LECTURE 1

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1 Relations

Let S be s a set (e.g. $\mathbb{N} = \{1, 2, 3, \dots\}$), integers: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
rational numbers $\mathbb{Q}$ i should know these ..
RECALL: cartesian product: A x $B = \{(a, b)|a \in A, b \in B\}$

Definition 1.1. A relation R on a set S is a subset $R \subseteq S \times S$. we write xRy (read "x is related to y") to denote an element $(x, y) \in R$

some examples of relations: "is orthogonal to" is a relation o the set of vectors in $\mathbb{R}^3$
another example "is congruent to" is a relation on the set of polygons
"divides" is a relation on the set of integers

WE say the relation R is

1. reflexive if xRx for all $x \in S$
2. symmetric if xRy implies yRx for all $x, y \in S$
3. antisymmetric if xRy and yRx implies $x = y$ for all $x, y \in S$
4. transitive if xRy and yRz implies xRz for all $x, y, z \in S$

(sometimes we note $\sim$ for the relation R)
going back to examples: "is congruent to" is reflexive, symmetric, and transitive

2 equivalence and partial order relations

Definition 2.1. two very important types of relations are:

1. equivalence relations
   (a) reflextive
   (b) symmetric
   (c) transitive
2. partial order relations
   (a) reflexive
   (b) antisymmetric
   (c) transitive

e.g. $R = \{(a, b)|b - a \in \mathbb{N}\} \subseteq \mathbb{Z} \times \mathbb{Z}$
is R an equivalence relation? a partial ordering ?

1. reflexive: aRa? yes since $a - a = 0 \in \mathbb{N}$
2. antisymmetric: aRb and bRa implies $a = b$? if aRb then $b - a \in \mathbb{N}$ and if bRa then $a - b \in \mathbb{N}$ so a and b are the same

3. transitive: aRb and bRc implies aRc? if aRb then $b-a \in \mathbb{N}$ and if bRc then $c-b \in \mathbb{N}$ so $c-a=(c-b)+(b-a) \in \mathbb{N}$

Exercise: check that aRb iff $a \leq b$

e.g. fix $n \in \mathbb{Z}^+$ $R=\{(a,b)|a-b \in n\mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$

Exercise this is an equivalece relation

aRb if $a-b=nk$ for some $k \in \mathbb{Z}$

we call this relation congruence module n

notation:if R is an equivalence relation, we often denote it by $\sim$

Definition 2.2 (equivalence class)**.** If $\sim$ is an equivalene relation on S we can fix and element $x \in S$ and consider all elements of S that is equivalent to x:

$$[x]=\{s \in S|s \sim x\} \subseteq S$$

and x is a representative of this equivalence class

e.g. $S=\{a,b,c,d,e\}$ and $\sim=\{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(b,a),(c,d),(d,c),(d,e),(e,d),(c,e),(e,c)\}$ $[a]=\{a,b\}=[b]$ and $[c]=\{c,d,e\}=[d]=[e]$

Q: are equivalence classes always disjoint?

A: yes, if $[x] \cap [y] \neq \emptyset$ then $[x]=[y]$

Lemma 2.1. Let $\sim$ be an equivalene relation on S and $x,y \in S$. Then $[x]=[y]$ iff $x \sim y$

proof of lemma 2.1. iff

$\rightarrow$: supose $[x]=[y]$ then $x \in [x]$, since $\sim$ is reflexive and so $x \in [y]$ Thus, $x \sim y$

$\leftarrow$: Let $s \in [x]$ since $x \sim x$ and $x \sim y$ (by assumption) then $s \sim y$ since it is transitive. hence $s \in [y]$ and so $[x] \subseteq [y]$

similarly, if $s \in [y]$, then $s \sim y$ and $x \sim y$ so since $\sim$ is symmetric and trasnstiive $s \sim x$ and so $xin[x]$ hence $[y] \subseteq [x]$ □

Corollary:

$$[x] \cap [y] = \emptyset \text{ if } [x] \neq [y]$$

Proof. suppose $[x] \cap [y] \neq \emptyset$ then there exists $s \in [x] \cap [y]$ then $s \sim x$ and $s \sim y$ by lemma 2.1 $x \sim y$ and so $[x]=[y]$ □

Definition 2.3. a partition on a set S is a collection $(S_i)_{i \in I}$ of nonempty subsets of S such that

1. $S=\bigcup_{i \in I} S_i$
2. $S_i \cap S_j = \emptyset$ for all $i,j \in I$ with $i \neq j$

e.g. $S = \{a, b, c, d, e\}$ and $S_1 = \{a, b\}$, $S_2 = \{c, d, e\}$ is a partition of S

Theorem 2.2. Let S be a set with an equivalece relation $\sim$: Then the set of equivalence classes $S/\sim = \{[x] | x \in S\}$ is a partition of S
conversely, if $(S_i)_{i \in I}$ is a partition of S then there exists an equivalence relation $\sim$ on S such that $S/\sim = (S_i)_{i \in I}$