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# LECTURE 7 SEP 11

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Cosets

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# 1 Bijective homomorphism

A Bijective homomorphism is called an isomorphism.

**Proposition 1.1** (2.4.9).

Let  $f : G \rightarrow K$  be a group homomorphism.

1. the image  $Imf \subseteq K$  is a subgroup of  $K$ .
2. The kernel  $Kerf \subseteq G$  is a normal subgroup of  $G$ .
3.  $f$  is injective if and only if  $Kerf = \{e\}$

Proof continues.

2. Normality Let  $N = Kerf$  for every  $g$  in  $G$  and  $n \in N$ ,  $gng^{-1} \in N$ .

$$f(gng^{-1}) = f(g)f(n)f(g)^{-1} \quad (1)$$

$$= f(g)f(g)^{-1} \quad (2)$$

$$= e \quad (3)$$

$$\text{so} \quad (4)$$

$$gng^{-1} \in Kerf = N \forall g \in G, n \in N \quad (5)$$

$$\text{Hence } gNg^{-1} \subseteq N \forall g \in G \quad (6)$$

3.  $\rightarrow$  Since  $f(e_G) = e_K$  and  $f$  is injective.  $kerf = \{e_G\}$

$\leftarrow$  Suppose  $kerf = \{e_G\}$  and  $f(g) = f(h)$  then  $f(gh^{-1}) = f(g)f(h)^{-1} = e$  so  $gh^{-1} \in kerf = \{e_G\}$  so  $g = h$  so  $f$  is injective.

**Theorem 1.2** (2.5.1 Isomorphism Theorem).

Let  $G$  and  $K$  be groups and  $f : G \rightarrow K$  a homomorphism with the kernel  $Kerf = N$ , then  $\tilde{f} : G/N \rightarrow f(G)$  given by  $\tilde{f}(gN) = f(g)$  is well defined and a group isomorphism.

*Proof.* Notice that  $f(x) = f(y) \iff f(y)^{-1}f(x) = e_K \iff f(y^{-1}x) = e_K \iff y^{-1}x \in N \iff xN = yN$  so  $\tilde{f}$  is well defined.

Recall lemma 2.2.6 -ii that  $y^{-1}x \in N \iff xN = yN$  so  $\tilde{f}$ . Hence

$$f(x) = f(y) \iff xN = yN$$

$\rightarrow$  gives that  $\tilde{f}$  is injective.

$\leftarrow$  give that well defined

□

**Proposition 1.3** (2.4.9).

states that  $\text{Ker } f = N$  is normal so  $\tilde{f}$  is a homomorphism since  $\tilde{f}((g_1N)(g_2N)) = \tilde{f}(g_1g_2N) = f(g_1g_2) = f(g_1)f(g_2) = \tilde{f}(g_1N)\tilde{f}(g_2N)$

Now  $\tilde{f}$  is surjective because  $f$  is surjective onto  $f(G)$  so  $\tilde{f}$  is an isomorphism.