LECTURE 6

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1 Useful Facts

1. L/K , $L = K \leftrightarrows [L : K] = 1$

2. L/K, $\alpha \in L$, $f(x) \in k(x), f(\alpha) = 0$ then f is irreducible $\leftrightarrows [k(\alpha) : k] = \partial f$

on number 2 specifically, f is irreducible then $k(\alpha) \cong k(x)/f(x), [k(x)/f(x) : k] = \partial f$
$I(\alpha) = \{p \in k(x) : p(\alpha) = 0\}$, $I(\alpha) = \langle m(\alpha)\rangle$

$m(\alpha) =$ minimal polynomial of α thus irreducible.
$[k(\alpha) : k] = \partial m(\alpha)$
$[k(\alpha) : k] = [k(x)/m(x) : k] = \partial m(x)$
So f must be a constant multiple thus f is irreducible.

2 ajoining multiple elements

$k(\alpha_1, \alpha_2, \ldots, \alpha_s)$ where α_i- minimal polynomial m_i, $d_i = \partial m_i$
3. $[k(\alpha_1, \alpha_2, \ldots, \alpha_s) : k] \leq d_1 \cdot d_2 \cdot \ldots \cdot d_s$

$[k(\alpha_1, \alpha_2) : k] = [k(\alpha_1, \alpha_2) : k(\alpha_1)] \cdot [k(\alpha_1) : k] = [k(\alpha_1, \alpha_2) : k(\alpha_1)] \cdot d_1$ .. tower theorem
want to show that $[k(\alpha_1, \alpha_2) : k] \leq d_2$
$d_2 =$ degree of the minimal polynomial $m_2(x)$ of d_2 in $k(x)$. $m_2(x)$ is irreducible in $k(x)$

$m_2(x)$ may become reducible in $k(d_1)[x]$
$\bar{m_2}(x)$ is the minimal polynomial of d_2 in $k(d_1)[x]$
we know $\partial \bar{m}_2 \leq d_2$

$[k(\alpha_1, \alpha_2, \alpha_3) : k] = [k(\alpha_1, \alpha_2, \alpha_3) : k(\alpha_1, \alpha_2)] \cdot [k(\alpha_1, \alpha_2) : k(\alpha_1)] \cdot [k(\alpha_1) : k] \leq d_3 \cdot d_2 \cdot d_1$

2.1 examlpes

$$[\mathbb{Q}(\sqrt[3]{2}, \sqrt{2}) : \mathbb{Q}]$$

before that lets see $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$

$x^2 - 2$ is irreducible in $\mathbb{Q}[x]$
thus, $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$
$[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$
$x^3 - 2$ Eisenstein criteria
thus, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$
$\rightarrowtail [\mathbb{Q}(\sqrt[3]{2}, \sqrt{2}) : \mathbb{Q}] \leq 6$ now trying to pvoe that it is equal to 6
tower theorem $[\mathbb{Q}(\sqrt[3]{2}, \sqrt{2}) : \mathbb{Q}]$ is divisible by 2 and 3
$\rightarrow [\mathbb{Q}(\sqrt[3]{2}, \sqrt{2}) : \mathbb{Q}] = 6$

$$[\mathbb{Q}(\sqrt{2}+\sqrt{3}):\mathbb{Q}]$$

$\mathbb{Q}(\sqrt{2},\sqrt{3}) \geq \mathbb{Q}(\sqrt{2}+\sqrt{3})$ we show that it is actually equal

$\dfrac{1}{\sqrt{2}+\sqrt{3}} = \dfrac{\sqrt{3}-\sqrt{2}}{(\sqrt{2}+\sqrt{3})(\sqrt{3}-\sqrt{2})} = \dfrac{\sqrt{3}-\sqrt{2}}{3-2} = \sqrt{3}-\sqrt{2} \in \mathbb{Q}(\sqrt{2},\sqrt{3})$

$\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$

$[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}] = \{2,4\}$

if the answer was 2, $[\mathbb{Q}(\sqrt{3},\sqrt{2}):\mathbb{Q}(\sqrt{2})] = 1$ making the fields the equal

Show that x^3-2 is irreducible over $\mathbb{Q}[i]$
x^3-2 is irreducible over $\mathbb{Q}[i] \leftrightarrows [Q[\sqrt[3]{2}]:\mathbb{Q}[i]] = 3$

3 squaring a circle