Probability Theory

IV Kolmogorov Extension, 0-1 law

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i.i.d Random Variables

A sequence $\{X_n\}_{n=1}^{\infty}$ of random variables $X_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B})$ if all the X_n are independent, and $\mu_{X_n} = \mu_{X_1} \forall \mathbb{N}$

Goal: Construct *Infinite* i.i.d sequences. For the finite case, it's not difficult.

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Lemma 1

Let $\mu_1, \ldots, \mu_N$ be probability measures on $(S_1, \mathcal{B}_1), \ldots (S_N, \mathcal{B}_N)$. Deinfe

$$\Omega = S_1 \times \cdots \times S_N,\ \mathcal{F} = \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_N,\ \mathbb{P} = \mu_1 \otimes \cdots \otimes \mu_N$$

Then the random variables $X_n : \Omega \to S_n, X_n = \pi_n(x) = x_n$ are independent, and $\mu_{X_n} = \mu_n$

Proof.

Kolmogorov's Extension Theorem

Setup: Want a probability measure on (Say) $\mathbb{R}^{\mathbb{N}} = \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{R} \forall n \in \mathbb{N}\}$
To take advantage of compactness results, we replace $\mathbb{R}$ with $[0,1]$. $Q := [0,1]^{\mathbb{N}}$

Definition 1

Q is given the topology of *pointwise convergence*:

$$x^1 = (x_n^1)_{n=1}^{\infty}, x^2, \dots x^k \in Q \text{ converge to } x \in Q \text{ if and only if } x_n^k \to x_n \ \forall n \in \mathbb{N}.$$

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Theorem 1 (Tychonoff)

Q is (sequentially) compact. If $(x^m)_{m=1}^{\infty}$ is a sequence in Q, it has a convergent subsequence $(x^{m_k})_{k=1}^{\infty}$.

Corollary 1 (Finite Intersection Property)

If $K_m \subseteq Q$ are closed subsets s.t. $\bigcap_{i=1}^{m} K_i \neq \emptyset \ \forall m \in \mathbb{N}$, then $\bigcap_{i=1}^{\infty} K_i \neq \emptyset$

Proof.

Regular Borel Measures

If Ω is a (locally compact Hausdorff) topological space, a measure μ on $\mathcal{B}(\Omega)$ is called

- *outer-regular* if $\mu(B) = \inf\{\mu(V) : B \subseteq V, V \text{ open }\}$
- *inner-regular* if $\mu(B) = \sup\{\mu(K) : K \subseteq B, k \text{ compact }\}$

A Borel measure μ is a **Radon measure** if it is locally finite: $\mu(K) < \infty\ \forall K \subseteq \Omega$ compact, and it is both outer- and inner-regular.

Theorem 2

All finite (e.g. probability) Borel measures on $\mathbb{R}^d$ are Radon measures.

Proof.

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Proof. Define
$\mathcal{F} := \left\{B \in \mathcal{B}\left(\mathbb{R}^d\right) \mid \forall \varepsilon > 0 \; \exists \text{ open } V, \text{ closed } C \text{ s.t. } C \subseteq B \subseteq V, \mu(V \setminus C) < \varepsilon\right\}$.
We will show that $\mathcal{F} = \mathcal{B}\left(\mathrm{r}^d\right)$. This suffices:
We can find $V_n \supseteq B$ s.t. $\mu(V_n \setminus B) < \frac{1}{n}$. In addition, we can find closed C_n s.t. $\mu(B) - \mu(C_n) < \frac{1}{n}$,

$$\mu(B) = \sup\{\mu(C) \mid C \subseteq B, C \text{ closed }\}$$

Also, $\bar{B}^d(0,n) \underset{n\to\infty}{\uparrow} \mathbb{R}^d$,

$$\therefore \bar{B}^d(0,n) \cap C \uparrow C \implies \mu(\bar{B}^d(0,n) \cap C) \to \mu(C).$$

Proof Cont.

We will show that $\mathcal{F}$ is a σ-field containing all closed sets.

- $\mathcal{F}$ is contains all closed sets: Let C be closed. Fix $\varepsilon > 0$, let $C_\varepsilon \bigcup_{x \in C} B(x, \varepsilon)$, C_ε is open and $C_\varepsilon \downarrow C$ as $\varepsilon \downarrow 0$ (in general, $C_\varepsilon \downarrow \bar{C}$).

- $\mathcal{F}$ is an algebra. Clearly, $\emptyset \in \mathcal{F}$ since $\emptyset \subseteq \emptyset \subseteq \emptyset \quad \mu(\emptyset \setminus \emptyset) = 0$.
If $A \in \mathcal{F}$, find $C \subseteq A \subseteq V$ with $\mu(V \setminus C) < \varepsilon$.
$V^c \subseteq A^c \subseteq C^c$. $C^c \setminus V^c = C^c \cap (V^c)^c = C^c \cap V = V \setminus C$.
$\therefore \mu(C^c \setminus V^c) = \mu(V \setminus C) < \varepsilon$.

- $\mathcal{F}$ is closed under countable disjoint union. Take disjoint $\{A_n\} \in \mathcal{F}$, find $C_n \subseteq A_n \subseteq V_n$ with $\mu(V_n \setminus C_n) < \frac{\varepsilon}{2^{n+1}}$
Fix $n \in \mathbb{N}$, let $D_n = C_1 \cup \cdots \cup C_N$ which is closed and $V = \bigcup_{n=1}^{\infty} V_n$ which is open.
Then, $D_n \subseteq \bigsqcup_{n=1}^{\infty} A_n \subseteq V$

$$\mu(V \setminus D_N) \leq \sum_{n=1}^{\infty} \mu(V_n \setminus D_N) \leq \sum_{n=1}^{N} \underbrace{\mu(V_n \setminus C_n)}_{<\varepsilon/2^{n+1}} + \sum_{n=N+1}^{\infty} \underbrace{\mu(V_n)}_{<\mu(A_n)+\varepsilon/2^{n+1}}$$

$$\leq \frac{\varepsilon}{2} + \sum_{n=N+1}^{\infty} \mu(A_n)$$

Now let $N \to \infty$, this shows that $\mathcal{F}$ is closed under countable disjoint union.

Kolmogorov Extension Theorem

Theorem 3 (Kolmogorov)

Let ν_n be a probability measure on $([0,1]^n, \mathcal{B}([0,1]^n))$, and suppose these measures satisfy the following consistency condition:

$$\nu_{n+1}(B \times [0,1]) = \nu_n(B), \quad \forall B \in \mathcal{B}([0,1]^n)$$

Then, there exists a unique probability measure $\mathbb{P}$ on $(Q, \mathcal{B}(Q))$ such that

$$\mathbb{P}(B \times Q) = \nu_n(B), \quad \forall B \in \mathcal{B}([0,1]^n)$$

Special Case: $\nu_n = \mu_1 \otimes \cdots \otimes \mu_n$, μ_j is a Borel probability measure on $[0,1]$

$\nu_{n+1} = \mu_1 \otimes \cdots \otimes \mu_{n+1} = \nu_n \otimes \mu_{n+1}$

$\nu_{n+1}(B \times [0,1]) = \nu_n \otimes \mu_{n+1}(B \times [0,1]) = \nu_n(B)\mu_{n+1}([0,1]) = \nu_n(B)$

Kolmogorov Extension Theorem

Corollary 2

Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $\pi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ the projection $\pi_n((x_k)_{k=1}^{\infty}) = x_n$ $\mathcal{B}_n := \sigma\{\pi_k : k \leq n\}$, $\mathcal{B} := \sigma(\mathcal{B}_n : n \in \mathbb{N})$

Let ν_n be Borel probabiliy measure on $\mathbb{R}^n$ such that

$$\nu_{n+1}(B \times \mathbb{R}) = \nu_n(B) \quad \forall B \in \mathcal{B}(\mathbb{R}^n)$$

Then $\exists!$ probability measure $\mathbb{P}$ on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B})$ s.t.

$$\mathbb{P}(B \times \mathbb{R}^{\mathbb{N}}) = \nu_n(B) \quad \forall B \in \mathcal{B} \in \mathcal{B}(\mathbb{R}^n)$$

Corollary 3

Let μ_n be Borel probability measures on $\mathbb{R}$. There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence $X_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ of independent random variables, such that

$$\mu_{X_n} = \mu_n \quad \forall n \in \mathbb{N}$$

Proof. Take $\Omega = \mathbb{R}^{\mathbb{N}}, \mathcal{F} = \sigma\{\pi_n : n \in \mathbb{N}\}$. Define $\nu_n = \mu_1 \otimes \cdots \otimes \mu_n$. Then $\nu_{n+1}(B \times \mathbb{R}) = \nu_n(B)$.

Proof of Theorem 3

Set $\mathcal{B}_n = \{B \times Q : B \in \mathcal{B}([0,1]^n)\} = \sigma\{\pi_1, \dots, \pi_n\}$, where
$\pi_k : Q \to [0,1], \pi_k((x_n)_{n=1}^{\infty}) = x_k$
Let $\mathcal{A} := \bigcup_{n \geq 1} \mathcal{B}_n$, not that $\mathcal{A}$ is an *algebra*. Also if $C \subseteq Q$ is closed, let
$B_n = \pi_1 \times \cdots \times \pi_n(C) \subseteq [0,1]^n$, closed.
Then $C = \bigcap_{n=1}^{\infty} (\pi_1 \times \dots \pi_n)^{-1}(B_n) \implies C \in \sigma\{\pi_n : n \in \mathbb{N}\} = \sigma(\mathcal{A})$
$\implies \mathcal{B}(Q) = \sigma(\mathcal{A})$

Now, define : $\mathbb{P}(A \times Q) := \nu_n(A) \quad \forall A \in \mathcal{A}$ (†).
Using the consistency condition, we see that $\mathbb{P}$ is a finitely-additive measure on $\mathcal{A}$.
Thus is suffices to show that $\mathbb{P}$ is a premeasure on $\mathcal{A}$. Then is extends to a measure $\bar{\mathbb{P}}$ on $\bar{\mathcal{A}}$. Set $\mathbb{P} := \bar{\mathbb{P}}|_{\sigma\mathcal{A} = \mathcal{B}(Q)}$. Then † will hold for all $A \in \sigma(\mathcal{A})$.

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Now, define : $\mathbb{P}(A \times Q) := \nu_n(A) \quad \forall A \in \mathcal{A}$ (**†**).
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Thus, suffice to show $\mathbb{P}(A_n) \downarrow 0$ whenever $A_n \downarrow \emptyset, A_n \in \mathcal{A}$.

We will prove the contraposition: if $B_n \in \mathcal{A}, B_n \downarrow 0$, and $\inf_n \mathbb{P}(B_n) = \varepsilon > 0$, then $B := \bigcap_n B_n \neq \emptyset$

Proof of Theorem 3

Claim: Suffices to assume $B_n \in \mathcal{B}_n$.
$B_n \in \mathcal{A} = \bigcup_n \mathcal{B}_n \implies B_n \in \mathcal{B}_{m_n}$ We can define a new sequence $(\tilde{B}_k)$ so that $\tilde{B}_k \in \mathcal{B}_k$ by spreading out the (B_n). The new sequence also $\tilde{B}_k$ also satisfy $\inf_k \mathbb{P}(\tilde{B}_k) = \varepsilon$, $\bigcap_k \tilde{B}_k = \bigcap_n B_n$.

So, we can set $B_n = B'_n \times Q,\ B'_n \in \mathcal{B}([0,1]^n)$. By regularity, find compact $K'_n \subseteq B'_n$ such that $\nu_n(B'_n \setminus K'_n) < \varepsilon/2^{n+1} \implies \mathbb{P}(B_n \setminus K_n) < \varepsilon/2^{n+1}$.

Thus, $\mathbb{P}(B_n \setminus \bigcap_{i=1}^n K_i) = \mathbb{P}(\bigcup_{i=1}^n (B_n \setminus K_i)) \leq \sum_{i=1}^n \mathbb{P}(B_n \setminus K_i) \leq \sum_{i=1}^n \mathbb{P}(B_i \setminus K_i) < \sum_{i=1}^n \frac{\varepsilon}{2^{i+1}} < \frac{\varepsilon}{2}$.

But we assumed $\inf_n \mathbb{P}(B_n) = \varepsilon > 0$. Thus

$$\mathbb{P}(\bigcap_{i=1}^n K_i) = \mathbb{P}(B_n) - \mathbb{P}(B_n \setminus \bigcap_{i=1}^n K_i) > \varepsilon - \frac{\varepsilon}{2} > 0$$

$\implies \bigcap_{i=1}^n K_i \neq \emptyset, \forall n$ and $\bigcap_{i=1}^\infty K_i \neq \emptyset$ by the finite intersection property for the closed K_i
$\implies \bigcap_{i=1}^\infty B_i \neq \emptyset$

Tail Events

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variabels defiend on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The **tail** σ**-field** τ of there r.v's is

$$\tau := \bigcap_{n=1}^{\infty} \sigma(X_n, x_{n+1}, X_{n+2}, \dots)$$

Example: $\{\lim_{n\to\infty} X_n \text{ exists}\} \in \tau$

Let $S_n = X_1 + \cdots + X_n$. $\{\lim_{n\to\infty} S_n \text{ exists }\} \in \tau$

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Theorem 4 (Kolmogorov's 0-1 Law)

If $\{X_n\}_{n=1}^{\infty}$ are independent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then for any tail event $E \in \tau(X_n : n \in \mathbb{N})$, $\mathbb{P}(E) = 0$ or 1.

Proof.

Tail Events

Let $\{X_n\}_{n=1}^{\infty}$ be independent rv's . Define $S_n = X_1 + \cdots + X_n$. Let $b_n \in (0, \infty)$ s.t. $b_n \uparrow \infty$ as $n \uparrow \infty$. Note that

$$\{\lim_{n\to\infty} \frac{S_n}{b_n} = c\} \in \tau(X_n : n \geq 1)$$

$\implies \mathbb{P}(S_n/b_n \to c) = 0$ or 1.

Question: What kind of random variables are τ-measurable?

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Question: What kind of random variables are τ-measurable?

Proposition 1

Let $\{X_n\}_{n=1}^{\infty}$ be random variables. Let $\varepsilon > 0$. if Y is $\sigma(X_1, X_2, \ldots)$-measurables and bounded, there is some $N \in \mathbb{N}$ and a Borel function $F : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\mathbb{E}[|Y - F(X_1, \ldots X_N)|] < \varepsilon$$

τ-measurable function

So, if Y is τ -measurable, it is $\sigma(X_n, X_{n+1}, \dots)$-measurable $\forall n$. This suggests that Y is a "function of nothing". If $\{X_n\}_{n=1}^{\infty}$ are independent, this is rigorous.

Proposition 2

Let $\{X_n\}_{n=1}^{\infty}$ be independent. If Y is a $\bar{\mathbb{R}}$ -valued random variables that is tail-measurable, then $\exists c \in \bar{\mathbb{R}}$ s.t. $Y = c$ a.s.