Useful real analysis results

1 Leibniz's Integral Rule

Let μ be a probability distribution with support Ω, let $I \subset \mathbb{R}$ be a nontrivial open interval, also let $f : \Omega \times I \to \mathbb{R}$ be a map with the following properties:

1. For any $x \in I$, $\mathbb{E}_{w\sim\mu}[|f(w,x)|] < \infty$ (Uniformly finite moment)
2. For almost all $w \in \infty$, the map $x \mapsto f(w,x)$ is differentiable with derivative $\frac{\partial}{\partial x} f(w,x)$ (Differentiability)
3. There is a map $h : \Omega \to \mathbb{R}$ with the property that $\mathbb{E}_{w\sim\mu}[|h(w)|] < \infty$, such that $\left|\frac{\partial}{\partial x} f(\cdot, x)\right| \le h$ (Derivative bound).

Then, for any $x \in I, E_{w\sim\mu}\left[\left|\frac{\partial}{\partial x} f(w,x)\right|\right] < \infty$ and the function $F : x \to \mathbb{E}_{w\sim\mu}[f(w,x)]$ is differentiable with derivative

$$F'(x) = \mathop{\mathbb{E}}_{w\sim\mu}\left[\frac{\partial}{\partial x} f(w,x)\right] \tag{1.1}$$