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Introduction:

Throughout this report I will refer to 'measure' often. A measure on a set systematically assigns a number to each suitable subset of the set. Intuitively this can be thought of as the size of the set or as a generalization of concepts like length, area, and volume.

History:

σ - algebra started as a means to abstractly view early theories of measurement. Emile Borel was the first to define measure succesful as a list of properties rather than a calculation, as opposed to the existing notions of P-measure by Giuseppe Peano and J-measure by Camille Jordan. Note that Jordan and Peano developed their theories seperately without knowledge of one another.

Definition: P-measure of a set: Let E be a set of points on $\mathbb{R}^2$. E is P-measureable if the two following areas are equal:

1. The inner area is the upper limit of the areas of the polygons that are the interior to E.
2. The outer area is the lower limit of polygon areas that contain E in their interior

Peano also used definitions for $\mathbb{R}$ and $\mathbb{R}^3$ that used linear segments and prisms. This approach, however, is limited to sets of points of the line, plane or space.

In contrast, Jordan did not have the same problem of interpretation of $\mathbb{R}^n$. Jordan proposed to divide $\mathbb{R}^n$ into n-cells. This created a grid that would allow him to use two sequences of squares: those squares stricly contained inside the set, and those that intersect with the set. From this it is possible to calculate the area and obtain an approximation of the inner and outer are of the figure.

Definition: Inner and outer expanses and the J-measure: Let E be a set bounded on $\mathbb{R}^n$. Then E is J-measurable if the inner extent and the outer extent are equal.

1. Let the sequence of n-cells be
$$\{P_k : r_k > r_{k+1}\}_{k=1}^{\infty}$$
as contained in all n-cells inside E for all k. Then the inner extent of E is found by
$$c_i(E) = \lim_{k\to\infty} V(P_k)$$
Where $V(P_k)$ is the volume of the n-cell.

2. Let the sequence of n-cells be

$$\{P_k + F_k : r_k > r_{k+1}\}_{k=1}^{\infty}$$

such that F_k containts all n-cells having at least one boundary. Then the outer extent of E is calculated as

$$c_e(E) = \lim_{k\to\infty} V(P_k + F_k)$$

where $V(P_k + F_k)$ is the volume of two n-cells.

Borel published his work in 1989 in a book called *Lesson on the Theory of Functions* in which he devoted a chapter to measurable sets and another to set theory. In these chapters he suggested the following definition for sigma-additivity.

Definition: B-measurable sets: Let E be a set of real numbers included in the interval $[0,1]$. Then if it is possible to attach a number $m(E)$, positive or zero, to E having the following properties, then we can say that E is $B-$ *measurable*:

1. If $E = \cup_{k=1}^{\infty} I_k \subset [0,1]$ with the intervals I_k disjoint two by two and that

$$\sum_{k=1}^{\infty} L(I_k) = s \text{ then } m(E) = s$$

2. If $E = \cup_{k=1}^{\infty} E_k \subset [0,1]$ with the intervals E_k disjoint two by two and the measure of E_k is s_k then

$$m(E) = \sum_{k=1}^{\infty} s_k$$

3. IF $E_1 \subset E_2 \subset [0,1]$ and the measure of E_1 and E_2 are s_1 and s_2 respectively, then the measure of the set

$$E_1 - E_2 = \{x : x \in E_2 \text{ and } x \notin E_1\} \text{ is } s_1 - s_2$$

$$\text{or, } m(E_1 - E_2) = s_1 - s_2$$

Borel used these rules 1,2,3 to prove interesting conclusions concerning the set toplogy. Borel showed that he could construct an uncountable set of zero measure, prove that any countable set is of zero measure, and that any perfect limited set on the interval $[0,1]$ is measurable. Borel did this to justify the need for all 3 of the properties. Borel wrote "[One defines] the new elements that one introduces, with the aid of their essential properties, that is to say those which are strictly indispensible for the reasonings which must follow." This view was significant by itself because it marked a change in the status of properties and the passage from calculations to the properties of calculation, which is characteristic to the process of abstraction.

Main Theory:

A σ- algebra is an algebraic sturcture that is used in both mathematical analysis and probability theory. More precisely, a σ-algebra on a set X is a collection σ of subsets of X, incuding X, that is closed under complement and countable union.

Definition: σ-Algebra Let X be a set. Then a σ-algebra $\mathcal{F}$ is a nonempty collection of subsets of X such that the following hold:

1) $X \in \mathcal{F}$
2) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
3) If A_n is a sequence of elements of $\mathcal{F}$, then the union of A_n's is in $\mathcal{F}$

For example let

$$X = \{0, 1\}$$

Then one possible σ-algebra on X is,

$$\sigma = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$$

In general, any finite algebra is always σ - algebra.

A Borel Space is also called a Measurable Space. A Borel Space consists of a set and a σ-algebra on the set. Consider an nonempty set X and a σ-algebra $\mathcal{F}$ on S. Then the tuple $(X, \mathcal{F})$ is called a Borel Space. For example consider the set $X = \{1, 2, 3\}$. A possible σ-algebra is $\mathcal{F} = \{X, \emptyset\}$. $\{X, \emptyset\}$ is called the Borel Space or Measurable space.

A Measure Space is not to be confused with a Measurable Space despite their close relation. A measure space contains information related to the underlying set, the subsets of the set that are possible for measuring(σ algebra) and the method of measuring(the measure). The measure space is a triple $(X, \mathcal{F}, \mu)$ where

- X is a nonempty set
- $\mathcal{F}$ is a σ-algebra on the set X
- μ is a measure on $(X, \mathcal{F})$

So in essence, a measure space can be thought of as a measure of a Borel space.

Examples and Why Should you Care:

The most important and relevant application (for this course) of a measure space is that a probability space is also a measure space. For example,let

$$X = \{0, 1\}$$

A σ-algebra of finite sets (like the one given) is usually a power set, or the set of all subsets. This is denoted as $P(\cdot)$. So now we set

$$\mathcal{F} = P(X)$$

We can explicitly write this as

$$\mathcal{F} = P(X) = \{\emptyset, \{0\}, \{1\}, X\}$$

Define the measure μ by

$$\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$$

So by additivity of measures we have

$$\mu(\{0\}) + \mu(\{1\}) = \mu(X) = 1, \text{ and } \mu(\emptyset) = 0$$

So we have the measure space $(X, P(X), \mu)$ which is a probability space since $\mu(X) = 1$. This ends up being a probability space for a Bernouli distribution for an event such as a fair coin flip. The probability space is a triple $(\Omega, \mathcal{F}, P)$. Where

- Ω is the sample space, or set of all possible outcomes
- $\mathcal{F}$ is a set of events where the events have zero or more outcomes
- P is a function that assigns probabilities to the events

There are other important related theorems that stem from measure spaces and probability spaces such as probability measure, which is important in conditional probability.

Definition: Probability Measure The function μ is said to be a probability measure on a probability space if

- μ returns results in the unit interval $[0, 1]$, returning 0 for the empty set and 1 for the entire space
- μ satisfies the countable additivity property for all countable collections $\{E_i\}$ of pairwise disjoint sets:

$$\mu(\bigcup_{i \in I} E_i) = \sum_{i \in I} \mu(E_i)$$

Example: I will show that $P_B(A) = \frac{P(B \cap A)}{P(B)}$ is a probability measure.

Proof. We let $A, B \in \Omega$ and $0 \leq P(A) \leq 1$ and $0 \leq P(B) \leq 1$

1. Since $P(A), P(B) \geq 0$ then $P(A \cap B) \geq 0$, so then $P_B(A) \geq 0$ and

$$P_B(A) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(B/A) + P(A \cap B)} \leq 1$$

2.
$$P_B(\Omega) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

3. Let $E_1, E_2, ...$ be independent events. Using P as a probability we have
$$P_B\big(\bigcup_{i=1}^{\infty} E_i\big) = \frac{P\big(\big(\bigcup_{i=1}^{\infty} E_i\big) \bigcap B\big)}{P(B)} = \frac{P\big(\bigcup_{i=1}^{\infty} \big(E_i \bigcap B\big)\big)}{P(B)} = \frac{\sum_{i=1}^{\infty} P(E_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P_B(E_i)$$

$\square$