Stochastic process

I Brownian Motion

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MILAB

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Brownian motion

The most common way to define a Brownian Motion is by the following properties:

Definition 1

A Brownian motion or Wiener process $(B_t)_{t \geq 0}$ is a real-valued stochastic process such that

1. $B_0 = 0$
2. Independent increments: the random variables $B_v - B_u$, $B_t - B_s$ are independent whenever $u \leq v \leq s \leq t$ (so the intervals $(u, v), (s, t)$ are disjoint.)
3. Normal increments: $B_{s+t} - B_s \sim \mathcal{N}(0, t)$ for all $s, t \geq 0$.
4. Continuous sample paths: with probability 1 (or almost surely), the function $t \mapsto W_t$ is continuous.

Remark. If the only properties $(1) \sim (3)$ holds, the process is called pre-Brownian motion.

Brownian motion

- For $0 = t_0 < t_1 < \cdots < t_n$, the finite-dimensional law of Brownian motion is given by

$\mathbb{P}((B_{t_0}, \ldots, B_{t_n}) \in A_0 \times \cdots \times A_n)$

$$= \mathbf{1}_{A_0}(0) \int_{A_1 \times \cdots \times A_n} \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_n}{(2\pi)^{n/2}\sqrt{t_1(t_2 - t_1)\ldots(t_n - t_{n_1})}} \exp\left(-\sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)$$

with convention $x_0 = 0$.

The vector $(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ has density

$$q(y_1, \ldots, y_n) = \frac{1}{(2\pi)^{n/2}\sqrt{t_1(t_2 - t_1)\ldots(t_n - t_{n-1})}} \exp\left(-\sum_{i=1}^{n} \frac{y_i^2}{2(t_i - t_{i-1})}\right),$$

and the change of variables $x_i = y_1 + \cdots + y_i$ for $i \in \{1, \ldots, n\}$ completes the argument.

Proposition 1

Let B be a Brownian motion. Then,

1. -B is also a Brownian motion (symmetric property)
2. for every $\lambda > 0$, the process $B_t^{\lambda} = \frac{1}{\lambda} B_{\lambda^2 t}$ is also a Brownian motion.
3. for every $s \geq 0$ the process $B_t^{(s)} = B_{s+t} - B_s$ is also Brownian motion and is independent of $\sigma(B_r, r \leq s)$ (simple Markov property).

Change of variable formula

- 1d case:

$$p_Y(y) = p_X(x) \cdot \left| \frac{\mathrm{d}x}{\mathrm{d}y} \right| = p_X(g(y))|g'(y)|$$

- Multi-dimensional case: Let $Y = \Phi(X)$, where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$. Define the Jacobian matrix as

$$\mathbf{J}_{X \to Y} := \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}.$$

If Φ is an invertible mapping, we can define the pdf of the transformed variables in terms of the original variables as follows:

$$p_Y(y) = p_X(x) \left|\mathsf{det}\mathbf{J}_{Y \to X}\right| = p_X(\Phi^{-1}(y)) \left|\mathsf{det}\mathbf{J}_{Y \to X}\right|$$

Properties of Brownian Sample paths

Properties of Brownian Sample paths

- In this section, we investigate the properties of sample paths of Brownian motion.

We fix a Brownian motion $(B_t)_{t\geq 0}$. For every $t \geq 0$, we set $\mathcal{F}_t = \sigma(B_s, s \leq t)$.
Note that $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \leq t$. We also set

$$\mathcal{F}_{0+} = \bigcap_{s>0} \mathcal{F}_s$$

We start by stating a useful $0 - 1$ law.

Properties of Brownian Sample paths

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Theorem 1 (Blumenthal's zero-one law)

The σ-field $\mathcal{F}_{0+}$ is trivial, in the sense that $\mathbb{P}(A) = 0$ or 1 for every event $A \in \mathcal{F}_{0+}$.

Proof. Let $0 < t_1 < t_2 < \cdots < t_k$ and $g : \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous function. Also fix $A \in \mathcal{F}_{0+}$. Then, by a continuity argument,

$$\mathbb{E}\,\mathbf{1}_A g(B_{t_1}, \ldots, B_{t_k} = \lim_{\varepsilon \downarrow 0} \mathbb{E}\,\mathbf{1}_A g(B_{t_1} - B_\varepsilon, \ldots, B_{t_k} - B_\varepsilon.$$

If $0 < \varepsilon < t_1$, the variables $B_{t_1} - B_\varepsilon, \ldots, B_{t_k} - B_\varepsilon$ are independent of $\mathcal{F}_\varepsilon$ by the simple Markov property and also of $\mathcal{F}_{0+} \subset \mathcal{F}_\varepsilon$. If follows that

$$\begin{aligned}\mathbb{E}\,\mathbf{1}_A g(B_{t_1}, \ldots, B_{t_k}) &= \lim_{\varepsilon \downarrow 0} \mathbb{P}(A)\,\mathbb{E}\, g(B_{t_1} - B_\varepsilon, \ldots, B_{t_k} - B_\varepsilon \\ &= \mathbb{P}(A)\,\mathbb{E}\, g(B_{t_1}, \ldots, B_{t_k})\end{aligned}$$

The above implies $\mathcal{F}_{0+}$ is independent of $\sigma(B_{t_1}, \ldots, B_{t_k})$. Since this holds for any finite collection $\{t_1, \ldots t_k\}$ of positive reals, $\mathcal{F}_{0+}$ is independent of $\sigma(B_t, t > 0)$.

Properties of Brownian Sample paths

However, $\sigma(B_t, t > 0) = \sigma(B_t, t \geq)$ since B_0 is the pointwise limit of B_t when $t \to 0$. Since $\mathcal{F}_{0+} \subset \sigma(B_t, t \geq 0)$, we conclude that $\mathcal{F}_{0+}$ is independent of itself, which yields the desired result.

Lemma 1

If the sigma field $\mathcal{F}$ is independent from itself, (i.e. for every $A, B \in \mathcal{F}$, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$). Then, every event from the sigma field $\mathcal{F}$ has probability 0 or 1.

Proposition 2

- We have, a.s. for every $\varepsilon > 0$,

$$\sup_{0 \leq s \leq \varepsilon} B_s > 0, \qquad \inf_{0 \leq s \leq \varepsilon} B_s < 0$$

- For every $a \in \mathbb{R}$, let $T_a = \{t \geq 0 \,|\, B_t = a\}$ (with the convention $\inf \emptyset = \infty$). Then,

$$a.s.\ \forall a \in \mathbb{R},\ T_a < \infty.$$

Consequently, we have a.s.

$$\limsup_{t \in \infty} B_t = +\infty, \qquad \liminf_{t \in \infty} B_t = -\infty$$

Properties of Brownian Sample paths

Corollary 1

Almost surely, the function $t \mapsto B_t$ is not monotone on any non-trivial interval.

Proposition 3

Let $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ be a sequence of subdivisions of $[0, t]$ whose mesh tends to 0 (i.e. $\sup_{1 \leq i \leq p_n}(t_i^n - t_{i-1}^n) \to 0$ as $n \to \infty$). Then,

$$\lim_{n \to \infty} \sum_{i=1}^{p_n} (B_{t_i^n} - B_{t_{i-1}^n})^2 = t. \text{ in } L^2$$

Proof. Recall that $X_n \to X$ in L^2 means that $\mathbb{E}\,(X_n - X)^2 \to 0$.

$$\begin{aligned}
&\mathbb{E}\left(\sum_{i=1}^{p_n} (B_{t_i^n} - B_{t_{i-1}^n})^2 - t\right)^2 \\
&= \mathbb{E}\left(\sum_{i=1}^{p_n} (B_{t_i^n} - B_{t_{i-1}^n})^2\right)^2 - 2t \sum_{i=1}^{p_n} (B_{t_i^n} - B_{t_{i-1}^n})^2 + t^2
\end{aligned}$$

Properties of Brownian Sample paths

$$= \mathbb{E}\sum_{i=1}^{p_n}(B_{t_i^n} - B_{t_{i-1}^n})^4 + \mathbb{E}\sum_{i\neq j}(B_{t_i^n} - B_{t_{i-1}^n})^2(B_{t_j^n} - B_{t_{j-1}^n})^2$$

$$- 2t \cdot \mathbb{E}\sum_{i=1}^{p_n}(B_{t_i^n} - B_{t_{i-1}^n})^2 + t^2$$

$$= 2\sum_i (t_i - t_{i-1})^2 + t^2 - 2t \cdot t + t^2 \leq \sup_i |t_i - t_{i-1}| \cdot t \to 0$$

Corollary 2

Almost surely, the function $t \mapsto B_t$ has infinite variation on any non-trivial interval (i.e. $\sum_{i=1}^{p_n} \left|B_{t_i^n} - B_{t_{i-1}^n}\right|$ tends to infinity).

Proof.

$$\sum_{i=1}^{p_n}(B_{t_i^n} - B_{t_{i-1}^n})^2 \leq \left(\sup_{1\leq i\leq p_n}\left|B_{t_i^n} - B_{t_{i-1}^n}\right|\right) \times \sum_{i=1}^{p_n}\left|B_{t_i^n} - B_{t_{i-1}^n}\right|.$$

(In general, L^2 convergence implies convergence in probability but not almost sure convergence. However, we can extract a subsequence that converges almost surely from an almost surely converging sequence.)

The Strong Markov Property of Brownian Motion

The Strong Markov Property of Brownian Motion

Definition 2

A random variable T with values in $[0, \infty]$ is a stopping time if, for every $t \geq 0$, $\{T \leq t\} \in \mathcal{F}_t$.

Q: If the time step t of $\mathcal{F}_t$ is replaced by the random time T, what would be the sigma field $\mathcal{F}_T$?

The Strong Markov Property of Brownian Motion

Definition 2

A random variable T with values in $[0, \infty]$ is a stopping time if, for every $t \geq 0$, $\{T \leq t\} \in \mathcal{F}_t$.

Q: If the time step t of $\mathcal{F}_t$ is replaced by the random time T, what would be the sigma field $\mathcal{F}_T$?

Definition 3

Let T be a stopping time. The σ-field of the past before T is

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty \,|\, \forall t \geq 0,\ A \cap \{T \leq t\} \in \mathcal{F}_t\}\,.$$

The Strong Markov Property of Brownian Motion

Theorem 2 (Strong Markov Property)

Let T be a stopping time. We assume that $\mathbb{P}(T < \infty) > 0$ and we set,for every $t \geq 0$,

$$B_t^{(T)} = \mathbf{1}_{T<\infty}(B_{T+t} - B_T).$$

Then under the probability measure $\mathbb{P}(\cdot|T < \infty)$, the process $(B_t^{(T)})_{t\geq 0}$ is a Brownian motion independent of $\mathcal{F}_T$.

The Strong Markov Property of Brownian Motion

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Using Strong Markov Property, we can prove the reflection principle.

Theorem 3 (Reflection principle)

For every $t > 0$, set $S_t = \sup_{s\leq t} B_s$. Then, if $a \geq 0$ and $b \in (-\infty, a]$, we have

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(B_t \geq 2a - b).$$

Moreover, S_t has the same distribution as $|B_t|$.

Remark. The event $\{S_t \geq a\}$ is same as the $\{T_a \leq t\}$.

The Strong Markov Property of Brownian Motion

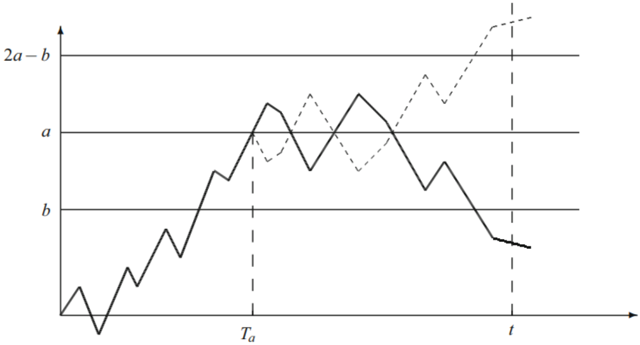

Fig. 2.2 Illustration of the reflection principle: the conditional probability, knowing that $\{T_a \leq t\}$, that the graph is below b at time t is the same as the conditional probability that the graph reflected at level a after time T_a (in *dashed lines*) is above $2a - b$ at time t

Figure 1:

The Strong Markov Property of Brownian Motion

Remark. Since $\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(B_t \geq 2a - b)$, the law of the pair (S_t, B_t) has density

$$g(a,b) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) \mathbf{1}_{a>0,b<a}.$$

Note that the relation between the CDF and density function:

$$g(a,b) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\,\mathbb{P}(S_t \leq x, B_t \leq y)\bigg|_{x=a,y=b}$$

Therefore, differentiating $-\mathbb{P}(B_t \geq 2a-b) = -\int_{2a-b}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \mathrm{d}x$ with respect to a, b gives the desired result.

Exercises

Exercises

- In all exercise, $(B_t)_{t\geq 0}$ is a real Brownian motion started from 0, and $S_t = \sup_{0\leq s\leq t} B_s$.

Exercise 2.25 (*Time inversion*)

1. Show that the process $(W_t)_{t\geq 0}$ defined by $W_0 = 0$ and $W_t = tB_{1/t}$ for $t > 0$ is (indistinguishable of) a real Brownian motion started from 0. (*Hint*: First verify that W is a pre-Brownian motion.)
2. Infer that $\lim_{t\to\infty} \frac{B_t}{t} = 0$ a.s.

Exercises

Tools for the Exercise 2.25:

Lemma 2 (Kolmogorov's Maximal inequality)

Let $X_1, \dots X_n$ be independent random variables defined on a common probability space with expected value $\mathbb{E}\, X_k = 0$ and finite variance for $k = 1, \dots, n$. Then, for each $\lambda > 0$,

$$\mathbb{P}(\max_{1 \le k \le n} |S_k| \ge \lambda) \le \frac{1}{\lambda^2} \mathrm{Var} S_n = \frac{1}{\lambda^2} \sum_{k=1}^{n} \mathbb{E}\, X_k^2,$$

where $S_k = X_1 + \cdots + X_k$.

Exercises

Exercise 2.27 (*Brownian bridge*)
We set $W_t = B_t - tB_1$ for every $t \in [0, 1]$.

1. Show that $(W_t)_{t\in[0,1]}$ is a centered Gaussian process and give its covariance function.

2. Let $0 < t_1 < t_2 < \cdots < t_p < 1$. Show that the law of $(W_{t_1}, W_{t_2}, \ldots, W_{t_p})$ has density

$$g(x_1, \ldots, x_p) = \sqrt{2\pi}\, p_{t_1}(x_1) p_{t_2-t_1}(x_2 - x_1) \cdots p_{t_p-t_{p-1}}(x_p - x_{p-1}) p_{1-t_p}(-x_p),$$

where $p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp(-x^2/2t)$. Explain why the law of $(W_{t_1}, W_{t_2}, \ldots, W_{t_p})$ can be interpreted as the conditional law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_p})$ knowing that $B_1 = 0$.

3. Verify that the two processes $(W_t)_{t\in[0,1]}$ and $(W_{1-t})_{t\in[0,1]}$ have the same distribution (similarly as in the definition of Wiener measure, this law is a probability measure on the space of all continuous functions from $[0, 1]$ into $\mathbb{R}$).

Exercises

Exercise 2.29 (*Non-differentiability*) Using the zero-one law, show that, a.s.,

$$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = +\infty \quad , \quad \liminf_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty \, .$$

Infer that, for every $s \geq 0$, the function $t \mapsto B_t$ has a.s. no right derivative at s.

Exercises

Exercise 2.32 (*Arcsine law*)
Set $T := \inf\{t \geq 0 : B_t = S_1\}$.

1. Show that $T < 1$ a.s. (one may use the result of the previous exercise) and then that T is not a stopping time.
2. Verify that the three variables S_t, $S_t - B_t$ and $|B_t|$ have the same law.
3. Show that T is distributed according to the so-called arcsine law, whose density is

$$g(t) = \frac{1}{\pi\sqrt{t(1-t)}} \mathbf{1}_{(0,1)}(t).$$

4. Show that the results of questions **1.** and **3.** remain valid if T is replaced by $L := \sup\{t \leq 1 : B_t = 0\}$.

Exercises

Tools for the Exercise 2.32:

Lemma 3 (Local maxima of Brownian paths)

Local maxima of Brownian motion are distinct: a.s., for any choice of the rational numbers $p, q, r, s \geq 0$ such that $p < q < r < s$ we have

$$\sup_{p \leq t \leq q} B_t \neq \sup_{r \leq t \leq s} B_t.$$

Remark. The Brownian motion attains its maximum at unique point $x^* \in [0, 1]$.

Lemma 4 (Zero set of Brownian motion)

Let $H := \{t \in [0, 1] \,|\, B_t = 0\}$. Then, H is a.s. a compact subset of [0,1] with no isolated point and zero Lebesgue measure.

Lemma 5 (Time reversal)

Let $B_t^R = B_1 - B_{1-t}$ for every $t \in [0, 1]$. Then, the two processes $(B_t)_{t\in[0,1]}$ and $(B_t^R)_{t\in[0,1]}$ have the same law.