LECTURE 7 SEP 11

Cosets

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Contents

1 Bijective homomorphism

A Bijective homomorphism is called an isomorphism.

Proposition 1.1 (2.4.9)**.**
Let $f : G \to K$ be a group homomorphism.

1. the image $Imf \subseteq K$ is a subgroup of K.
2. The kernel $Kerf \subseteq G$ is a normal subgroup of G.
3. f is injective if and only if $Kerf = \{e\}$

Proof continutes.
2. Normality Let $N = Kerf$ for every $g\ inG$ and $n \in N$, $gng^{-1} \in N$.

$$\begin{aligned} f(gng^{-1}) &= f(g)f(n)f(g)^{-1} & (1)\\ &= f(g)f(g)^{-1} & (2)\\ &= e & (3)\\ &\text{so} & (4)\\ gng^{-1} &\in Kerf = N \forall g \in G, n \in N & (5)\\ \text{Hence } gNg^{-1} &\subseteq N\ \forall g \in G & (6) \end{aligned}$$

3. $\to$ Since $f(e_G) = e_K$ and f is injective. $kerf = \{e_G\}$
$\leftarrow$ Suppose $kerf = \{e_G\}$ and $f(g) = f(h)$ then $f(gh^{-1}) = f(g)f(h)^{-1} = e$ so $gh^{-1} \in kerf = \{e_G\}$ so $g = h$ so f is injective.

Theorem 1.2 (2.5.1 Isomorphism Theorem)**.**
Leg G and K be groups and $f : G \to K$ a homomorphism with the kernel $Kerf = N$, then $\tilde{f} : G/N \to f(G)$ given by $\tilde{f}(gN) = f(g)$ is well defined and a group isomorphism.

Proof. Notice that $f(x) = f(y) \iff f(y)^{-1}f(x) = e_K \iff f(y^{-1}x) = e_K \iff y^{-1}x \in N \iff xN = yN$ so $\tilde{f}$ is well defined.

Recall lemma 2.2.6 -ii that $y^{-1}x \in N \iff xN = yN$ so $\tilde{f}$. Hence

$$f(x) = f(y) \iff xN = yN$$

$\to$ gives that $\tilde{f}$ is injective.
$\leftarrow$ give that well defined $\square$

Proposition 1.3 (2.4.9)**.**
states that $Kerf = N$ is normal so $\tilde{f}$ is a homomorphism since $\tilde{f}((g_1N)(g_2N)) = \tilde{f}(g_1g_2N) = f(g_1g_2) = f(g_1)f(g_2) = \tilde{f}(g_1N)\tilde{f}(g_2N)$
Now $\tilde{f}$ is surjective because f is surjective onto $f(G)$ so $\tilde{f}$ is an isomorphism.

example: D_3 is the symmetries of a triangle; define $f : D_3 \to \mathbb{Z}/2\mathbb{Z}$ by sending rotations (including e) to [0] and [1]
$Kerf = \{e, r_1, r_2\} = N$ so by Isomorphism Theorem, $\tilde{f} : D_3/N \to \mathbb{Z}/2\mathbb{Z}$ is an isomorphism.
$\{e, r_1, r_2\} = eN \rightarrowtail [0]$
$\{s_1, s_2, s_3\} = s_1N \rightarrowtail [1]$

Examples: det $(GL_2(\mathbb{R}), \cdot) \to (\mathbb{R}^*, \cdot)$ is a homomorphism. the kernel is the set of matrices with determinant 1.

$kerdet = \{A \in GL_2(\mathbb{R}) | detA = 1\} = SL_2(\mathbb{R})$ is a normal subgroup of $GL_2(\mathbb{R})$ and $SL_2(\mathbb{R}) \cong \mathbb{R}^*$

$$\tilde{det} : GL_2(\mathbb{R}/SL_2(\mathbb{R})) \to \mathbb{R}^*$$

is an isomorphism

Another example. Let $K = \{z \in \mathbb{C} || z| = 1\}$ be the unit circle in $\mathbb{C}$ and $G = \mathbb{R}$ be the group of real numbers under addition. $(K, \cdot)$ is a group and define $f : (\mathbb{R}, +) \to (K, \cdot)$ by $f(x) = e^{2\pi ix}$ By Isom Theorem

$$\mathbb{R}/2\pi\mathbb{Z} \cong K$$

Consider $g \in G, n \in \mathbb{Z}^+$, $g^n = \underbrace{g \cdot g \cdot \ldots \cdot g}_{n \text{ times}}$ and $g^{-n} = \underbrace{g^{-1} \cdot g^{-1} \cdot \ldots \cdot g^{-1}}_{n \text{ times}}$ so $g^n \cdot g^{-n} = e_G$

Proposition 1.4 (2.6.1)**.**
Let G be a group and $g \in G$
$f_g : \mathbb{Z} \to G$
$f_g(n) \rightarrowtail g^n$
is a homomorphism

Definition 1.1.
1. the image of f_g is the cyclic subgroup of G generated by g and is denoted $\langle g \rangle$
2. we will call —$\langle g \rangle$— the order of g and write $|g|$ or ord G
3. Alternativley the order of element $g \in G$ is the smallest posirie integer n such that $g^n = e$

<u>Note</u> If $g^n \neq e$ for only $n \in \mathbb{Z}^+$ then $|g| = \infty$

Definition 1.2. D_3 order of $r_1 = 3$ and order of $s_1 = 2$

Proposition 1.5 (2.6.3)**.**
Let G be a finite group and $g \in G$.

1. the order ord g of g divides the order of G
2. $g^{|G|} = e$
3. if $g^n = e$ then for some positive n, $ord(g)|n$

Proof.

1. $ord(g) = |\langle g \rangle|$ and BY Lagrange's Theorem $|G| = |\langle g \rangle| \cdot [G : \langle g \rangle]$ so $ord(g)||G|$
2. $g^{|G|} = g^{|\langle g \rangle| \cdot [G:\langle g \rangle]} = (g^{ord(g)})^{[G:\langle g \rangle]} = e^{[G:\langle g \rangle]} = e$
3. If $g^n = e$ then $n \in kerf_g = n_g\mathbb{Z}$ for some $n_g \neq 0$ so we have that $n_g = ord(g)$ so $ord(g)|n$

$\square$