Stochastic process

1 Predictable process

1.1 How to understand the predictable process?

Fact 1. The predictable sigma-algebra is generated by continuous and adapted processes.

Fact 2. The predictable sigma-algebra is generated by the sets of the form

$$\{(s,t] \times A : t > x \geq 0, A \in \mathcal{F}_s\} \cup \{\{0\} \times A : A \in \mathcal{F}_0\} \tag{.1}$$

2 Local martingale

2.1 Quadratic variation

Fact. If X is a continuous local martingale, then $[X]_t < \infty$ a.s. for every $t \geq 0$, where $[X]$ denote the quadratic variation of the process X.

2.2 Stochastic integral

Proposition. For any continuous L^2-martingale M where $M_0 = 0$, and any predictable step process V where $|V| \leq 1$, the process $(V \cdot M)$ is an L^2-martingale with $\mathbb{E}(V \cdot M)_t^2 \leq \mathbb{E}\, M_t^2$.

3 Doob's h transform

Set

$$h(x) = \mathop{\mathbb{P}}_x(\tau_A < \tau_B) \tag{.2}$$

Then, $h(x)$ is the probability, starting from x to hit A before hitting B. Then h is positive on $\mathcal{X} \setminus (A \cup B)$. Furthermore, for $x \notin A \cup B$

$$\hat{P}(x,y) = \mathop{\mathbb{P}}_x[X_1 = y | \tau_A < \tau_B] \tag{.3}$$

Finally, $h(x) = \mathbb{P}_x(\tau_A < \tau_B)$ satisfies both

1. $h(x) = 1$ for $x \in A$ and $h(z) = 0$ for $z \in B$
2. h is harmonic at x for every $x \notin A \cup B$

and $h(\cdot)$ is the unique solution of linear system given by 1 and 2 above. (Good source)