LECTURE 1 (MAY 13)

Fields, Induction, Binomial Theorem:
TEXTBOOK pg 2 - approx 16 (up to 1.2)

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1 Natural Numbers and Real Numbers

Natural numbers are a set of positive integers:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

Real numbers are a set of all rational and irrational numbers:

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

1.1 Addition and Multiplication as a function on $\mathbb{R}$

1. Addition: $+ : \mathbb{R}^2 \to \mathbb{R}$
2. Multiplication: $\cdot : \mathbb{R}^2 \to \mathbb{R}$

2 Fields

Axiom 2.1 (Field Axiom)**.** A field is a set of numbers $\mathbb{F}$ with two operations: addition and multiplication. A field must satisfy the following properties:

1. Closure property under addition and multiplication: $a + b \in \mathbb{F}$ and $a \cdot b \in \mathbb{F}$
2. Associative: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. Commutative: $a + b = b + a$ and $a \cdot b = b \cdot a$
4. Distributive: $a \cdot (b + c) = a \cdot b + a \cdot c$
5. Unique Additive Identity: There exists a unique element $0 \in \mathbb{F}$ such that $\forall x \in \mathbb{F}$, $0 + x = x$.
6. Unique Multiplicative Identity: There exists a unique element $1 \in \mathbb{F}$ such that $\forall x \in \mathbb{F}$, $1 \cdot x = x$.
7. Additive Inverse: For every $x \in \mathbb{F}$, there exists a unique element $-x \in \mathbb{F}$ such that $x + (-x) = 0$.
8. Multiplicative Inverse: For every $x \in \mathbb{F}, x \neq 0$, there exists a unique element $x^{-1} \in \mathbb{F}$ such that $x \cdot x^{-1} = 1$.

A field is a set of numbers with two operations: addition and multiplication. A field must satisfy the following properties:

1. Associative: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. Commutative: $a + b = b + a$ and $a \cdot b = b \cdot a$
3. Distributive: $a \cdot (b + c) = a \cdot b + a \cdot c$

We can see that $\mathbb{R}$ is a field since it satisfies all three properties.

2.1 Additive Identity

There exists a unique element $0 \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $0 + x = x$.

2.2 Multiplicative Identity

There exists a unique element $1 \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $1 \cdot x = x$.

2.3 Additive Inverse

For every $x \in \mathbb{R}$, there exists a unique element $-x \in \mathbb{R}$ such that $x + (-x) = 0$.

2.4 Multiplicative Inverse

For every $x \in \mathbb{R}$, there exists a unique element $x^{-1} \in \mathbb{R}$ such that $x \cdot x^{-1} = 1$.

Using these additional properties, we can prove that $(-1)^2 = 1$.

3 Proof: $(-1)^2 = 1$

Proof 3.1. Let 1_L be the multiplicative identity and -1 be the additive inverse of the multiplicative identity. We want to show that $\forall x \in \mathbb{R} : (-1) \cdot (-1)x = x$. Then by the uniqueness of the multiplicative identity, $(-1)^2$ must be the multiplicative identity. We will show:

$$
\begin{gathered}
(-1)^2 x = x \\
(-1) + (1) = 0 \\
((-1) + 1) \cdot x = 0 \cdot x \\
(-1) \cdot x + 1 \cdot x = 0 \\
(-1) \cdot x + x = 0 \\
(-1)^2 \cdot x - x = 0 \\
(-1)^2 \cdot x - x + x = 0 + x \\
(-1)^2 \cdot x = x
\end{gathered}
$$

Thus we have shown that $(-1)^2$ is the multiplicative identity and thus by the uniqueness of the multiplicative identity, $(-1)^2$ must be 1.

4 Ordered Real Numbers

Definition 4.1.

$$<: \mathbb{R}^2 \longrightarrow \{\mathit{True}, \mathit{False}\} \text{ (or using binary) } \{0, 1\}$$

The set of real numbers $\mathbb{R}$ is ordered by a relation $<: \mathbb{R}^2 \to \{0,1\}$ with the following properties:

1. Trichotomy: $\forall x, y \in \mathbb{R} : (x < y) \vee (x = y) \vee (y < x)$
2. Transitivity: $\forall x, y, z \in \mathbb{R} : (x < y) \wedge (y < z) \to (x < z)$
3. Additivity: $\forall x, y, z \in \mathbb{R} : (x < y) \to (x + z < y + z)$
4. Multiplicative: $\forall x, y, z \in \mathbb{R} : (x < y) \wedge (0 < z) \to (x \cdot z < y \cdot z)$ and $(x < y) \wedge (z < 0) \to (y \cdot z < x \cdot z)$

5 Prove the Binomial Expansion Formula

The binomial expansion formula states that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Axiom 5.1. Given any non-empty subset $E \subseteq \mathbb{N}$, there is a least element $\alpha \in E$ such that $\alpha \leq x \; \forall x \in E$.

Theorem 5.2 (Theorem of Induction)**.** Consider a statement dependent on $\mathbb{N}$:

$$S : \mathbb{N} \longrightarrow \{0,1\}$$

If:

1. $S(1) = 1$
2. $\forall n \in \mathbb{N} : S(n) \to S(n+1)$
3. then $\forall n \in \mathbb{N} : S(n)$

Proof 5.3. Consider $E = \{n \in \mathbb{N} : S(n) = 0\}$. Suppose $E \neq \emptyset$. Then by the axiom of $\mathbb{N}$, there is a least element $\alpha \in E$. $\alpha \neq 1$ because $S(1) = 1$. Consider $\alpha - 1$. $S(\alpha - 1)$ must be true because α is the least element of E. Because $S(\alpha - 1) \to S(\alpha)$ and $S(\alpha) = 0$. Thus our supposition is wrong. Therefore $E = \emptyset$ and $\forall n \in \mathbb{N} : S(n)$. Q.E.D

Lemma 5.4 (Binomial Theorem)**.**

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

N ow we can prove the binomial expansion formula using induction.

5.1 Proof of the Binomial Theorem by Induction

Proof 5.5. We will prove the binomial theorem using mathematical induction.
Base Case: For $n = 0$, the binomial theorem states that $(a+b)^0 = 1$. This is true since any number raised to the power of 0 is equal to 1.
Inductive Hypothesis: Assume that the binomial theorem holds for some positive integer k, i.e., $(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$.
Inductive Step: We need to show that the binomial theorem holds for $k+1$, i.e., $(a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$.
Expanding $(a+b)^{k+1}$ using the distributive property, we get:

$$(a+b)^{k+1} = (a+b)(a+b)^k$$

Using the inductive hypothesis, we can rewrite this as:

$$(a+b)^{k+1} = (a+b)\left(\sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i\right)$$

Expanding the product, we have:

$$(a+b)^{k+1} = \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^{i+1}$$

Now, let's focus on the terms in the second sum. We can rewrite them as:

$$\sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^{i+1} = \sum_{i=1}^{k+1} \binom{k}{i-1} a^{k+1-i} b^i$$

Combining the two sums, we get:

$$(a+b)^{k+1} = \sum_{i=0}^{k+1} \left(\binom{k}{i} + \binom{k}{i-1}\right) a^{k+1-i} b^i$$

Using the lemma $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, we can simplify the expression further:

$$(a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$$

This completes the inductive step.
By the principle of mathematical induction, the binomial theorem holds for all positive integers n.
Therefore, the binomial expansion formula

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

is proven.

6 Last Property of $\mathbb{R}$

Given any bounded non-empty subset $E \subseteq \mathbb{R}$, there exists a supremum (the least upper bound). (continued next lecture)