Useful real analysis results

1 Leibniz's Integral Rule

Let μ be a probability distribution with support Ω, let $I \subset \mathbb{R}$ be a nontrivial open interval, also let $f : \Omega \times I \to \mathbb{R}$ be a map with the following properties:

1. For any $x \in I$, $\mathbb{E}_{w \sim \mu}[|f(w,x)|] < \infty$ (**Uniformly finite moment**)

2. For almost all $w \in \infty$, the map $x \mapsto f(w,x)$ is differentiable with derivative $\frac{\partial}{\partial x} f(w,x)$ (**differentiability**)

3. There is a map $h : \Omega \to \mathbb{R}$ with the property that $\mathbb{E}_{w \sim \mu}[|h(w)|] < \infty$, such that $\left| \frac{\partial}{\partial x} f(\cdot, x) \right| \leq h$ (**derivative bound**).

Then, for any $x \in I$, $E_{w \sim \mu}\left[\left| \frac{\partial}{\partial x} f(w,x) \right| \right] < \infty$ and the function $F : x \to \mathbb{E}_{w \sim \mu}[f(w,x)]$ is differentiable with derivative

$$F'(x) = \mathop{\mathbb{E}}_{w \sim \mu} \left[\frac{\partial}{\partial x} f(w,x) \right] \tag{.1}$$