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Lecture Notes

Author
Tom Jeong

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1 Power series Fourier series

Definition 1.1. A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n \tag{1}$$

where a_n are constants and x is a variable.

Definition 1.2. Define $R \ in \mathbb{R} \cup \{\pm\infty\}$ the radious of convergence of $S(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ if $S(x)$ convergent abs for $|x - x_0| < R$ and divergence $|x - x_0| > R$

How to find R?

1. $R = \dfrac{1}{\limsup_{k\to\infty} |a_k|^{1/k}}$

$|\sum_{k=0}^{\infty} a_k (x - x_0)^k| \leq \sum_{k=0}^{\infty} R^k \leq \sum_{k=0}^{\infty} |a_k (x - x_0)^k| \leq \sum_{k=0}^{\infty} |a_k| R^k$ this is how we get the equation for R

2. if $R = \lim_{k\to\infty} \dfrac{|a_k|}{|a_{k+1}|}$ exists. then R is the radious of convergence of $S(x)$

Infomrations we get from the radious of convergence :

1. if $S(x)$ has radious of Convergence $R > 0$ what do we know about $S(x)$ on $B_R(x_0)$?

2 smoothness of functions 7.4

continuous functions on $\mathbb{R}$ are labeled as $C^0(\mathbb{R})$ functions. (0 means no deriavtive)
$C^1(\mathbb{R})$ functions have continuous and the derivative is continuous
$C^2(\mathbb{R})$ functions have continuous and the first and second derivative is continuous
$C^\infty(\mathbb{R})$ all derivative are conttinuous

When does the taylor series converge? What means to be analytic C^μ - analytic: taylor series convergence and ROC is ∞
example: Consider $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

f ix C^∞ but not analytic on any interval containing 0. Proof: look up but also can take c infinity outside of 0. maybe induction

Proof. we want to check if f continuous at 0.

$$\lim_{x\to 0} e^{-\frac{1}{x^2}} = 0$$

$\forall \epsilon \exists N$ such that $e^{-N} < \epsilon$
take $x << 1$

out iside of x = 0:

$$f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$$

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

lhosptial ? $\square$

Lemma 2.1. $lim_{h \to 0} \frac{e^{-1/h^2}}{P_n(h)} = 0$ ($P_n(h)$ = polynomial of degree N)

$$lim_{N to \infty} \frac{1}{P_n(1/N) e^{N^2}} = 0$$

want to show $lim_{N \to \infty} e^{N^2} P_n(1/N) = \infty$

$$e^{N^2} P_n(1/N) = e^{N^2} \sum_{k=0}^{n} a_k \frac{1}{N^k}$$

Theorem 2.2. taylor w/ remainder

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$f(x) = P_n(x) + R_n(x)$ where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ for some $c \in (0, x)$

Definition 2.1 (Trig series)**.** A trig. series is a series like

$$T(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

$$T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$$