# Stochastic process

## IV Stochastic Differential Equations

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# Examples and Some Solution Methods

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To solve the SDEs, Itô formula is the key.

**Example 1.** Given $N_0$, solve

$$\mathrm{d}N_t = rN_t\,\mathrm{d}t + \alpha N_t\,\mathrm{d}B_t \tag{1.1}$$

$$\begin{aligned}
&\frac{\mathrm{d}N_t}{N_t} = r\,\mathrm{d}t + \alpha\,\mathrm{d}B_t \\
\implies &\int_0^t \frac{\mathrm{d}N_s}{N_s} = rt + \alpha B_t
\end{aligned}$$

$(\ln(x))' = \frac{1}{x}$ 이므로 Itô formula를 적용할 g로 $g(t,x) = \ln x$가 적절하다.

$$\begin{aligned}
\mathrm{d}\ln N_t &= \frac{1}{N_t}\cdot\mathrm{d}N_t + \frac{1}{2}\left(-\frac{1}{N_t^2}\right)(\mathrm{d}N_t)^2 = \frac{\mathrm{d}N_t}{N_t} - \frac{1}{2}\alpha^2\,\mathrm{d}t \\
\implies \ln\frac{N_t}{N_0} &= (r - \frac{1}{2}\alpha^2)t + \alpha B_t
\end{aligned}$$

We get $N_t = N_0\exp((r - \frac{1}{2}\alpha^2)t + \alpha B_t)$

# Cont.

**Remark.**

$$\mathbb{E}\, N_t = \mathbb{E}\, N_0 e^{rt} \tag{1.2}$$

Apply Itô formula to $Y_t = e^{\alpha B_t}$

$$\mathrm{d}Y_t = \alpha e^{\alpha B_t}\,\mathrm{d}B_t + \frac{1}{2}\alpha^2 e^{\alpha B_t}\,\mathrm{d}t \tag{1.3}$$

$$Y_t = Y_0 + \alpha \int_0^t e^{\alpha B_s}\,\mathrm{d}B_s + \frac{1}{2}\alpha^2 \int_0^t e^{\alpha B_s}\,\mathrm{d}s \tag{1.4}$$

Since $\mathbb{E}\int_0^t e^{\alpha B_s}\,\mathrm{d}B_s = 0$ We get

$$\mathbb{E}\, Y_t = \mathbb{E}\, Y_0 + \frac{1}{2}\alpha^2 \int_0^t \mathbb{E}\, Y_s\,\mathrm{d}s \tag{1.5}$$

which implies $\mathbb{E}\, Y_t = e^{\frac{1}{2}\alpha^2 t}$
**Remark.** Since $B_t = \tilde{\mathcal{O}}(\sqrt{t})$,

$$N_t \to \begin{cases} \infty & \text{if } r > \frac{1}{2}\alpha^2 \\ \text{fluctuate} & \text{if } r = \frac{1}{2}\alpha^2 \\ 0 & \text{if } r < \frac{1}{2}\alpha^2 \end{cases} \tag{1.6}$$

a.s. as $t \to \infty$.

**Example 2.** Solve

$$\mathrm{d}X = AX_t\,\mathrm{d}t + H_t\,\mathrm{d}t + K\,\mathrm{d}B_t \tag{1.7}$$

where

$$\mathrm{d}X = \begin{pmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix}, H_t = \begin{pmatrix} 0 \\ \frac{1}{L}G_t \end{pmatrix}, K = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix} \tag{1.8}$$

Rewrite the eq. (1.8) as

$$\exp(-At)\,\mathrm{d}X_t - \exp(-At)AX_t\,\mathrm{d}t = \exp(-At)(H(t)\,\mathrm{d}t + K\,\mathrm{d}B_t) \tag{1.9}$$

Apply the Itô formula to the function $g : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g(t, x_1, x_2) = \exp(-At)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.10}$$

then we obtain that

$$\mathrm{d}(\exp(-At)X_t) = -A\exp(-At)X_t\,\mathrm{d}t + \exp(-At)\,\mathrm{d}X_t \tag{1.11}$$

This gives

$$X_t = \exp(At)(X_0 + \int_0^t \exp(-As)H_s\,\mathrm{d}s + \int_0^t \exp(-As)K\,\mathrm{d}B_s) \tag{1.12}$$

# Examples and Some Solution Methods

**Example 3.** Let $(B_1, \ldots, B_n)$ be Brownian motion in $\mathbb{R}^n, \alpha_1, \ldots, \alpha_n$ constants. Solve the stochastic differential equation

$$\mathrm{d}X_t = rX_t\,\mathrm{d}t + X_t\left(\sum_{k=1}^{n}\alpha_k\,\mathrm{d}B_k(t)\right); \quad X_0 > 0. \tag{1.13}$$

Divide by $X_t$:

$$\frac{\mathrm{d}X_t}{X_t} = r\,\mathrm{d}t + \sum_{k=1}^{n}\alpha_k\,\mathrm{d}B_k(t) \tag{1.14}$$

Note that $\mathrm{d}(\log X_t) = \frac{1}{X_t}\,\mathrm{d}X_t - \frac{1}{2X_t^2}(\,\mathrm{d}X_t)^2$ and $(\,\mathrm{d}X_t)^2 = X_t^2\sum_{k=1}^{n}\alpha_k^2\,\mathrm{d}t$ which implies

$$\begin{aligned}
\mathrm{d}(\log X_t) &= r\,\mathrm{d}t + \sum_{k=1}^{n}\alpha_k\,\mathrm{d}B_k(t) - \frac{1}{2}\sum_{k=1}^{n}\alpha_k^2\,\mathrm{d}t \\
\log(X_t/X_0) &= \left(r - \frac{1}{2}\sum_{k=1}^{n}\alpha_k^2\right)t + \sum_{k=1}^{n}\alpha_k B_k(t) \\
\implies X_t &= X_0\exp\left(\left(r - \frac{1}{2}\sum_{k=1}^{n}\alpha_k^2\right)t + \sum_{k=1}^{n}\alpha_k B_k(t)\right)
\end{aligned}$$

# An Existence and Uniqueness Result

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We now turn to the existence and uniqueness question (A) above.

**Theorem 1 (Existence and uniqueness theorem for stochastic differential equations)**

Let $T > 0$ and $b(\cdot,\cdot) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^n, \sigma(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying

$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|); \quad x \in \mathbb{R}^n, t \in [0,T] \tag{2.1}$$

where $|\sigma|^2 = \sum_{i,j} |\sigma_{ij}|^2$ and such that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le D|x-y|; \quad x,y \in \mathbb{R}^n, t \in [0,T] \tag{2.2}$$

Let $Z$ be a random variable which is independent of $\sigma$-algebra $\mathcal{F}_\infty$ generated by $B_s(\cdot), s \ge 0$ and such that $\mathbb{E}\,|Z|^2 < \infty$. **Then** the SDE

$$\mathrm{d}X_t = b(t,X_t)\,\mathrm{d}t + \sigma(t,X_t)\,\mathrm{d}B_t, \quad 0 \le t \le T, X_0 = Z \tag{2.3}$$

has a unique continuous in $t$ solution $X_t(w)$ with the property

- $X_t(w)$ is adapted to the filtration generated by $B_s(\cdot), s \le t$ and $Z$
- $$\mathbb{E}\int_0^T |X_t|^2\,\mathrm{d}t < \infty \tag{2.4}$$

# An Existence and Uniqueness Result

**Lemma 1 (Gronwall's ineqaulity)**

Let $v(t)$ be a nonnegative function such that

$$v(t) \leq C + A \int_0^t v(s)\,\mathrm{d}s \ \text{ for } 0 \leq t \leq T \text{ and some constants } C, A \tag{2.5}$$

then

$$v(t) \leq C \exp(At) \ \text{ for } 0 \leq t \leq T \tag{2.6}$$

**Proof** For non trivial case, assume $A \neq 0$. Let $\phi(t) = \int_0^t v(s)\,\mathrm{d}s$, then $\phi'(t) \leq C + A\phi(t)$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-At}\phi(t)) = e^{-At}(\phi'(t) - A\phi(t)) \leq Ce^{-At}. \tag{2.7}$$

Integrating both sides give us $\phi(t) \leq \frac{C}{A}(e^{At} - 1)$, plugging into the $\phi'(t) \leq C + A\phi(t)$ gives us $v(t) \leq C\exp(At)$.

# Proof.

- **Uniqueness** Let $X_1(t,w), X_2(t,w)$ be solutions with initial values $Z, \hat{Z}$ resp. We will show that

$$\mathbb{P}(\{X_1(t,w) = X_2(t,w)\}, t \in [0,T]) = 1 \tag{2.8}$$

Put $a(s,w) = b(s,X_s) - b(s,\hat{X}_s)$ and $\gamma(s,w) = \sigma(s,X_s) - \sigma(s,\hat{X}_s)$. Then

$$\begin{aligned}
\mathbb{E}\,|X_t - \hat{X}_t|^2 &= \mathbb{E}\left(Z - \hat{Z} + \int_0^t a\,\mathrm{d}s + \int_0^t \gamma\,\mathrm{d}B_s\right)^2 \\
&\le 3\,\mathbb{E}\,|Z - \hat{Z}|^2 + 3\,\mathbb{E}\left(\int_0^t a\,\mathrm{d}s\right)^2 + 3\,\mathbb{E}\left(\int_0^t \gamma\,\mathrm{d}B_s\right)^2 \\
&\le 3\,\mathbb{E}\,|Z - \hat{Z}|^2 + 3t\,\mathbb{E}\int_0^t a^2\,\mathrm{d}s + 3\,\mathbb{E}\int_0^t \gamma^2\,\mathrm{d}s \\
&\le 3\,\mathbb{E}\,|Z - \hat{Z}|^2 + 3(1+t)D^2 \int_0^t \mathbb{E}\,|X_s - \hat{X}_s|^2\,\mathrm{d}s
\end{aligned}$$

Let $v(t) = \mathbb{E}\,|X_t - \hat{X}_t|^2; \quad 0 \le t \le T$ satisfies

$$v(t) \le F + A\int_0^t v(s)\,\mathrm{d}s \quad \text{where } F = 3\,\mathbb{E}\,|Z - \hat{Z}|^2 \text{ and } A = 3(1+T)D^2 \tag{2.9}$$

# Cont.

By the Gronwall inequality, we conclude that

$$v(t) \leq F \exp(At) \tag{2.10}$$

Now assume that $Z = \hat{Z}$. Then $F = 0$ and $v(t) = 0$ for each $t \geq 0$. By the continuity of $t \mapsto |X_t - \hat{X}_t|$ the argument follows.

- **Existence**

Define $Y_t^{(0)} = X_0$ and $Y_t^{(k)}$ inductively as follows

$$Y_t^{(k+1)} = X_0 + \int_0^t b(s, Y_s^{(k)})\,\mathrm{d}s + \int_0^t \sigma(s, Y_s^{(k)})\,\mathrm{d}B_s \tag{2.11}$$

Then, a similar computation as for the uniqueness above gives

$$\mathbb{E}\,|Y_t^{(k+1)} - Y_t^{(k)}|^2 \leq (1+T)3D^2 \int_0^t \mathbb{E}\,|Y_s^{(k)} - Y_s^{(k-1)}|^2\,\mathrm{d}s, \tag{2.12}$$

for $k \geq 1, t \leq T$ and

$$\begin{aligned} \mathbb{E}\,|Y_t^{(1)} - Y_t^{(0)}|^2 &= \mathbb{E}\left|\int_0^t b(s, X_0)\,\mathrm{d}s + \int_0^t \sigma(s, X_0)\,\mathrm{d}B_s\right|^2 \\ &\leq 2C^2t^2(1 + \mathbb{E}\,|X_0|^2) + 2C^2t(1 + \mathbb{E}\,|X_0|^2) \leq A_1 t \end{aligned}$$

where the constant $A_1$ only depends on $C, T$ and $\mathbb{E}\,|X_0|^2$.

# Cont (2).

So by induction on $k$, we obtain

$$\mathbb{E}\left|Y_t^{(K+1)} - Y_t^{(k)}\right|^2 \leq \frac{A_2^{k+1}t^{k+1}}{(k+1)!}; \quad k \geq 0, t \in [0,T] \qquad (2.13)$$

for some suitable constant $A_2$ depending only on $C, D, T$ and $\mathbb{E}\,|X_0|^2$. Now

$$\begin{aligned}\sup_{0\leq t\leq T}|Y_t^{(k+1)} - Y_t^{(k)}| \leq & \int_0^T \left|b(s,Y_s^{(k)}) - b(s,Y_s^{(k-1)}\right| \mathrm{d}s \\ & + \sup_{0\leq t\leq T}\left|\int_0^t (\sigma(s,Y_s^{(k)}) - \sigma(s,Y_s^{(k-1)}))\,\mathrm{d}B_s\right|\end{aligned}$$

Since $\left\{A+B > 2^{-k}\right\} \subseteq \left\{A > 2^{-k-1}\right\} \cup \left\{B > 2^{-k-1}\right\}$,

$$\begin{aligned}&\mathbb{P}(\sup_{0\leq t\leq T}|Y_t^{(k+1)} - Y_t^{(k)}| > 2^{-k}) \\ &\leq \mathbb{P}\left(\left(\int_0^T |b(s,Y_s^{(k)}) - b(s,Y_s^{(k-1)})|\,\mathrm{d}s\right)^2 > 2^{-2k-2}\right) \\ &+ \mathbb{P}\left(\sup_{0\leq t\leq T}\left|\int_0^t (\sigma(s,Y_s^{(k)}) - \sigma(s,Y_s^{(k-1)}))\,\mathrm{d}B_s\right| > 2^{-k-1}\right) \\ &\leq 2^{2k+2}D^2(T+1)\int_0^T \frac{A_2^k t^k}{k!}\,\mathrm{d}t \leq \frac{(4A_2T)^{k+1}}{(k+1)!}, \ \text{ if } A_2 \geq D^2(T+1)\end{aligned}$$

# Cont (3).

Therefore, by the Borel-Cantelli Lemma,

$$\mathbb{P}\left(\sup_{0\le t\le T}|Y_t^{(k+1)}-Y_t^{(k)}|>2^{-k}\ \text{ for infinitely many } k\right)=0 \tag{2.14}$$

Therefore, almost surely there exists $k_0(w)$ such that

$$\sup_{0\le t\le T}|Y_t^{(k+1)}-Y_t^{(k)}|\le 2^{-k}\quad \text{ for } k\ge k_0 \tag{2.15}$$

Therefore the sequence

$$Y_t^{(n)}=Y_t^{(0)}+\sum_{k=0}^{n-1}(Y_t^{(k+1)}(w)-Y_t^{(k)}(w)) \tag{2.16}$$

is uniformly convergent in $[0,T]$ almost surely. Denotes the limit process by $X_t$. Then, $X_t$ is $t$-continuous and $\mathcal{F}_t^Z$-measurable for all $t$.

Moreover using eq. (2.13), we can deduce that

$$\mathbb{E}\,|Y_t^{(m)}-Y_t^{(n)}|^{2\,1/2}\le\sum_{k=n}^{\infty}\left(\frac{(A_2t)^{k+1}}{(k+1)!}\right)^{1/2}\to 0\ \text{ as } n\to\infty \tag{2.17}$$

So $\{Y_t^{(n)}\}_{n=1}^{\infty}$ converges in $L^2(P)$ to a limit $Y_t$, then we must have $Y_t=X_t$.

# Cont (4).

It remains to show $X_t$ satisfies eq. (2.3). For all $n$ we have

$$Y_t^{(n+1)} = X_0 + \int_0^t b(s, Y_s^{(n)})\,\mathrm{d}s + \int_0^t \sigma(s, Y_s^{(n)})\,\mathrm{d}B_s. \tag{2.18}$$

LHS goes to $X_t$ as $n \to \infty$ uniformly almost surely. By eq. (2.17) and the Fatou lemma we have

$$\mathbb{E}\int_0^T |X_t - Y_t^{(n)}|^2\,\mathrm{d}t \le \liminf \mathbb{E}\int_0^T |Y_t^{(m)} - Y_t^{(n)}|^2\,\mathrm{d}t \to 0 \tag{2.19}$$

as $n \to \infty$. It follows by the Itô isometry that

$$\int_0^t \sigma(s, Y_s^{(n)})\,\mathrm{d}B_S \to \int_0^t \sigma(s, X_s)\,\mathrm{d}B_S \tag{2.20}$$

in $L^2(P)$ and by the Hölder inequality that

$$\int_0^t b(s, Y_s^{(n)})\,\mathrm{d}s \to \int_0^t b(s, X_s)\,\mathrm{d}s \tag{2.21}$$

in $L^2(p)$. By taking the limit, we obtain eq. (2.3) for $X_t$.

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- Why do we need that independent $Z$ of $B$?

We need $B$ to be a Brownian motion with respect to the filtration $\mathcal{F}_t^Z$ to define $\int_0^t \sigma(s, Z)\,\mathrm{d}B_s$.

# Weak and Strong Solutions

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- If We are only given the functions $b(t,x)$ and $\sigma(t,x)$ and ask for a pair of processes $((\tilde{X}_t, \tilde{B}_t), \mathcal{H}_t)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the solution $(\tilde{X}_t, \tilde{B}_t)$ is called a weak solution. Here $\mathcal{H}_t$ is an increasing family of $\sigma$-algebras such that $\tilde{X}_t$ is $\mathcal{H}_t$-adapted and $\tilde{B}_t$ is an $\mathcal{H}_t$-Brownian motion.

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**Lemma 2**

If $b$ and $\sigma$ satisfy the conditions of Theorem 1 then we have

A solution (weak or strong) of eq. (2.3) is weakly unique.

# Weak and Strong Solutions

- There are stochastic differential equations which have no strong solutions but still a (weakly) unique weak solution

**Example 3 (The Tanaka equation).** Consider the $1$-dimensioanl stochastic differential equation

$$dX_t = \mathsf{sign}(X_t)\,\mathrm{d}B_t; \quad X_0 = 0 \tag{3.1}$$

where

$$\mathsf{sign}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Note that $\sigma(t, x) = \mathsf{sign}(x)$ does not satisfy the Lipschitz condition, so theorem 1 does not apply. Indeed, the eq. (3.1) has no strong solution. To see this, let $\hat{B}_t$ be a Brownian motion generating the filtration $\hat{\mathcal{F}}_t$ and define

$$Y_t = \int_0^t \mathsf{sign}(\hat{B}_s)\,\mathrm{d}\hat{B}_s$$

By the Tanaka formula we have

$$Y_t = |\hat{B}_t| - |\hat{B}_0| - \hat{L}_t(w),$$

where $\hat{L}_t(w)$ is the local time for $\hat{B}_t(w)$ at $0$.

# Cont (2).

It follows that $Y_t$ is measurable w.r.t the $\sigma$-algebra $\mathcal{G}_t$ generated by $|\hat{B}_s(\cdot)|; s \leq t$, which is strictly contained in $\hat{\mathcal{F}}_t$. Hence the $\sigma$-algebra $\mathcal{N}_t$ generated by $Y_s(\cdot); s \leq t$ is also strictly contained in $\hat{\mathcal{F}}_t$

Now suppose $X_t$ is a strong solution of eq. (3.1). It follows that $X_t$ is a Brownian motion w.r.t the measure $\mathbb{P}$. Let $\mathcal{M}_t$ be the $\sigma$-algebra generated by $X_s(\cdot); s \leq t$. We can write eq. (3.1) as

$$\mathrm{d}B_t = \mathsf{sign}(X_t)\,\mathrm{d}X_t \tag{3.2}$$

By the above argument applied to $\hat{B}_t = X_t, Y_t = B_t$ we conclude that $\mathcal{F}_t$ is strictly contained in $\mathcal{M}_t$. But this contradicts that $X_t$ is a strong solution. Hence strong solutions of eq. (3.1) do not exist.

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To find a weak solution of eq. (3.1) we simply choose $X_t$ to be *any* Brownian motion $\hat{B}_t$. Then we define $\tilde{B}_t$ by

$$\tilde{B}_t = \int_0^t \mathsf{sign}(\hat{B}_s)\,\mathrm{d}\hat{B}_s = \int_0^t \mathsf{sign}\,\mathrm{d}X_s$$

i.e. $\mathrm{d}\tilde{B}_t = \mathsf{sign}(X_t)\,\mathrm{d}X_t$. Then

$$\mathrm{d}X_t = \mathsf{sign}(X_t)\,\mathrm{d}\tilde{B})_t$$

so $X_t$ is a weak solution.