LECTURE NUMBER (DATE)

Author
Tom Jeong

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1 Completion

$R \approxeq$ Completion of Q
w.r.t $e/\sim = \{[(a_n)] : a_n \in e\}$ where $e = \{(a_n) : a_n \in Q\}$ and (a_n) is cauchy, i.e. $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $n, m \geq N \implies |a_n - a_m| < \epsilon$
we want to show
1. (a_n) is cauchy $\forall a \in \mathbb{R} \to a \cdot (a_n)$ is also cauchy. What does it mean to multiply a number to a sequence? it means to multiply a number to each element of the sequence. i.e. $(a \cdot a_n)$
2. (a_n) and (b_n) are cauchy $\to (a_n + b_n)$ is also cauchy.
3. if $(a_n) + (b_n)$ is cauchy, then $(a_n)(b_n)$ are cauchy.

Proof 1.1 (proof of 2, 3)**.** (a_n) and (b_n) are cauchy $\to (a_n + b_n)$ is also cauchy.
let $\epsilon > 0, \exists N$ such that $\forall n_1, n_2, m_1, m_2 > N \implies |a_{n_1} - a_{m_1}| < \epsilon$ and $|b_{n_2} - b_{m_2}| < \epsilon$

w.t.s $\exists M$ such that $\forall n_3, m_3 > M \implies |(a_{n_3} + b_{n_3}) - (a_{m_3} + b_{m_3})| < \epsilon$
(using triangle inequality we can pair a's and b's together)
$|(a_{n_3} + b_{n_3}) - (a_{m_3} + b_{m_3})| = |(a_{n_3} - a_{m_3}) + (b_{n_3} - b_{m_3})| \leq |a_{n_3} - a_{m_3}| + |b_{n_3} - b_{m_3}| < 2\epsilon$

3. WTS $\exists M$ such that $\forall n_3, m_3 > M \implies |(a_{n_3} \cdot b_{n_3}) - (a_{m_3} \cdot b_{m_3})| < \epsilon$
$= |a_{n_3} \cdot b_{n_3} - a_{m_3} \cdot b_{m_3}| = |a_{n_3} \cdot b_{n_3} - a_{n_3} \cdot b_{m_3} + a_{n_3} \cdot b_{m_3} - a_{m_3} \cdot b_{m_3}|$ (we add an subtract)
$= |a_{n_3} \cdot (b_{n_3} - b_{m_3}) + b_{m_3} \cdot (a_{n_3} - a_{m_3})|$ (... we need a lemma) using lemma 1.2 we see that the sequence converges

Lemma 1.2. if (a_n) is cauchy, then the sequence (a_n) is bounded.

$$\forall \epsilon_0 > 0, \exists N \text{ such that } \forall n, m > N \implies |a_n - a_m| < \epsilon$$

in particular,if we look at $|a_n - a_N| < \epsilon$ for $n \geq N$

$$-\epsilon_0 < a_n - a_N < \epsilon_0 \implies a_N - \epsilon < a_n < \epsilon + a_N$$

?? reverse inequality: $|a_n| - |a_N| < |a_n - a_N| < \epsilon_0 \implies |a_n| < |a_N| + \epsilon_0$ we see that the sequence is bounded by some number; in mathematical terms M such that $\forall n \in \mathbb{N} \implies |a_n| < M$ we can choose $M = \max\{|a_1|, |a_2|, \ldots, |a_N| + \epsilon_0\}$

Definition 1.1. define

$$[(b_n)][(a_n)] = [(a_n \cdot b_n)]$$

$$[(a_n)]([(b_n)] + [c_n]) = [(a_n)] \cdot [(b_n + c_n)] = [(a_n \cdot b_n + a_n \cdot c_n)]$$

Theorem 1.3.

$$(a_n) \subseteq \mathbb{R}$$

cauchy $\longleftrightarrow$ (a_n) converges (there is a specific number that the sequence converges to) $\exists a$ such that $\forall \epsilon \exists N$ such that $\forall n \geq N \implies |a_n - a| < \epsilon$

Proof. (a_n) bounded, due to B.W Theorem. there exists a convergence subsequence. consider $(a_{n_j})_{j \geq 1}$ then a such that $\forall \epsilon > 0, \exists J \in \mathbb{N}$ such that $\forall j \geq J \implies |a_{n_j} - a| < \epsilon$
$\exists N$, such that $\forall n, m \geq N \implies |a_n - a_m| < \epsilon$ we pick j large enough such that $j \geq J$ and $n_j \geq N$ we know that $|a_{n_j} - a| < \epsilon$ and $|a_{n_{j+1}} - a| < \epsilon$ we can use the triangle inequality to show that $|a_n - a| < 2\epsilon$ □

2 liminf and limsup

they are combination of words limit infimum and limit supremum.

Definition 2.1 (Limsup)**.** given $(a_n)_{n\geq 1}$ define $limsup(a_n)$ which is not the same concept as $\sup\{a_n\}$ they are not equal in general. $limsup(a_n) = \sup\{inf\{a_{n_j} : a_{n_j}$ is a subsequence of $(a_n)\}\}$ (a_n) is bounded then $= \sup\{\lim_{n\to\infty}\sup\{a_{n_j}\}\}$ thus $= \limsup\{a_n\} = \lim_{n\to\infty}\{\sup\{a_m\}\}$ $m \geq N$

Definition 2.2 (Liminf)**.** $liminf(a_n) = \inf\{\sup\{a_{n_j} : a_{n_j}$ is a subsequence of $(a_n)\}\}$ (a_n) is bounded then $= \inf\{\lim_{n\to\infty}\inf\{a_{n_j}\}\}$ thus $= \liminf\{a_n\} = \lim_{n\to\infty}\{\inf\{a_m\}\}$ $m \geq N$

Definition 2.3 (Cluster Point)**.** a is a cluster point of $(a_n)_{n\geq 1}$ if $\exists$ subsequence $(a_{n_j})_{j\geq 1}$ such that $\lim_{j\to\infty} a_{n_j} = a$

Here is an example, make a subsequence with cluster points 1, 2, 3, and ∞

$$a_n = \begin{cases} 1 & \text{if } n = 1 \mod 4 \\ 2 & \text{if } n = 2 \mod 4 \\ 3 & \text{if } n = 3 \mod 4 \\ n & \text{if } n = 0 \mod 4 \end{cases}$$

then $lim\sup(a_n) = \infty$ and $lim\inf(a_n) = 1$

Theorem 2.1. $\forall (a_n)_{n\geq 1}$ cluster points of (a_n) are in the interval $[\liminf(a_n), \limsup(a_n)]$

Proof. Recall the definiition, $a_n = liminf(a_n) = \inf\{\sup\{a_{n_j} : a_{n_j}$ is a subsequence of $(a_n)\}\}$ and , $\bar{a_n} = limsup(a_n) = \sup\{\inf\{a_{n_j} : a_{n_j}$ is a subsequence of $(a_n)\}\}$ this means $a_n = inf\{\sup\{a_{n_j} : a_{n_j}$ is a subsequence of $(a_n)\} \leq \sup\{a_{n_j} : a_{n_j}$ is a subsequence of $(a_n)\} \leq \sup\{a_m\}$ for $m \geq N$ and $a_n \leq \sup\{a_m\}$ for $m \geq N$ let a, be a cluster point $\implies \exists$ subsequence $(a_{n_j})_{j\geq 1}$ s.t. $\lim_{j\to\infty} a_{n_j} = a$ $\square$

Corollary: if $lim\sup(a_n) = lim\inf(a_n)$ then (a_n) converges. in this case all three are equal