

# Fundamental Algebra

## 1 Series

### 1.1 Geometric

$$\sum_{n=a}^b r^n = r^a \frac{1 - r^{b-a+1}}{1 - r} = \frac{r^a - r^{b+1}}{1 - r} \quad (1.1)$$

$$\sum_{j=1}^{\infty} \frac{j^2}{\rho^j} = \frac{\rho(\rho+1)}{(\rho-1)^3} \quad (1.2)$$

### 1.2 Quotient stack

$$\sum_{k=1}^n \lfloor \frac{n}{k} \rfloor = 2 \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \lfloor \frac{n}{k} \rfloor - \lfloor \sqrt{n} \rfloor^2 \quad (1.3)$$

## 2 Inequalities

### 2.1 Ratio of Summation

$$\frac{\sum_i a_i}{\sum_i b_i} \leq \max_i \frac{a_i}{b_i} \quad (2.1)$$

### 2.2 Weighted AM-GM Inequality

Let the nonnegative numbers  $x_1, x_2, \dots, x_n$  and the nonnegative weights  $w_1, w_2, \dots, w_n$  be given. Set  $w = w_1 + w_2 + \dots + w_n$ . If  $w > 0$ , then the inequality

$$\frac{w_1 x_1 + w_2 x_2 + \dots + w_n x_n}{w} \geq \sqrt[w]{x_1^{w_1} x_2^{w_2} \dots x_n^{w_n}} \quad (2.2)$$

holds.

### 2.3 Titu's Lemma

#### 2.3.1 Summation Form

For any real numbers  $a_1, a_2, \dots, a_n$  and positive reals  $b_1, b_2, b_3, \dots, b_n$ , we have

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n} \quad (2.3)$$

#### 2.3.2 Probabilistic Form

Let  $X$  be a real random variable and  $Y$  be a positive random variable such that  $\mathbb{E}[|X|]$  and  $\mathbb{E}[Y]$  are well defined. Then

$$\mathbb{E}[X^2/Y] \geq \mathbb{E}[|X|^2]/\mathbb{E}[Y] \geq \mathbb{E}[X]^2/\mathbb{E}[Y] \quad (2.4)$$

### 2.4 Cauchy-Schwarz Inequality

For any non-zero vector  $\mathbf{x}$ ,

$$\|\mathbf{x}\|_2^2 \leq \|\mathbf{x}\|_1^2 \leq \|\mathbf{x}\|_0 \|\mathbf{x}\|_2^2 \quad (2.5)$$

(Note: Useful in binary matrix multiplication.)

## 2.5 Squared norm of sum of vectors bound

Applying  $n$  times Cauchy-Schwartz inequality:

$$\left\| \sum_{i=1}^n \mathbf{x}_i \right\|^2 \leq n \sum_{i=1}^n \|\mathbf{x}_i\|^2$$

## 2.6 Chebyshev's Sum Inequality

If  $a_1 \geq a_2 \geq \dots \geq a_n$  and  $b_1 \geq b_2 \geq \dots \geq b_n$ , then

$$\frac{1}{n} \sum_{k=1}^n a_k b_k \geq \left( \frac{1}{n} \sum_{k=1}^n a_k \right) \left( \frac{1}{n} \sum_{k=1}^n b_k \right) \quad (2.6)$$

## 2.7 Symmetric Parametric Inequality

$$\left( 1 - p + \frac{p}{x} \right)^{\alpha-2} \cdot (1 - p + px)^{\alpha-2} \geq 1 \quad (2.7)$$

## 2.8 We love Jensen

By convexity of  $(u, v) \mapsto u^\lambda v^{1-\lambda}$

$$\mathbb{E}[U]^\lambda \mathbb{E}[V]^{1-\lambda} \leq \mathbb{E}[U^\lambda V^{1-\lambda}] \quad (2.8)$$

# 3 Bounds and Approximations

## 3.1 Exponential Bound on Hyperbolic Ratio

For  $0 \leq y < x \leq 2$ ,

$$\frac{\sinh(x) - \sinh(y)}{\sinh(x - y)} \leq e^{\frac{1}{2}xy} \quad (3.1)$$

## 3.2 Exponential Inequalities

For all  $t \in \mathbb{R}$  and  $0 \leq p \leq 1$ ,

$$1 - p + p \cdot x \leq e^{p(x-1)} \quad (3.2)$$

Additionally:

$$\forall x, y \geq 0 \quad \frac{1 + e^{x+y}}{e^x + e^y} \leq e^{xy/2} \quad (3.3)$$

$$\frac{1}{2}(e^x + e^{-x}) \leq e^{x^2/2} \quad (3.4)$$

## 3.3 Softplus Quadratic Bound

For all  $a, x \in \mathbb{R}$  with  $a \neq 0$ , we have

$$\log(1 + e^x) \leq \log(1 + e^a) + \frac{x - a}{1 + e^{-a}} + \frac{(e^a - 1) \cdot (x - a)^2}{4 \cdot a \cdot (e^a + 1)} \quad (3.5)$$

## 3.4 Miscellaneous upper Bounds

### 3.4.1 Linear-Rational function

For all  $x > 0$ ,  $t \in [0, 1]$ :

$$\frac{1}{1 - t + tx} \leq 1 - t(1 - t)(1 + 3t)(x - 1) + t^2 \left( (1 - t)x^2 + \frac{t}{x} - 1 \right) \quad (3.6)$$

### 3.4.2 Inverse logarithmic

For all  $u > 0$ :

$$\frac{1}{\log(1 + 1/u)} \leq u + \frac{1}{2} \quad (3.7)$$

### 3.4.3 $x \log_+ y$ decoupling

For non negative reals  $x$  and  $y$ ,

$$x \log_+ y + 1 \wedge +1 \leq x \log x + e^{-1}y + 1 \quad (3.8)$$

### 3.4.4 Weighted reciprocal

For all  $p \in [0, 1]$  and  $x \in (0, \infty)$ ,

$$\frac{1}{1 - p + p/x} \leq 1 - p + p \cdot x \quad (3.9)$$

## 3.5 Order of Rademacher Sums

Let  $\sigma \in \{-1, 1\}^n$  be a random Rademacher sequence and let  $a \in \mathbb{R}^n$  be an arbitrary real vector with sorted entries  $|a_1| \geq |a_2| \geq \dots \geq |a_n|$ . Then

$$\|\langle a, \sigma \rangle\|_{L^p} \sim \sum_{i \leq p} a_i + \sqrt{p}(\sum_{i > p} a_i^2)^{1/2} \quad (3.10)$$

## 3.6 Sum Approximation

$$\max(a, b) \leq a + b \leq 2 \max(a, b), \quad a, b \geq 0 \quad (3.11)$$

# 4 Combinatorics

## 4.1 Expansion

$$\binom{2n}{m} = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{n}{j} \binom{n-j}{m-2j} 2^{m-2j} \quad (4.1)$$

## 4.2 Vandermonde's Identity

$$\sum_{i=0}^r \binom{m}{i} \binom{n}{r-i} = \binom{n+m}{r} \quad (4.2)$$

$$\sum_{m=0}^n \binom{m}{j} \binom{n-m}{k-j} = \binom{n+1}{k+1} \quad (4.3)$$

Special form ([hockey-stick identity](#)):

$$\sum_{m=k}^n \binom{m}{k} = \binom{n+1}{k+1} \quad (4.4)$$

(c.f.  $\prod_{\ell=0}^{k-1} \binom{\ell+\eta}{\ell+1} = \binom{k+\eta-1}{k}$ )