Probability Theory

I. Construction of Probability Measure

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Sigma Fields

We want to define a collection of subsets with necessary properties that fit with probability.

Definition 1 (Field)

A collection $\mathcal{F} \subset 2^{\Omega}$ is a **field** if

1. $\Omega \in \mathcal{F}$
2. If $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$
3. If $E_1, \ldots, E_n \in \mathcal{F}$, then $\bigcup_{j=1}^{n} E_j \in \mathcal{F}$

If, instead of 3, we have the stronger $3'$:

$$3': \text{If } \{E_j\}_{j=1}^{\infty} \text{ is a countable set of events in } \mathcal{F}, \text{ then } \bigcup_{j=1}^{\infty} E_j \in \mathcal{F}$$

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Examples:

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Examples:

- $\mathcal{F} = 2^{\Omega}$
- $\mathcal{F} = \{\emptyset, \Omega\}$
- $\Omega = \{1, 2, 3\}$, $\mathcal{F} = \{\emptyset, \{1\}, \{2, 3\}, \Omega\}$
- $\mathcal{F} = \{B \subset \Omega : B$ is countable or B^c is countable $\}$

Sigma fields

Lemma 1 (σ fields are closed in intersection.)

If I is any index set and $\{\mathcal{F}_i : i \in I\}$ are σ-fields over Ω, then $\bigcap_{i \in I} \mathcal{F}_i$ is a σ-field.

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Proposition 1 (Uniqueness of the smallest σ-field)

Let $\mathcal{E} \subseteq 2^{\Omega}$ be any collection of subsets of Ω. There is a unique smallest σ-field $\sigma(\mathcal{E})$ that contains $\mathcal{E}$. It is called the $\underline{\sigma\text{-field generated by } \mathcal{E}}$.

- **Example**: $\Omega = \{1, 2, 3\}$, $\mathcal{E} = \{\emptyset, \{1, 2\}, \Omega\} \implies \sigma(\mathcal{E}) = \{\emptyset, \{1, 2\}, \{3\}, \Omega\}$
- **Exercise**: Let $\mathcal{E}_1, \mathcal{E}_2 \subseteq 2^{\Omega}$. Show that $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$ if and only if $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$

The Borel σ-field

Let X be a topological space (e.g. Euclidean space $\mathbb{R}^d$. The **Borel σ-field** $\mathcal{B}(X)$ is the σ-field generated by the **open** subsets in X.

$$\mathcal{B}(X) = \sigma\{\text{open subsets of} X\} \tag{0.1}$$

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Fun Fact: For $X = \mathbb{R}^d$, every open set u is a countable union of open balls

$$u = \bigcup_{i=1}^{\infty} B(x_i, r_i) \tag{0.2}$$

$\implies \mathcal{B}(\mathbb{R}^d) = \sigma\{\text{open balls in } \mathbb{R}^d\}$
If $d = 1$, $\mathcal{B}(\mathbb{R}) = \sigma\{(a, b) : a < b\} = \sigma\{[a, b) : a < b\} = \sigma\{(a, b] : a < b\}$

Measures

Definition 2 (Measure)

A function $\mu : \mathcal{F} \to [0, \infty]$ is called a measure if it is countably additive. I.E. If $E_1, E_2, E_3, \ldots$ are all disjoint, then

$$\mu(\bigsqcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n) \tag{0.3}$$

- The triple $(\Omega, \mathcal{F}, \mu)$ is then called a **measure space**.
- If $\mu(\Omega) < \infty$, we say μ is a **finite measure**.
- If $\mu(\Omega) = 1$, we say μ is a **probability measure**, and the triple $(\Omega, \mathcal{F}, \mu)$ is a **probability space**.

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Example

- Point mass on $(\Omega, 2^{\Omega})$, fix a point $w_0 \in \Omega$, and define $\delta_{w_0} : 2^{\Omega} \to \{0, 1\}$ by

$$\delta_{w_0}(E) = \begin{cases} 1 & \text{if } w_0 \in E \\ 0 & \text{if } w_0 \notin E \end{cases} \tag{0.4}$$

  Exercise: Verify that it is a **measure** on 2^{Ω}.

Basic properties of measure

- **Monotone**: If $A, B \in \mathcal{F}$ and $A \subseteq B$, $\mu(A) \leq \mu(B)$.
- **Rule of addition**: If $A, B \in \mathcal{F}$, $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.
- **Subadditive**: If $\{B_n\}_{n=1}^{\infty}$ are in $\mathcal{F}$, then

$$\mu(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mu(B_n) \tag{0.5}$$

Genuine measures defined on full σ-fields are often difficult to construct. To approach this problem, we first start with weaker notions of "measure" that we later build up to the full measure.

Finitely-Additive Measure

Proposition 2 (Super-additivity of F.A. measure)

Let $(\Omega, \mathcal{A}, \chi)$ be a finitely-additive measure space. If $\{A_i\}_{i=1}^{\infty}$ are disjoint in $\mathcal{A}$, and $A = \bigsqcup_{i=1}^{\infty} A_i \in \mathcal{A}$, then $\chi(A) \geq \sum_{i=1}^{\infty} \chi(A_i)$

Proof.

A "Borel Filed" $\mathcal{B}_{(]}(\mathbb{R})$

Among the many natural generating sets for the Borel σ-field $\mathcal{B}(\mathbb{R}^d)$ is

$$\ell_{(]} = \{(a, b] : -\infty \leq a \leq b \leq \infty\} \tag{0.6}$$

Then, what about the **filed** generated by these intervals?

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$$\mathcal{A}(\ell_{(]}) = \{\text{finite } \textbf{disjoint} \text{ unions of intervals in } \ell_{(]}\} = \mathcal{B}_{(]}(\mathbb{R}) \tag{0.7}$$

Semi-Algebras of Sets

Definition 3

A collection $\mathcal{S} \subseteq 2^{\Omega}$ is a semi-algebra or elementary-class if

1. $\emptyset \in \mathcal{S}$
2. If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$
3. If $A \in \mathcal{S}$, then A^c is a finite disjoint union of elements from $\mathcal{S}$.

In particular, $\Omega = \emptyset^c$ is a finite disjoint union of elements in $\mathcal{S}$.

Proposition 3 (Field generated by Semi-algebras)

If $\mathcal{S}$ is a semi-algebra over Ω, then the field $\mathcal{A}(\mathcal{S})$ it generates is equal to

$$\{ \text{ all finite disjoint unions of sets from } \mathcal{S} \text{ } \}$$

Proof.

F.A. Measures and Semi-Algebras

Proposition 4 (Extension of F.A. measure on semi-algebras)

Let $\mathcal{S}$ be a semi-algebras over Ω.
Let $\chi : \mathcal{S} \to [0, \infty]$ be finitely additive: $\chi(E \bigsqcup F) = \chi(E) + \chi(F), E, F \in \mathcal{S}$.
Then, χ extends to a uniquely finitely-additive measure on $\mathcal{A}(\mathcal{S})$, defined by

$$A = \bigsqcup_{i=1}^{n} E_i \implies \chi(A) := \sum_{i=1}^{n} \chi(E_i)$$

Proof

Stieltjes (pre) Measures on $\mathcal{B}_{(]}(\mathbb{R})$

Let $F : \mathbb{R} \to \mathbb{R}$ be a on-decreasing function. On the semi-algegbra $\ell_{(]} = \{(a, b] : -\infty \leq a \leq b \leq \infty\}$, define

$$\chi_F((a, b]) = F(b) - F(a) \geq 0 \tag{0.8}$$

This is additive on the semi-algebra $\ell_{(]}$. Then by prop 4, χ_F extends to a finitely-additive measure on $\mathcal{A}(\ell_{(]}) = \mathcal{B}_{(]}(\mathbb{R})$

Question: Is it a premeasure? Is it countably additive?

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Question: Is it a premeasure? Is it countably additive?

Answer: If $F(a+) = F(a)$, then χ_F is countably additive.

Theorem 1 (Stieltjes Premeasure)

The finitely-additive measure χ_F is a premeasure on $\mathcal{B}_{(]}(\mathbb{R})$ if and only if F is right-continuous on $\mathbb{R}$:

$$\lim_{\delta \downarrow 0} F(a + \delta) = F(a)$$

Stieltjes (pre) Measures on $\mathcal{B}_{(]}(\mathbb{R})$

Proposition 5 (When finitely-additive measure becomes a premeasure?)

Let $\mathcal{S} \subseteq 2^{\Omega}$ be a semi-algebra. A finitely-additive measure $\chi : \mathcal{A}(\mathcal{S}) \to [0, \infty]$ is a premeasure if and only it is countably subadditive on $\mathcal{S}$:

$$E = \bigsqcup_{j=1}^{\infty} E_j \in \mathcal{S} \implies \chi(E) \leq \sum_{j=1}^{\infty} \chi(E_j)$$

Proof.

χ_F is a premeasure

We now show that $\chi_F : \mathcal{A}(\ell_{(]}) \to [0,\infty)$ is a premeasure by showing it is countably subadditive on the semi-algebra $\ell_{(]}$

Choose some $\delta > 0$, by compactness:

$$[a+\delta, b] \subset (a,b] = \bigsqcup_{j=1}^{\infty} (a_j, b_j] \subseteq \bigcup_{j=1}^{N} (a_j, b_j + \delta_j) \subseteq \bigcup_{j=1}^{N} (a_j, b_j + \delta_j]$$

for some $N < \infty$. Since χ_F is subadditive:

$$\chi_F(a+\delta, b] \leq \sum_{j=1}^{N} \chi_F(a_j, b_j + \delta_j] \leq \sum_{j=1}^{\infty} \chi_F(a_j, b_j + \delta_j]$$

$$\underbrace{\chi_F(a+\delta, b]}_{F(b)-F(a+\delta)} \leq \sum_{j=1}^{\infty} \chi_F(a_j, b_j] + \sum_{j=1}^{\infty} \underbrace{(\chi_F(b_j, b_j+\delta_j])}_{F(b_j+\delta_j)-F(b_j)<\varepsilon/2^j}$$

Now, let $\delta \to 0$, show that χ_F is countably subadditive on the semi-algebra.

Measure Extension

Caratheodory's Extension

Let Ω be a set and $\mathcal{E} \subseteq 2^{\Omega}$ such that $\emptyset, \Omega \in \mathcal{E}$.
Let $\rho : \mathcal{E} \to [0, \infty]$ such that $\phi(\emptyset) = 0$.
Deinfe $\phi^* : 2^{\Omega} \to [0, \infty]$ as follows:

$$\rho^*(A) = \inf\{\sum_{j=1}^{\infty} \phi(E_j) : E_j \in \mathcal{E}, A \subseteq \bigcup_{j=1}^{\infty} E_j\}$$

Theorem 2

If $\mathcal{E}$ is a field and ρ is a premeasure, then $\rho^*|_{\sigma(\mathcal{E})}$ is a measure.

Measure Extension

Proposition 6 (Properties of ρ^*)

Fix $\rho : \mathcal{E} \subset 2^\Omega \to [0, \infty]$ $(\emptyset, \Omega \in \mathcal{E}, \rho(\emptyset) = 0)$:

1. $\rho^*(\emptyset) = 0$
2. ρ^* is monotone: $A \subseteq B \implies \rho^*(A) \le \rho^*(B)$
3. ρ^* is countably subadditive: $\rho^*(\cup_{n=1}^\infty A_n) \le \sum_{n=1}^\infty \rho^*(A_n)$

Proof.

Outer measure

Definition 4

Let Ω be a nonempty set. A function $\nu : 2^{\Omega} \to [0, \infty]$ is an outer measure if:

1. $\nu(\emptyset)$ =0
2. ν is monotone: $A \subseteq B \implies \nu(A) \leq \nu(B)$
3. ν is countably subadditive: $\nu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \nu(A_n)$

Thus, Caratheodory's extension ρ^* of a set function is an outer measure. We can use it to distinguish finitely additive measures from premeasure.

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Lemma 2 (Outer measure dominates any measure extension)

If $(\Omega, \mathcal{A}, \mu)$ is a premeasure space, and $(\Omega, \sigma(\mathcal{A}), \nu)$ is a measure space extending it, then

$$\nu \leq \mu^* \quad \text{on} \quad \sigma(\mathcal{A}).$$

Outer measure

Proposition 7 (Distinguish premeasure from F.A measure via outer measure)

If $(\Omega, \mathcal{A}, \chi)$ is a finitely-additive measure space, then $\chi^* \leq \chi$ on $\mathcal{A}$, and $\chi^* = \chi$ on $\mathcal{A}$ if and only if χ is a premeasure.

Proof.

From premeasure to measure

• So far we have developed the premeasure, how do we extend it to the genuine measure?
Goal: Define a measure μ on sigma algebra

Approach:

1. Make 2^{Ω} into a topological space.
2. Define $\bar{\mathcal{A}}$ to be the closure of $\mathcal{A}$
3. Prove $\mu : \mathcal{A} \to [0, \infty)$ is sufficiently continuous, thus extends to closure $\bar{\mathcal{A}}$
4. Use topological tools to show $\bar{\mathcal{A}}$ is a σ-field, and $\bar{\mu}$ is a measure. It will turn out that $\bar{\mu}$ equals to the outer measure μ^* on $\bar{\mathcal{A}}$

Measure extension

Pseudo-Metric Spaces $d : X \times X \to [0, \infty)$

$$\text{1}\;\; d(x, y) = 0 \Longleftarrow x = y$$

$$\text{2}\;\; d(x, y) = d(y, x)$$

$$\text{3}\;\; d(x, z) \leq d(x, y) + d(y, z)$$

- A sequence $(x_n)_{n=1}^{\infty}$ in X has a limit x if
$$\forall \varepsilon > 0, \exists N \in \mathbb{N}\, \forall n \;\geq N \; d(x_n, x) < \varepsilon \tag{0.9}$$
- Given $V \subseteq X$, the closure $\mathrm{Var}V$ is the set of limits of sequences in V
- A set V is closed if $\bar{V} = V$.
- A function $f : V \to \mathbb{R}$ is Lipschitz if $\exists K \in (0, \infty)$ such that $|f(x) - f(y)| \leq K d(x, y)$

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Proposition 8

If f is Lipschitz on a nonempty $V \subseteq X$, then there is a unique Lipschitz extension $\bar{f} : \bar{V} \to \mathbb{R}$ with the same Lipschitz constant K

The outer Pseudo-Metric

• Let $(\Omega, \mathcal{A}, \mu)$ be finite premeasure space. $\mu^* : 2^\Omega \to [0, \mu(\Omega)]$ Caratheodory outer measure

$$\mu^*(E) = \inf\{\sum_{j=1}^{\infty} \mu(A_j) : A_j \in \mathcal{A}, E \subseteq \bigcup_{j=1}^{\infty} A_j\}$$

Definition 5

Define $d_\mu : 2^\Omega \times 2^\Omega \to [0, \mu(\Omega)]$,

$$d_\mu(E, F) = \mu^*(E \triangle F) := \mu^*((E \setminus F) \cup (F \setminus E))$$

Exercise: The d_μ is a pseudo-metric on 2^Ω

Key properties of the Outer Pseudo-Metric

1. $\forall A, B \in 2^{\Omega}$, $d_\mu(A, B) = d_\mu(A^c, B^c)$
2. $\forall \{A_n\}_{n=1}^{\infty} \{B_n\}_{n=1}^{\infty} \in 2^{\Omega}$

$$d_\mu\Big(\bigcup_{n=1}^{\infty} A_n, \bigcup_{n=1}^{\infty} B_n\Big) \leq \sum_{n=1}^{\infty} d_\mu(A_n, B_n)$$

$$d_\mu\Big(\bigcap_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} B_n\Big) \leq \sum_{n=1}^{\infty} d_\mu(A_n, B_n)$$

Proof.

Outer Pseudo Metric

Lemma 3 (Continuity of Pseudo Metric)

If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, and $A_n \in \mathcal{A}$ with $A_n \uparrow A$, then $d_\mu(A_n, A) = \mu^*(A) - \mu(A_n) \to 0$

Corollary 1

If $A_n \in \mathcal{A}$ and $A_n \uparrow A$, then $A \in \bar{\mathcal{A}}$

Proof of Lemma .

Mesure Extension

Theorem 3 (The closure is a σ-field)

If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, then the closure $\bar{\mathcal{A}}$ of the filed $\mathcal{A}$ in the pseudo-metric space $2^\Omega, d_\mu$ is a σ-field.

Remark. For $A, B \in \mathcal{A}$,
$d_\mu(A, B) = \mu^*(A \triangle B) = \mu(A \triangle B) = \mu(A \cup B) - \mu(A \cap B) \geq |\mu(A) - \mu(B)|$
$\implies$ μ is Lip-1 on $\mathcal{A}$. Note that there is unique Lip-1 function $\bar{\boldsymbol{\mu}} : \bar{\mathcal{A}} \to [0, \mu(\Omega)]$
Proof.

Measure Extension

Definition 6

Given $\mathcal{E} \subseteq 2^{\Omega}$, $\mathcal{E}_\sigma := \{$ countable unions of elements of $\mathcal{E}\}$

Note: $\mathcal{E}_\sigma$ is automatically closed under countable unions. If $\mathcal{E}$ is closed under finite intersections, so is $\mathcal{E}_\sigma$.

Restatement of Lemma
If $(\Omega, \mathcal{A}, \mu)$ is a finite premeausre space, then $\bar{\mathcal{A}}_\sigma = \bar{\mathcal{A}}$, and $\bar{\mu} = \mu^*$ on $\mathcal{A}_\sigma$
Proof.
If $A \in \mathcal{A}_\sigma$, $A = \bigcup_{n=1}^{\infty} B_n$, each $B_n \in \mathcal{A}$
Define $A_n = \bigcup_{j=1}^{n} B_j$, then $A_n \uparrow A$, $d_\mu(A_n, A) \to 0$
This implies $\mathcal{A}_\sigma \subseteq \bar{\mathcal{A}}$ and note that $\mathcal{A} \subseteq \mathcal{A}_\sigma$. Therefore, $\bar{\mathcal{A}}_\sigma = \bar{\mathcal{A}}$

Measure Extension

Proposition 9 (Conditions for a set $\in \bar{\mathcal{A}}$)

Let $(\Omega, \mathcal{A}, \mu)$ be a finite premeasure space. For $B \in 2^\Omega$, The followings are equivalent:

1. $B \in \bar{\mathcal{A}}$
2. $\forall \varepsilon > 0$, $\exists C \in \mathcal{A}_\sigma$ such that $B \subseteq C$ and $\mu^*(C \setminus B) = d_\mu(B, C) < \varepsilon$

Proof.

Measure Extension

Corollary 2

Let $(\Omega, \mathcal{A}, \mu)$ be a finite premeasure space. Then $\mu^* = \bar{\mu}$ on $\bar{\mathcal{A}}$.

Proof.

Let $B \in \bar{\mathcal{A}}$. $\bar{\mu}(B) = |\bar{\mu}(B) - \bar{\mu}(\emptyset)| \leq d_\mu(B, \emptyset) = \mu^*(B \triangle \emptyset) = \mu^*(B)$

For reverse inequality: fix $\varepsilon > 0$. By proposition **9**, we can choose $C \in \mathcal{A}_\sigma$ such that $B \subseteq C$ and $d_\mu(B, C) < \varepsilon$.

$\implies |\bar{\mu}(B) - \bar{\mu}(C)| \leq d_\mu(B, C) < \varepsilon$, $\bar{\mu}(C) \leq \bar{\mu}(B) + \varepsilon$

$$\implies \bar{\mu}(B) \leq \mu^*(B) \leq \mu^*(C) \stackrel{\dagger}{=} \bar{\mu}(C) \leq \bar{\mu}(B) + \varepsilon$$

$\dagger$ holds since $\bar{\mu} = \mu^*$ on $\mathcal{A}_\sigma$. Take $\varepsilon \downarrow 0$, $\bar{\mu}(B) = \mu^*(B)$.

Measure Extension

Theorem 4 ($\bar{\mu}$ is a genuine measure on $\bar{\mathcal{A}}$)

If $(\Omega, \mathcal{A}, \mu)$ is a finite premeasure space, then $\bar{\mu} : \bar{\mathcal{A}} \to [0, \mu(\Omega)]$ is a measure.

Proof.

If we show that $\bar{\mu}$ is **finitely-additive** on $\bar{\mathcal{A}}$, then it is a finitely additive measure on the σ-field $\bar{\mathcal{A}}$. Note that $\bar{\mu}$ is countably super-additive. But by the previous corollary, $\bar{\mu} = \mu^*$ on $\bar{\mathcal{A}}$, μ^* is countably subadditive.

Measure Extension

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Let $A, B \in \bar{\mathcal{A}}$. Find $A_n \to A, B_n \to B$, $A_n, B_n \in \mathcal{A}$.
Then, $d_\mu(A_n \bigcup B_n, A \cup B)$ and $d_\mu(A_n \cap B_n, A \cap B)$ is bounded by $d_\mu(A_n, A) = d_\mu(B_n, B)$ which converges to 0 This implies

$$\bar{\mu}(A \cup B) + \bar{\mu}(A \cap B) = \lim_{n \to \infty} [\mu(A_n \cup B_n) + \mu(A_n \cap B_n)] = \bar{\mu}(A) + \bar{\mu}(B)$$

Uniqueness Theorem

So far, we have constructed **finite** measure on $\bar{\mathcal{A}}$.

Theorem 5 (Uniqueness of Extension)

If $\mathcal{F}$ is a σ-field with $\mathcal{A} \subseteq \mathcal{F} \subseteq \bar{\mathcal{A}}$. and ν is a measure on $\mathcal{F}$ with $\nu|_{\mathcal{A}} = \mu$, then $\nu = \bar{\mu}|_{\mathcal{F}}$

Proof.

Extension to σ-Finite Measures

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite premeasure space. $\Omega = \bigcup_{n=1}^{\infty} A_n$ such that
$A_n \in \mathcal{A}, \mu(A_n) < \infty$
Take $\Omega_1 = A_1$, $\Omega_n = A_n \setminus A_{n=1}$, thus $\mu(\Omega_n) \leq \mu(A_n) < \infty$, $\Omega = \bigsqcup_{n=1}^{\infty} \Omega_n$
Define $\mu_n \to [0, \infty) : \mu_n(A) := \mu(A \cap \Omega_n)$
Then $(\Omega_n, \mathcal{A}, \mu_n)$ is a finite premeasure space

$\implies$ Extends to a finite measure $\bar{\mu}_n$ on $\bar{\mathcal{A}}^n \supseteq \sigma(\mathcal{A})$

Extension to σ-Finite Measures

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Define $\mu_n \to [0, \infty) : \mu_n(A) := \mu(A \cap \Omega_n)$
Then $(\Omega_n, \mathcal{A}, \mu_n)$ is a finite premeasure space

$$\implies \text{Extends to a finite measure } \bar{\mu}_n \text{ on } \bar{\mathcal{A}}^n \supseteq \sigma(\mathcal{A})$$

Theorem 6

$\bar{\mu} := \sum_{n=1}^{\infty} \bar{\mu}_n$ is the unique measure on $\sigma(\mathcal{A})$ extending μ

Proof. Easy to check that $\bar{\mu}$ is a countably-additive measure since Ω_n are disjoint. We need to check uniqueness.

Extension to σ-finite Measures

Proposition 10

Let $(\Omega, \mathcal{A}, \mu)$ σ-finite premeausre space.

1. $\bar{\mu} = \mu^*$ on $\sigma(\mathcal{A})$
2. If $B \in \sigma(\mathcal{A})$ and $\varepsilon > 0$, $\exists C \in \mathcal{A}_\sigma$ such that $B \subseteq C$ and $\bar{\mu}(C \setminus B) < \varepsilon$
3. Moreover, if $\bar{\mu}(B) < \infty$, $\exists A \in \mathcal{A}$ such that $A \in \mathcal{A}$ such that $\bar{\mu}(A \triangle B) < \varepsilon$

Radon Measures

• If Ω is a topological space, any measure on $\mathcal{B}(\Omega)$ will be referred to as a **Borel Measure**.

Definition 7 (Radon measure on $\mathbb{R}$)

A Borel measure $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ on $\mathbb{R}$ is called a **Radom Measure** if

$$\mu([a, b]) < \infty \quad \forall a < b \in \mathbb{R}$$

Example: The Stieltjes premeasures $\mu_F((a, b]) = F(b) - F(a)$ for $F \to \mathbb{R} \to \mathbb{R}$ increasing and right cotinuous

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Theorem 7

If μ is a Radon measure on $\mathbb{R}$, then there exists a non-decreasing, right-continuous function $F : \mathbb{R} \to \mathbb{R}$ (unique up to an additive constant) such that

$$\mu((a, b]) = F(b) - F(a), \quad -\infty \leq a \leq b \leq \infty$$

Radon Measure

Proposition 11 (Continuity of measure)

Let μ be a finitely additive measure on $(\Omega, \mathcal{A})$. The following are equivalents:

1. μ is a premeasure on $\mathcal{A}$
2. If $A_n, A \in \mathcal{A}$ and $A_n \to A$, then $\mu(A_n) \to \mu(A)$. Moreover, in the case of finite measure, the following are also equivalent
3. If $A_n \downarrow A$ in $\mathcal{A}$, then $\mu(A_n) \downarrow \mu(A)$
4. If $A_n \uparrow \Omega$ in $\mathcal{A}$, then $\mu(A_n) \uparrow \mu(\Omega)$.
5. If $A_n \downarrow \emptyset$ in $\mathcal{A}$, then $\mu(A_n) \downarrow 0$

Proof.

Radon Measure

Definition 8

Let μ be a Borel probability measure on $\mathbb{R}$.

$$F_\mu : \mathbb{R} \to \mathbb{R};\ F_\mu(x) = \mu((-\infty, x])$$

is the cumulative distribution function (CDF) of μ

By the Radon measure theorem, Borel probability measures on $\mathbb{R}$ are characterized by their CDF.

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Corollary 3

Any right-continuous, non-decreasing function $F : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{x \to -\infty} F(x) = 0, \qquad \lim_{x \to +\infty} F(x) = 1$$

is the cumulative distribution function (CDF) of a unique Borel probability measure on $\mathbb{R}$.

Lebesgue Measure

The Radon measure on $\mathbb{R}$ satisfying

$$\lambda((a,b]) = b - a, \quad -\infty < a < b < \infty$$

is called **Lebesgue measure**
Note that λ and the outer measure λ^* is both translation invariant, i.e.,
$\lambda(A+\tau) = \lambda(A)$, $\lambda^*(E+\tau) = \lambda^*(E)$ for $\tau \in \mathbb{R}, \forall A \in \mathcal{B}_{(]}(\mathbb{R}), \forall E \subseteq \mathbb{R}$

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Theorem 8

λ is the unique translation invariant Borel measure such that $\lambda((0,1]) = 1$; if μ is another translation invariant Borel measure, then $\mu = \alpha\lambda$ for some $\alpha \geq 0$

Null complete

In a measure space $(\Omega, \mathcal{F}, \mu)$, a measurable set $N \in \mathcal{F}$ is a null set if $\mu(N) = 0$. There are many sets $N \in \mathcal{B}(\mathbb{R})$ of Lebesgue measure 0 that contain non-Borel sets $\Lambda \subseteq N$. This can sometimes cause technical problems.

Definition 9

A measure space $(\Omega, \mathcal{F}, \mu)$ is called **null-complete** if, for every $N \in \mathcal{F}$ with $\mu(N) = 0$, every subset $\Lambda \subseteq N$ is in $\mathcal{F}$ and $\mu(\Lambda) = 0$.

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Theorem 9

For any measure space $(\Omega, \mathcal{F}, \mu)$, there is an extension $\tilde{\mathcal{F}} \supseteq \mathcal{F}$, $\tilde{\mu}|_{\mathcal{F}} = \mu$ such that $(\Omega, \tilde{\mathcal{F}}, \tilde{\mu})$ is null-complete.