LECTURE 5

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Contents

1 Degree of polynomials in fields

$K[x]/f(x)$ where f(x) is irreducible. What is $[L:K]$? where $d = \delta f(x)$

$\bar{1}, \bar{x}, \bar{x}^2, \ldots, \bar{x}^{d-1}$ basis for L

$\bar{1} = 1 + < f >$

$\bar{x} = x + < f >$ spn L over K

$k(x)/ < f >$ is isomorphic to L

$\alpha_0 + \alpha_1 x + \ldots + \alpha_{d-1} x^{d-1} \in L$ where $\alpha_i \in K$

$\alpha_0 + \alpha_1 x + \ldots \alpha_{d-1} x^{d-1} \in < f >$ which is a contradiction the only way this can happen

$\rightarrow \alpha_0 = \alpha_1 = \ldots = \alpha_{d-1} = 0$

so $\bar{1}, \bar{x}, \ldots, \bar{x}^{d-1}$ are linearly independent

so $\dim_K L = d$ so $[L:K] = d$

we take L/K where $\alpha \in L$ smallest field containing K and L.
$K[\alpha]$ = polynomials in alpha with coefficients in K

$$c_0 + c_1\alpha + c_2\alpha^2 + \ldots + c_{d-1}\alpha^{d-1} \quad \text{where } c_i \in K$$

$$K[\alpha] \subseteq L \text{ subring of L}$$

$$K(\alpha) = \text{ rationals functions in } \alpha = \{\frac{p(\alpha)}{q(\alpha)} : q(\alpha \neq 0, p, q \in K[x])\}$$

$K(\alpha)$ is the smallest subfield of L
Smallest subfield of L containing K and α

Theorem 1.1. $L/K, \alpha \in L$ then

1. If there is no polynomial in K[x] s.t. $p(\alpha) = 0$ then $K(\alpha) \cong K(x)$.
2. if there is a polynomial in $K[x]$ such that $p(\alpha) = 0$ then $K(\alpha) = K[\alpha] \equiv \langle m(x) \rangle$

where $m(x)$ is m-irreducible polyonimla m(d) = 0.
In case (1) α is called transcendental over K.
$\mathbb{R}/\mathbb{Q}, \pi, e$ is trascendental. what about $\pi + e, \pi \cdot e$? it is not easy to show that a number is transcendental In Case(2) we say that alpha is algebraic over K. for example $\mathbb{R}/\mathbb{Q}$ algebraic number are countable. R is uncountable

L/K , $d \in L$
$I(\alpha) = \{p \in K(x) : p(\alpha) = 0\}$,
$p \in I(\alpha), p \cdot r, r \in K(x)$
$p \cdot r(\alpha) = p(\alpha) \cdot r(\alpha) = 0$
case 1: $I(\alpha) = 0$
case 2: $I(\alpha) \neq 0$
I is principale $I = \langle f \rangle$

f makes it monic – $m = \frac{f}{\text{lead coeff of f}}$
this makes uniequely defind $I(\alpha) = \langle m \rangle$
m - minimal polynomial of α.

Proof. 1. case 1: $I(\alpha) = 0$
then $K(\alpha) = K(x)$
$K(\alpha) \cong K(x)$
$\phi : K(x) \to K(\alpha)$
$\frac{p(x)}{q(x)} \to \frac{p(\alpha)}{q(\alpha)}$
ϕ is an onto isomorphism.
$ker\phi = 0?$ $\frac{p(x)}{q(x)} \in ker\phi$ since $p(\alpha) = 0$

2. case 2: $I(\alpha) = \langle m \rangle$
$K(\alpha) = K[\alpha] = \langle m \rangle$
$K(\alpha) \cong K[x]/\langle m \rangle$
$\phi : K[x] \to K(\alpha)$
$p(x) \to p(\alpha)$
ϕ is onto
$ker\phi = \langle m \rangle$
$p(x) \in ker\phi \implies p(\alpha) = 0 \implies p(x) = m(x) \cdot q(x)$
$p(x) \in \langle m \rangle$
$K[x]/\langle m \rangle \cong K(\alpha)$

□

$K(x)/\langle f(x) \rangle \cong K(\alpha)$
where f(x) is irreduciоble.

$$x^3 - 2 \text{ is irreducible using Eisenstein p} = 2$$
$$\text{the roots are } \sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2} \text{ where } \omega = e^{\frac{2\pi i}{3}}$$
$$\mathbb{Q}[\sqrt[3]{2}] \cong \mathbb{Q}(x)/\langle x^3 - 2 \rangle \cong \mathbb{Q}[\omega\sqrt[3]{2}] \cong \mathbb{Q}[\omega^2\sqrt[3]{2}]$$

Smallest field obtaining all roots of f is called splitting field of f.