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1 Question 1

Prove that $\lim_{p\to\infty} \|f\|_{L^p(\mathbb{R})} = \|f\|_{C^0(\mathbb{R})}$, assuming f is such that both norms exist. **Proof:**

$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}$$

As $p \to \infty$,

$$\left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p} \approx \left(\sup_{x\in\mathbb{R}} |f(x)|^p \right)^{1/p} = \sup_{x\in\mathbb{R}} |f(x)| = \|f\|_{C^0(\mathbb{R})}$$

Thus,

$$\lim_{p\to\infty} \|f\|_{L^p(\mathbb{R})} = \|f\|_{C^0(\mathbb{R})}$$

2 Question 2

Prove that if $E \subseteq \mathbb{R}$ is connected, then so is E^0. Prove that this is false if you replace $\mathbb{R}$ with $\mathbb{R}^2$.
Proposition 1: If $E \subseteq \mathbb{R}$ is connected, then so is E^0.

Proof: Assume $E \subseteq \mathbb{R}$ is connected. To prove E^0 is connected, assume the contrary, i.e., suppose E^0 is not connected. Then there exist disjoint non-empty open sets U and V in $\mathbb{R}$ such that

$$E^0 = (U \cap E^0) \cup (V \cap E^0)$$

and

$$(U \cap E^0) \cap (V \cap E^0) = \emptyset.$$

Since U and V are open and $E^0 \subseteq E \subseteq \mathbb{R}$, we have that $U \cap E$ and $V \cap E$ are disjoint open sets in E. Moreover,

$$E = (U \cap E) \cup (V \cap E).$$

Given that E is connected and we have a decomposition of E into two disjoint non-empty open sets, this leads to a contradiction. Therefore, our initial assumption that E^0 is not connected must be false. Hence, E^0 is connected.

Proposition 2: The statement is false if you replace $\mathbb{R}$ with $\mathbb{R}^2$.

Proof: Consider $E \subseteq \mathbb{R}^2$ as the set defined by

$$E = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

E is connected because it is the closed unit disk in $\mathbb{R}^2$.

Now consider E^0:

$$E^0 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

E^0 is the open unit disk in $\mathbb{R}^2$, which is also connected.

To show that the statement is false in general, consider $E \subseteq \mathbb{R}^2$ defined by

$$E = \{(x, y) \in \mathbb{R}^2 \mid y = 0 \text{ or } y = 1\}.$$

E is connected since it is the union of two horizontal lines which are connected in $\mathbb{R}$, making E connected in $\mathbb{R}^2$.

However, E^0 is given by:

$$E^0 = \emptyset,$$

because there are no interior points in E in $\mathbb{R}^2$. The empty set is not connected, hence showing that E^0 is not connected in $\mathbb{R}^2$. Thus, the statement is false in $\mathbb{R}^2$.

3 Question 3

Suppose $E \subseteq \mathbb{R}^n$ is connected, and $f : E \to \mathbb{R}$ is continuous. Suppose also that there exist $a, b \in E$ such that $f(a) \neq f(b)$, and $y \in \mathbb{R}$ is such that $f(a) \leq y \leq f(b)$. Prove that there exists $c \in E$ such that $f(c) = y$.

Proof:

Since f is continuous on the connected set E, the image $f(E) \subseteq \mathbb{R}$ is an interval (by the intermediate value theorem).

Given $a, b \in E$ such that $f(a) \leq y \leq f(b)$ and $f(a) \neq f(b)$, without loss of generality, assume $f(a) < f(b)$.

Define the sets:

$$A = \{x \in E \mid f(x) \leq y\}, \quad B = \{x \in E \mid f(x) \geq y\}.$$

Both A and B are non-empty since $a \in A$ and $b \in B$.

Also, A and B are closed in E because f is continuous.

Since E is connected, $A \cap B \neq \emptyset$.

Therefore, there exists $c \in E$ such that $c \in A \cap B$.

Thus, $f(c) = y$

4 qestion 4

Definition: Let $n \in \mathbb{N}$. A subset $E \subseteq \mathbb{R}^n$ is sequentially compact if every sequence $\{x_k\}$ in E has a convergent subsequence whose limit is in E.

1. **1:** Every compact set is sequentially compact.

   Proof: Let $E \subseteq \mathbb{R}^n$ be compact. Consider any sequence $\{x_k\}$ in E. Since E is compact, E is both closed and bounded. By the Bolzano-Weierstrass theorem, every bounded sequence in

$\mathbb{R}^n$ has a convergent subsequence. Therefore, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ that converges to some limit $x \in \mathbb{R}^n$. Since E is closed, $x \in E$. Thus, E is sequentially compact.

2. **2:** Every sequentially compact set is closed and bounded.

   Proof: Let $E \subseteq \mathbb{R}^n$ be sequentially compact. To show that E is bounded, assume the contrary. Suppose E is unbounded. Then there exists a sequence $\{x_k\}$ in E such that $\|x_k\| \to \infty$ as $k \to \infty$. Since E is sequentially compact, there exists a convergent subsequence $\{x_{k_j}\}$ with limit $L \in E$. However, $\|x_{k_j}\| \to \infty$ contradicts the boundedness of the convergent subsequence. Therefore, E must be bounded.

   To show that E is closed, let $\{x_k\}$ be a sequence in E converging to some $x \in \mathbb{R}^n$. Since E is sequentially compact, there exists a convergent subsequence $\{x_{k_j}\}$ with limit $L \in E$. Since $\{x_k\}$ converges to x and any subsequence of a convergent sequence converges to the same limit, we have $L = x$. Thus, $x \in E$, and E is closed.

3. **3:** $E \subseteq \mathbb{R}^n$ is sequentially compact if and only if E is compact.

   Proof: ($\Rightarrow$) Assume $E \subseteq \mathbb{R}^n$ is sequentially compact. By Proposition 2, E is closed and bounded. Since $\mathbb{R}^n$ is a metric space, the Heine-Borel theorem states that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Therefore, E is compact.

   ($\Leftarrow$) Assume $E \subseteq \mathbb{R}^n$ is compact. By Proposition 1, every compact set is sequentially compact. Thus, E is sequentially compact.

5 question 5

Question 5: Let X and Y be metric spaces, $E \subseteq X$, and $f : X \to Y$.

1. **Proposition:** If f is uniformly continuous on E and $\{x_n\} \subseteq E$ is Cauchy in X, then $\{f(x_n)\}$ is Cauchy in Y.

   Proof:

   $$f \text{ uniformly continuous on } E \implies \forall \epsilon > 0, \exists \delta > 0 : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

   $$\begin{aligned} \{x_n\} \text{ Cauchy in } X &\implies \forall \epsilon > 0, \exists N \in \mathbb{N} : \forall m, n \geq N, d_X(x_m, x_n) < \delta. \\ &\implies d_Y(f(x_m), f(x_n)) < \epsilon. \\ &\therefore \{f(x_n)\} \text{ is Cauchy in } Y. \end{aligned}$$

2. **Proposition:** Suppose D is a dense subspace of X ($D \subset X$ and $\overline{D} = X$). If Y is complete and $f : D \to Y$ is uniformly continuous on D, then f has a continuous extension to X.

   Proof:

   $$\text{Define } g : X \to Y \text{ by } g(x) = \lim_{n \to \infty} f(x_n),$$

   where $\{x_n\} \subseteq D$ and $x_n \to x$.

   To show g is well-defined, let $\{x_n\}$ and $\{y_n\}$ be sequences in D such that $x_n \to x$ and $y_n \to x$.

   $$f \text{ uniformly continuous on } D \implies \forall \epsilon > 0, \exists \delta > 0 : d_X(u, v) < \delta \implies d_Y(f(u), f(v)) < \epsilon.$$

$$x_n \to x \text{ and } y_n \to x \implies \exists N \in \mathbb{N} : \forall n \geq N, d_X(x_n, x) < \frac{\delta}{2} \text{ and } d_X(y_n, x) < \frac{\delta}{2}.$$

$$\implies d_X(x_n, y_n) \leq d_X(x_n, x) + d_X(x, y_n) < \delta.$$

$$\implies d_Y(f(x_n), f(y_n)) < \epsilon.$$

$$\therefore \{f(x_n)\} \text{ and } \{f(y_n)\} \text{ are Cauchy in } Y.$$

Since Y is complete, $\{f(x_n)\}$ and $\{f(y_n)\}$ converge to the same limit, i.e., g is well-defined.

To show g is continuous, let $x_k \to x$ in X. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $u, v \in D$,

$$d_X(u, v) < \delta \implies d_Y(f(u), f(v)) < \epsilon.$$

Choose $u = x_k$ and $v = x$,

$$d_X(x_k, x) < \delta \implies d_Y(g(x_k), g(x)) < \epsilon.$$

Thus, g is continuous.