LECTURE 2 (MAY 15)

supremums, completeness, approximation theorem

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Contents

1 $\mathbb{R}$ is Complete

Definition 1.1.

$$\forall E \subseteq \mathbb{R}$$

$$E \neq \emptyset \text{ and E is bounded above } \rightarrow E \text{ has a finite supremum}$$

A set E is bounded above if there exists a number M such that $x \leq M$ for all $x \in E$. The supremum of E is the least upper bound of E.

$$\exists M \in \mathbb{R} \text{ such that } \forall e \in E, e < M$$

Definition 1.2. A set E is complete if every nonempty subset of E that is bounded above has a supremum in E.

If E is bdd above, $\rightarrow Sup(E) =$ smallest upper bound of E.

1.1 prove the supremum of [0,1] is 1

$[0, 1] = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$

Proof 1.1. w.t.s. 1 is an upperbound, and 1 is smaller than any other upper bound.

1. 1 is an upperbound: this is true by definition of the interval [0, 1]
2. let M be an upper bound of [0, 1].

$$\begin{aligned}
&\text{let } M \text{ be an upper bound of [0, 1]} \\
&\rightarrow \forall x \in [0, 1], x \leq M \\
&\rightarrow 1 \leq M \\
&\rightarrow 1 \text{ is the smallest upper bound of [0, 1]}
\end{aligned}$$

1.2 prove the supremum of [0,1) is 1

$[0, 1) = \{x \in \mathbb{R} | 0 \leq x < 1\}$

Proof 1.2. w.t.s. 1 is an upperbound, and 1 is smaller than any other upper bound.

1. 1 is an upperbound: this is true by definition of the interval [0, 1)

2. let M be an upper bound of [0, 1).

$$\begin{aligned}
&\text{let } M \text{ be an upper bound of } [0,\,1)\\
&x \in [0,1) \to x \le M\\
&\text{suppose } M < 1 \text{ (in particular } M \in [0,1))\\
&1 - M > 0 \text{ (we take } 0 < \epsilon < 1 - M)\\
&\to \exists \epsilon > 0 \text{ such that } M + \epsilon < 1\\
&\to M + \epsilon \in [0,1)\\
&\to \text{ then } M + \epsilon < M (\text{contradiction})
\end{aligned}$$

1.3 Completeness of $\mathbb{R}$

Theorem 1.3. $\mathbb{R}$ is complete.

$\forall E \subseteq \mathbb{R}$, E bdd above and nonempty $\to$ E has a supremum in $\mathbb{R}$.
$\mathbb{Q}$ does not have this property e.g. $E = \{x | x \in \mathbb{Q} | x^2 < 2\}$ in $\mathbb{Q}$, E is bdd above, but E does not have a supremum in $\mathbb{Q}$.

1.4 Prove that $\sqrt{2}$ is not rational

Proof 1.4.

$$\begin{aligned}
&\text{assume that } \sqrt{2} \text{ is rational}\\
&\to \sqrt{2} = \frac{a}{b} \text{ where } a \text{ and } b \text{ are integers that are relatively prime}\\
&\to 2 = \frac{a^2}{b^2}\\
&\to 2b^2 = a^2\\
&\to a^2 \text{ is even}\\
&\to a \text{ is even}\\
&\to a = 2k \text{ where } k \text{ is an integer}\\
&\to 2b^2 = 4k^2\\
&\to b^2 = 2k^2\\
&\to b^2 \text{ is even}\\
&\to b \text{ is even}\\
&\to a \text{ and } b \text{ are both even}\\
&\to \text{they are not relatively prime: contradiction}
\end{aligned}$$

1.5 Approximation theorem for supremums

Theorem 1.5 (approximation for supremums)**.** Let $E \subseteq \mathbb{R}$ be nonempty and bounded above.
Let $\sup(E) = s$
$\to \forall \epsilon > 0, \exists e \in E$ such that $s - \epsilon \leq e \leq s$

Proof 1.6 (proof of Theorem 1.5)**.** we will use proof by inversion

$$\begin{aligned}
&\text{suppose not}\\
&\to \exists \epsilon > 0 \text{ such that } \forall e \in E, e < s - \epsilon\\
&\to \forall e \in E, e < s - \epsilon\\
&\to s - \epsilon \text{ is an upper bound of E}\\
&\Rightarrow\!\Leftarrow
\end{aligned}$$

Note: if you take $\epsilon = \frac{1}{2^j}$ and $s = \sup(E)$, the approximation theorem says that:
$\forall \epsilon_j > 0, \exists e_j \in E$ such that $s - \epsilon_j \leq e_j \leq s$.
(e_j) has a limit: $\lim_{j \to \infty} e_j = s$ (more on this later)

1.6 Archimedian Principle

Lemma 1.7. If $F \subseteq \mathbb{N}(\subseteq \mathbb{R})$ has a supremum: $s = \sup(F) \to s \in F$

Proof 1.8. $s = \sup(E) \in \mathbb{N} : (s+1)a > br$
$s = \sup(F)$ let $0 < \epsilon < 1$
by approximation theorem: $\exists f \in F$ such that $s - \epsilon \leq f \leq s$
suppose for contradiction $s \notin F \; ifs \notin F$ then $f < s$
if $f = s$ then $s \in F$
$s - \epsilon < f \leq s$

Theorem 1.9 (Archimedian Principle). given $a, b \in \mathbb{R}_{>0}$, $a > 0$, $\exists n \in \mathbb{N}$ such that $na > b$

Proof 1.10. either $a > b$ or $b < a$
Case 1: $a > b$

$$\text{let } n = 1$$
$$\rightarrow 1a > b$$

Case 2: $a < b$
$1 < \frac{b}{a}$

Consider $E = \{na | n \in \mathbb{N} \wedge na < b\}$

Q: does E have a supremem (i.e does E have a least upper bound)?

can you find $M \in \mathbb{R}$ s.t. $\forall na \in E, na < M$

yes take M = b

$\rightarrow$ E is bounded above.

$\rightarrow$ E has a supremum call it s $s = \sup(E)$

let $\epsilon > 0$ by the **approximation theorem**,

$\exists na \in \mathbb{E}$ such that $s - \epsilon \leq na \leq s = b$

take ϵ very small, and consider

$\rightarrow a(n_\epsilon + 1) > b$ we want $an_\epsilon + a > b$

suppoe for any $\epsilon > 0, an_\epsilon + a \leq b$

$\rightarrow s + a - \epsilon \leq an_\epsilon + a \leq b$

$\rightarrow s + a = \epsilon \leq b$

$\rightarrow s + a - b \leq 0$

$\rightarrow$ if $s + a - b > 0$by taking $\epsilon <<< 1$ contradiction

$\because$ s is a supremum of E: $na < s \leq b$

$\forall n \in \mathbb{N}$ such that $na < s \leq b$

$\rightarrow 0 < (n + 1)a - b < s + a - b \leq a$