Lecture 10

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February 10, 2025

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1 From Previous Lecture

Discussion of $\mathbb{Q}(\sqrt[4]{2})$:

- $\sqrt[4]{2}$ is a root of $x^4 - 2$
- Consider $-\sqrt[4]{2}$

Interesting maps $\mathbb{Q}(\sqrt[4]{2}) \to \mathbb{Q}(\sqrt[4]{2})$ that fix $\mathbb{Q}$:

1. $\sqrt[4]{2} \to \sqrt[4]{2}$ (identity homomorphism)
2. $\sqrt[4]{2} \to -\sqrt[4]{2}$

Intermediate subfields:

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[4]{2})$$

1.1 L/K Field Extension

$Aut(L:K)$ or $Aut(L/K)$ or $Aut(L,K)$ = Automorphisms of L fixing K

Definition 1.1 (In case you forgot)**.** Automorphisms are isomorphisms to itself:

$$Aut(L:K) = \{\sigma : L \to L : \sigma \text{ is an isomorphism and } \sigma(x) = x \text{ for all } x \in K\}$$

Note: $Aut(L,K)$ is a group under composition (the book calls this the Galois group of L/K).

In our example:

$$Aut(\mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}) = \mathbb{Z}_2$$

Working with finite degrees $[L:K] < \infty$:

- $\alpha \in L$ has a minimal polynomial $f_\alpha \in K[x]$
- f_α has same roots in L
- For $\sigma \in Aut(K,L)$, σ permutes the roots of $f_\alpha \in L$

$$f_\alpha(r) = 0 \Rightarrow f_\alpha(\sigma(r)) = 0$$

For $f(x) \in K[x]$ and $f(r) = 0$, then

$$\begin{aligned}
f(\sigma(r)) &= 0 \text{ for all } \sigma \in Aut(L:K) \\
f(x) &= a_0 + a_1 x + \cdots + a_n x^n \\
f(r) &= a_0 + a_1 r + \cdots + a_n r^n = 0 \\
f(\sigma(r)) &= a_0 + a_1\sigma(r) + \cdots + a_n\sigma(r)^n = 0 \\
&= a_0 + a_1\sigma(r) + \cdots + a_n\sigma(r)^n = 0 \quad \text{(by the definition of } \sigma) \\
&= \sigma(a_0) + \sigma(a_1)\sigma(r) + \sigma(a_2)\sigma(r^2) + \cdots + \sigma(a_n)\sigma(r^n) = 0 \\
&= \sigma(a_0 + a_1 r + \cdots + a_n r^n) = \sigma(0) = 0
\end{aligned}$$

Since we have that $\sigma(r^k) = [\sigma(r)]^k$ because $\sigma(r^2) = \sigma(r) \cdot \sigma(r)$

Question:

1. How big can this group be ?
2. For now we will show that $|Aut(L : K)|$ is finite if $[L : K] < \infty$

Theorem 1.1. If $[L : K] < \infty$, then $|Aut(L : K)| < \infty$

Proof.

$L = K(\alpha_1, \dots, \alpha_n)$

$\sigma \in Aut(L : K)$ has minimal polynomial f_{α_i}

$\forall \alpha \in Aut(L : K)$is sepcific if we know $\sigma(\alpha_1), \dots, \sigma(\alpha_n)$

σ permutes the roots of f_{α_i} in L

$\Rightarrow \forall \alpha_i$ there are infiniitely many possiblities for $\sigma(\alpha_i)$

$\Rightarrow$ finitely many possibilities for $\sigma \in Aut(L : K)$

$\Rightarrow$ Not every possible permutation of roots leads to a field automorphism

$\square$