RINGS

Cosets

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1 RINGS

Q: Given a ring R, is there a field F such that $R \subseteq F$?
ex. $\mathbb{Z} \subseteq \mathbb{Q}$
the field of Fractions
Q: How do we build $\mathbb{Q}$ from $\mathbb{Z}$

$$\frac{p}{q} \in \mathbb{Q}, p \in \mathbb{Z}, q \in \mathbb{Z} \backslash \{0\}$$

$$\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} - \{0\}) / \sim$$

$(p, q) \sim (r, s)$ if $ps = qr$ More generally, for a domain R consider

$R \times (R - \{0\})$
and define an equivalence relation
$(a, s) \sim (b, t)$ iff $at = sb$

exercise Check that $\sim$ is an equivalence relation.
define the field of fractions Q of a domain R as $Q = R \times (R - \{0\}) / \sim$
define $(a, s) = \frac{a}{s}$
Exercise: Q is a ring with operations:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \text{ or } (a, s) + (b, t) = (at + bs, st)$$
$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

1. additive identity in Q is $\frac{0}{1} = \frac{0}{s}$, $\forall s \in R - \{0\}$
2. multiplication in Q is $1 = \frac{1}{1} = \frac{s}{s}$, $\forall s \in R - \{0\}$

Lemma 1.1.
Q is a field

Proof.
If $\frac{a}{s} \neq 0$, then $a \neq 0$
and then $\frac{s}{a} \cdot \frac{a}{s} = 1$ □

Exercise How to define this injective ring homomorphisms
$i : R \to Q$, and $i(a) = \frac{a}{1}$

Proposition 1.2 (3.4.1)**.**
Let R be a domain with field of fractions Q. Let L be a field and let

$$\varphi : R \to L$$

be an injectivre ring homomorphisms, then, $\exists!$ injective ring homomorphism

$$\bar{\varphi} : Q \to L$$

such that $\bar{\varphi} \cdot i = \varphi$
$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "\varphi"'] & Q \arrow[d, "\exists!\bar{\varphi}"] \\ & L \end{tikzcd}$$

EX

$$R = \mathbb{Z}$$
$$Q = \mathbb{Q}$$
$$L = \mathbb{R}$$

Proposition 1.3 (3.4.2 COROLLARY not prop)**.**
Let R be a domain contained in a field L. The smallest subfield in L containing R is

$$K = \{as^{-1} | a \in R, s \in R - \{0\}\}$$

The field of fractino is isomorphic to K.

Proof.
Let $a, b \in R$ and $s, t \in R - \{0\}$
Then $(as^{-1})(bt^{-1}) = abs^{-1}t^{-1}$
And $as^{-1} + bt^{-1} = (at + bs)(st)^{-1}$
and $(as^{-1})^{-1} = sa^{-1}$ if $a \neq 0$
Thus K is a subfield of L
Note that any subfield of L containing R must contain K, and we have $a \in R, s^{-1} \forall s \in R - \{0\}$

Let Q be the field of fractions of R and ocntains the injection $R \to L$ sending $r \in R$ to $r \in L$
By Proposition (3.4.1) $\bar{\varphi}(\frac{a}{s}) = as^{-1}$ is an injective homomorphism. and is clearly injective onto K.

□

Sket Proof of Proposition

since $\varphi \cdot i = \varphi$ we must have for $s \in R - \{0\}$,

$$\begin{aligned} 1 &= \bar{\varphi}(1) \\ &= \bar{\varphi}(\frac{s}{1} \cdot \frac{1}{s}) \\ &= \bar{\varphi}(\frac{s}{1}) \cdot \bar{\varphi}(\frac{1}{s}) \\ &= \bar{\varphi}(i(s)) \cdot \bar{\varphi}(\frac{1}{s}) \\ &= \bar{\varphi}(s) \cdot \bar{\varphi}(\frac{1}{s}) \end{aligned}$$

Hence

$$\bar{\varphi}(\frac{a}{s}) = \bar{\varphi}(\frac{a}{1})\bar{\varphi}(\frac{1}{s}) = \varphi(a)\varphi(s)^{-1}$$

We need to check $\bar{\varphi}$ is well-defined:
if $\frac{a}{s} = \frac{b}{t}$ then $at = sb$, and so
$\varphi(a)\varphi(t) = \varphi(s)\varphi(b)$ and so
$\varphi(a)\varphi(s)^{-1} = \varphi(b)\varphi(t)^{-1}$
i.e. $\varphi(\bar{\frac{a}{s}}) = \varphi(\bar{\frac{b}{t}})$

exercise Check that $\bar{\varphi}$ is a ring homomorphism and injective

Plans for the rest of the semester: Given a ring R, if R is a field, every non-zero element has a multi. inverse
If R is a domain, we can build its field of fractions.
Q: What properites of $\mathbb{Z}$ Hold in other rings?
Let R be a ring

1. Can we factor elements in R?
2. Are factorizations unique?
3. Is there a notion of "prime" in R?
4. Does the division algorithm work in r do we have unique remainder

Assume R is a domain for the next few classes Factoring in R Deifine: let $x, z \in R$ if $x = ry$ for some $r \in R$ we say y is a divisor of x denoted $y \mid x$

Notes if R is a PID then for any a,b $\in R, \exists d$ such that

$$< a, b >= \{\lambda_1 a + \lambda_2 b | \lambda_1, \lambda_2 \in R\}$$