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1 Question 1

Prove the following statement: For all $x \in R$, $0 \cdot x = 0$, where 0 is the additive identity of R. (Hint: consider $1 + 0$.)

Proof 1.1. $\mathbb{R}$ is a field, so it follows the distributive property. We also know $1 + 0 = 1$ since 0 is the additive identity:

$$\begin{aligned} 1 + 0 &= 1 \\ x(1 + 0) &= x(1) \text{ .. using the distributive property of fields} \\ 1 \cdot x + 0 \cdot x &= 1 \cdot x \end{aligned}$$

1 is the multiplicative identity,

$$\begin{aligned} x + 0 \cdot x &= x \\ x - x + 0 \cdot x &= x - x \text{ .. } -x \text{ is the additive inverse of } x \\ 0 \cdot x &= 0 \end{aligned}$$

We show that $0 \cdot x = 0$ for all $x \in \mathbb{R}$.

2 Question 2

Write a clear proof of the Archimedian Principle, i.d. the statement: if $a, b \in \mathbb{R}_{>0}$, then there exists $n \in \mathbb{N}$ such that $a \cdot n > b$. You may use the following lemma without proving it: If $E \subset \mathbb{N}$ has a supremum s, then $s \in E$. hint : consider the set $E = \{n \in \mathbb{N} : n < \frac{b}{a}\}$. Then use the lemma above and consider $a \cdot (s + 1)$.

Proof 2.1. To prove: If $a, b \in \mathbb{R}_{>0}$, then there exists $n \in \mathbb{N}$ such that $a \cdot n > b$.
Consider the equivalent statement: If $a, b \in \mathbb{R}_{>0}$, then there exists $n \in \mathbb{N}$ such that $n > \frac{b}{a}$.
We will consider two cases:

1. **Case 1:** $a \geq b$

   Let $n = 1$. Then,
   $$a \cdot n = a \geq b.$$
   Thus, $a \cdot n > b$ is satisfied for $n = 1$.

2. **Case 2:** $a < b$

   Let $E = \{n \in \mathbb{N} : n < \frac{b}{a}\}$.

   We know that E is non-empty since $1 \in E$ (because $a < b$ implies $\frac{b}{a} > 1$).

   We also know that E is bounded above by $\frac{b}{a}$. Thus, $E \subset \mathbb{R}$ and it has a supremum $s = \sup(E)$. By the given lemma, $s \in E$.

   By definition of the supremum, for all $e \in E$, we have $e \leq s$. Also note that $s \in E \to s \in \mathbb{N}$

   Now, consider $k = s + 1$. Since s is the supremum of E, $k \notin E$. This implies that $k > \frac{b}{a}$. Therefore,
   $$a \cdot k > b.$$
   Since $k = s + 1 \in \mathbb{N}$ (Natural numbers closed under addition), we have shown that there exists $n \in \mathbb{N}$ such that $a \cdot n > b$.

Thus, the Archimedean Principle is proved.

3 Question 3

Prove that
$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$
using mathematical induction.

Proof 3.1. We will prove the statement by induction on n.

1. **Base Case:** $n = 1$

$$\text{LHS: } \sum_{k=1}^{1} k^3 = 1^3 = 1$$
$$\text{RHS: } \left(\frac{1(1+1)}{2}\right)^2 = 1^2 = 1$$

   We get $1 = 1$ so the base case holds.

2. **Inductive Hypothesis:** Assume that the statement holds for $n = m$. That is,

$$\sum_{k=1}^{m} k^3 = \left(\frac{m(m+1)}{2}\right)^2$$

3. **Inductive Step:** We will show that the statement holds for $n = m + 1$.

$$\begin{aligned}
\sum_{k=1}^{m+1} k^3 &= \sum_{k=1}^{m} k^3 + (m+1)^3 \\
&= \left(\frac{m(m+1)}{2}\right)^2 + (m+1)^3 \\
&= \frac{m^2(m+1)^2}{4} + (m+1)^3 \\
&= \frac{m^2(m+1)^2 + 4(m+1)^3}{4} \\
&= \frac{(m+1)^2(m^2+4m+4)}{4} \\
&= \frac{(m+1)^2(m+2)^2}{4} \\
&= \left(\frac{(m+1)(m+2)}{2}\right)^2
\end{aligned}$$

Thus, by induction, we have shown that

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

4 Question 4

Prove that for any pair of integers $m, n \in \mathbb{Z}$ with $n \geq 1$, there exists a pair of unique integers $q, r \in \mathbb{Z}$ such that $m = qn + r$ and $0 \leq r \leq n - 1$.

Proof 4.1. I will prove two things: first, the existence of q and r and second, the uniqueness of q and r.

1. **Existence:** Let $m, n \in \mathbb{Z}$ with $n \geq 1$ which means $n \in \mathbb{N}$. We will show that there exists $q, r \in \mathbb{Z}$ such that $m = qn + r$ and $0 \leq r \leq n - 1$.
   Consider the set $S = \{m + an : a \in \mathbb{Z}\} \cap \mathbb{Z}_{\geq 0}$. Since $n \in \mathbb{N}$, S is non-empty. By the Well-Ordering Principle, S has a least element r. if $m \geq 0$ then we can take a to be 1. we see that $n > 0$ and $m + an = m + n \geq 0$ thus $m + n \in S$. If $m < 0$ then we can take a to be $-m$. we see that $m + an = m(1 - n) \geq 0$ thus $m(1 - n) \in S$.

   Using the well ordring principle, we see that S has a least element r. and $r \in S$ implies $\exists a \in \mathbb{Z}$ such that $r = m + an$. We can write $m = qn + r$ where $q = a$ and $r = m + an$.

   since $r \in S$, we know that $r \geq 0$. $S \in \mathbb{Z}_{\geq 0}$

   Now we will show that $r \leq n - 1$. But since $n \in \mathbb{N}$ we can re-write this inequality as $r < n$ which is easier for this part. For contradiction, assume $r \geq n$ then we see that $r - n \in S$ because $r - n = m - nq - n = m - n(q + 1) \geq 0$. $r - n \in S \rightarrow r - n \geq r$ which is a contradiction. Thus, $r < n$ and if we write this in the terms of the question $r \leq n - 1$.

2. **Uniqueness:** We will show injection between q, r to m. Assume that there exists $q_1, q_2, r_1, r_2 \in \mathbb{Z}$ such that $m = q_1 n + r_1 = q_2 n + r_2$ and $0 \leq r_1, r_2 \leq n - 1$.

   We can write $q_1 n + r_1 = q_2 n + r_2$ as $q_1 n - q_2 n = r_2 - r_1$. This implies that $(q_1 - q_2)n = r_2 - r_1$. Since $0 \leq r_1, r_2 \leq n - 1$, we have $-n < r_2 - r_1 < n$. This implies that $-1 < q_1 - q_2 < 1$. Since $q_1, q_2 \in \mathbb{Z}$, we have $q_1 - q_2 = 0 \rightarrow q_1 = q_2$. Therefore we have $q_1 n = q_2 n \rightarrow r_1 = r_2$. Thus, q and r are unique.

Thus, we have shown that for any pair of integers $m, n \in \mathbb{Z}$ with $n \geq 1$, there exists a pair of unique integers $q, r \in \mathbb{Z}$ such that $m = qn + r$ and $0 \leq r \leq n - 1$.

5 Question 5

Suppose (a_n) is a sequence in $\mathbb{R}$. Prove that (a_n) converges to a number $a \in \mathbb{R}$ if and only if every subsequence of (a_n) converges to a.

Proof 5.1. We will prove the statement by proving both directions.

1. **If:** Suppose (a_n) converges to a number $a \in \mathbb{R}$. We will show that every subsequence of (a_n) converges to a.

   Let (a_{n_k}) be a subsequence of (a_n). Since (a_n) converges to a, we have $\lim_{n\to\infty} a_n = a$. This implies that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N$.

   Since (a_{n_k}) is a subsequence of (a_n), we have $n_k \geq k$ for all $k \in \mathbb{N}$. Thus, for all $k \geq N$, we have $n_k \geq k \geq N$. This implies that $|a_{n_k} - a| < \epsilon$ for all $k \geq N$. Therefore, (a_{n_k}) converges to a.

2. **Only If:** Suppose every subsequence of (a_n) converges to a. We will show that (a_n) converges to a.

   For contradiction, assume that (a_n) does not converge to a. This implies that there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $n \geq N$ such that $|a_n - a| \geq \epsilon$.

   Consider the subsequence (a_{n_k}) defined as follows: $n_1 = 1$ and $n_{k+1} > n_k$ such that $|a_{n_k} - a| \geq \epsilon$. Since (a_{n_k}) is a subsequence of (a_n), we have $n_k \geq k$ for all $k \in \mathbb{N}$. This implies that

   $|a_{n_k} - a| \geq \epsilon$ for all $k \in \mathbb{N}$. Therefore, (a_{n_k}) does not converge to a, which is a contradiction.

Thus, we have shown that (a_n) converges to a number $a \in \mathbb{R}$ if and only if every subsequence of (a_n) converges to a.