Convexity

1 Characterization

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable over an open domain. Then, the following are equivalent

1. f is convex
2. $f(y) \geq f(x) + \nabla f(x)^\top (y - x)$, for all $x, y, \in \mathrm{dom}(f)$
3. $\nabla^2 f(x) \succeq 0$, for all $x \in \mathrm{dom}(f)$

여기서 Condition 3은 모든 점에서 non-negative curvature를 가지고 있다는 의미.

1.1 Characterization of Strict Convexity

1. $\nabla^2 f(x) \succ 0$, $\forall \in \Omega$ (The converse is not true)
2. A function f is strictly convex on $\Omega \subseteq \mathbb{R}^n$ if and only if

$$f(y) > f(x) + \nabla^\top f(x)(y - x),\ \forall x, y, \in \Omega, x \neq y$$

3. f is strongly convex if and only if there exists $m > 0$ such that

$$\begin{aligned} & f(y) \geq f(x) + \nabla^\top f(x)(y - x) + m \left\| y - x \right\|^2, \forall x, y, \in \mathrm{dom}(f) \\ \iff & \nabla^2 f(x) \succeq m\boldsymbol{I}, \forall x \in \mathrm{dom}(f) \end{aligned}$$

1.2 When can we assume equal variables?

If the constraints and the function to be optimized are both symmetric with respect to a group of permutations of the variables, then the solution set will also be symmetric with respect to this group.

2 Dual Problem

$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)$ is concave and lower bound of the optimal value.

Example

$$\begin{aligned} \text{maxmize} \quad & \sum_{i=0}^{m} \mathbb{P}_F[F(t) = i] \cdot \mathbb{P}_{\check{W}}[\check{W}(t) \geq v - i] \\ \text{subject to} \quad & \sum_{i=0}^{m} \mathbb{P}[F(t) = i] \cdot i \leq 2m \cdot \delta \\ & \sum_{i=0}^{m} \mathbb{P}_F[F(t) = i] = 1, \text{and} \mathbb{P}_F[F(t) = i] \geq 0\ \forall i \in \{0, 1, \cdots, m\} \end{aligned}$$

Step 1. Find Lagrangian

$$L(F, \alpha, \beta, \lambda) = \sum_{i=0}^{m} \mathbb{P}_F[F(t) = i](\mathbb{P}_{\check{W}}[\check{W}(t) \geq v - i] - \alpha i - \beta - \lambda_i) + 2m\delta\alpha + \beta \quad (2.1)$$

with $\alpha \geq 0, \lambda \geq 0$. Then, $g(\alpha, \beta, \lambda) := \sup_F L(F, \alpha, \beta, \lambda) \geq \sup_{F \in C} L(F, \alpha, \beta, \lambda) \geq f^*$. To drop the first term, add the constraint: $\alpha \cdot i + \beta \geq \mathbb{P}_{\check{W}}[\check{W}(t) \geq v - i]\ \forall i \in \{0, 1, \cdots, m\}$.

Step 2. Optimize the Dual function

$$\begin{aligned} \text{minimize} \quad & 2m\delta\alpha + \beta \\ \text{subject to} \quad & \alpha \cdot i + \beta \geq \mathbb{P}_{\check{W}}[\check{W}(t) \geq v - i] \quad \forall i \in \{0, 1, \ldots, m\} \\ & \alpha \geq 0 \end{aligned}$$

Make α, β as small as possible!

$$\begin{aligned} \beta &= \mathbb{P}_{\check{W}^*}[\check{W} \geq v], \\ \alpha &= \max\left(\{0\} \cup \left\{ \frac{1}{i}\left(\mathbb{P}_{\check{w}^*}[\check{W} \geq v - i] - \beta\right) : i \in \{1, 2, \cdots, m\} \right\}\right) \end{aligned}$$

where $\hat{W}^*$ is a distribution on $\mathbb{R}$ such that $\mathbb{P}[\check{W}^* \geq v - i] \geq \mathbb{P}[\check{W}(t) \geq v - i]$ for all $i \in \{0, 1, \cdots, m\}$ and all t in the support of T.

Remark. Dual problem을 구하는 과정에서 constraint 추가해도 괜찮음.