Probability Theory

II Random variables $\sim$ Radon-Nikodym

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Measurable Functions

Definition 1

Let $(\Omega, \mathcal{F})$and $(S, \mathcal{B})$ be measurable spaces. The *pull-back* of $\mathcal{B}$ to Ω is

$$f^*\mathcal{B} = \{f^{-1}(B) \subseteq \Omega : B \subset \mathcal{B}\}$$

The *push-forward* of $\mathcal{F}$ to S is

$$f_*\mathcal{F} = \{E \subseteq S : f^{-1}(E) \in \mathcal{F}\}$$

- Both of them are σ-fields.
- $\sigma(\cdot)$ and pull-back operations commute, i.e., $\sigma(f^*\mathcal{E}) = f^*(\sigma(\mathcal{E}))$

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Definition 2

Let $(\Omega, \mathcal{F}), (S, \mathcal{B})$ measurable spaces. $f : \Omega \to S$ is $\mathcal{F}/\mathcal{B}$-measurable if $f^*\mathcal{B} \subseteq \mathcal{F}$.

Example: Indicator functions $\mathbf{1}_A : \Omega \to \mathbb{R}, A \subseteq \Omega$ is measurable if and only $A \in \mathcal{F}$.

Measurable Functions

Proposition 1

Let $\mathcal{E} \subseteq \mathcal{B}$ such that $\sigma(\mathcal{E}) = \mathcal{B}$. Then f is measurable if and only if $f^*\mathcal{E} \subseteq \mathcal{F}$.

Examples: $X : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable if and only if

- $X^{-1}(B) \in \mathcal{F}$, $\forall B \in \mathcal{B}(\mathbb{R})$
- $X^{-1}(a, b] \in \mathcal{F}$, $\forall a < b \in \mathbb{R}$
- $X^{-1}(-\infty, t] \in \mathcal{F}$, $\forall t \in \mathbb{R}$

Definition 3

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a (Borel) random variable is a $\mathcal{F}/\mathcal{B}(\mathbb{R})$ measurable function $X : \Omega \to \mathbb{R}$

Properties

- Composition of measurable functions are measurable.
- Let $X_1, X_2, \ldots, X_d$ be random variables on $(\Omega, \mathcal{F})$. If $f : \mathbb{R}^d \to \mathbb{R}$ is continuous, then $Y = f(X_1, \ldots, X_d)$ is a random variable.
- Given random variable X, $\mu_X = \mathbb{P} \circ X^{-1} = X^*\mathbb{P}$ is a probability measure on $(S, \mathcal{B})$

Robustness of Measurability

Proposition 2

If $f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ are measurable, then so are

$$\sup_n f_n, \ \inf_n f_n, \ \limsup_{n \to \infty} f_n, \ \liminf_{n \to \infty} f_n$$

Remark. In $C(\mathbb{R})$, this does not work.

Proof.

Simple Approximation

Theorem 1

If $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable, there is a sequence φ_n of simple measurable functions such that

$$\lim_{n\to\infty} \varphi_n(x) = f(x) \quad \forall x \in \Omega$$

In addition, $\varphi \to f$ uniformly on $f^{-1}[-M, M]\ \forall M > 0$

Proof.

Set $\varphi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbf{1}_{\{k/2^n < f \leq (k+1)/2^n\}} + 2^n \mathbf{1}_{\{f > 2^n\}}(x)$. Note that
$\varphi_n \leq \varphi_{n+1} \leq f$

Doob-Dynkin Representation

Corollary 1

Let $X_1, X_2, \ldots, X_d$ $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable and $Y : \Omega \to \mathbb{R}$ $\sigma(X_1, \ldots, X_d)/\mathcal{B}(\mathbb{R})$ measurable. Then, there exists Borel measurable $f : \mathbb{R}^d \to \mathbb{R}$ such that $Y = f(X_1, \ldots, X_d)$

Proof.

Simple integration

Proposition 3

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let

$$S_{\mathcal{F}} := \{f : \Omega \to \mathbb{R} \mid f \text{ is simple and } \mathcal{F}\text{-measurable}\}$$

denote the set of real-valued, $\mathcal{F}$-measurable simple functions. Then $S_{\mathcal{F}}$ is a real vector space, and the mapping

$$\int \cdot \, \mathrm{d}\mu : S_{\mathcal{F}} \to \mathbb{R}$$

is a positive linear functional.

Lebesgue Integral

Define L^+ on $(\Omega, \mathcal{F}, \mu)$ by

$$L^+(\mathcal{F}) = \{f : \Omega \to [0, \infty), f \text{ is } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable } \}$$

Definition 4

For $f \in L^+$, the Lebesgue integral is

$$\mu(F) = \int f \,\mathrm{d}\mu = \int_\Omega f(w) \,\mathrm{d}\mu(w) = \sup\{\int \varphi \,\mathrm{d}\mu : \varphi \leq f, \varphi \text{ simple, measurable } \}$$

Properties

1. If $f \in L^+, \alpha > 0$, $\int \alpha f \,\mathrm{d}\mu = \alpha \int f \,\mathrm{d}\mu$.
2. If $f, g, \in L^+$, $\int (f+g) \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu$
3. If $0 \leq f \leq g$, $\int f \,\mathrm{d}\mu \leq \int g \,\mathrm{d}\mu$
4. If $f_n \in L^+$, then
$$\int (\sum_{n=1}^{\infty} f_n) \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int f_n \,\mathrm{d}\mu$$

Monotone Convergence Theorem

Theorem 2

If $f_n \in L^+, f_n \uparrow f$, then $\int f_n \,\mathrm{d}\mu \uparrow \int f \mathrm{d}\mu$

Proof.

Lebesgue Integral

Proposition 4

Let $f, g \in L^+$.

1. If $f \leq g$ μ-a.s. then $\int f \,\mathrm{d}\mu \leq \int g \,\mathrm{d}\mu$
2. If $f = g$ μ-a.s. then $\int f \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu$
3. $\int f \,\mathrm{d}\mu = 0$ then $f = 0$ a.s.

Proof.

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Proof.

Fatou's Lemma

If $f_n \in L^+$, $\int \liminf_{n\to\infty} f_n \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int f_n \, \mathrm{d}\mu$

Borel-Cantelli Lemma

Lemma 1

If $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu\{A_n \text{ i.o } \} = 0$

Proof.

Dominated Convergence Theorem

Theorem 3

Suppose $f_n, g_n, g \in L^1$, with

1. $f_n \to f$ a.s. and $g_n \to g$ a.s.
2. $g_n \geq 0$ and $|f_n| \leq g_n$ a.s.
3. $\int g_n \,\mathrm{d}\mu \to \int g \,\mathrm{d}\mu < \infty$.

Then, $f \in L^1$ and $\int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu$

For finite measure case, bounded convergence theorem works.

When do integral and derivative commute?

Proposition 5

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $f : (a,b) \times \Omega \to \mathbb{R}$ such that

1. $w \mapsto f(t,w)$ is measurable for each $t \in (a,b)$
2. $f(t_0, \cdot) \in L^1(\Omega, \mathcal{F}, \mu)$ for some $t_0 \in (a,b)$
3. $\frac{\partial f}{\partial t}(t,w)$ exists for μ-a.e. w and for every $t \in (a,b)$
4. There is $g \in L^1(\Omega, \mathcal{F}, \mu)$ such that $|\frac{\partial f}{\partial t}(t,w)| \leq g(w)$ for μ-a.e. w and for every $t \in (a,b)$

Then, $f(t, \cdot) \in L^1$ for all $t \in (a,b)$, $t \mapsto \int f(t,w)\, \mathrm{d}\mu$ is differentiable on (a,b) and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f(t,w)\, \mu(\mathrm{d}w) = \int \frac{\partial f}{\partial t}(t,w)\, \mu(\mathrm{d}w)$$

Remark. Almost sure statements must hold *independently* of t.

Radon-Nikodym

Definition 5

Say $\nu << \mu$, ν is *absolutely continuous* w.r.t. μ if $\mu(A) \implies \nu(A)$, $\forall A \in \mathcal{F}$

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Theorem 4

Let μ, ν be σ-finite measures on $(\Omega, \mathcal{F})$.
Then $\nu << \mu$ if and only if $\exists \rho : \Omega \to [0, \infty)$ measurable such that $\nu(A) = \int_A \rho \, \mathrm{d}\mu$, $\forall A \in \mathcal{F}$. Moreover, the density ρ is uniquely defined up to a ν-null set. It is called the *Radon-Nikodym derivative* $\rho = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$.

Theorem 5

Let μ, ν be σ-finite measures on $(\Omega, \mathcal{F})$. Then ν has a unique *Lebesgue decomposition* $\nu = \nu_a + \nu_s$. where

- $\nu_a << \mu$
- $\nu_s \perp \mu$: ν_s and μ are *mutually singular*, meaning $\exists A \in \mathcal{F}$ such that $\nu_s(A) = 0$ and $\mu(A^c) = 0$