Fundamental Algebra

1 Series

1.1 Geometric

$$\sum_{n=a}^{b} r^n = r^a \frac{1-r^{b-a+1}}{1-r} = \frac{r^a - r^{b+1}}{1-r} \tag{.1}$$

$$\sum_{j=1}^{\infty} \frac{j^2}{\rho^j} = \frac{\rho(\rho+1)}{(\rho-1)^3} \tag{.2}$$

1.2 Quotient stack

$$\sum_{k=1}^{n} \lfloor \frac{n}{k} \rfloor = 2 \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \lfloor \frac{n}{k} \rfloor - \lfloor \sqrt{n} \rfloor^2 \tag{.3}$$

2 Inequalities

2.1 Ratio of Summation

$$\frac{\sum_i a_i}{\sum_i b_i} \leq \max_i \frac{a_i}{b_i} \tag{.4}$$

2.2 Weighted AM-GM Inequality

Let the nonnegative numbers $x_1, x_2, \ldots, x_n$ and the nonnegative weights $w_1, w_2, \ldots w_n$ be given. Set $w = w_1 + w_2 + \ldots w_n$. If $w > 0$, then the inequality

$$\frac{w_1 x_1 + w_2 x_2 + \cdots + w_n x_n}{w} \geq \sqrt[w]{x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}} \tag{.5}$$

holds.

2.3 Titu's Lemma

Summation Form. For any real numbers $a_1, a_2, \ldots a_n$ and positive reals $b_1, b_2, b_3, \ldots b_n$, we have

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \ldots b_n} \tag{.6}$$

Probabilistic Form. Let X be a real random variable and Y be a positive random variable such that $\mathbb{E}[|X|]$ and $\mathbb{E}[Y]$ are well defined. Then

$$\mathbb{E}[X^2/Y] \geq \mathbb{E}[|X|^2]/\,\mathbb{E}[Y] \geq \mathbb{E}[X]^2/\,\mathbb{E}[Y] \tag{.7}$$

2.4 Cauchy-Schwarz Inequality

For any non-zero vector $\boldsymbol{x}$,

$$\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{x}\|_1^2 \leq \|\boldsymbol{x}\|_0 \, \|\boldsymbol{x}\|_2^2 \tag{.8}$$

(Note: Useful in binary matrix multiplication.)

2.5 Chebyshev's Sum Inequality

If $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, then

$$\frac{1}{n}\sum_{k=1}^{n} a_k b_k \geq \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \tag{.9}$$

2.6 Symmetric Parametric Inequality

$$\left(1 - p + \frac{p}{x}\right)^{\alpha-2} \cdot (1 - p + px)^{\alpha-2} \geq 1 \tag{.10}$$

2.7 We love Jensen

By convexity of $(u, v) \mapsto u^\lambda v^{1-\lambda}$

$$\mathbb{E}[U]^\lambda \,\mathbb{E}[V]^{1-\lambda} \leq \mathbb{E}[U^\lambda V^{1-\lambda}] \tag{.11}$$

3 Bounds and Approximations

3.1 Exponential Bound on Hyperbolic Ratio

For $0 \leq y < x \leq 2$,

$$\frac{\sinh(x) - \sinh(y)}{\sinh(x-y)} \leq e^{\frac{1}{2}xy} \tag{.12}$$

3.2 Exponential Inequalities

For all $t \in \mathbb{R}$ and $0 \leq p \leq 1$,

$$1 - p + p \cdot x \leq e^{p(x-1)} \tag{.13}$$

Additionally:

$$\forall x, y \geq 0 \quad \frac{1 + e^{x+y}}{e^x + e^y} \leq e^{xy/2} \tag{.14}$$

$$\frac{1}{2}(e^x + e^{-x}) \leq e^{x^2/2} \tag{.15}$$

3.3 Softplus Quadratic Bound

For all $a, x \in \mathbb{R}$ with $a \neq 0$, we have

$$\log(1 + e^x) \leq \log(1 + e^a) + \frac{x - a}{1 + e^{-a}} + \frac{(e^a - 1) \cdot (x - a)^2}{4 \cdot a \cdot (e^a + 1)} \tag{.16}$$

3.4 Miscellaneous upper Bounds

Linear-Rational function. For all $x > 0$, $t \in [0, 1]$:

$$\frac{1}{1 - t + tx} \leq 1 - t(1-t)(1+3t)(x-1) + t^2\left((1-t)x^2 + \frac{t}{x} - 1\right) \tag{.17}$$

Inverse logarithmic. For all $u > 0$:

$$\frac{1}{\log(1 + 1/u)} \leq u + \frac{1}{2} \tag{.18}$$

$x \log_+ y$ **decoupling.** For non negative reals x and y,

$$x \log_+ y + 1 \wedge +1 \le x \log x + e^{-1} y + 1 \tag{.19}$$

Weighted reciprocal. For all $p \in [0,1]$ and $x \in (0, \infty)$,

$$\frac{1}{1 - p + p/x} \le 1 - p + p \cdot x \tag{.20}$$

3.5 Order of Rademacher Sums

Let $\sigma \in \{-1,1\}^n$ be a random Rademacher sequence and let $a \in \mathbb{R}^n$ be an arbitrary real vector with sorted entries $|a_1| \ge |a_2| \ge \cdots \ge |a_n|$. Then

$$\|\langle a, \sigma \rangle\|_{L^p} \sim \sum_{i \le p} a_i + \sqrt{p} (\sum_{i > p} a_i^2)^{1/2} \tag{.21}$$

3.6 Sum Approximation

$$\max(a,b) \le a + b \le 2 \max(a,b), \quad a, b \ge 0 \tag{.22}$$

4 Combinatorics

4.1 Expansion

$$\binom{2n}{m} = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{n}{j} \binom{n-j}{m-2j} 2^{m-2j} \tag{.23}$$

4.2 Vandermonde's Identity

$$\sum_{i=0}^{r} \binom{m}{i} \binom{n}{r-i} = \binom{n+m}{r} \tag{.24}$$

$$\sum_{m=0}^{n} \binom{m}{j} \binom{n-m}{k-j} = \binom{n+1}{k+1} \tag{.25}$$

Special form (**hockey-stick identity**):

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1} \tag{.26}$$

(c.f. $\prod_{\ell=0}^{k-1} \left(\frac{\ell+\eta}{\ell+1} \right) = \binom{k+\eta-1}{k}$)