LECTURE 5

Cosets

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Definition 1.1. Let H be a subgroup of G and $g \in G$ then the subset $gH = \{gh|h \in H\}$ This is called the left coset of H containing g

We de note the set of left cosets of$H$ by G/H that means $G/H = \{gH|\forall g \in G\}$

Definition 1.2. similarly we define the right coset of H containing g as $Hg = \{hg|h \in H\}$

Note if G is abelian then $gH = Hg \forall g \in G, H \subseteq G$
e.g. $G = (\mathbb{Z}, +)$ and $H = 3\mathbb{Z}$ we have

1. $0 + 3\mathbb{Z}$
2. $1 + 3\mathbb{Z}$
3. $2 + 3\mathbb{Z}$

Then $G/H = \mathbb{Z}/3\mathbb{Z} = \{0 + 3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}\}$
e.g. $G = \mathbb{Z}/6\mathbb{Z}$ and $H = \{[0], [3]\}$
$[0] + H = \{[0], [3]\}$,
$[1] + H = \{[1], [4]\}$,
$[2] + H = \{[2], [5]\}$,
$[3] + H = \{[0], [3]\}$, same thing
$[4] + H = \{[1], [4]\}$, same thing
$[5] + H = \{[2], [5]\}$ same thing
so $G/H = \{[0] + H, [1] + H, [2] + H\}$ (all the example is abelian group)

now lets look at non abelian group $G = D_3$ then
$H = \{e, s_3\}$ left cosets
$eH = \{e, s_3\} = s_3H$
$r_1H = \{r_1, s_2\} = s_2H$
$r_2H = \{r_2, s_1\} = r_1H$

Right cosets
$He = \{e, s_3\} = Hs_3$
$Hr_1 = \{r_1, s_2\} = Hs_2$
$Hr_2 = \{r_2, s_1\} = Hs_1$
NOTE: $D_3/H = \{eH, r_1H, r_2H\} \neq H\backslash D_3 = \{He, Hr_1, Hr_2\}$
exercise: find a group G and a subgroup H such that $G/H = H\backslash G$

Lemma 1.1 (2.2.6)**.** let H be a subgroup of G and $x, y \in G$; Then

1. $x \in xH$
2. $xH = yH$ if and only if $x^{-1}y \in H$
3. $xH \cap yH \neq \emptyset$ if and only if $xH = yH$ (intersection empty if xH not equal yH)
4. The map $phi : H \to xH$ given by $\phi(h) = xh$ is a bijection

Proof.

1. H is a subgroup so $e \in H$ and $x = xe$ so $x \in xH$
2. (a) $(\to)$ if $xH = yH$ then $y\ inxH$ that is, for some $h \in H, y = xh$ So Then $x^{-1}y = h \in H$
   (b) $(\leftarrow)$ if $x^{-1}y \in H$ then $y = x(x^{-1}y) \in xH$ so $yH \subseteq xH$ similarly $xH \subseteq yH$ so $xH = yH$
3. Suppose $g \in xH \cap yH$ Then $g = xh_1 = yh_2$ for some $h_1, h_2 \in H$ so $x = yh_2h_1^{-1} \in yH$ so $xH \subseteq yH$ similarly $yH \subseteq xH$ so $xH = yH$
4. $\phi : H \to xH, \phi(h) = xh$ is a multiplication on G restriction to H so its a bijection.

□

<u>Observe</u> The set of cosets of a subgroup H forms a partition on G. Cor(2.27) Then $G = \bigcup_{g \in G} gH$ and $gH \cap g'H = \emptyset$ if $g \neq g'$

Theorem 1.2 (lagrange)**.**
If H is a subgroup of a finite group G then $|G| = |G/H||H|$ sometimes $|G/H|$ is notated with $[G : H]$
in English, the order of a subgoroup divides the order of the group

Proof. let $gH \in G/H$ by the lemma 2.2.6 there is a bijection $\phi : H \to gH, \phi(h) = gh$ so $|H| = |gH|$
Sicne the set of left cosets of H forms a partition of G, and also each coset has the same number of elements, we have $|G| = |G/H||H|$

□

Definition 1.3 (2.2.9)**.**
The numnber of cosets $|G/H|$ is called the index of H in G and is denoted by $[G : H]$

Examples: Consider $2\mathbb{Z}$ as a subgroup of $\mathbb{Z}$ then $|\mathbb{Z}| = 2|2\mathbb{Z}|$ so $|\mathbb{Z}/2\mathbb{Z}| = 2$ so $[\mathbb{Z} : 2\mathbb{Z}] = 2$ generalizing this consider $n\mathbb{Z}$ as a subgroup of $\mathbb{Z}$ then $[\mathbb{Z} : n\mathbb{Z}] = n$

Very important questions :When does G/H form a group ??? (ex $G = \mathbb{Z}$ and $H = n\mathbb{Z}$)

Given $X, Y \subseteq G$ defin $XY = \{xy | x \in X, y \in Y\}$
and given left cosets xH and gH is $(xH)(gH) = (xg)H$?
Consider $H = \{e, s_3\}$ in D_3 then $r_1H = \{r_1, s_2\}$ and $s_2H = \{s_2, r_1\}$ so $(r_1H)(s_2H) = \{r_1s_2, r_1r_1, s_2s_2, s_2r_1\} = \{s_1, e, e, s_2\} = \{s_1, e, s_2\}$ but $(r_1s_2)H = \{r_1s_2, s_2s_2\} = \{s_1, e\}$ so $(r_1H)(s_2H) \neq (r_1s_2)H$ (DOESNT EVEN HAVE THE RIGHT NUMBER OF ELEMTNS LOL)

Proposition 1.3 (2.3.1)**.** Let H be a subgroup of G If $gH = Hg \forall g \in G$ then G/H is a group under the operation $(gH)(g'H) = (gg')H$

Proof.

1. $(xy)H \subseteq (xH)(yH)$ then $xyh = xeyh \in (xH)(yH)$
2. $(xH)(yH) \subseteq (xy)H$ then $xyh = x(yh) \in (xy)H$

□

Definition 1.4. A subgroup N of a group G is called normal if $gNg^{-1} = \{gng^{-1} | n \in N\} = N \forall g \in G$

Notice if G is abelian,$gNg^{-1} = gg^{-1}N = N$ so all subgroups of an abelian group are normal.
If N is a normal subgroup of G then we write $N \trianglelefteq G$

exercise $N \trianglelefteq G$ iff $gN = Ng \forall g \in G$
ex: $H = \{e, s_3\}$ in D_3 is NOT normal. $r_1H = \{r_1, s_2\}$ but $s_2H = \{s_2, r_1\}$ so $r_1H \neq s_2H$ so H is not normal.