Intro to Ordinary Differential Equations

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What is a differential equation? It's an equation that our function $y(t)$ must satisfy, involving derivatives, and our goal is to find a solution, meaning find y.

Ex: #1 most simplest diff. eq'n, $\frac{dy}{dt} = ky$

We can solve this one by hand: solution is $y(t) = c \cdot e^{kt}$ for any constant c we like

kinda like $\int x^2 dx = \frac{1}{3}x^3 + c$

We often write $y' = ky$ or $\dot{y} = ky$ especially when t represents time

A $\underbrace{\text{first-order, ordinary differential equation}}_{\text{ODE}}$ is of the form

$$y' = f(t, y)$$ where f is a given (known) function.

independent variable (t), dependent variable (y)

(other common notations besides $y(t)$ are $y(x)$ and $u(t)$ or $u(x)$)

Recall there are classifications of 1st order ODE

1) separable means $f(t,y) = \phi(t) \cdot \psi(y)$ [Ex. $f(t,y) = 3t^2 \cdot (1+y)$]

and easy to solve analytically via separation of variables

i.e. $\frac{dy}{dt} = \phi(t)\psi(y) \Rightarrow \frac{dy}{\psi(y)} = \phi(t)\,dt$

and integrate both sides

A special subcase is autonomous, when $f(t,y)$ has no dependence on t

2) linear means (rewriting from our standard form)

$$\underbrace{a_2(t)}\cdot y' + \underbrace{a_1(t)}\, y = \underbrace{a_0(t)}$$

if both a_1 and a_2 are constants (so no dependence on t) we call it a constant coefficient linear ODE

if the constant term is 0, we call it a homogeneous linear ODE

These are also easy to solve analytically, using either the Euler-Lagrange (variation of parameters) method or the integrating factor method.

Most other 1^{st} order ODE are not easy to solve analytically.
or impossible

Physicists, engineers and mathematicians in the 1960s would have studied many special tricks (eg. clever change-of-variables) to be able to solve a few more ODE.

Beyond 1^{st} order ODE

① Systems of ODE, like Lotka-Volterra $\underbrace{\text{predator}}_{F\text{ (fox)}}-\underbrace{\text{prey}}_{R\text{ (rabbit)}}$

$$\dot{R} = a_R R - c_R R\cdot F$$
$$\dot{F} = -a_F F + c_F R\cdot F$$
coupling

② Higher-order ODE, like $y'' + 3y' + y = \cos(t)$

Special techniques for linear ones (variation of parameters, method of undetermined coefficients, guessing $y = e^{kt}$, Laplace transform)

but main technique is to convert it to a system by introducing $v = y'$ ("velocity")

③ Partial differential equations, like

$u = u(t, x_1, x_2, x_3)$

Heat equation "parabolic" $\quad \frac{\partial u}{\partial t} = \Delta u$, $\quad \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^3 u}{\partial x_3^2}$ (in 3D, similar in other dimensions)

Laplacian $\Delta = \nabla\cdot\nabla = \nabla^2$ divergence of gradient

Poisson equation "elliptic" $\quad \Delta u = h$, $u = u(x_1, x_2, x_3)$ no time dependence

and Laplace equation $\Delta u = 0$, the homogeneous version

and if $h(\vec{x}) = -k^2 \cdot u(\vec{x})$ this is the Helmholtz equation

Wave equation "hyperbolic" $\quad \frac{\partial^2 u}{\partial t^2} = c \cdot \Delta u$

Motivation

PDEs describe the natural world

1) Hamiltonian mechanics for classical mechanics as well as things like $F = m\cdot a$, statics, weather/climate — Aerospace, Mechanical Engineering

2) Quantum mechanics — Physics, chemistry

3) Maxwell's equations for electricity & magnetism Electrical Engineering

4) Waves e.g., sound waves, light (microscopy, astronomy, x-rays)
Earthquakes

At CU, differential eq'n form a significant portion of the research topics

Engineering Depts: Aerospace ✓ Chem. Bio. ✓ Civil, Env., Architectural ✓ Computer Science ✓ (numerical methods)
Electrical ✓ Mechanical ✓

A&S Natural Science Depts: ATOC ✓ Chemistry ✓ Geology ✓ (ice flows...) I. Phys. ? MCDB ?
Appl. Math ✓ Math ✓ (theory) Astro. ✓ Biochem. ✓ Eco.+Ev. Bio ?
Env. Studies ? Geography ✓ Physics ✓ Psych. ?

A&S Social Science Depts.
Economics ✓ Sociology ? other depts. not so much

Business School: ✓(yes! Finance uses PDE a lot)

ODE cover many phenomenon too, especially in math. bio.
(e.g. growth of bacteria), or in chemical reactions, or
for epidemiology (ex. "SIR" model for disease modeling),
and often in misc. physics and engineering

Our class will focus on numerically solving ODE, building on
chapters 3 (interpolation) and 4 (integration, differentiation)

Techniques for PDE are briefly discussed in 2nd semester, ch. 12
Very active research topic!
We always want to do better (more resolution, more complicated models)

Theory

1) linear ODE: Thm the set of all solutions to a homogeneous linear ODE forms a vector space
(of dimension 1, for a 1st order eq'n.
of dimension 2, for a 2nd order eq'n ...)

If not homogeneous, write it as a particular solution + a homogeneous solution

2) What are we trying to solve?

ODE, $y' = f(t,y)$

Initial conditions (I.C.). $y(a) = \alpha$

Initial Value Problem (IVP) aka Cauchy Problem

Sometimes instead we might have

ODE, $y'' + 3y' + 2y = t^2$

Boundary conditions $y(a) = \alpha$, $y(b) = \beta$

Boundary Value Problem chill

3) do solutions even exist? are there more than one?

Very famous classical theorems (and since old, many different names, "Cauchy", "Picard", ...)

Background:

Def: Lipschitz A function $f(t,y)$ is Lipschitz on the variable y on a set $D \subseteq \mathbb{R}^2$ with constant $L > 0$ if

$$|f(t,y_1) - f(t,y_2)| \leq L \cdot |y_1 - y_2|$$

if (t, y_1) and (t, y_2) are in D.

Sometimes we say this is "uniformly in t" to make it clear that L does not depend on t

Def: Convex set A set $D \subseteq \mathbb{R}^n$ is convex if it contains all the lines drawn between any 2 elements of D

Not convex   Convex

Usually we have a simple way to show if a function is Lipschitz (same as for the fixed-point theorem aka contraction mapping theorem)

Fact (Thm 5.3) If $f(t,y)$ defined on a convex set $D \subseteq \mathbb{R}^2$, then if $\left| \frac{\partial f}{\partial y}(t,y) \right| \leq L \ \ \forall (t,y) \in D$ then f is Lipschitz continuous with respect to y, with constant L, uniformly in t

Theorem (main ODE existence + uniqueness thm, Thm 5.4)

Let $D = \{ (t,y) : a \leq t \leq b, \ -\infty < y < \infty \}$ and f is (jointly) continuous on D, and let f be Lipschitz with respect to y on D.

Then the IVP $\quad y'(t) = f(t, y) \quad a \le t \le b$

$\qquad\qquad\qquad y(a) = \alpha$

has a unique solution for $a \le t \le b$.

existence   uniqueness

4) What's our numerical goal? Is it well-conditioned?

We'll numerically solve ODE for which we don't have an analytic solution.

So, we'll approximate it at a discrete set of time points

But that's not a function! To get the function, we can always interpolate, e.g. Hermite interpolation since we'll know derivatives

In Matlab, this is done for you with deval

In python (w/ scipy.integrate), use solve_ivp and then use the OdeSolution output

We'll define a well-posed IVP $y' = f(t, y)$, $y(a) = \alpha$ to mean

① a unique solution exists

② $\exists\, \varepsilon_0$ and $k > 0$ s.t. $\forall\, 0 < \varepsilon < \varepsilon_0$, if $\delta(t)$ is continuous and $|\delta(t)| < \varepsilon \;\; \forall\, t \in [a, b]$, and $|\delta_0| < \varepsilon$, then there is a unique sol'n z to the perturbed IVP

$$z' = f(t, z) + \delta(t), \quad z(a) = \alpha + \delta_0$$

and $\quad |z(t) - y(t)| < k \cdot \varepsilon \quad \forall\, t \in [a, b]$.

Theorem 5.6 Under the same conditions as Thm 5.4, the IVP is well-posed

In fact, if $\delta(t) = 0$ and we only consider δ_0, we can take $K = \delta_0 \cdot e^{L \cdot (b-a)}$ where L is the Lipschitz constant

(i.e., longer time intervals are harder) *

* For long periods of time, we slowly get errors, and it can happen that we violate laws like conservation of momentum.

(energy drift)

There are special methods to preserve quantities like this: symplectic integrators (Verlet, etc.)

Ex Show $f(t,y) = y - t^2 + 1$ is Lipschitz in y on $0 \le t \le 2$

(hence the IVP is well-posed)

Sol'n: Take $\left|\frac{df}{dy}\right| = \left|\frac{d}{dy}(y - t^2 + 1)\right| = |1|$ ✓

Next time... our 1st numerical method

This chapter covers many methods, the best choice depending on your exact ODE and how much accuracy, speed and convenience you want

eg. stiff or not

Two main types:
1. Runge-Kutta
2. linear multistep