

Stability continued (multi-step methods)

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Recall our 3 notions: (all for $h \rightarrow 0$)

- 1) Consistency Does the method locally approximate the ODE?
- 2) Convergence Does the approximation w_i converge to $y(t_i)$? MAIN GOAL
- 3) Stability Links consistency to convergence

Recap for 1-step methods (RK, etc.):

Define a general notion of stability to mean w_i depends continuously on the initial data w_0 .

Thm 5.20 (abbreviated) Most reasonable 1-step methods are

- (i) stable
- (ii) convergent iff consistent
- (iii) global error is proportional (in terms of h) to the local truncation error $\mathcal{E}(h)$

This lecture: multistep methods (AB, AM, BD, midpoint, trapezoidal, Simpson's, etc.)

More complicated picture than 1-step.

Not all reasonable multistep methods are stable!

Setup IVP: $y' = f(t, y)$, $a \leq t \leq b$, $y(a) = \alpha$

$w_0 = \alpha$, $w_1 = \alpha_1$, $w_2 = \alpha_2$, ..., $w_{m-1} = \alpha_{m-1}$] initialization

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m}$$

$$+ h \cdot \underbrace{\left(b_m f_{i+1} + b_{m-1} f_i + \dots + b_0 f_{i+1-m} \right)}_{F(t_i, h, w_{i+1}, w_i, \dots, w_{i+1-m})}, \quad f_i := f(t_i, w_i)$$

so we can express the local truncation error as

$$y_i := y(t_i)$$

$$\mathcal{E}_{i+1}(h) = \frac{1}{h} \left(y_{i+1} - a_{m-1} y_i - \dots - a_0 y_{i+1-m} \right) - F(t_i, h, y_{i+1}, y_i, \dots, y_{i+1-m})$$

Our analysis will mostly ignore the F part

- (why? ch 5.10 is about $h \rightarrow 0$ so $h \cdot F(\dots)$ has less effect)
as long as F is reasonable, namely
- ① $f = 0 \Rightarrow F = 0$
 - ② F satisfies a Lipschitz-like condition (see book)

Can we prove things like we did in Thm 5.20?

- For (iii) "global error is proportional (in terms of h) to the local truncation error $\mathcal{E}(h)$ "

Yes, we can prove a result. The book gives a very specific result:

Thm 5.21 Global rate of convergence for Adams predictor-corrector methods.

Use a m -step Adams-Basforth predictor, w_i local truncation error $\mathcal{E}_{i+1}(h)$ and a $m-1$ -step Adams-Moulton corrector, w_i local truncation error $\tilde{\mathcal{E}}_{i+1}(h)$, and suppose $f(t_i, y)$ and $\frac{\partial}{\partial y} f(t_i, y)$ are continuous ($a \leq b \leq t$) and $\frac{\partial}{\partial y} f(t_i, y)$ bounded ($\Rightarrow f$ Lipschitz in y), then

(paraphrasing) (i) the local truncation error of the combined predictor-corrector scheme is $\sigma_{i+1}(h) = O(\mathcal{E}_{i+1}(h) + \tilde{\mathcal{E}}_{i+1}(h))$

(ii) the global error $|w_i - y_i|$ ($\forall i=1, \dots, n$) is $O(h)$

- What about more general statements?

yes, this will be our focus.

The characteristic polynomial $P(\lambda)$

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m} + h F(t_i, h, w_{i+1}, w_i, \dots, w_{i+1-m})$$

An observation:

For Adams methods, $a_{m-1} = 1$, all other $a_j = 0$

For backward differentiation,

Name	Order	Steps m	a_{m-1}	a_{m-2}	a_{m-3}	a_{m-4}	b_m
BD1 = Backward Euler	1	1	1				1
BD2	2	2	$4/3$	$-1/3$	$\Sigma = 1$		$2/3$
BD3	3	3	$18/11$	$-9/11$	$2/11$	$\Sigma = 1$	$6/11$
BD4	4	4	$48/25$	$-36/25$	$16/25$	$-3/25$	$12/25$

... so for AB, AM, BD, we have

$$(*) \quad \sum_{j=1}^m a_{m-j} = 1$$

and in fact you can show that $(*)$ is a necessary condition

for Consistency (e.g., assuming $\mathcal{O} f=0 \Rightarrow F=0$, then for the ODE $y'=0, y(a)=\alpha$
the local truncation error is $\Xi_{i+1}(h) = \frac{1}{h} \left(1 - \sum_{j=1}^m a_{m-j}\right) \alpha$)

Let's look at the IVP $y'=0, y(a)=\alpha$ ($\Rightarrow y(t)=\alpha$ is the unique soln)
 $f=0 \Rightarrow F=0$ so our numerical method is

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m} \quad \begin{pmatrix} \text{i.e. we can} \\ \text{ignore the} \\ F \text{ term} \end{pmatrix}$$

By the property $(*)$, if $w_i = w_{i-1} = w_{i-2} = \dots = \alpha$
then $w_{i+1} = \alpha$ too.

But is this stable to perturbations? (in our simple ODE, all perturbations
are due to roundoff, but generally
also due to $\Xi_i(h) \neq 0$)

Rewrite our method as

$$(**) \quad w_{i+1} - (a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m}) = 0$$

A "difference
equation"
which shares many
features of
a "differential eqn"
(aka ODE)

$$m=3 \text{ example: } w_{i+1} - a_2 w_i - a_1 w_{i-1} - a_0 w_{i-2} = 0$$

Reminiscent of 3rd order linear homogeneous ODEs?

$$y''' - a_2 y'' - a_1 y' - a_0 y = 0$$

Ansatz: $y(t) = e^{\lambda t}$ so solve $\lambda^3 - a_2 \lambda^2 - a_1 \lambda - a_0 = 0$
(and special considerations if get a root w/ multiplicity > 1)

We'll do something similar:

Def The characteristic polynomial $P(\lambda)$ associated w/ the numerical
method $(**)$ (it does not depend on what F looks like) is

$$P(\lambda) = \lambda^m - a_{m-1} \lambda^{m-1} - a_{m-2} \lambda^{m-2} - a_{m-3} \lambda^{m-3} - \dots - a_0$$

$$m=3 \text{ example } P(\lambda) = \lambda^3 - a_2 \lambda^2 - a_1 \lambda - a_0$$

Observe if λ is a root of P then $w_i = \lambda^i$ is a solution
to the difference equation (not yet worrying about initial conditions)

To see this, let's work with our $m=3$ example:



$$\begin{aligned} w_{i+1} - a_2 w_i - a_1 w_{i-1} - a_0 w_{i-2} &= 0, \quad \text{plug in } w_i = \lambda^i \\ \lambda^{i+1} - a_2 \lambda^i - a_1 \lambda^{i-1} - a_0 \lambda^{i-2} &= 0 \\ \lambda^{i-2} (\lambda^3 - a_2 \lambda^2 - a_1 \lambda - a_0) &= 0 \end{aligned}$$

$$= P(\lambda) = 0 \text{ since } \lambda \text{ is a root}$$

The difference equation is homogeneous \Rightarrow if w_i and \tilde{w}_i are both solutions
then so is $w_i + \tilde{w}_i$ (superposition)

So, if P has m distinct* roots $\{\lambda_j\}_{j=1}^m$ then all sol'n to
the difference equation can be written as

$$w_i = \sum_{j=1}^m c_j \lambda_j^i \quad \text{for some coefficients } c_j$$

(* If not all roots are distinct, see Eq. 5.63
Ex: λ is a double root, then $w_i = \lambda^i$ and $w_i = i \lambda^{i-1}$
are solutions)

Now, throw in initial condition. $\underbrace{\text{call this } \gamma_1}_{\text{call this } \gamma_1}$

Also, via (*), we observe $\lambda=1$ is always a root (if the method is consistent)

$$\text{ex: } m=3 \quad P(\lambda) = \lambda^3 - a_2 \lambda^2 - a_1 \lambda - a_0$$

$$a_2 + a_1 + a_0 = 1 \text{ via (*)}$$

\Rightarrow if $\lambda=1$,

$$P(1) = 1 - a_2 - a_1 - a_0 = 0$$

So $w_i = 1^i = 1$ is a solution to the difference equation,

and $w_i = 1 \cdot \alpha$ is a solution to the difference eqn and initial condition.

So all soln look like $w_i = \alpha + \sum_{j=2}^m c_j \lambda_j^i$
 $\underbrace{= 0}_{\text{i.e., } c_2 = c_3 = \dots = c_m = 0}$

But, due to roundoff error, we might have $c_j \neq 0$ ($j \geq 2$)

Is this "catastrophic"? If $|\lambda_j| < 1$ then $\lim_{i \rightarrow \infty} c_j \lambda_j^i = 0$ so no big deal.

but if $|\lambda_j| > 1$ then $\lim_{i \rightarrow \infty} c_j \lambda_j^i$ diverges! Big deal
(and bad news)

and if $|\lambda_j| = 1$

CASE: not a simple root, then have $c_j \cdot i \lambda_j^{i-1} \rightarrow \infty$
 \uparrow $\underbrace{\text{no big deal}}_{\text{dead breaker}}$

CASE: $\lambda_j = 1$ (and simple), then that's good since this
is the true solution, not an error

CASE: $|\lambda_j| = 1$ but $\lambda_j \neq 1$ (and simple)

then perturbation doesn't grow but also doesn't go away

This motivation was based on the ODE $y' = 0$

What about other ODEs? $y' = f(t, y)$

For linear ODEs, we always can write $y = \underbrace{y_{\text{homogeneous}}}_{\text{solves } y' = 0} + y_{\text{particular}}$

and it turns out, even for nonlinear ODEs,
our analysis is still useful (we won't prove it though)

Def Root condition and **strongly stable** and **weakly stable**

Let $\{\lambda_j\}_{j=1}^m$ be the roots of the characteristic polynomial P associated with a multi-step scheme. Then we say the multi-step method satisfies the **root condition** if **(A)** $|\lambda_j| \leq 1 \quad \forall j$

and **(B)** $|\lambda_j| = 1 \Rightarrow \lambda_j$ is a simple root

and if a method satisfies the root condition, then we call it either
strongly stable if $|\lambda_j| < 1 \Rightarrow \lambda_j \neq 1$

weakly stable otherwise.

If it doesn't satisfy the root condition, we call it **unstable**

added after
lecture

Examples	Roots	Stability
i.e. e.g. $P(\lambda) = (\lambda - 1)^2$	$\{1, -3\}$	unstable
	$\{1, -1\}$	weakly stable
	$\{1, \frac{1}{2}, \frac{4}{3}\}$	unstable
	$\{1, -3i, 9\}$	strongly stable
	$\{1, i, 4i, -2\}$	weakly stable

Note: Sometimes you might hear of **zero-stability** which is that $\exists B$

such that $\exists h_0 > 0$ and $\forall 0 < h < h_0$, $n := \frac{b-a}{h}$, $|w_i| \leq B$
 $\forall i = 1, \dots, n$

i.e., the numerical solution is bounded $\forall a \leq t_i \leq b$

regardless of how small h is (some texts use slightly different definitions)

This **zero-stability** is basically equivalent to the **root condition**

Thm 11.4 (Quarteroni et al.)

For a **consistent** multistep method, the **root condition** is equivalent to **zero-stability**

Main result

Stability links consistency to convergence

Thm 5.24 "Equivalence Thm"

ie. output depends continuously on input data

① A multistep method is stable iff it satisfies the root condition

② If the multistep method is consistent, then it is convergent iff it's stable

("iff" = if and only if)

This is a version of the Dahlquist equivalence theorem

(mid 1950s, for all ODEs, linear and nonlinear)

very similar to the Lax (aka Lax-Richtmeyer) equivalence theorem

(mid 1950s, also for PDEs but only linear ones)

and related

Thm First Dahlquist Stability Barrier

cf. Driscoll +Brown Thm 6.8.3

If a linear multistep method (eg AB, AM, BD) has global error $O(h^P)$
then if it is a stable method,

$$P \leq \begin{cases} m+2 & m \text{ even} \\ m+1 & m \text{ odd} \\ m & \text{if method is explicit} \end{cases}$$

not on exam