\mathbb{E}^x\left[f(X_t)\right]\Big|_0^\infty + \int_0^\infty e^{-\alpha t}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}^x\left[f(X_t)\right]\,\mathrm{d}t - \int_0^\infty e^{-\alpha t}\mathbb{E}^x\left[Af(X_t)\right] \\
&= \mathbb{E}^x\left[f(X_0)\right]
\end{aligned}
$$

b) If $g \in C_b(\mathbb{R}^n)$ then by the strong Markov property

$$
\begin{aligned}
&\mathbb{E}^x\left[R_\alpha g(X_t)\right] = \mathbb{E}^x\left[\mathbb{E}^{X_t}\left[\int_0^\infty e^{-\alpha s} g(X_s)\,\mathrm{d}s\right]\right] \\
&= \mathbb{E}^x\left[\mathbb{E}^x\left[\theta_t\left(\int_0^\infty e^{-\alpha s} g(X_s)\,\mathrm{d}S\right)\mathcal{F}_t\right]\right] = \mathbb{E}^x\left[\mathbb{E}^x\left[\int_0^\infty g(X_{t+s})\,\mathrm{d}s\mathcal{F}_t\right]\right] \\
&= \mathbb{E}^x\left[\int_0^\infty e^{-\alpha s} g(X_{t+s})\,\mathrm{d}s\right] = \int_0^\infty e^{-\alpha s}\mathbb{E}^x\left[g(X_{t+s})\right]\,\mathrm{d}s
\end{aligned}
$$

Integration by parts gives

$$
\mathbb{E}^x\left[R_\alpha g(X_t)\right] = \alpha \int_0^\infty e^{-\alpha s}\int_t^{t+s}\mathbb{E}^x\left[g(X_v)\right]\,\mathrm{d}v\,\mathrm{d}s. \tag{1.12}
$$

This identity implies that $R_\alpha g \in \mathcal{D}_A$ and

$$
A(R_\alpha g) = \alpha R_\alpha g - g. \tag{1.13}
$$

Cont.

$R_\alpha g(x)$ 를 integral by parts 해주면

$$
\begin{aligned}
R_\alpha g(x) &= \int_0^\infty e^{-\alpha s} \mathbb{E}^x \left[g(X_s) \right] \mathrm{d}s \\
&= e^{-\alpha s} \int_t^s \mathbb{E}^x \left[g(X_u) \right] \mathrm{d}u \Big|_{s=0}^{s=\infty} + \alpha \int_0^\infty e^{-\alpha s} \int_t^s g(X_u) \, \mathrm{d}u \, \mathrm{d}s \\
&= \int_0^t \mathbb{E}^x \left[g(X_u) \right] \mathrm{d}u + \alpha \int_0^\infty e^{-\alpha s} \int_t^s \mathbb{E}^x \left[g(X_u) \right] \mathrm{d}u \, \mathrm{d}s
\end{aligned}
$$

Therefore,

$A(R_\alpha g)(x)$

$$
\begin{aligned}
&= \lim_{t \to \infty} \frac{\alpha}{t} \int_0^\infty e^{-\alpha s} \left(\int_t^{t+s} \mathbb{E}^x \left[g(X_v) \right] - \int_t^s \mathbb{E}^x \left[g(X_v) \right] \right) \mathrm{d}v \, \mathrm{d}s - \frac{1}{t} \int_0^t \mathbb{E}^x \left[g(X_u) \right] \mathrm{d}u \\
&= \lim_{t \to \infty} \alpha \int_0^\infty e^{-\alpha s} \underbrace{\frac{1}{t} \int_s^{t+s} \mathbb{E}^x \left[g(X_v) \right] \mathrm{d}v}_{\to \mathbb{E}^x [g(X_s)]} \mathrm{d}s - \underbrace{\frac{1}{t} \int_0^t \mathbb{E}^x \left[g(X_u) \right] \mathrm{d}u}_{\to g(x)} \\
&= \alpha R_\alpha g(x) - g(x)
\end{aligned}
$$

Lemmas for the Resolvent

Lemma 1

$R_\alpha g$ is a bounded continuous function.

Proof. Directly followed by below lemma, since $R_\alpha g(x) = \int_0^\infty e^{-\alpha t}\mathbb{E}^x\left[g(X_t)\right] \mathrm{d}t$.

Lemma 2

Let g be a lower bounded, measurable function on $\mathbb{R}^n$ and define, for fixed $t \geq 0$

$$u(x) = \mathbb{E}^x\left[g(X_t)\right]. \tag{1.14}$$

1. If g is lower semi-continuous, then u is lower-semicontinous.
2. If g is bounded and continuous, then u is continuous.

Proof. Recall from the Chapter 5,

$$\mathbb{E}\left|X_t^x - X_t^y\right|^2 \leq |y-x|^2 C(t), \tag{1.15}$$

where $C(t)$ does not depend on x and y. Let $\{y_n\}$ be a sequence of points converging to x. Then $X_t^{y_n} \to X_t^x$ in $L^2(\mathbb{P})$ as $n \to \infty$. So we can take a subsequence $\{z_n\}$ that converges a.s. to $X_t^x(w)$.

Cont.

a) If g is lower bounded and lower semicontinuous, then by the Fatou lemma

$$u(x) = \mathbb{E}\, g(X_t^x) \leq \mathbb{E} \liminf_{n\to\infty} g(X_t^{z_n}) \leq \liminf_{n\to\infty} \mathbb{E}\, g(X_t^{z_N}) = \liminf_{n\to\infty} u(z_n). \tag{1.16}$$

which proves that u is lower semi-continuous.

b) If g is bounded and continuous, the result in a) can be applied both to g and $-g$. Hence both u and $-u$ are lower semicontinuous and we conclude that u is continuous.

The Feynman-Kac Formula. Killing

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We can obtain the following useful generalization of Kolmogorov's backward equation:

Theorem 3 (The Feynman-Kac formula)

Let $f \in C_0^2(\mathbb{R}^n)$ and $q \in C(\mathbb{R}^n)$. Assume that q is lower bounded

1. Put
$$v(t,x) = \mathbb{E}^x \left[\exp\left(-\int_0^t q(X_s)\,\mathrm{d}s \right) f(X_t) \right] \tag{2.1}$$
Then
$$\frac{\partial v}{\partial t} = Av - qv; \quad t > 0, x \in \mathbb{R}^n \tag{2.2}$$
$$v(0,x) = f(x); \quad x \in \mathbb{R}^n \tag{2.3}$$
2. Moreover, if $w(t,x) \in C^{1,2}(R \times \mathbb{R}^n)$ is bounded on $K \times \mathbb{R}^n$ for each compact $K \subset \mathbb{R}$ and w solves eq. (2.2), then $w(t,x) = v(t,x)$ given by eq. (2.1).

Proof.

Part a. Let $Y_t = f(X_t), Z_t = \exp(-\int_0^t q(X_s)\,\mathrm{d}s)$. Then $\mathrm{d}Y_t$ is given by

$$\mathrm{d}f(X_t) = Lf + \sum_{i,k} v_{ik}\frac{\partial f}{\partial x_i}\,\mathrm{d}B_k \tag{2.4}$$

and $\mathrm{d}Y_t Z_t = Y_t\,\mathrm{d}Z_t + Z_t\,\mathrm{d}Y_t$, since $\mathrm{d}Z_t \cdot \mathrm{d}Y_t = 0$.
Note that since $Y_t Z_t$ is an Ito process it follows from Lemma 7.3.2 that $v(t,x) = \mathbb{E}^x\left[Y_t Z_t\right]$ is differentiable w.r.t t, therefore we get

$$\frac{1}{r}(\mathbb{E}^x\left[v(t,X_r)\right] - v(t,x)) = \frac{1}{r}\mathbb{E}^x\left[\mathbb{E}^{X_r}\left[Z_t f(X_t)\right] - \mathbb{E}^x\left[Z_t f(X_t)\right]\right]$$

$$= \frac{1}{r}\mathbb{E}^x\left[\mathbb{E}^x\left[f(X_{t+r})\exp\left(-\int_0^t q(X_{s+r})\,\mathrm{d}s\right)|\mathcal{F}_r\right] - \mathbb{E}^x\left[Z_t f(X_t)|\mathcal{F}_r\right]\right]$$

$$= \frac{1}{r}\mathbb{E}^x\left[Z_{t+r}\cdot\exp\left(\int_0^r q(X_s)\,\mathrm{d}s\right)f(X_{t+r}) - Z_t f(X_t)\right]$$

$$= \frac{1}{r}\mathbb{E}^x\left[f(X_{t+r})Z_{t+r} - f(X_t)Z_t\right] + \frac{1}{r}\mathbb{E}^x\left[f(X_{t+r})Z_{t+r}\cdot\left(\exp\left(\int_0^r q(X_s)\,\mathrm{d}s\right) - 1\right)\right]$$

$$\to \frac{\partial}{\partial t}v(t,x) + q(x)v(t,x) \quad \text{as } r \to 0,$$

since

$$\frac{1}{r}f(X_{t+r})Z_{t+r}\left(\exp\left(\int_0^r q(X_s)\,\mathrm{d}s\right) - 1\right) \to f(X_t)Z_t q(X_0) \tag{2.5}$$

pointwise boundedly. This completes the part (a).

Proof.

Part b. Assume that $w(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ satisfies eq. (2.2) and that $w(t,x)$ is bounded on $K \times \mathbb{R}^n$ for each compact $k \subset \mathbb{R}$. Then

$$\hat{A}w(t,x) := -\frac{\partial w}{\partial t} + Aw - qw = 0 \quad \text{for } t > 0, x \in \mathbb{R}^n \tag{2.6}$$

and

$$w(0,x) = f(x); \quad x \in \mathbb{R}^n. \tag{2.7}$$

Fix $(s,x,z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ and define $Z_t = z + \int_0^t q(X_s)\,\mathrm{d}s$ and $H_t = (s-t, X_t^{0,x}, Z_t)$. Then H_t is an Ito diffusion with generator

$$A_H\phi(s,x,z) = -\frac{\partial \phi}{\partial s} + A\phi q(x) + \frac{\partial}{\partial z}; \quad \phi \in C_0^2(\mathbb{R} \times \mathbb{R}^n \mathbb{R}^n). \tag{2.8}$$

Hence by eq. (2.6) and Dynkin's formula we have, for all $t \geq 0, R > 0$ and with $\phi(s,x,z) = \exp(-z)w(s,x)$:

$$\mathbb{E}^{s,x,z}\left[\phi(H_{t\wedge\tau_R})\right] = \phi(s,x,z) + \mathbb{E}^{s,x,z}\left[\int_0^{t\wedge\tau_R} A_H\phi(H_r)\,\mathrm{d}r\right], \tag{2.9}$$

where $\tau_R = \inf\{t > 0 \,|\, |H_t| \geq R\}$.

Proof.

With the choice of ϕ and by eq. (2.6)

$$A_H\phi(s,x,z) = \exp(-z)\left[-\frac{\partial w}{\partial s} + Aw - q(x)w\right] = 0. \tag{2.10}$$

Hence

$$\begin{aligned} w(s,x) = \phi(s,x,0) &= \mathbb{E}^{s,x,0}\left[\phi(H_{t\wedge\tau_R})\right] \\ &= \mathbb{E}^x\left[\exp\left(-\int_0^{t\wedge\tau_R} q(X_r)\,\mathrm{d}r\right) w(s-t\wedge\tau_R, X_{t\wedge\tau_R}|\right] \\ &\to \mathbb{E}^x\left[\exp\left(-\int_0^t q(X_r)\,\mathrm{d}r\right) w(s-t, X_t)\right] \quad \text{as } R\to\infty \end{aligned}$$

since $w(r,x)$ is bounded for $(r,x)\in K\times\mathbb{R}^n$. In particular, choosing $t=s$ we get

$$w(s,x) = \mathbb{E}^x\left[\exp\left(-\int_0^s q(X_r)\,\mathrm{d}r\right) w(0, X_s^{0,x})\right] = v(s,x) \text{ as claimed .} \tag{2.11}$$

The Martingale Problem

The Martingale Problem

If $\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$ is an Ito diffusion in $\mathbb{R}^n$ with generator A and if $f \in C_0^2(\mathbb{R}^n)$ then

$$f(X_t) = f(x) + \int_0^t Af(X_s)\,\mathrm{d}s + \int_0^t \nabla f^\top(X_s)\sigma(X_s)\,\mathrm{d}B_s \tag{3.1}$$

Define

$$M_t := f(X_t) - \int_0^t Af(X_r)\,\mathrm{d}r = f(x) + \int_0^t \nabla f^\top(X_r)\sigma(X_r)\,\mathrm{d}B_r \tag{3.2}$$

It follows that

$$\mathbb{E}^x\left[M_s|\mathcal{F}_t\right] = M_t \tag{3.3}$$

$$\mathbb{E}^x\left[M_s|\mathcal{M}_t\right] = \mathbb{E}^x\left[\mathbb{E}^x\left[M_s|\mathcal{F}_t\right]|\mathcal{M}_t\right] = \mathbb{E}^x\left[M_t|\mathcal{M}_t\right] = M_t \tag{3.4}$$

since M_t is $\mathcal{M}_t$-measurable. We have shown the following:

Theorem 4

If X_t is an Ito diffusion in $\mathbb{R}^n$ with generator A, then for all $f \in C_0^2(\mathbb{R}^n)$ the process

$$M_t = f(X_t) - \int_0^t Af(X_r)\,\mathrm{d}r \tag{3.5}$$

is a martingale w.r.t $\{\mathcal{M}_t\}$.

When is an Ito process a Diffusion?

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The answer is no in general, but it may be yes in some cases.

Example. Let $n \geq 2$. Note that the process $R_t(w) = |B(t, w)|$ satisfies the equation

$$\mathrm{d}R_t = \sum_{i=1}^{n} \frac{B_i \,\mathrm{d}B_i}{R_t} + \frac{n-1}{R_t}\,\mathrm{d}t \tag{4.1}$$

If we show that 1-dim Brownian motion $\tilde{B}_t$ has same law as the process

$$Y_t := \int_0^t \sum_{i=1}^{t} \frac{B_i}{|B|}\,\mathrm{d}B_i, \tag{4.2}$$

then by weak uniqueness, R_t is an Ito diffusion with generator

$$Af(x) = \frac{1}{2} f''(x) + \frac{n-1}{2x} f'(x). \tag{4.3}$$

When is an Ito process a Diffusion?

To verify the claim, we may use the following result:

Theorem 5

An Ito process

$$\mathrm{d}Y_t = v\,\mathrm{d}B_t; \quad Y_0 = 0 \text{ with } v(t,w) \in \nu_{\mathcal{H}}^{n\times m} \tag{4.4}$$

coincides (in law) with n-dimensional Brownian motion if and only if

$$vv^\top(t,w) = I_n \;\; \mathrm{d}t \times \mathrm{d}P \text{ for a.e } (t,w) \tag{4.5}$$

Note that in the example above we have

$$Y_t = \int_0^t v\,\mathrm{d}B \tag{4.6}$$

with

$$v = \left[\frac{B_1}{|B|}, \dots \frac{B_n}{|B|}\right], \quad B = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} \tag{4.7}$$

and since $vv^\top = 1$, we get that Y_t is a 1-dim Brownian motion.

When is an Ito process a Diffusion?

Theorem 5 is a special case of the following result, which gives a necessary and sufficient condition for an Ito process to coincide in law with a given diffusion.

Theorem 6

Let X_t be an Ito diffusion given by

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t,\ b \in \mathbb{R}^n,\ \sigma \in \mathbb{R}^{n\times m},\ X_0 = x, \tag{4.8}$$

and let Y_t be an Ito process given by

$$Y_t = u(t,w)\,\mathrm{d}t + v(t,w)\,\mathrm{d}B_t,\ u \in \mathbb{R}^n,\ v \in \mathbb{R}^{n\times m},\ Y_0 = x. \tag{4.9}$$

Then, X_t and Y_t equal in law if and only if

$$\mathbb{E}^x\left[u(t,\cdot)|\mathcal{N}_t\right] = b(Y_t^x) \text{ and } vv^\top(t,w) = \sigma\sigma^\top(Y_t^x) \tag{4.10}$$

for a.s. $\mathrm{d}t \times \mathrm{d}P$ (t,w), where $\mathcal{N}_t$ is the σ-algebra generated by $Y_s, s \le t$.

Remark.

1. $u(t,\cdot)$ need not be $\mathcal{N}_t$-measurable, and $v(t,w)$ need not be $\mathcal{N}_t$-adpated either.
2. $\phi(X_t)$ and Z_t equal in law if and only if

$$A[f \circ \phi] = \hat{A}[f] \circ \phi \tag{4.11}$$

for all $f \in C_0^2$ where A and $\hat{A}$ are the generators of X_t and Z_t respectively.

When is an Ito process a Diffusion?

Corollary 1 (How to recognize a Brownian motion)

Let

$$\mathrm{d}Y_t = u(t,w)\,\mathrm{d}t + v(t,w)\,\mathrm{d}B_t \tag{4.12}$$

be an Ito process in $\mathbb{R}^n$. Then Y_t is a Brownian motion if and only if

$$\mathbb{E}^x\left[u(t,\cdot)|\mathcal{N}_t\right] = 0 \text{ and } vv^\top = I_n \tag{4.13}$$

for a.s. (t,w).

Random Time Change

Random Time Change

Let $c(t,w) \geq 0$ be an $\mathcal{F}_t$-adpated process. Define

$$\beta_t = \beta(t,w) = \int_0^t c(s,w)\,\mathrm{d}s. \tag{5.1}$$

We will say that β_t is a (random) time change with time change rate $c(t,w)$.

Define $\alpha_t = \alpha(t,w)$ by

$$\alpha_t = \inf\{s \mid \beta_s > t\}. \tag{5.2}$$

Then α_t is a right-inverse of β_t, for each w:

$$\beta(\alpha(t,w),t) = t \quad \text{for all } t \geq 0. \tag{5.3}$$

Moreover, $t \mapsto \alpha_t(w)$ is right-continuous.

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Moreover, $t \mapsto \alpha_t(w)$ is right-continuous.

Proposition 1 (Random Time Change)

Given above definition

1. If $c(s, w) > 0$ for a.s. (s, w) then $t \mapsto \beta_t(w)$ is strictly increasing, $t \mapsto \alpha_t(w)$ is continuous and α_t is also a left-inverse of β_t:

$$\alpha(\beta(t, w), w) = t \quad \text{for all } t \geq 0. \tag{5.4}$$

2. $w \mapsto \alpha(t, w)$ is an $\{\mathcal{F}_s\}$-stopping time for each t, since

$$\{ w \,|\, \alpha(t, w) < s \} = \{ w \,|\, t < \beta(s, w) \} \in \mathcal{F}_s. \tag{5.5}$$

Random Time Change

Question: Suppose X_t is an Ito process and Y_t is an Ito process. When does exist a time change β_t such that Y_{α_t} and X_t equal in law?

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Theorem 7

Let X_t, Y_t be as in theorem 6 and let β_t be a time change with right inverse α_t as the above. Assume that

$$u(t,w) = c(t,w)b(Y_t) \text{ and } vv^\top(t,w) = c(t,w) \cdot \sigma\sigma^\top(Y_t) \tag{5.6}$$

for a.s. (t,w). Then Y_{α_t} and X_t equal in law.

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This result allows us to recognize time changes of Brownian motion:

Theorem 8

Let $\mathrm{d}Y_t = v(t,w)\,\mathrm{d}B_t$, $v \in \mathbb{R}^{n\times m}, B_t \in \mathbb{R}^m$ be an Ito integral in $\mathbb{R}^n$, $Y_0 = 0$ and assume that

$$vv^\top(t,w) = c(t,w)I_n \tag{5.7}$$

for some process $c(t,w) \geq 0$. Let α_T, β_t as the above, Then Y_{α_t} is an n-dimensional Brownian motion.

Random Time Change

Corollary 2

Let $\mathrm{d}Y_t = \sum_{i=1}^{n} v_i(t,w)\,\mathrm{d}B_i(t,w), Y_0 = 0$, where $B = (B_1, \ldots, B_n)$ is a Brownian motion in $\mathbb{R}^n$. Then Y_{α_t} is a 1-dimensional Brownian motion, where

$$\beta_s = \int_0^s \left(\sum_{i=1}^{n} v_i^2(r,w) \right) \mathrm{d}r. \tag{5.8}$$

Corollary 3

Let Y_t, β_s be as in the above, Assume that

$$\sum_{i=1}^{n} v_i^2(r,w) > 0 \text{ for a.s. } (r,w). \tag{5.9}$$

Then there exists a Brownian motion $\hat{B}_t$ such that

$$Y_t = \hat{B}_{\beta_t}. \tag{5.10}$$

Random Time Change

Corollary 4

Let $c(t, w) \geq 0$ be give and define

$$\mathrm{d}Y_t = \int_0^t \sqrt{c(s, w)}\, \mathrm{d}B_s \tag{5.11}$$

where B_s is an n-dimensional Brownian motion. Then Y_{α_t} is also an n-dimensional Brownian motion.

a time change of an Ito integral is again an Ito integral, but driven by a different Brownian motion $\tilde{B}_t$. First we construct $\tilde{B}_t$.

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Lemma 3

Suppose $s \mapsto \alpha(s,w)$ is continous, $\alpha(0,w) = 0$ for a.s. w. Fix $t > 0$ such that $\beta_t < \infty$ a.s. and assume that $\mathbb{E}\,\alpha_t < \infty$. For $k = 1, 2, \ldots$ put

$$t_j = \begin{cases} j \cdot 2^{-k} & \text{if } j \cdot 2^{-k} \leq \alpha_t \\ t & \text{if } j \cdot 2^{-k} > \alpha_t \end{cases} \tag{5.12}$$

and choose r_j such that $\alpha_{r_j} = t_j$. Suppose $f(s,w) \geq 0$ is $\mathcal{F}_s$-adpated, bounded and s-continuous for a.s. w. Then

$$\lim_{k\to\infty} \sum_j f(\alpha_j, w)\Delta B_{\alpha_j} = \int_0^{\alpha_t} f(s,w)\,\mathrm{d}B_s \ \text{ in } L^2(\mathbb{P}) \text{ a.s.} \tag{5.13}$$

where $\alpha_j = \alpha_{r_j}, \Delta B_{\alpha_j} = B_{\alpha_{j+1}} - B_{\alpha_j}$.

Theorem 9 (Time change formula for Ito Integrals)

Suppose $c(s,w)$ and $\alpha(s,w)$ are s-continuous, $\alpha(0,w)=0$ for a.s. w and that $\mathbb{E}\,\alpha_t < \infty$. Let B_s be an m-dimensional Brownian motion and let $v(s,w) \in \nu_{\mathcal{H}}^{n\times m}$ be bounded and s-continuous. Define

$$\tilde{B}_s := \lim_{k\to\infty} \sum_j \sqrt{c(\alpha_j,w)}\Delta B_{\alpha_j} = \int_0^{\alpha_t} \sqrt{c(s,w)}\,\mathrm{d}B_s \tag{5.14}$$

Then $\tilde{B}_t$ is an m-dimensional $\mathcal{F}_{\alpha_t}$-Brownian motion (i.e. $\tilde{B}_t$ is a martingale w.r.t $\mathcal{F}_{\alpha_t}$) and

$$\int_0^{\alpha_t} v(s,w)\,\mathrm{d}B_s = \int_0^t v(\alpha_r,w)\sqrt{\alpha_r'(w)}\,\mathrm{d}\tilde{B}_r \ \ \mathbb{P}-\text{a.s.} \tag{5.15}$$

where $\alpha_r'(w)$ is the derivative of $\alpha(r,w)$ w.r.t. r, so that

$$\alpha_r'(w) = \frac{1}{c(\alpha_r,w)} \ \text{ for a.s. } r \geq 0, \ \ \text{a.s. } w \in \Omega. \tag{5.16}$$

Random Time Change

Example : Brownian motion the unit sphere in $\mathbb{R}^n; n > 2$. . Apply the function $\phi : \mathbb{R}^n \setminus \{0\} \to S$ defined by

$$\phi(x) = x \cdot |x|^{-1}; \quad x \in \mathbb{R}^n \setminus \{0\} \tag{5.17}$$

to n-dim Brownian motion $B = (B_1, \dots, B_n)$. The result is a stochastic integral $Y = \phi(B)$ given by

$$\mathrm{d}Y = \frac{1}{|B|} \cdot \sigma(Y)\,\mathrm{d}B + \frac{1}{|B|^2} b(Y)\,\mathrm{d}t, \tag{5.18}$$

where

$$\sigma = [\sigma_{ij}] \in \mathbb{R}^{n \times n}, \text{ with } \sigma_{ij}(Y) = \delta_{ij} - Y_i Y_j \;; 1 \leq i, j \leq n \tag{5.19}$$

$$b(y) = -\frac{n-1}{2} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \tag{5.20}$$

Now perform the following time change: Define $Z_t(w) = Y_{\alpha(t,w)}(w)$ where

$$\alpha_t = \beta^{-1}, \; \beta(t, w) = \int_0^t \frac{1}{|B|^2}\,\mathrm{d}s. \tag{5.21}$$

Random Time Change

Then Z is again an Ito process such that

$$\mathrm{d}Z = \sigma(Z)\,\mathrm{d}\tilde{B} + b(Z)\,\mathrm{d}t. \tag{5.22}$$

Hence Z is a diffusion with the characteristic operator

$$\mathcal{A}f(y) = \frac{1}{2}\left(\Delta f(y) - \sum_{ij} y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j}\right) - \frac{n-1}{2} \cdot \sum_i y_i \frac{\partial f}{\partial y_i};\ |y| = 1. \tag{5.23}$$

Note that Z is invariant under orthogonal transformation in $\mathbb{R}^n$ (since B is). It is reasonable to call Z Brownian motion on the unit sphere S.

The Girsanov Theorem

The Girsanov Theorem

First we state (without proof) the useful Levy characterization of Brownian motion.

Proposition 2 (The Levy characterization of Brownian motion)

Let $X(t) = (X_1(t), \ldots, X_n(t))$ be a continuous stochastic process on a probability space $(\Omega, \mathcal{H}, Q)$ with values in $\mathbb{R}^n$. TFAE:

1. $X(t)$ is a Brownian motion w.r.t. Q, i.e. the law of $X(t)$ w.r.t Q is the same as the law of an n-dimensional Brownian motion.
2. $X(t) = (X_1(t), \ldots, X_n(t))$ is a martingale w.r.t Q (and w.r.t its own filtration) and
   $X_i(t)X_j(t) - \delta_{ij}t$ is a martingale w.r.t Q (and w.r.t. its own filtration) for all $i, j \in \{1, 2, \ldots, n\}$.

Remark. One may replace the condition as
The cross-varation process $\langle X_i, X_j \rangle_t$ satisfy the identity

$$\langle X_i, X_j \rangle_t (w) = \delta_{ij} t \quad \text{a.s. } 1 \le i, j \le n \tag{6.1}$$

where

$$\langle X_i, X_j \rangle_t = \frac{1}{4}(\langle X_i + X_j, X_i + X_j \rangle_t - \langle X_i - X_j, X_i - X_j \rangle_t) \tag{6.2}$$

$\langle Y, Y \rangle_t$ being the quadratic variation process.

The Girsanov Theorem

Next we need an auxiliary result about conditional expectation:

Lemma 4

Let μ and ν be two probability measures on a measurable space $(\Omega, \mathcal{G})$ such that $\mathrm{d}\nu(w) = f(w)\,\mathrm{d}\mu(w)$ for some $f \in L^1(\mu)$. Let X be a random variable on $(\Omega, \mathcal{G})$ such that

$$\mathbb{E}^{\nu}\left[|X|\right] = \int_{\Omega} |X(w)| f(w)\,\mathrm{d}\mu(w) < \infty \tag{6.3}$$

Let $\mathcal{H}$ be a σ-algebra, $\mathcal{H} \subset \mathcal{G}$. Then,

$$\mathbb{E}^{\nu}\left[X|\mathcal{H}\right] \cdot \mathbb{E}^{\mu}\left[f|\mathcal{H}\right] = \mathbb{E}^{\mu}\left[fX|\mathcal{H}\right] \text{ a.s.} \tag{6.4}$$

The Girsanov Theorem

Theorem 10 (The Girsanov theorem I)

Let $Y(t) \in \mathbb{R}^n$ be an Ito process of the form

$$\mathrm{d}Y_t = a(t,w)\,\mathrm{d}t + \mathrm{d}B(t); \quad t \leq T, Y_0 = 0. \tag{6.5}$$

where $T \leq \infty$ is a given constant and $B(t)$ is n-dimensional Brownian motion. Put

$$M_t = \exp\left(-\int_0^t a(s,w)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t a^2(s,w)\,\mathrm{d}s\right); \quad t \leq T. \tag{6.6}$$

Assume that $a(s,w)$ satisfes Novikov's condition

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^T a^2(s,w)\,\mathrm{d}s\right) < \infty \tag{6.7}$$

where $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ is the expectation w.r.t $\mathbb{P}$. Define the measure Q on $(\Omega, \mathcal{F}_T^{(n)})$ by

$$\mathrm{d}Q(w) = M_T(w)\,\mathrm{d}P(w) \tag{6.8}$$

Then $Y(t)$ is an n-dimensional Brownian motion w.r.t. the probability law Q, for $t \leq T$.

Remark. Note that since M_t is a martingale we actually have that

$$M_T\,\mathrm{d}P = M_t\,\mathrm{d}P \quad \text{on } \mathcal{F}_t; t \leq T \tag{6.9}$$

The Girsanov Theorem

Theorem 11 (The Girsanov theorem II)

Let $Y(t) \in \mathbb{R}^n$ be an Ito process of the form

$$\mathrm{d}Y(t) = \beta(t,w)\,\mathrm{d}t + \theta(t,w)\,\mathrm{d}B(t); \quad t \leq T \tag{6.10}$$

where $B(t) \in \mathbb{R}^m$, $\beta(t,w) \in \mathbb{R}^n$ and $\theta(t,w) \in \mathbb{R}^{n\times m}$. Suppose there exist processes $u(t,w) \in W_{\mathcal{H}}$ and $\alpha(t,w) \in W_{\mathcal{H}}$ such that

$$\theta(t,w)u(t,w) = \beta(t,w) - \alpha(t,w) \tag{6.11}$$

and assume that $u(t,w)$ satisfies Novikov's condition

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^T u^2(s,w)\,\mathrm{d}s\right) < \infty \tag{6.12}$$

Put

$$M_t = \exp\left(-\int_0^t u(s,w)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t u^2(s,w)\,\mathrm{d}s\right); \quad t \leq T \tag{6.13}$$

$$\mathrm{d}Q(w) = M_T(w)\,\mathrm{d}P(w) \text{ on } \mathcal{F}_T \tag{6.14}$$

Then,

$$\hat{B}(t) := \int_0^t u(s,w)\,\mathrm{d}s + B(t);\ t \leq T \tag{6.15}$$

is a Q-Brownian motion and in terms of $\hat{B}(t)$ the process $Y(t)$ has the stochastic integral representation

$$\mathrm{d}Y(t) = \alpha(t,w)\,\mathrm{d}t + \theta(t,w)\,\mathrm{d}\hat{B}(t). \tag{6.16}$$

The Girsanov Theorem

Finally, we formulate a diffusion version:

Theorem 12 (The Girsanov theorem III)

Let $X(t) = X^x(t) \in \mathbb{R}^n$ and $Y(t) = Y^x(t) \in \mathbb{R}^n$ be an Ito diffusion and an Ito process, resp, of the forms

$$\mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t); \quad t \leq T, X(0) = x$$

$$\mathrm{d}Y(t) = [\gamma(t,w) + b(Y(t))]\,\mathrm{d}t + \sigma(Y(t))\,\mathrm{d}B(t); \quad t \leq T, Y(0) = x$$

where the functions $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy the conditions of Theorem 5.2.1 and $\gamma(t,w) \in W_{\mathcal{H}}, x \in \mathbb{R}^n$. Suppose there exists a process $u(t,w) \in W_{\mathcal{H}}$ such that

$$\sigma(Y(t))u(t,w) = \gamma(t,w) \tag{6.17}$$

and assume that $u(t,w)$ satisfies Novikov's condition

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^t u^2(s,w)\,\mathrm{d}s\right) < \infty \tag{6.18}$$

Define M_t, Q and $\hat{B}(t)$ as in theorem 11. **Then,**

$$\mathrm{d}Y(t) = b(Y(t))\,\mathrm{d}t + \sigma(Y(t))\,\mathrm{d}\hat{B}(t). \tag{6.19}$$

Therefore, the Q-law of $Y^x(t)$ is the same as the P-law of $X^x(t)$; $t \leq T$.

The Girsanov Theorem

Example 8.6.6. Let $a : \mathbb{R}^n \to \mathbb{R}^n$ be a bounded, measurable function. Then we can construct a weak solution $X_t = X_t^x$ of the stochastic differential equation

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + \mathrm{d}B_t; \quad X_0 = x \in \mathbb{R}^n. \tag{6.20}$$

We proceed according to the procedure above, with $\sigma = I, b = 0$ and

$$\mathrm{d}Y_t = \mathrm{d}B_T; \quad Y_0 = x. \tag{6.21}$$

Choose $u_0 = \sigma^{-1}(b - a) = -a$ and define

$$M_t = \exp\left(-\int_0^t u_0(Y_s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t u_0^2(Y_s)\,\mathrm{d}s\right) \tag{6.22}$$

i.e.

$$M_t = \exp\left(\int_0^t a(B_s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t a^2(B_s)\,\mathrm{d}s\right) \tag{6.23}$$

Fix $T < \infty$ and put $\mathrm{d}Q = M_T\,\mathrm{d}P$ on $\mathcal{F}_T$. Then,

$$\hat{B}_t := -\int_0^t a(B_s)\,\mathrm{d}s + B_t \tag{6.24}$$

is a Q-Brownian motion and

$$\mathrm{d}B_t = \mathrm{d}Y_t = a(Y_t)\,\mathrm{d}T + \mathrm{d}\hat{B}_t. \tag{6.25}$$

The Girsanov Theorem

If we set $Y_0 = x$ the pair $(Y_t, \hat{B}_t)$ is a weak solution of the SDE for $t \leq T$. By weak uniqueness the Q-law of $Y_t = B_t$ coincides with the P-law of X_t^x, so that

$$\begin{aligned} \mathbb{E}\, f_1(X_{t_1}^x) \ldots f_k(X_{t_k}^x) &= \mathbb{E}^Q\left[f_1(Y_{t_1}) \ldots f_k(Y_{t_k})\right] \\ &= \mathbb{E}\, M_T f_1(B_{t_1}) \ldots f_k(B_{t_k}) \end{aligned}$$

for all $f_1, \ldots, f_k \in C_0(\mathbb{R}^n)$; $t_1, \ldots, t_k \leq T$.