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1 Question 1

Let $a_1 = 1$ and $a_{n+1} = 1 + \frac{1+a_n}{1+a_n}$. Find $\lim_{n\to\infty} a_n$. hint: Observe that $a_{n+1} = 1 + \frac{2a_n}{1+a_n}$. Then write $a_{n+k} = \dfrac{A_k + B_k a_n}{C_k + D_k a_n}$ and find a formula for A_n, B_n, C_n, D_n. Then observe you can write $a_{n+1} = \dfrac{A_n + B_n}{C_n + D_n}$. finding the $\lim a_n$

$$\begin{aligned}
a_{n+1} &= 1 + \frac{2a_n}{1+a_n} \\
a_{n+2} &= 1 + \frac{2a_{n+1}}{1+a_{n+1}} \\
&= 1 + \frac{2(1+\frac{2a_n}{1+a_n})}{1+1+\frac{2a_n}{1+a_n}} \\
&= 1 + \frac{2+\frac{4a_n}{1+a_n}}{2+\frac{2a_n}{1+a_n}} \\
&= 1 + \frac{2+4a_n}{2+2a_n} \\
a_{n+3} &= 1 + \frac{2+4a_{n+1}}{2+2a_{n+1}} \\
&= 1 + \frac{2+4(1+\frac{2a_n}{1+a_n})}{2+2(1+\frac{2a_n}{1+a_n})} \\
&= 1 + \frac{2+4+\frac{8a_n}{1+a_n}}{2+2+\frac{4a_n}{1+a_n}} \\
&= 1 + \frac{6+8a_n}{4+4a_n}
\end{aligned}$$

generalizing the pattern we get

$$a_{n+k} = 1 + \frac{2k + 2a_n}{k + a_n}$$

we can write $a_{n+k} = \dfrac{A_k + B_k a_n}{C_k + D_k a_n}$ by setting $A_k = 2k + 2, B_k = 1, C_k = k, D_k = 1$.

$$a_{n+k} = \frac{2k + 2 + a_n}{k + a_n}$$

then $a_{n+1} = \dfrac{A_n + B_n}{C_n + D_n}$ by setting $A_n = 2 + 2, B_n = 1, C_n = 1, D_n = 1$.

$$a_{n+1} = \frac{4 + a_n}{1 + a_n}$$

to find the limit of a_n we can set $a_{n+1} = a_n = a$ and solve for a.

$$\begin{aligned} a &= \frac{4 + a}{1 + a} \\ a + a^2 &= 4 + a \\ a^2 &= 4 \\ a &= 2, -2 \end{aligned}$$

since $a_1 = 1$ we have $a = 2$.

$$\lim_{n \to \infty} a_n = 2$$

2 question 2

Let $0 < x_1 \le x_2 \le \cdots \le x_k$ prove that $\lim_{n\to\infty}(x_1^n + x_2^n + \cdots + x_k^n)^{1/n} = x_k$.

Proof 2.1. Consider the sequence $\{a_n\}$ defined by

$$a_n = (x_1^n + x_2^n + \cdots + x_k^n)^{1/n}.$$

We want to show that

$$\lim_{n\to\infty} a_n = x_k.$$

First, observe that since $0 < x_1 \leq x_2 \leq \cdots \leq x_k$, we have:

$$x_k^n \leq x_1^n + x_2^n + \cdots + x_k^n \leq kx_k^n.$$

Taking the n-th root of each part of the inequality, we get:

$$(x_k^n)^{1/n} \leq (x_1^n + x_2^n + \cdots + x_k^n)^{1/n} \leq (kx_k^n)^{1/n}.$$

Simplifying:

$$x_k \leq (x_1^n + x_2^n + \cdots + x_k^n)^{1/n} \leq k^{1/n} \cdot x_k.$$

lhs, $\lim_{n\to\infty} x_k = x_k$ and rhs, $\lim_{n\to\infty} k^{1/n} \cdot x_k = x_k$. Thus, by squeeze theorem the limit of a_n is x_k.

3 question 3

Let $k \in \mathbb{N}$ and suppose $\sum_{n=1}^{\infty} |a_{n+k} - a_n|$ converges. Prove that (a_n) has a convergent subsequence.Hint: Use Bolzano Weierstrass, and that the tail end of a convergent series must go to 0

given that $\sum_{n=1}^{\infty} |a_{n+k} - a_n|$ converges, we can wrie te partial sums by $S_N = \sum_{n=1}^{N} |a_{n+k} - a_n|$. Since this series converges and (S_N) is bounded
there exists some $M > 0$ s.t.:

$$S_N \leq M \quad \forall N.$$

since the series $\sum_{n=1}^{\infty} |a_{n+k} - a_n|$ converges, $|a_{n+k} - a_n| \to 0$

$$\lim_{n\to\infty} |a_{n+k} - a_n| = 0.$$

The BW states that every bounded sequence in $\mathbb{R}$ has a convergent subsequence. To apply this we show that (a_n) is bounded.

Since $\sum_{n=1}^{\infty} |a_{n+k} - a_n|$ converges, and we know $\lim_{n\to\infty} |a_{n+k} - a_n| = 0$, we have:

$$|a_{n+k} - a_n| < \epsilon \quad \text{for large large } n.$$

This means that for large n, a_{n+k} becomes closer too a_n. we have

1. The first k terms $a_1, a_2, \ldots, a_k$ which form a finite set and bounded. 2. For $n > k$, each a_n is close to some a_{n-k} due to the convergence to 0 of the differences.

We can cover the sequence by finite bounds up to a_k and then note that subsequent terms stay close to these, which makes the entire sequence bounded.

Given that (a_n) is bounded, by BW, there exists a convergent subsequence (a_{n_j}) of (a_n). thus, the sequence (a_n) has a convergent subsequence.

$\blacksquare$

4 question 4

Prove that a sequence in a finite set has a constant subsequence.

Let S be a finite set. Then $S = \{a_1, a_2, \dots, a_n\}$ for some $n \in \mathbb{N}$. Consider a sequence (a_n) in S. Since S is finite, the sequence (a_n) will repeat. so there exists some $N \in \mathbb{N}$ st $a_n = a_{n+k}$ $\forall n \geq N$ and some $k \in \mathbb{N}$.

5 question 5

let (n_k) be a sequence of positive integers which contains every positive integer exactly once. prove that $\lim_k \sup a_{n_k} = \lim_n \sup a_n$ and $\lim_k \inf a_{n_k} = \lim_n \inf a_n$.

let (n_k) be sequence of positive integers that contains every positive integer once.we show that

1.
$$\lim_k \sup a_{n_k} = \lim_n \sup a_n$$

2.
$$\lim_k \inf a_{n_k} = \lim_n \inf a_n.$$

so, we show that

$$\lim_k \sup a_{n_k} \leq \lim_n \sup a_n.$$

Let $M = \lim_n \sup a_n$. Then, $\forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that

$$a_n < M + \epsilon \quad \forall n \geq N.$$

Since (n_k) contains every positive integer exactly once, there exists $K \in \mathbb{N}$ such that

$$n_k \geq N \quad \forall k \geq K.$$

thus, $\forall k \geq K$, we have

$$a_{n_k} < M + \epsilon.$$

Taking the supremum over all $k \geq K$, we get

$$\sup_{k \geq K} a_{n_k} \leq M + \epsilon.$$

$$\lim_k \sup a_{n_k} \leq \lim_n \sup a_n.$$

we show that

$$\lim_k \sup a_{n_k} \geq \lim_n \sup a_n.$$

Let $M = \lim_n \sup a_n$. $\forall\ \epsilon > 0$, by the definition of $\lim\sup$, there are infinitely many n such that

$$a_n > M - \epsilon.$$

Since (n_k) contains every positive integer exactly once, there are infinitely many k such that

$$a_{n_k} > M - \epsilon.$$

Taking the supremum the subsequence,

$$\sup a_{n_k} \geq M - \epsilon.$$

Since $\epsilon > 0$ we have

$$\lim_k \sup a_{n_k} \geq \lim_n \sup a_n.$$

Combining the two inequalities, we have

$$\lim_k \sup a_{n_k} = \lim_n \sup a_n.$$

Now, we show that

$$\lim_k \inf a_{n_k} \leq \lim_n \inf a_n.$$

Let $m = \lim_n \inf a_n$. Then $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$a_n > m - \epsilon \quad \forall n \geq N.$$

Since (n_k) contains every positive integer exactly once, there exists $K \in \mathbb{N}$ such that

$$n_k \geq N \quad \forall k \geq K.$$

Therefore, $\forall k \geq K$, we have

$$a_{n_k} > m - \epsilon.$$

Taking the infimum over all $k \geq K$, we get

$$\inf_{k \geq K} a_{n_k} \geq m - \epsilon.$$

Since $\epsilon > 0$, we have

$$\lim_k \inf a_{n_k} \leq \lim_n \inf a_n.$$

now we show that

$$\lim_k \inf a_{n_k} \geq \lim_n \inf a_n.$$

Let $m = \lim_n \inf a_n$. And $\forall \epsilon > 0$, by definition of $\liminf$, there are infinitely many n such that

$$a_n < m + \epsilon.$$

Since (n_k) contains every positive integer exactly once, there are infinitely many k st

$$a_{n_k} < m + \epsilon.$$

Taking the infimum over these values, we get

$$\inf a_{n_k} \leq m + \epsilon.$$

Since $\epsilon > 0$ we have

$$\liminf_k a_{n_k} \leq \liminf_n a_n.$$

Combining the two inequalities, we get

$$\liminf_k a_{n_k} = \liminf_n a_n.$$

thus, we have proved that

$$\limsup_k a_{n_k} = \limsup_n a_n$$

and

$$\liminf_k a_{n_k} = \liminf_n a_n.$$