LECTURE 7

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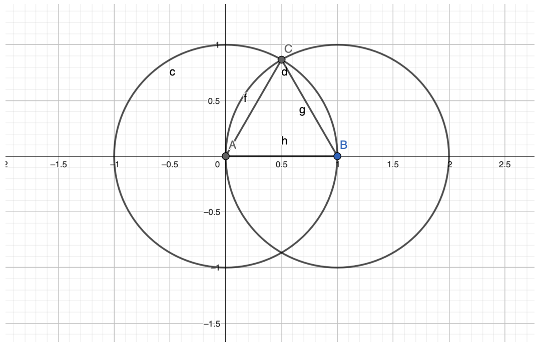
1 continuing from last lecture



a field generated by 0, 1 is $\mathbb{Q}$. with this equilateral trinalge we generated $\mathbb{Q}(\sqrt{3})$

1.1 intersecting lines

$$\begin{aligned}
&\alpha_1 x + \beta_1 y = \gamma_1 \\
&\alpha_2 x + \beta_2 y = \gamma_2 \\
&\alpha_i, \beta_i, \gamma_i \in K \\
&x, y \in K
\end{aligned}$$

1.2 intersecting line with circle

$$\begin{aligned}
&\alpha x + \beta y = \gamma \\
&(x - c_1)^2 + (y - c_2)^2 = r^2 \\
&\text{quadratic in x}
\end{aligned}$$

We may need to add square roots to the field. The degree of extension = 1 or 2. This is because the degree of the extension is the degree of the minimal polynomial.
Degree $[K : \mathbb{Q}]$ either stays the same or doubles.
$[K_{i+1} : \mathbb{Q}] = [K_{i+1} : K_i] \cdot [K_i : \mathbb{Q}]$

1.3 Intersecting two cricles

$$\begin{aligned}
&(x - c_1)^2 + (y - c_2)^2 = r_1^2 \\
&(x - d_1)^2 + (y - d_2)^2 = r_2^2 \\
&c_i, d_i, r_i \in K_s
\end{aligned}$$

Solving this system of equations will give us the intersection points.

$$
\begin{aligned}
&x^2 - 2c_1 x + c_1^2 + y^2 - 2c_2 y + c_2^2 = r_1^2 \\
&x^2 - 2d_1 x + d_1^2 + y^2 - 2d_2 y + d_2^2 = r_2^2 \\
&(x - c_1)^2 + (y - c_2)^2 = r_1^2 \\
&\text{linear equation x of y} \\
&\rightarrow K_o = \mathbb{Q} \\
&[K_s : \mathbb{Q}] = 2^j \quad j \in \mathbb{Z}, j \geq 0
\end{aligned}
$$

We must construct $\sqrt[3]{2}$ Using Eisenstein criteria $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ This is not possible because:

$$[K_s : \mathbb{Q}](= 2^j) = [K_s : \mathbb{Q}(\sqrt[3]{2})] \cdot [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}](= 3)$$

2 Trisecting an angle

$$
\begin{aligned}
&\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta) \\
&3\theta \text{is described by a cubic equation} \\
&3\theta = \frac{\pi}{3} = 60^\circ \\
&\cos(\frac{\pi}{3}) = \frac{1}{2} \\
&\cos(\frac{\pi}{9}) = x \\
&4x^3 - 3x - \frac{1}{2} = 0 \quad \text{if this is reducible we are done} \\
&\text{if yes, then} \quad [\mathbb{Q}(\cos(\frac{\pi}{3})) : \mathbb{Q}] = 3 \\
&8x^3 - 6x - 1 = u^3 - 3u - 1 \\
&\text{where } x = \frac{u}{2} \quad u = 2x \\
&u^3 - 3u - 1 = (au + b)(cu^2 + du + e) \\
&ac1, be = 1 \quad a, b, c, d, e \in \mathbb{Z} \\
&a, c = \pm 1, \quad b, e = \pm 1 \\
&\rightarrow u = \pm 1 \text{ is a root} \quad \text{Gauss' lemma} \\
&\text{but this is not the case}
\end{aligned}
$$

3 Splitting Fields

$K, f(x) \in K(x)$ L is a splitting field of $f(x)$ if

1. $f(x)$ splits into linear factors as polynomial in $L(x)$
2. $f(x)$ doesn't split into linear factors over any subfield of L

We want to show that a splitting field exists and they are unique.

Lemma 3.1 (Existence of Spltting fields)**.**
$f(x), K..$ $f = q_1 \dots q_s$ where q_i is irreducible in $K(x)$
$K_1 = K(t)/\langle q_1, (t)\rangle$
$t + \langle q_1(t)\rangle$ is a root of q_1 in K_1
factor f as a polynomial in $K_1(x)$. It will have a linear factor
$f = r_1 \dots r_m$ if $\partial r_i > 1$
$K_2 = K_1/\langle r_i(x)\rangle$
AFter at most n steps we terminate.
Built Field M such that $f(x)$ factors into leinear factors in $M(x) \leftrightarrows$ All roots of $f(x)$ lies in M
$\alpha_1, \dots \alpha_n$ roots of $f(x)$ in M
$L = K(\alpha_1, \dots \alpha_n) \leq M$
$L-$ splitting field of $f(x)$
$[L : K] \leqq n!$ where $n\partial f$
$K(\alpha_1) : K \leq n$
$K(\alpha_1, \alpha_2) : K(\alpha_1) \leq n - 1$

$f(x) = x^3 - 1$
L splitting fiekd $f(x)$ over $\mathbb{Q}$
$L = \mathbb{Q}(\sqrt[3]{2}, w\sqrt[3]{2}, w^2\sqrt[3]{2})$
$[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$
$[\mathbb{Q}(w) : \mathbb{Q}] = 2$
$[L : \mathbb{Q}] = 6$