Stochastic process

V Diffusions: Basic Properties

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The Markov Property

The Markov Property

This will give us the necessary background for the applications in the remaining chapters.

Definition 1

A time-homogeneous Itô diffusion is a stochastic process $X_t(w) = X(t, w) : [0, \infty) \times \Omega \to \mathbb{R}^n$ satisfying a stochastic differential equation of the form

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t, \quad t \geq s;\ X_s = x \tag{1.1}$$

where B_t is m-dimensional Brownian motion and $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ satisfy the conditions in Theorem 5.2.1.

Denote the unique solution by $X_t = X_t^{s,x}; t \geq s$. Note that

$$\begin{aligned} X_{s+h}^{s,x} &= x + \int_s^{s+h} b(X_u^{s,x})\,\mathrm{d}u + \int_s^{s+h} \sigma(X_u^{s,x})\,\mathrm{d}B_u \\ &= x + \int_0^h b(X_{s+v}^{s,x})\,\mathrm{d}v + \int_0^h \sigma(X_{s+v}^{s,x})\,\mathrm{d}\tilde{B}_v,\ (u = s + v) \end{aligned}$$

where $\tilde{B}_v = B_{s+v} - B_s;\ v \geq 0$.

The Markov Property

The solution of the stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t; \quad X_0 = x \tag{1.2}$$

that $\left\{X^{s,x}_{s+h}\right\}_{h\geq 0}$ and $\left\{X^{0,x}_{h}\right\}_{h\geq 0}$ have the same P^0-distributions, i.e. $\{X_t\}_{t\geq 0}$ is time-homogeneous.

We now want to define the probability laws Q^x of $\{X_t\}_{t\geq 0}$, for $x \in \mathbb{R}^n$. Intuitively, Q^x gives the distribution of $\{X_t\}_{t\geq 0}$ assuming that $X_0 = x$. To express this let M_∞ be the σ-algebra (of subsets of Ω) generated by the random variables $w \mapsto X_t(w) = X^y_t(w)$, where $t \geq 0, y \in \mathbb{R}^n$. Define Q^x on the members of $\mathcal{M}$ by

$$Q^x[X_{t_1} \in E_1, \ldots, X_{t_k} \in E_k] = P^0[X^x_{t_1} \in E_1, \ldots, X^x_{t_k} \in E_k] \tag{1.3}$$

where $E_i \subset \mathbb{R}^n$ are Borel sets; $1 \leq i \leq k$.

We know that X_t is measurable with respect $\mathcal{F}^{(m)}_t$, the filtration generated by the Brownian motion up to time t. We now prove that X_t satisfies the Markov property: The future behaviour of the process given what has happened up to time t is the same as the behaviour obtained when starting the process at X_t.

The Markov Property

Theorem 1 (The Markov property for Itô diffusions)

Let f be a bounded Borel function from $\mathbb{R}^n \to \mathbb{R}$. Then, for $t, h \geq 0$

$$\mathbb{E}^x \left[f(X_{t+h}) | \mathcal{F}_t^{(m)} \right] (w) = \mathbb{E}^{X_t(w)} \left[f(X_h) \right]. \tag{1.4}$$

Remark. $\mathbb{E}^x$ denotes the expectation w.r.t. the probability measure Q^x and $\mathbb{E}$ denotes the expectation w.r.t. the measure P^0. The right hand side means the function $y \mapsto \mathbb{E}^y \left[f(X_h) \right]$ evaluated at $y = X_t(w)$.

Proof. Since, for $r \geq t$,

$$X_r(w) = X_t(w) + \int_t^r b(X_u) \, \mathrm{d}u + \int_t^r \sigma(X_u) \, \mathrm{d}B_u, \tag{1.5}$$

we have by uniqueness $X_r(w) = X_r^{t,X_t}(w)$. If we define

$$F(x, t, r, w) = X_r^{t,x}(w) \quad \text{for } r \geq t \tag{1.6}$$

we have $X_r(w) = F(X_t, t, r, w)$; $r \geq t$. Note that $W \mapsto F(x, t, r, w)$ is independent of $\mathcal{F}_t^{(m)}$. We may rewrite eq. (1.4) as

$$\mathbb{E} \, f(F(X_t, t, t+h, w)) | \mathcal{F}_t^{(m)} = \mathbb{E} \, f(F(x, 0, h, w)) \Big|_{x = X_t} \tag{1.7}$$

Cont.

Put $g(x,w) = f \circ F(x,t,t+h,w)$. Then $(x,w) \mapsto g(x,w)$ is measurable, thus we can approximate g pointwise boundedly by functions of the form

$$\sum_{k=1}^{m} \phi_k(x)\psi_k(w). \tag{1.8}$$

Using the properties of conditional expectation

$$\begin{aligned}
\mathbb{E}\, g(X_t,w)|\mathcal{F}_t &= \mathbb{E} \lim_{m\to\infty} \sum_{k=1}^{m} \phi_k(X_t)\psi_k(w)|\mathcal{F}_t \\
&= \lim_{m\to\infty} \sum \phi_k(X_t) \cdot \mathbb{E}\, \psi_k(w)|\mathcal{F}_t \\
&= \lim_{m\to\infty} \sum \mathbb{E}\, \phi_k(y)\psi_k(w)|\mathcal{F}_{t\,y=X_t} \\
&= \mathbb{E}\, g(y,w)|\mathcal{F}_{t\,y=X_t} = \mathbb{E}\, g(y,w)_{y=X_t}
\end{aligned}$$

Therefore, since $\{X_t\}$ is time-homogeneous,

$$\begin{aligned}
\mathbb{E}\, f(F(X_t,t,t+h,w))|\mathcal{F}_t &= \mathbb{E}\, f(F(y,t,t+h,w))_{y=X_t} \\
&= \mathbb{E}\, f(F(y,0,h,w))_{y=X_t}
\end{aligned}$$

Remark. X_t is a Markov process w.r.t the σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ implies X_t is also a Markov process w.r.t. the σ-algebras $\{\mathcal{M}_t\}_{t\geq 0}$.

The Strong Markov Property

The Strong Markov Property

Theorem 2 (The strong Markov property for Itô diffusions)

Let f be a bounded Borel function on $\mathbb{R}^n$, τ a stopping time w.r.t $\mathcal{F}_t$, $\tau < \infty$ a.s. Then

$$\mathbb{E}^x \left[f(X_{\tau+h}) | \mathcal{F}_\tau \right] = \mathbb{E}^{X_\tau} \left[f(X_h) \right] \quad \text{for all } h \geq 0. \tag{2.1}$$

Application

Example 1. Let $H \subset \mathbb{R}^n$ be measurable and let τ_H be the first exit time from H for an Itô diffusion X_t. Let α be another stopping time, g a bounded continuous function on $\mathbb{R}^n$ and put

$$\eta = g(X_{\tau_H}) \mathbf{1}_{\tau_H < \infty}, \; \tau_H^\alpha = \inf \{ t > \alpha \,|\, X_t \notin H \} \tag{2.2}$$

Then we have

$$\theta_\alpha \eta \cdot \mathbf{1}_{\alpha < \infty} = g(X_{\tau_H^\alpha} \mathbf{1}_{\tau_H^\alpha < \infty}) \tag{2.3}$$

In particular, if $\alpha = \tau_G$ with $G \subset\subset H$ measurable, $\tau_H < \infty$ Q^x-a.s, then $\tau_H^\alpha = \tau_H$ and so $\theta_{\tau_G} g(X_{\tau_H}) = g(X_{\tau_H})$. This leads to

$$\mathbb{E}^x \left[f(X_{\tau_H}) \right] = \mathbb{E}^x \left[\mathbb{E}^{X_{\tau_G}} \left[f(X_{\tau_H}) \right] \right] = \int_{\partial G} \mathbb{E}^y \left[f(X_{\tau_H}) \right] \cdot Q^x [X_{\tau_G} \in \mathrm{d}y] \tag{2.4}$$

Proof.

For a.s. w we have

$$X^{\tau,x}_{\tau+h} = x + \int_{\tau}^{\tau+h} b(X^{\tau,x}_u)\,\mathrm{d}u + \int_{\tau}^{\tau+h} \sigma(X^{\tau,x)}_u\,\mathrm{d}B_u. \tag{2.5}$$

Interval을 바꾸기 위해 $\mathcal{F}_\tau$에 independent한 brownian motion $\tilde{B}_v = \tilde{B}_{\tau+v} - B_\tau$을 정의하자. Then,

$$X^{\tau,x}_{\tau+x} = x + \int_0^h b(X^{\tau,x}_{\tau+v})\,\mathrm{d}v + \int_0^h \sigma(X^{\tau,x}_{\tau+v})\,\mathrm{d}\tilde{B}_v. \tag{2.6}$$

Since $\left\{X^{0,x}_h\right\}_{h\geq 0}$ is solution of

$$X_h = x + \int_0^h b(X_v)\,\mathrm{d}v + \int_0^h \sigma(Y_v)\,\mathrm{d}B_v. \tag{2.7}$$

$\left\{X^{\tau,x}_{\tau+h}\right\}_{h\geq 0}$ has the same law as $\left\{X^{0,x}_h\right\}_{h\geq 0}$. Let $F(x,t,r,w) = X^{t,x}_r(w)$ for $r \geq t$. Equation (2.1) can be written

$$\mathbb{E}\, f(F(x,0,\tau+h,w))|\mathcal{F}_\tau = \mathbb{E}\, f(F(x,0,h,w))_{x=X^{0,x}_\tau}. \tag{2.8}$$

Proof.

Now, with $X_t = X_t^{0,x}$,

$$
\begin{aligned}
F(x,0,\tau+h,w) &= X_{\tau+h}(w) + x + \int_0^{\tau+h} b(X_s)\,\mathrm{d}s + \int_0^{\tau+h} \sigma(X_s)\,\mathrm{d}B_s \\
&= x + \int_0^{\tau} b(X_s)\,\mathrm{d}s + \int_0^{\tau} \sigma(X_s)\,\mathrm{d}B_s + \int_\tau^{\tau+h} b(X_s)\,\mathrm{d}s + \int_\tau^{\tau_h} \sigma(X_s)\,\mathrm{d}B_s \\
&= X_\tau + \int_\tau^{\tau+h} b(X_s)\,\mathrm{d}s + \int_\tau^{\tau+h} \sigma(X_s)\,\mathrm{d}B_s \\
&= F(X_\tau, \tau, \tau+h, w).
\end{aligned}
$$

which implies

$$\mathbb{E}\, f(F(X_\tau,\tau,\tau+h,w))|\mathcal{F}_\tau = \mathbb{E}\, f(F(x,0,h,w))_{x=X_\tau}. \tag{2.9}$$

$$\mathbb{E}^x\left[f(X_{\tau_h})|\mathcal{F}_\tau\right] = \mathbb{E}^{X_\tau}\left[f(X_h)\right] \quad \text{for all } h \geq 0 \tag{2.10}$$

More generally, for all $\mathcal{M}_\infty$-measurable function η,

$$\mathbb{E}^x\left[\theta_\tau \eta|\mathcal{F}_\tau\right] = \mathbb{E}^{X_\tau}\left[\eta\right] \tag{2.11}$$

where θ_τ is the shift operator defined as follows: if $\eta = g_1(X_{t_1})\dots g_k(X_{t_k})$ we put

$$\theta_t \eta = g_1(X_{t_1+t})\dots g_k(X_{t_k+t}) \tag{2.12}$$

Generator of an Itô Diffusion

Generator of an Itô Diffusion

We can associate a second order partial differential operator A to an Itô diffusion X_t

Definition 2

Let $\{X_t\}$ be a (time-homogeneous) Itô diffusion in $\mathbb{R}^n$. The infinitesimal generator A of X_t is defined by

$$Af(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}^x\left[f(X_t)\right] - f(x)}{t}; \; x \in \mathbb{R}^n \tag{3.1}$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exists at x is denoted by $\mathcal{D}_A(x)$, while $\mathcal{D}_A$ denotes the set of functions for which the limit exists for all $x \in \mathbb{R}^n$

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The generator A can be written w.r.t drift b and diffusion term σ:

Theorem 3

Let X_t be the Itô diffusion

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t \tag{3.2}$$

If $f \in C_0^2(\mathbb{R}^n)$ then $f \in \mathcal{D}_A$ and

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{ij}(\sigma\sigma^\top)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{3.3}$$

Generator of an Itô Diffusion

Example (The graph of Brownian motion). Let B denote 1-dim Brownian motion and let $\mathsf{X} = (X_1, X_2)^\top$ be the solution of the SDE

$$\begin{cases} \mathrm{d}X_1 = \mathrm{d}t;\ X_1(0) = t_0 \\ \mathrm{d}X_2 = \mathrm{d}B;\ X_2(0) = x_0 \end{cases} \tag{3.4}$$

The generator A of X is given by

$$Af = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}; \quad f = f(t,x) \in C_0^2(\mathbb{R}^n). \tag{3.5}$$

The Dynkin Formula

The Dynkin Formula

Theorem 4

Let $f \in C_0^2(\mathbb{R}^n)$. Suppose τ is a stopping time, $\mathbb{E}^x[\tau] < \infty$. Then

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)\,\mathrm{d}s\right]. \tag{4.1}$$

Remark. If τ is the first exit time of a bounded set, $\mathbb{E}^x[\tau] < \infty$, then eq. (4.1) holds for any function $f \in C^2$.

Example . Consider n-dim Brownian motion $B = (B_1, \dots, B_n)$ starting at $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and assume $|a| < R$. What is the expected value of the first exist time τ_K of B from the ball

$$K = K_R = \{x \in \mathbb{R}^n \mid |x| < R\}? \tag{4.2}$$

Choose an integer k and apply Dynkin 's formula with $X = B, \tau = \sigma_k = k \wedge \tau_K$, and $f \in C_0^2$ such that $f(x) = |x|^2$ for $|x| < R$:

$$\begin{aligned}\mathbb{E}^a[f(B_{\sigma_k})] &= f(a) + \mathbb{E}^a\left[\int_0^{\sigma_k} \frac{1}{2}\Delta f(B_s)\,\mathrm{d}s\right] \\ &= |a|^2 + \mathbb{E}^a\left[\int_0^{\sigma_k} n \cdot \mathrm{d}s\right] = |a|^2 + n \cdot \mathbb{E}^a[\sigma_k].\end{aligned}$$

Cont.

Example . Consider n-dim Brownian motion $B = (B_1, \ldots, B_n)$ starting at $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and assume $|a| < R$. What is the expected value of the first exist time τ_K of B from the ball

$$K = K_R = \{x \in \mathbb{R}^n \,|\, |x| < R\}? \tag{4.3}$$

This implies $\mathbb{E}^a[\sigma_k] \leq \frac{1}{n}(R^2 - |a|^2)$ for all k. k를 무한대로 보내면 $\tau_K < \infty$ a.s.
정의에 의해 $B_{\tau_K} = R$, therefore

$$\mathbb{E}^a[\tau_K] = \frac{1}{n}(R^2 - |a|^2) \tag{4.4}$$

Next we assume that $b \geq 2$ and $|b| < R$. What is the probability that B starting at b ever hits K?
Let α_k be the first exit time from the annulus

$$A_k = \left\{x : R < |x| < 2^k R\right\}; \quad k = 1, 2, \ldots \tag{4.5}$$

and put

$$T_K = \inf\{t > 0 : B_t \in K\}. \tag{4.6}$$

Cont (2).

Let $f = f_{n,k}$ be a C^2 function with compact support such that, if $R \le |x| < 2^k R$,

$$f(x) = \begin{cases} -\log|x| & \text{when } n = 2 \\ |x|^{2-n} & \text{when } n > 2 \end{cases} \tag{4.7}$$

Then, since $\Delta f = 0$ in A_k, we have by Dynkin's formula

$$\mathbb{E}^b\left[f(B_{\alpha_k})\right] = f(b) \text{ for all } k. \tag{4.8}$$

Put

$$p_k = \overset{b}{\mathbb{P}}(|B_{\alpha_k}| = R),\ q_k = \overset{b}{\mathbb{P}}(|B_{\alpha_k}| = 2^k R). \tag{4.9}$$

- $n = 2$ We get from eq. (4.8)

$$-\log R \cdot p_k - (\log R + k \cdot \log 2)q_k = -\log|b| \quad \text{for all } k. \tag{4.10}$$

This implies that $q_k \to 0$ as $k \to 0$, thus

$$\overset{b}{\mathbb{P}}(T_K < \infty) = 1 \tag{4.11}$$

- $n > 2$. In this case, eq. (4.8) gives

$$p_k \cdot R^{2-n} + q_k \cdot (2^k R)^{2-n} = |b|^{2-n}. \tag{4.12}$$

Since $0 \le q_k \le 1$ we get by letting $k \to \infty$

$$\lim_{k\to\infty} p_k = \overset{b}{\mathbb{P}}(T_k < \infty) = \left(\frac{|b|}{R}\right)^{2-n}. \tag{4.13}$$

i.e. Brownian motion is transient in $\mathbb{R}^n$ for $n > 2$.

The Characteristic Operator

The Characteristic Operator

Definition 3

Let $\{X_t\}$ be an Itô diffusion. The characteristic operator $\mathcal{A} = \mathcal{A}_X$ of $\{X_t\}$ is defined by

$$\mathcal{A}(x) = \lim_{U \downarrow x} \frac{\mathbb{E}^x\left[f(X_{\tau_U})\right] - f(x)}{\mathbb{E}^x\left[\tau_U\right]}, \tag{5.1}$$

where the U's are open sets U_k decreasing to the point x, in the sense that $U_{k+1} \subset U_k$ and $\bigcap_k U_k = \{x\}$, and $\tau_U = \inf\{t > 0 : X_t \notin U\}$ is the first exit time from U for X_t. The set of functions f such that the limit eq. (5.1) exists for all $x \in \mathbb{R}^n$ (and all $\{U_k\}$) is denoted by $\mathcal{D}_{\mathcal{A}}$. If $\mathbb{E}^x\left[\tau_U\right] = \infty$ for all open $U \ni x$, we define $\mathcal{A}f(x) = 0$.

Remark. It turns out that $\mathcal{D}_A \subseteq \mathcal{D}_{\mathcal{A}}$ always and that

$$Af = \mathcal{A}f \quad \text{for all } f \in \mathcal{D}_A \tag{5.2}$$

The Characteristic Operator

Definition 4

A point $x \in \mathbb{R}^n$ is called a trap for $\{X_t\}$ if

$$Q^x(\{X_t = x \text{ for all } t\}) = 1 \tag{5.3}$$

For example, if $b(x_0) = \sigma(x_0) = 0$, then x_0 is a trap for X_t.

Lemma 1

If x is not a trap for X_t, then there exists an open set $U \ni x$ such that

$$\mathbb{E}^x\left[\tau_U\right] < \infty. \tag{5.4}$$

The Characteristic Operator

Theorem 5

Let $f \in C^2$. Then $f \in \mathcal{D}_{\mathcal{A}}$ and

$$\mathcal{A}f = \sum_i b_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^\top)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{5.5}$$

Remark. We have now obtained that an Itô diffusion is a continuous, strong Markov process.

Example (Brownian motion on the unit circle). The characteristic operator of the process $\mathsf{Y} = (Y_1, Y_2)^\top$ satisfying the SDE

$$\begin{cases} \mathrm{d}Y_1 = -\frac{1}{2}Y_1\,\mathrm{d}t - Y_2\,\mathrm{d}B \\ \mathrm{d}Y_2 = -\frac{1}{2}Y_2\,\mathrm{d}t + Y_1\,\mathrm{d}B \end{cases} \tag{5.6}$$

is

$$\mathcal{A}f(y_1, y_2) = \frac{1}{2}\left[y_2^2\frac{\partial^2 f}{\partial y_1^2} - 2y_1y_2\frac{\partial^2 f}{\partial y_1 \partial y_2}\right] \tag{5.7}$$

The Characteristic Operator

Let D be an open subset of $\mathbb{R}^n$ such that $\tau_D < \infty$ Q^x-a.s. for all x. Let ϕ be a bounded & measurable function on ∂D and define [1]

$$\tilde{\phi}(x) = \mathbb{E}^x \left[\phi(X_{\tau_D})\right] \tag{5.8}$$

Then if U is open, $x \in U \subset\subset D$, we have that

$$\mathbb{E}^x \left[\tilde{\phi}(X_{\tau_U})\right] = \mathbb{E}^x \left[\mathbb{E}^{X_{\tau_U}} \left[\phi(X_{\tau_D}]\right] = \mathbb{E}^x \left[\phi(X_{\tau_D}\right] = \tilde{\phi}(x). \tag{5.9}$$

So $\tilde{\phi} \in \mathcal{D}_{\mathcal{A}}$ and

$$\mathcal{A}\tilde{\phi} = 0 \text{ in } D \tag{5.10}$$

Proof of theorem 5. If x is a trap for $\{X_t\}$ then $\mathcal{A}f(x) = 0$. Choose a bounded open set V such that $x \in V$. Modify f to f_0 outside V such that $f_0 \in C_0^2(\mathbb{R}^n)$. Then $f_0 \in \mathcal{D}_A(x)$ and $0 = Af_0(x) = Lf_0(x) = Lf(x)$. Hence $\mathcal{A}f(x) = Lf(x) = 0$ in this case.
if x is not a trap, choose a bounded open set $x \in U$ such that $\mathbb{E}^x\left[\tau_U\right] < \infty$. Then by Dynkin's formula with $\tau_u = \tau$

$$\begin{aligned} \left|\frac{\mathbb{E}^x\left[f(X_\tau)\right] - f(x)}{\mathbb{E}^x\left[\tau\right]} - Lf(x)\right| &= \frac{\left|\mathbb{E}^x\left[\int_0^\tau ((Lf)(X_s) - Lf(x))\,\mathrm{d}s\right]\right|}{\mathbb{E}^x\left[\tau\right]} \\ &\le \sup_{y \in U} |Lf(x) - Lf(y)| \to 0 \quad \text{as } U \downarrow x \end{aligned}$$

[1] $\tilde{\phi}$ is called the X-harmonic extension of ϕ