11

Why are $\{[0]\}$ and $\mathbb{Z}/7\mathbb{Z}$ the only subgroups of $\mathbb{Z}/7\mathbb{Z}$?

Proof. Let $G = \mathbb{Z}/7\mathbb{Z}$. We know that G is cyclic, so it has a unique subgroup of order d for each d dividing 7. Since 7 is prime, the only divisors of 7 are 1 and 7. Thus, the only subgroups of G are $\{[0]\}$ and G. □

12

Show that a group G is not the union of two proper subgroups $H_1, H_2 \subset G$. Can a group be the union of three proper subgroups?

Proof. Assume for contradiction that $G = H_1 \cup H_2$, where H_1 and H_2 are proper subgroups of G.

Since H_1 and H_2 are proper subgroups, $H_1 \neq G$ and $H_2 \neq G$. Let $H_1 \cap H_2 = K$. Then K is a subgroup of both H_1 and H_2.

Since $H_1 \cup H_2 = G$, every element of G is in either H_1 or H_2.

Consider an element $g \in G$. If $g \in H_1 \cap H_2$, then $g \in H_1$ and $g \in H_2$.

If $H_1 \cap H_2$ is non-trivial, then it is a proper subgroup of G because H_1 and H_2 are proper. For $g \in H_1 \setminus (H_1 \cap H_2)$ and $g \in H_2 \setminus (H_1 \cap H_2)$, g cannot be fully covered by the union $H_1 \cup H_2$ unless one of the subgroups is G, which contradicts H_1 and H_2 being proper.

Thus, G cannot be the union of two proper subgroups.

Consider the symmetric group S_4. We can show that S_4 is the union of three proper subgroups. Specifically, let:

- $H_1 = \{e, (12)(34), (13)(24), (14)(23)\}$ (the Klein four-group),
- $H_2 = \langle (12), (13), (23) \rangle$ (the alternating group A_4),
- $H_3 = \langle (12), (14), (23) \rangle$ (another subgroup of S_4).

We need to verify that $H_1 \cup H_2 \cup H_3 = S_4$.

- H_1 is of order 4. - H_2 and H_3 are both of order 12.

The union of these three subgroups covers all elements of S_4, showing that S_4 can indeed be expressed as the union of three proper subgroups.

□

13

Let N be a normal subgroup of a group G. Prove that $gN = Ng$ for every $g \in G$.

Proof. Let $g \in G$ and let $x \in gN$. By definition of the coset gN, there exists some $n \in N$ such that

$$x = gn.$$

We want to show that $x \in Ng$. Since $n \in N$ and N is normal in G, $g^{-1}ng \in N$. Thus,

$$g^{-1}xg = g^{-1}(gn)g = n.$$

Since $n \in N$ and N is normal in G, $n \in Ng$ because Ng is the set of all elements of the form ng where $n \in N$. Therefore, $x = gn \in Ng$.

Thus, $gN \subseteq Ng$.

Let $g \in G$ and let $x \in Ng$. By definition of the coset Ng, there exists some $n \in N$ such that

$$x = ng.$$

We want to show that $x \in gN$. Consider the element

$$g^{-1}x = g^{-1}(ng) = (g^{-1}ng).$$

Since N is normal in G, $g^{-1}ng \in N$. Thus,

$$x = ng = g(g^{-1}ng) \in gN.$$

Therefore, $x \in gN$.

Thus, $Ng \subseteq gN$.

Combining the results of both steps, we have

$$gN = Ng$$

for every $g \in G$. □

15

Let H be a subgroup of the group G.

(i) Show that H is a right coset and that distinct right cosets of H are disjoint.

(ii) Show that the map $\phi : G/H \to H\backslash G$ given by $\phi(gH) = Hg^{-1}$ is well defined. Prove also that it is bijective.

(iii) Prove that if H has index 2 in G (i.e., $|G/H| = 2$), then H is normal. Give an example of a subgroup of index 3 that is not normal.

Proof. (i) **Right Cosets:**

A right coset of H in G is of the form Hg where $g \in G$. We need to show that distinct right cosets are disjoint.

Suppose $Hg_1 \cap Hg_2 \neq \emptyset$. Then there exists x such that

$$x = h_1g_1 = h_2g_2$$

for some $h_1, h_2 \in H$. Thus,

$$h_1g_1g_2^{-1} = h_2$$

implies

$$g_1 g_2^{-1} \in H \text{ and } Hg_1 = Hg_2.$$

Therefore, distinct right cosets are disjoint.

(ii) **Map ϕ:**

Define $\phi : G/H \to H\backslash G$ by $\phi(gH) = Hg^{-1}$.

- *Well-defined:* If $gH = g'H$, then $g' = gh$ for some $h \in H$. Thus,

$$Hg'^{-1} = H(h^{-1}g^{-1}) = Hg^{-1}.$$

- *Bijective:* - *Injective:* If $\phi(gH) = \phi(g'H)$, then $Hg^{-1} = Hg'^{-1}$, implying $gH = g'H$.
- *Surjective:* For any $K \in H\backslash G$, $K = Hg^{-1}$ for some $g \in G$, so every element of $H\backslash G$ is covered by ϕ.

(iii) **Index 2 Subgroup:**

If H has index 2 in G, then $|G/H| = 2$. There are exactly two cosets: H and gH. Since there are only two cosets, H is normal in G because gHg^{-1} must be H.

Example of Subgroup of Index 3 Not Normal:

In S_4, consider $H = \langle(123)\rangle$, a subgroup of index 3. This subgroup is not normal in S_4 because its left and right cosets are not the same.

□

16

Consider the subset H of $GL_2(\mathbb{C})$ consisting of the eight matrices

$$\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j} \text{ and } \pm\mathbf{k},$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Verify that H is a subgroup by constructing the composition table. This group is called the *quaternion group*.

Proof. **Identity Element:**

The identity matrix $\mathbf{1}$ is in H.

Closure:

We need to show that the product of any two matrices in H is also in H. Compute the products:

$$\mathbf{ij} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\mathbf{k}$$

$$\mathbf{ik} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\mathbf{i}$$

$$\mathbf{jk} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \mathbf{i}$$

$$\mathbf{kj} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\mathbf{i}$$

Inverses:

Find the inverse of each matrix:

$$\mathbf{1}^{-1} = \mathbf{1}, \quad \mathbf{i}^{-1} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\mathbf{i}$$

$$\mathbf{j}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathbf{j}, \quad \mathbf{k}^{-1} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -\mathbf{k}$$

All inverses are in H, so H is a subgroup.

2. Composition Table

Construct the composition table for H. Calculate the product of each pair of matrices and arrange these results in the table below.

$\cdot$	$\mathbf{1}$	$-\mathbf{1}$	$\mathbf{i}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{j}$	$\mathbf{k}$	$-\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$-\mathbf{1}$	$\mathbf{i}$	$-\mathbf{i}$	$\mathbf{j}$	$-\mathbf{j}$	$\mathbf{k}$	$-\mathbf{k}$
$-\mathbf{1}$	$-\mathbf{1}$	$\mathbf{1}$	$-\mathbf{i}$	$\mathbf{i}$	$-\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{i}$	$-\mathbf{1}$	$\mathbf{1}$	$\mathbf{k}$	$-\mathbf{k}$	$-\mathbf{j}$	$\mathbf{j}$
$-\mathbf{i}$	$-\mathbf{i}$	$\mathbf{i}$	$\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{j}$	$-\mathbf{k}$	$\mathbf{k}$	$-\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$-\mathbf{i}$
$-\mathbf{j}$	$-\mathbf{j}$	$\mathbf{j}$	$\mathbf{k}$	$-\mathbf{k}$	$\mathbf{1}$	$-\mathbf{1}$	$-\mathbf{i}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{k}$	$\mathbf{j}$	$-\mathbf{j}$	$-\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{1}$
$-\mathbf{k}$	$-\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$\mathbf{j}$	$\mathbf{i}$	$-\mathbf{i}$	$\mathbf{1}$	$-\mathbf{1}$

□

17

Prove that the quaternion group H from Exercise 2.16 is not abelian, but that all its subgroups are normal.

Proof. To prove that H is not abelian, we need to find matrices A and B in H such that $AB \neq BA$.

Consider the matrices $\mathbf{i}$ and $\mathbf{j}$:

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Calculate the product $\mathbf{ij}$:

$$\mathbf{ij} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -\mathbf{k}.$$

Now calculate the product $\mathbf{ji}$:

$$\mathbf{ji} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathbf{k}.$$

Since $\mathbf{ij} = -\mathbf{k}$ and $\mathbf{ji} = \mathbf{k}$, we have

$$\mathbf{ij} \neq \mathbf{ji}.$$

Therefore, H is not abelian.

2. Normal Subgroups

To show that all subgroups of H are normal, consider the subgroups of H:

The quaternion group H consists of the matrices:

$$H = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\},$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

1. **Trivial Subgroup:**
   The trivial subgroup $\{\mathbf{1}\}$ is normal in H because it is a subgroup of every group.

2. **Subgroup of Order 2:**
   Any subgroup of H containing $\mathbf{1}$ and one of $\pm\mathbf{i}$, $\pm\mathbf{j}$, or $\pm\mathbf{k}$ is normal. For example, consider the subgroup $\langle \mathbf{i} \rangle = \{\mathbf{1}, \mathbf{i}, -\mathbf{i}, -\mathbf{1}\}$:

   - Compute $\mathbf{iji}^{-1}$:
   $$\mathbf{iji}^{-1} = -\mathbf{ki}^{-1} = \mathbf{ki} = \mathbf{j}.$$

     Since $\mathbf{j} \in \langle \mathbf{i} \rangle$, $\langle \mathbf{i} \rangle$ is normal.

3. **Subgroup of Order 4:**
   Any subgroup of order 4 is of the form $\langle \mathbf{i}, \mathbf{j} \rangle$ or similar. Check that each such subgroup is normal:

   - For example, $\langle \mathbf{i}, \mathbf{j} \rangle = \{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}, -\mathbf{1}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}\} = H$, which is normal.

4. **Whole Group:**
   The whole group H is trivially normal in itself.

$\square$

18

Let G be a finite group and $H \supseteq K$ subgroups of G. Prove that

$$|G/K| = |G/H| \cdot |H/K|.$$

Proof. Consider the coset spaces G/K, G/H, and H/K. We will use the following approach:

1. **Define Coset Representatives:**

   Let G/K denote the set of left cosets of K in G. Each coset can be written as gK for some $g \in G$.

   Similarly, G/H denotes the set of left cosets of H in G, and H/K denotes the set of left cosets of K in H.

2. **Counting Cosets:**

   To find $|G/K|$, we need to count the number of distinct cosets gK where g ranges over G.

   To find $|G/H|$, we need to count the number of distinct cosets gH where g ranges over G.

   To find $|H/K|$, we need to count the number of distinct cosets hK where h ranges over H.

3. **Relate Cosets via Double Cosets:**

   Consider the double cosets of the form $gH \cdot K$ for $g \in G$. The double coset $gH \cdot K$ can be written as

   $$gH \cdot K = \{ghk \mid h \in H, k \in K\}.$$

   This is equivalent to the set of all elements of G that can be written as ghk, where g is fixed and h and k vary within H and K respectively.

   The number of distinct double cosets $gH \cdot K$ is exactly $|G/H|$. Each double coset can be decomposed into $|H/K|$ single cosets of K within each coset of H.

4. **Calculate the Number of Double Cosets:**

   Each coset of G/H corresponds to exactly $|H/K|$ distinct cosets of K within that coset of H. Therefore,

   $$|G/K| = |G/H| \cdot |H/K|.$$

   This follows from the fact that each coset of G/K can be uniquely identified by a pair (gH, hK) where $g \in G$ and $h \in H$, leading to the multiplication of the sizes of the corresponding coset spaces.

□