V SLLN

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The Law of Large Numbers: Revisited

Recall the weak Law of Large Numbers:
Let $\{X_n\}_{n=1}^{\infty}$ be uncorrelated L^2 random variables, $\mathrm{Cov} x_n x_m = 0$, $\forall n \neq m$ and suppose that $\mathbb{E}[X_n] = \alpha \ \forall n$, $\mathbb{E}[X_n^2] = S^2 \ \forall n$
Set $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n}{n} \xrightarrow[\mathbb{P}]{} \alpha$$

There are at least two ways we could improve the result

1. Weaken the hypothesis that $X_n \in L^2$, $X_n \in L^1$ should suffice.
2. Strengthen the convergence to almost sure convergence.

We are aiming to prove the following result:

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2. Strengthen the convergence to almost sure convergence.

We are aiming to prove the following result:

Theorem 1 (Kolmogorov's Strong Law of Large Numbers)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d L^1 random variables with $\mathbb{E}[X_n] = \alpha$. Let $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n}{n} \xrightarrow[\mathbb{P}]{} \alpha \; a.s.$$

The Law of Large Numbers

Corollary 1

If $X_n \notin L^1$ but $X_n^- \in L^1$, then $\frac{S_n}{n} \to +\infty$ a.s.

Definition 1 (Tail Equivalence)

Two sequences $\{X_n\}_{n=1}^{\infty}, \{X'_n\}_{n=1}^{\infty}$ on a common probability space are called *tail equivalent* if

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq X'_n) < \infty$$

By the Borel-Cantelli Lemma I, setting $A_n = \{X_n \neq X'_n\}$, we have $\mathbb{P}(A_n \text{ i.o }) = 0$
I.e. $\exists$ null set N s.t. $\forall w \in N^c$,

$$X_n(w) = X'(w) \quad \forall \text{ but finitely many } n$$

Tail equivalence

Corollary 2

If $\{X_n\}_{n=1}^{\infty}, \{X'_n\}_{n=1}^{\infty}$ are tail equivalent , and $b_n \uparrow \infty$, if $\exists$ r.v. X s.t. $\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{n} X'_j = X$ a.s, then

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{j=1}^{n} X_j = X \text{ a.s.}$$

We would like to find a sequence of cut-offs $X'_n = X_n \mathbf{1}_{|X_n| \le M_n}$ so that $\{X_n\}_{n=1}^{\infty}, \{x'\}_{n=1}^{\infty}$ are tail equivalent. To this end, we have:

Tail equivalence

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Lemma 1

If $X_1 \in L^1$ and $\varepsilon > 0$, then $\sum_{n=1}^{\infty} \mathbb{P}(|X| \ge n\varepsilon) \le \frac{1}{\varepsilon}\mathbb{E}[|X|]$

Proof. Note that $x \mapsto \sum_{n=1}^{\infty} \mathbf{1}_{[n,\infty)}(x) \le x$.
$\mathbb{E}\left(\sum_{n=1}^{\infty} \mathbf{1}_{[n,\infty)}(x)\right) \le \mathbb{E}\left(\frac{|x|}{\varepsilon}\right) \implies \mathbb{E}(\mathbf{1}_{\{|X| \ge n\varepsilon\}}) \le \sum_{n=1}^{\infty} \mathbb{E}\left(\frac{|X|}{\varepsilon}\right)$

Tail equivalence

Corollary 3

If $\{X_n\}_{n=1}^{\infty}$ are i.i.d and L^1, they are tail equivalent to $X_n' = X_n \mathbf{1}_{|X_n| \leq n}$

Proof.

$\sum_{n=1}^{\infty} \mathbb{P}(X_n' \neq X_n) = \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n) = \sum_{n=1}^{\infty} \mathbb{P}(|X_1| > n) \leq \mathbb{E}[|X_1|] < \infty$

Thus, in order to prove the SLLN, it suffices to prove: If $\{X_n\}_{n=1}^{\infty}$ is an iid sequence of L^1 random variables with $\mathbb{E}[X_n] = \alpha$, and $S_n' = \sum_{k=1}^{n} X_k \mathbf{1}_{\{|X_k| \leq k\}}$, then $\frac{S_n'}{n} \to \alpha$ a.s.
Advantages: bounded in L^2
Disadvantage: X_n' not identically distributed.

L^2-convergence

Let $\{n\} Y_n$ be uncorrelated random variables in L^2.

Proposition 1

If $\{Y_n\}_{n=1}^{\infty}$ are uncorrelated, and $\sum_{n=1}^{\infty} \mathrm{Var} Y_n < \infty$, then $\sum_{n=1}^{\infty} (Y_n - \mathbb{E}[Y_n])$ converges in L^2.

Proof.

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Proof. $\mathrm{Var} Y_n = \mathrm{Cov} Y_n Y_n = \mathbb{E}[\mathring{Y}_n^2] = ||\mathring{Y}_n||_{L^2}^2$

$\mathring{S}_n = \sum_{j=1}^{n} \mathring{Y}_j$, $||\mathring{S}_n - \mathring{S}_m||_{L^2}^2 = ||\sum_{j=m+1}^{n} \mathring{Y}_j||_{L^2}^2 = \sum_{j=m+1}^{n} \left\|\mathring{Y}_j\right\|_{L^2}^2 \to 0$ as $n, m \to \infty$

$\therefore \mathring{S}_n = \sum_{j=1}^{n} (Y_j - \mathbb{E}[Y_j])$ is Cauchy in L^2.

a.s. convergence

We would like to upgrade the convergence from L^2 to a.s. We upgrade the orthogonality to independence.

Theorem 2 (Kolmogorov's Convergence Criterion)

Let $\{Y_n\}_{n=1}^{\infty}$ be independent L^2 random variables. If $\sum_{n=1}^{\infty} \mathrm{Var} Y_n < \infty$, then $\sum_{n=1}^{\infty}(Y_n - \mathbb{E}[Y_n])$ converges a.s. In particular, if in addition $\sum_{n=1}^{\infty} \mathbb{E}[Y_n] < \infty$, then $\sum_{n=1}^{\infty} Y_n$ converges a.s. and in L^2.

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Let $\{Y_n\}_{n=1}^{\infty}$ be independent random variables, with $\mathbb{E}[Y_n] = 0$. Set $S_n = Y_1 + \cdots + Y_n$. If $Y_n \in L^2$, then Markov tell us

$$\mathbb{P}(|S_n| \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\mathbb{E}[S_n^2] = \frac{1}{\varepsilon^2}\sum_{j=1}^{b}\mathbb{E}[Y_j^2]$$

What can we say about the running maximum $S_n^* = \max_{1 \leq j \leq n} |S_j|$?

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What can we say about the running maximum $S_n^* = \max_{1 \leq j \leq n} |S_j|$?
Turns out: the Markov conclusion still applies

Bound for the running maximum

Theorem 3 (Kolmogorov's Maximal Inequality)

With Y_n, S_n as above,

$$\mathbb{P}(S_n^* \geq \varepsilon) \leq \frac{1}{\varepsilon}\mathbb{E}[S_n^2 \mathbf{1}_{\{S_n^* \geq \varepsilon\}}] \leq \frac{1}{\varepsilon^2}\mathbb{E}[S_n^2] = \frac{1}{\varepsilon^2}\sum_{j=1}^n \mathbb{E}[Y_j^2]$$

Proof. Fix $\varepsilon > 0$, and set $\tau := \inf\{j \in \mathbb{N} : |S_j| \geq \varepsilon\}$ with convention $(\int \emptyset = \infty$
τ is a random variables, note that

$$\{\tau = j\} = \{|S_1| < \varepsilon, |S_2| < \varepsilon, \ldots, |S_{j-1}| < \varepsilon, |S_j| \geq \varepsilon\}$$

Now, $\{S_n^* \geq \varepsilon\} = \{\exists j \in [n] : |S_j| \geq \varepsilon\} = \{\tau \leq n\}$
Notation: $\mathbb{E}[X : A] = \mathbb{E}[X\mathbf{1}_A]$

$$\mathbb{E}[S_n^2 : S_n^* \geq \varepsilon] = \mathbb{E}[S_n^2 : \tau \leq n] = \sum_{j=1}^n \mathbb{E}[S_n^2 : \tau = j]$$

Now a trick:

$$S_n^2 = (S_j + S_n - S_j)^2 = S_j^2 = +(S_n - S_j)^2 + 2S_j(S_n - S_j)$$

$$\mathbb{E}[S_n^2 : S_n^* \geq \varepsilon] = \sum_{j=1}^n \mathbb{E}[\left(S_j^2 + (S_n - S_j)^2 + 2S_j(S_n - S_j)\right)\mathbf{1}_{\{\tau=j\}}]$$

Note that the third term inside the expectation$= 0$ by the independence.
$\mathbb{E}[(S_n - S_j)S_j\mathbf{1}_{\{\tau=j\}}] = \mathbb{E}[S_n - S_j]\mathbb{E}[S_j : \tau = j]$

Bound for the running maximum

$$\cdots = \sum_{j=1}^{n} \big(\mathbb{E}[S_j^2 : \tau = j] + \mathbb{E}[(S_n - S_j)^2 : \tau = j]\big) \geq \sum_{j=1}^{n} \mathbb{E}[S_j^2 : \tau = j],\ \{\tau = j\} \subseteq \{|S_j| \geq \varepsilon\}$$

$$\geq \sum_{j=1}^{n} \mathbb{E}[\varepsilon^2 \mathbf{1}_{\{\tau = j\}}] = \varepsilon^2 \sum_{j=1}^{n} \mathbb{P}(\tau = j) = \varepsilon^2 \mathbb{P}(\tau \leq n) = \varepsilon^2 \mathbb{P}(S_n^* \geq \varepsilon)$$

Proof for 2

Let $S_n = \sum_{j=1}^{n} \mathring{Y}_j$. For $m < n$, we have

$$S_n - S_m = \mathring{Y}_{m+1} + \cdots + \mathring{Y}_n.$$

Apply the **Kolmogorov maximal inequality**:

$$\mathbb{P}\left(\max_{m<j\leq n} |S_j - S_m| \geq \frac{\varepsilon}{2}\right) \leq \frac{1}{(\varepsilon/2)^2}\mathbb{E}[(S_n - S_m)^2] = \frac{4}{\varepsilon^2}\sum_{j=m+1}^{n}\mathbb{E}[\mathring{Y}_j^2] = \frac{4}{\varepsilon^2}\sum_{j=m+1}^{n}\mathrm{Var}(Y_j)$$

Now letting $n \to \infty$, we obtain:

$$\mathbb{P}\left(\sup_{j\geq m} |S_j - S_m| \geq \frac{\varepsilon}{2}\right) \leq \frac{4}{\varepsilon^2}\sum_{j=m+1}^{\infty}\mathrm{Var}(Y_j) \to 0 \quad \text{as } m \to \infty.$$

Furthermore,

$$\begin{aligned}
\sup_{j,k\geq m} |S_j - S_k| &= \sup_{j,k\geq m} |S_j - S_m + S_m - S_k| \\
&\leq \sup_{j\geq m} |S_j - S_m| + \sup_{k\geq m} |S_k - S_m| \\
&= 2\sup_{j\geq m} |S_j - S_m|.
\end{aligned}$$

Proof for 2

Thus,

$$\left\{ \sup_{j,k \geq m} |S_j - S_k| \geq \varepsilon \right\} \subseteq \left\{ 2 \sup_{j \geq m} |S_j - S_m| \geq \varepsilon \right\}.$$

So,

$$\mathbb{P}\left(\sup_{j,k \geq m} |S_j - S_k| \geq \varepsilon \right) \to 0 \quad \text{as } m \to \infty.$$

Define $\delta_m := \sup_{j,k \geq m} |S_j - S_k|$. Then $\delta_m \to 0$ in probability. Since $\delta_m \downarrow \delta$ almost surely and $\delta \geq 0$, it follows that $\delta = \lim_{m \to \infty} \delta_m = 0$ almost surely.

Therefore, $\{S_j\}_{j \in \mathbb{N}}$ is a.s. Cauchy and $\sum_{k=1}^{j} \mathring{Y}_k$ converges almost surely.

Example: Let $\{X_n\}_{n \in \mathbb{N}}$ be i.i.d. Rademacher random variables, i.e., $\mathbb{P}(X_n = \pm 1) = \frac{1}{2}$. Does the series $\sum_{n=1}^{\infty} \frac{X_n}{n}$ converge?

Kronecker Lemma

We now have some tools to prove a.s. convergence of a sum $\sum_{n=1}^{\infty} Y_n$, given information about $\mathrm{Var} Y_n$. Not well-adapted to $\frac{1}{n}\sum_{j=1}^{n} X_j$; more adapted to $\sum_{n=1}^{\infty} \frac{X_n}{n}$

Lemma 2 (Kronecker)

Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$ (or any normed space) and let $\{b_k\}_{k=1}^{\infty} \subset (0,\infty)$ be an increasing sequence $b_k \uparrow \infty$. If $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{x_k}{b_k}$ exists in $\mathbb{R}$, then $\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} x_k = 0$

Proof.

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Proof. Let $y_k := \frac{x_k}{b_k}, S_n := \sum_{k=1}^{n} y_k (S_0 := 0), \lim_{n\to\infty} S_n := s$

Then, $$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} b_k y_k = \sum_{k=1}^{n} b_k (S_k - S_{k-1})$$

$$= \sum_{k=1}^{n} b_k S_k - \sum_{k=0}^{n-1} b_{k+1} S_k = b_n S_n + \sum_{k=1}^{n-1} (b_k - b_{k+1}) S_k$$

$$\therefore \frac{1}{b_n}\sum_{k=1}^{n} x_k = S_n - \frac{1}{b_n}\sum_{k=1}^{n-1}(b_{k+1} - b_k) S_k$$

$$= S_n - \frac{1}{b_n}\sum_{k=1}^{n-1}(b_{k+1} - b_k) s + R_n, \quad R_n = \frac{1}{b_n}\sum_{k=1}^{n-1}(b_{k+1} - b_k)(s - s_k)$$

Kronecker Lemma

$$\cdots = S_n - \left(1 - \frac{b_1}{b_n}\right) s + R_n$$

$$|R_n| \leq \frac{1}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k)|s - s_k| \leq \frac{M}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_K) = M(1 - \frac{b_1}{b_n}) \leq M$$

This shows that the first $N-1$ term divided by b_n goes to 0 as $n \to \infty$.

$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \frac{1}{b_n} \sum_{k=N}^{n} (b_{k+1} - b_k)|s - S_k| \leq \lim_{n\to\infty} \sup_{k\geq N} |s - S_k| \cdot \frac{1}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k)$$

$$\implies \lim_{n\to\infty} |R_n| = 0$$

SLLN

Theorem 4 (SLLN)

Let $\{X_n\}_{n=1}^{\infty}$ be iid L^1 random variables with $\mathbb{E}[X_n] = \alpha$.
Let $S_n = X_1 + \cdots + X_n$. Then,

$$\frac{S_n}{n} \longrightarrow \alpha \quad \text{a.s.}$$

We already showed that it suffices to show $\frac{S_n'}{n} \to \mu$ a.s., where

$$S_n' = \sum_{j=1}^{n} X_j', \quad X_j' = X_j \mathbf{1}_{\{|X_k| \leq j\}}$$

Proof.

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We already showed that it suffices to show $\frac{S'_n}{n} \to \mu$ a.s., where

$$S'_n = \sum_{j=1}^{n} X'_j, \quad X'_j = X_j \mathbf{1}_{\{|X_k| \le j\}}$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathrm{Var}\frac{X'_n}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \mathrm{Var} X'_n &\le \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}\,|X'_n|^2 \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}\,|X_1|^2 \mathbf{1}_{\{|X_1| \le n\}} \\
&= \mathbb{E}\,|X_1|^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbf{1}_{|X_1| \le n}
\end{aligned}$$

SLLN

Observe that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \mathbf{1}_{x \le n} = \sum_{n \ge x} \le \int_x^{\infty} \left(\sum_{n=2}^{\infty} \frac{1}{n^2} \mathbf{1}_{\{n \le t < n+1\}} \right) \mathrm{d}t \quad \text{for } x > 1$$

$$\overset{\dagger}{\le} \int_x^{\infty} \frac{1}{(t-1)^2} \, \mathrm{d}t = \frac{1}{x-1} \le \frac{2}{x}$$

$\dagger$: $\frac{1}{\lfloor x \rfloor} \le \frac{1}{(t-1)^2}$ for fixed t.

For $x \le 1$, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \le 2 \therefore \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbf{1}_{x \le n} \le \min(2, \frac{2}{x})$

$$\mathbb{E}\left[|X_1|^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbf{1}_{|X_1| \le n} \right] \le \mathbb{E}\left[|X_1|^2 \min(2, 2/|X_1|) \right] \le 2\, \mathbb{E}\, |X_1| < \infty$$

$\implies \sum_{n=1}^{\infty} \mathrm{Var} \frac{X'_n}{n} < \infty$, by Kolmogorov's convergence criterion

$\sum_{n=1}^{\infty} \left(\frac{X'_n}{n} - \mathbb{E}\, \frac{X'_n}{n} \right)$ converges a.s.

$\sum_{n=1}^{\infty} \left(\frac{X'_n}{n} - \mathbb{E}\, \frac{X'_n}{n} \right) = \sum_{n=1}^{\infty} \frac{1}{n}(X'_n - \mathbb{E}\, X'_n)$ converges a.s. By kronecker's Lemma,

$$\implies \mathring{S}'_n := \frac{1}{n} \sum_{k=1}^{n} (X'_k - \mathbb{E}\, X'_k) \to 0 \text{ a.s}$$

SLLN

For each k, let $\alpha_k = \mathbb{E}\, X_1 \mathbf{1}_{|X_1| \leq n} \to \alpha$ by DCT.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \alpha_k = \alpha$$

$\implies \frac{1}{n} \sum_{k=1}^{n} X'_k = \frac{S'_n}{n} \to \alpha$ a.s.

Rates of Convergence

Question: What is the fastest growing $\alpha_n \uparrow \infty$ s.t.

$$\limsup_{n\to\infty} \alpha_n \cdot \left| \frac{S_n}{n} - \mathbb{E}\, X_1 \right| < \infty$$

Theorem 5 (Marcinkiewciz, Zygmund)

Suppose $\{X_n\}_{n=1}^{\infty}$ are iid in L^p for some $p \in (1, 2)$. Then,

$$n^{1-\frac{1}{p}} \left(\frac{S_n}{n} - \alpha \right) \longrightarrow 0 \text{ a.s.}$$

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Theorem 6 (L^2-SLLN)

Let $\{X_n\}_{n=1}^{\infty}$ be independent L^2 random variables, with common mean $\mathbb{E}\, X_n = \alpha$ and variance $\mathrm{Var} X_n \leq s^2$. Let $S_n = X_1 + \cdots + X_n$, and let $b_n > 0$ s.t. $\sum_{n=1}^{\infty} \frac{1}{b_n^2} < \infty$
Then,

$$\frac{S_n - n\alpha}{b_n} \to 0 \text{ a.s. and in } L^2$$