LECTURE 3 (MAY 20)

Comparison Theorem, Triangle Inequalities, Bolzao Weirstrass Thoerem (TEXTBOOK pg 2 -)

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1 Comparison Theorem

Theorem 1.1 (Comparison Theorem)**.** if (a_n), (b_n) are sequences of real numbers such that $a_n \leq b_n$ for all n, then

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

example: let $k \in \mathbb{N}$, which we will define (a_n), s.t $a_{n+k} - a_n = c \in \mathbb{R}$. Prove that $lim_{n\to\infty} \frac{a_n}{n} = \frac{c}{k}$. (hint examin case k = 1 first.)

Let $\epsilon > 0, \exists N \in \mathbb{N}$ s.t $\forall n \geq N \implies |a_{n+1} - a_n - c| < \epsilon$. we cwant to examin $\frac{a_n}{n}$ c

$$|\frac{a_{n+1}}{n} - \frac{a_n}{n} - \frac{c}{n}| < \frac{\epsilon}{n}$$

want to exmine divergent sequences: we say $(a+n)_{n\geq 1}$ diverges to ∞ written as $a_n \to \infty$ if $forall M \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t $a_n > M$ for all $n \geq N$.

Proof 1.2.

$$
\begin{aligned}
an+1-an&=c\\
\to a_{n+2}-a_{n+1}&=c\\
a_{n+2}-c-a_n&=c\\
a_{n+2}&=a_n+2c\\
&\vdots\\
a_{n+k}&=a_n+kc
\end{aligned}
$$

for $k \geq 1$, and $forall n \geq N$, we examin $|a_{N+k} - a_N - kc| < k\epsilon$

$$= |(a_{N+k} - a_{N+a-1}) + (a_{N+k-1} - a_{N+k-2}) + \cdots + (a_{N+1} - a_N) - kc| < k\epsilon$$

by the giga triangle inequality this is less than equal to

$$\leq |a_{N+k} - a_{N+k-1} - c| + |a_{N+k-1} - a_{N+k-2} - c| + \cdots + |a_{N+1} - a_N - c| < k\epsilon$$

$\forall k \geq 1 \longrightarrow |\frac{a_{N+k}}{N+k} - \frac{a_N}{N} - c\frac{k}{N+k}| < k\frac{\epsilon}{N+k}$ and we take limit as $k \to \infty$
$\forall \delta > 0, \exists M$ s.t. $forall k \geq M$

$$
\begin{aligned}
|\frac{a_N}{N+k}| &< \delta|\\
|c\frac{k}{N+k} - c| &< \delta\\
|k\frac{\epsilon}{N+k} - \epsilon| &< \delta
\end{aligned}
$$

note that $|a| - |b| \leq |a - b|$

$$
\begin{aligned}
&\leq |(b_{N+k} - c) - \frac{a_N}{N+k} - (\frac{ck}{N+k} - c)| < \frac{\epsilon k}{N+k} \cdots (1*)\\
|b_{N+k} - c| - |\frac{a_n}{N+k} - (\frac{ck}{N+k} - c)| &< \frac{\epsilon k}{N+k}
\end{aligned}
$$

$$(1*) \to |b_{N+k} - c| < \frac{\epsilon k}{N+k} + \frac{a_N}{N+k} + \frac{ck}{N+k} - c \leq \epsilon + 3\delta$$

1.1 Bolzeno - weirstrass Tbeorem

Theorem 1.3 (Bolzeno - Weirstrass Theorem)**.** if $(a_n)_{n\geq 1}$ is a sequence in $\mathbb{R}$ s.t. (a_n) is bounded, then $\exists$ a convergent subsequence.

Definition 1.1 (Subsequence). $(b_n)_{n\geq 1}$ is a subsequence of $(a_n)_{n\geq 1}$ if $\exists$ a strictly increasing sequence of natural numbers $(n_k)_{k\geq 1}$ s.t. $b_k = a_{n_k}$ for all $k \geq 1$.

Definition 1.2. a sequence (a_n) is monotone (stricly) increaseing if $a_{n+1} \geq a_n$ for all $n \geq 1$ (strictly increasing if $a_{n+1} > a_n$ for all $n \geq 1$).

it is useful for the monotone convergence theorem.

Theorem 1.4 (Monotone Convergence Theorem). if $(a_n)_{n\geq 1}$ is a monotone increasing sequence in $\mathbb{R}$ that is **bounded above**, then (a_n) has a finite limit. (convergence)

Consider set $E = a_1, a_2, \ldots$, and E is bounded above, then $sup(E)$ exists. We will define $a = \sup(E)$
Let $\epsilon > 0$ and a is the supremum implies that $\exists a_N \in E$ s.t. $a - \epsilon < a_N \leq a$
in order to prove this we want to end up with the equation $-\epsilon < a - a_N < 0 < \epsilon$

$$a - \epsilon \leq a_n \leq a,\ \forall n \geq N \text{ (as } a_n \text{ is increasing)}$$
$$\rightarrow |a_n - a| \leq \epsilon,\ \forall n \geq N \text{ (as } a_n \text{ is increasing)}$$

2 Proof of Bolzao -Weirstrass Theorem

We will prove the theorem until we need a lemma and we will then introduce couple of lemmas. WE can have some simple observations. First, observe that if $E = A \cup B$ (This means this is a disjoint union). And (a_n) has values in E which implies that $a_n \in E | \forall n$

$$\exists M \in \mathbb{R} \text{ s.t. } |a_n| \leq M,\ \forall n \geq 1$$

which implies that a_n must be in A or initinite numbers in B: if not, only fininte number of terms in A and B which is a contradiction.

$$I_o = [-M, M]$$

observe that $I_o = [-M, 0] \cup [0, M] \rightarrow$ there is infinite nuber of terms in one of these call that I_1. Then $I_1 - = [-M, 0]$ or $[0, M]$.
Inductively obtain I_n: $I_{n+1} = I_n -$ finite number of terms, I_{n+1} is a closed interval.
$M(I_n) = \dfrac{2M}{2^n}$ (M here means 'the measure of')
for each n, we can take an element of $\{a_1, \ldots\}$ which is in I_n which satisfies:

1. $a_{n_m} \in I_M$
2. $n_m > n_{m-1}$

Consider a subsequence $(a_{n_m})_{m\geq 1}$ s.t. $a_{n_m} \in I_m$ for all $m \geq 1$. Claim: $(a_{n_m})_{m\geq 1}$ is a Cauchy sequence (converges).

Definition 2.1 (Cauchy Sequence)**.** $(a_n)_{n\geq 1}$ is a Cauchy sequence if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_n - a_m| < \epsilon$ for all $n, m \geq N$.

Here is an important lemma for the Bolzab - Weirstrass

Lemma 2.1 (Nested Interval Property)**.** If (I_n) is a sequence of Intervals such that

1. $I_{n+1} \subset I_n$ for all $n \geq 1$
2. I_n is bdd
3. I_n if closed

$$\rightarrow \bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$$

More over if $\lim_{n\to\infty} M(I_n) = \lim_{n\to\infty}(b_n - a_n) = 0$ then $\bigcap_{n\in\mathbb{N}} I_n = \{c\}$ where c is a single point.

$$\exists a \in \bigcap_{n\in\mathbb{N}} I_n$$

$$\text{consider } |a_{n_m} - a| \leq \frac{2M}{2^m}$$

$$a \in I_m, \text{ and } a_{n_m} \in I_m, \ \forall m \geq 1$$