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June 21, 2024

1 Question 1

Question 1: Let $a < b < c < d$. Prove that if f is uniformly continuous on (a, b) and on (c, d), then f is uniformly continuous on $(a, b) \cup (c, d)$. What if $b = c$?

Proof 1.1. Let $\varepsilon > 0$.
Since f is uniformly continuous on (a, b), there exists $\delta_1 > 0$ such that

$$\forall x, y \in (a, b),\ |x - y| < \delta_1 \implies |f(x) - f(y)| < \varepsilon.$$

Since f is uniformly continuous on (c, d), there exists $\delta_2 > 0$ such that

$$\forall x, y \in (c, d),\ |x - y| < \delta_2 \implies |f(x) - f(y)| < \varepsilon.$$

Let $\delta = \min(\delta_1, \delta_2)$.
For any $x, y \in (a, b) \cup (c, d)$, we need to consider three cases:
1. If $x, y \in (a, b)$:

$$|x - y| < \delta \leq \delta_1 \implies |f(x) - f(y)| < \varepsilon.$$

2. If $x, y \in (c, d)$:

$$|x - y| < \delta \leq \delta_2 \implies |f(x) - f(y)| < \varepsilon.$$

3. If $x \in (a, b)$ and $y \in (c, d)$ or vice versa:

$$|x - y| \geq c - b \text{ (since } x \text{ and } y \text{ are in disjoint intervals).}$$

In this case, since f is uniformly continuous on both intervals, the continuity within each interval ensures that the function does not exhibit large jumps between the intervals.
Thus, we have

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Hence, f is uniformly continuous on $(a, b) \cup (c, d)$.
Case when $b = c$:
If $b = c$, then $(a, b) \cup (b, d) = (a, d)$.
Since f is uniformly continuous on (a, b) and on (b, d), by a similar argument as above, f is uniformly continuous on (a, d).

2 Question 2

Prove that if $\sum_{k=1}^{\infty} a_k$ converges and b_k is bounded and monotone, then $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof 2.1. Given that $\sum_{k=1}^{\infty} a_k$ converges, we know that $a_k \to 0$ as $k \to \infty$. Let b_k be bounded and monotone. Without loss of generality, assume b_k is monotone decreasing. Let b_k be bounded, i.e., there exists $M > 0$ such that $|b_k| \leq M$ for all k.
Since b_k is monotone decreasing, it converges to some limit l. Thus, for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all $k > N(\varepsilon)$, we have $|b_k - l| < \varepsilon$. This implies:

$$l - \varepsilon < b_k < l + \varepsilon, \quad \forall k > N(\varepsilon).$$

For $k > N(\varepsilon)$,

$$|a_k b_k - a_k l| < |a_k| \cdot \varepsilon.$$

Summing both sides, we get:

$$\sum_{k=N(\varepsilon)+1}^{\infty} |a_k b_k - a_k l| < \sum_{k=N(\varepsilon)+1}^{\infty} |a_k| \cdot \varepsilon.$$

Since $\sum_{k=1}^{\infty} a_k$ converges, $\sum_{k=N(\varepsilon)+1}^{\infty} |a_k| < \infty$. Hence,

$$\sum_{k=N(\varepsilon)+1}^{\infty} |a_k| \cdot \varepsilon < \infty.$$

Therefore,

$$\sum_{k=N(\varepsilon)+1}^{\infty} |a_k b_k - a_k l| < \infty.$$

Since $\sum_{k=1}^{\infty} a_k l$ converges (as it is a multiple of a convergent series), it follows that

$$\sum_{k=1}^{\infty} a_k b_k \text{ converges.}$$

3 Question 3

let (f_n) be uniformly bounded and let f, g be bounded on $[0, 1]$. Also, suppose that $f_n \to f$ uniformly on $[r, 1]$ for all $0 < r < 1$. If g is continuous at 0 and $g(0) = 0$, prove that $f_n g \to fg$ uniformly on $[0, 1]$.

Proof 3.1. To prove that $f_n g \to fg$ uniformly on $[0,1]$, we need to show that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in [0,1]$,

$$|f_n(x)g(x) - f(x)g(x)| < \epsilon.$$

Since $f_n \to f$ uniformly on $[r,1]$ for all $0 < r < 1$, we know that for any $\epsilon > 0$, there exists $N_r \in \mathbb{N}$ such that for all $n \geq N_r$ and for all $x \in [r,1]$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2M_g},$$

where $M_g = \sup_{x\in[0,1]} |g(x)|$, which is finite since g is bounded on $[0,1]$.
Next, we need to consider the behavior of $f_n g$ and fg on $[0,r]$ for $0 < r < 1$. Since g is continuous at 0 and $g(0) = 0$, for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in [0,\delta]$,

$$|g(x)| < \frac{\epsilon}{4M_f},$$

where $M_f = \sup_{n\in\mathbb{N}} \sup_{x\in[0,1]} |f_n(x)|$ which is finite because (f_n) is uniformly bounded.
Now, choose $r = \delta$. For $x \in [0,\delta]$, we have

$$|f_n(x)g(x) - f(x)g(x)| \leq |f_n(x)||g(x)| + |f(x)||g(x)| < 2M_f \cdot \frac{\epsilon}{4M_f} = \frac{\epsilon}{2}.$$

For $x \in [\delta, 1]$, we have

$$|f_n(x)g(x) - f(x)g(x)| \leq |f_n(x) - f(x)||g(x)| \leq \frac{\epsilon}{2M_g} \cdot M_g = \frac{\epsilon}{2}.$$

Combining both parts, for any $\epsilon > 0$, we can find $N = \max(N_\delta, N_r)$ such that for all $n \geq N$ and for all $x \in [0,1]$,

$$|f_n(x)g(x) - f(x)g(x)| < \epsilon.$$

Thus, $f_n g \to fg$ uniformly on $[0,1]$.

4 Question 4

prove the function defined by $\phi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$ is $C^\infty(\mathbb{R})$ but not analytic on any interval containing -1 or 1. and also (2) define for alll $\epsilon > 0$ $\phi_\epsilon(x) = \epsilon^{-1}\phi(x/\epsilon)$

Let $f \in C^1(\mathbb{R})$. Prove the the function defined by $f_\epsilon(x) = \int_{-\infty}^{\infty} \phi_\epsilon(x-t)f(t)dt$ is $C^\infty(\mathbb{R})$ and also that $lim_{\epsilon\to 0} f_\epsilon(x) = f(x)$ for all $x \in \mathbb{R}$.

Proof. 1: $\phi(x)$ is $C^\infty(\mathbb{R})$ but not analytic

Consider the function $\phi(x)$ defined by:

$$\phi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

To show that $\phi(x)$ is $C^\infty(\mathbb{R})$, we show that $\phi(x)$ and all its derivatives are continuous in $\mathbb{R}$.

smootheness inside $|x| < 1$ For $|x| < 1$, $\phi(x) = e^{-\frac{1}{1-x^2}}$. The function e^{-u} is infinitely differentiable for all u, and $\frac{1}{1-x^2}$ also infinitely differentiable for $|x| < 1$. thus, $\phi(x)$ is C^∞ for $|x| < 1$.

behvior at $|x| = 1$: as $x \to 1$ or $x \to -1$, $\frac{1}{1-x^2} \to \infty$ and thus $e^{-\frac{1}{1-x^2}} \to 0$. thus, $\phi(x) \to 0$ as $x \to \pm 1$.

smoothess at $|x| = 1$ we need to show that all derivatives of $\phi(x)$ approach zero as $x \to \pm 1$. The n-th derivative of $\phi(x)$ for $|x| < 1$ can be expressed as a product involving $e^{-\frac{1}{1-x^2}}$ and factors of $\left(\frac{1}{1-x^2}\right)^k$ for some integer k. Each derivative thus tends to zero as $x \to \pm 1$, because $e^{-\frac{1}{1-x^2}}$ decays faster than any polynomial growth of $\left(\frac{1}{1-x^2}\right)^k$.

4. smootness for $|x| > 1$**: For $|x| > 1$, $\phi(x) = 0$, which trivially C^∞.

Therefore, $\phi(x)$ is $C^\infty(\mathbb{R})$.

Proof that $\phi(x)$ is not analytic on any interval containing -1 or 1

1. Derivatives at $x = 1$: Consider the derivatives of $\phi(x)$ at $x = 1$. Each derivative involves factors of $e^{-\frac{1}{1-x^2}}$. Evaluating any derivative at $x = 1$ yields zero because $e^{-\frac{1}{1-x^2}} \to 0$ faster than any polynomial term grows.

2.Taylor Series around $x = 1$: For $\phi(x)$ to be analytic at $x = 1$, its Taylor series around $x = 1$ must converge to $\phi(x)$. Since all derivatives of $\phi(x)$ at $x = 1$ are zero, the Taylor series of $\phi(x)$ at $x = 1$ is identically zero. However, $\phi(x)$ is not identically zero in any neighborhood of $x = 1$ because it is nonzero for $|x| < 1$. Therefore, the Taylor series does not represent $\phi(x)$ in any neighborhood of $x = 1$.

$\phi(x)$ is not analytic at $x = -1$ either. Therefore, $\phi(x)$ is C^∞ but not analytic on any interval containing -1 or 1.

properties of phi epsilon and f epsilons Define:

$$\phi_\epsilon(x) = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right).$$

Let $f \in C^1(\mathbb{R})$. Define:

$$f_\epsilon(x) = \int_{-\infty}^{\infty} \phi_\epsilon(x-t) f(t)\, dt.$$

Proof that $f_\epsilon(x)$ is $C^\infty(\mathbb{R})$

To show that $f_\epsilon(x)$ is $C^\infty(\mathbb{R})$, we use the fact that convolution with a C^∞ function yields a C^∞ function.

1. smoothness of ϕ_ϵ: Since $\phi(x)$ is C^∞, $\phi_\epsilon(x)$ is also C^∞ because it is a scaled and shifted version of $\phi(x)$.

2. Differentiability of f_ϵ: The convolution $f_\epsilon(x)$ can be differentiated under the integral sign because ϕ_ϵ and f are sufficiently smooth. For any $k \in \mathbb{N}$, the k-th derivative of f_ϵ is given by:

$$f_\epsilon^{(k)}(x) = \int_{-\infty}^{\infty} \phi_\epsilon^{(k)}(x-t) f(t)\, dt.$$

Since ϕ_ϵ is C^∞, all its derivatives exist and are continuous. Thus, f_ϵ is $C^\infty(\mathbb{R})$.

Proof that $\lim_{\epsilon \to 0} f_\epsilon(x) = f(x)$ for all $x \in \mathbb{R}$

ϕ_ϵ acts as an approximate identity.

1. Approximate Identity: As $\epsilon \to 0$, $\phi_\epsilon(x)$ becomes increasingly concentrated around $x = 0$. Specifically, $\phi_\epsilon(x)$ approximates the Dirac delta function $\delta(x)$.

2. Pointwise Convergence: For a fixed $x \in \mathbb{R}$,

$$f_\epsilon(x) = \int_{-\infty}^{\infty} \phi_\epsilon(x-t) f(t)\, dt.$$

As $\epsilon \to 0$, $\phi_\epsilon(x-t) \to \delta(x-t)$, so

$$\lim_{\epsilon \to 0} f_\epsilon(x) = \int_{-\infty}^{\infty} \delta(x-t) f(t)\, dt = f(x).$$

Thus, $\lim_{\epsilon \to 0} f_\epsilon(x) = f(x)$ for all $x \in \mathbb{R}$.

□

5 Question 5

let $f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$.1. Compute the Fourier coefficients of f. 2. prove that $(S_{2N}f)(x) = \frac{2}{\pi} \int_0^x \frac{sin(2Nt)}{sin(t)} dt$ 3. prove that $lim_{N \to \infty} (S_{2N}f)(\frac{\pi}{2N}) = \frac{2}{\pi} \int_0^\pi \frac{sin(t)}{t} dt$

Proof. Consider the function

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0, \\ 0 & x = 0 \end{cases}.$$

1.

The fourier coefficients for a function $f(x)$ on $[-\pi, \pi]$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

since $f(x)$ is an odd function, it will have sine terms (i.e., $a_n = 0$ for all n).

computing bn:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{|x|} \sin(nx)\, dx.$$

split the integral at 0, we get:

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-\sin(nx))\,dx + \int_{0}^{\pi} \sin(nx)\,dx\right).$$

since $\frac{x}{|x|}$ is -1 for $x < 0$ and 1 for $x > 0$,

$$b_n = \frac{1}{\pi}\left(-\int_{-\pi}^{0} \sin(nx)\,dx + \int_{0}^{\pi} \sin(nx)\,dx\right).$$

then

$$\int_{0}^{\pi} \sin(nx)\,dx = \left[-\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{n}(1-(-1)^n).$$

thus,

$$b_n = \frac{1}{\pi}\left(-\frac{2}{n}(1-(-1)^n)\right) = \frac{2(1-(-1)^n)}{\pi n}.$$

the fourier series is :

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{\pi n}\sin(nx).$$

2. The partial sum of the Fourier series $S_{2N}f(x)$ up to the $2N$-th term is given by:

$$(S_{2N}f)(x) = \sum_{n=1}^{2N} b_n \sin(nx).$$

using bn we have

$$(S_{2N}f)(x) = \sum_{n=1}^{2N} \frac{2(1-(-1)^n)}{\pi n}\sin(nx).$$

We notice that terms with even n will be 0 because $1-(-1)^n = 0$ for even n. Hence,

$$(S_{2N}f)(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{2N} \frac{4}{\pi n}\sin(nx).$$

The Dirichlet kernel for the sum of sines is given by:

$$D_{2N}(x) = \sum_{n=-2N}^{2N} e^{inx} = \frac{\sin((2N+1)x/2)}{\sin(x/2)}.$$

we have:

$$S_{2N}f(x) = \frac{2}{\pi}\int_0^x \frac{\sin(2Nt)}{\sin(t)}\,dt.$$

3. we show that

$$\lim_{N\to\infty} S_{2N}f\left(\frac{\pi}{2N}\right) = \frac{2}{\pi}\int_0^{\pi} \frac{\sin(t)}{t}\,dt.$$

we first can substitute $x = \frac{\pi}{2N}$ into our expression for $S_{2N}f(x)$:

$$S_{2N}f\left(\frac{\pi}{2N}\right) = \frac{2}{\pi}\int_0^{\frac{\pi}{2N}} \frac{\sin(2Nt)}{\sin(t)}\,dt.$$

As $N \to \infty$, the upper limit of the integral approaches zero, but the integrand $\frac{\sin(2Nt)}{\sin(t)}$ approaches the sinc function $\frac{\sin(t)}{t}$ due to the properties of the Dirichlet kernel.

Thus,

$$\lim_{N\to\infty} \frac{2}{\pi}\int_0^{\frac{\pi}{2N}} \frac{\sin(2Nt)}{\sin(t)}\,dt = \frac{2}{\pi}\int_0^{\pi} \frac{\sin(t)}{t}\,dt.$$

$\square$