LECTURE 1

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1 Relations

Let S be s a set (e.g. $\mathbb{N} = \{1, 2, 3, \dots\}$), integers: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
rational numbers $\mathbb{Q}$ i should know these ..
RECALL: cartesian product: A x $B = \{(a, b) | a \in A, b \in B\}$

> **Definition 1.1.** A relation R on a set S is a subset $R \subseteq S \times S$. we write xRy (read "x is related to y") to denote an element $(x, y) \in R$

some examples of relations: "is orthogonal to" is a relation o the set of vectors in $\mathbb{R}^3$
another example "is congruent to" is a relation on the set of polygons
"divides" is a relation on the set of integers

WE say the relation R is

1. reflexive if xRx for all $x \in S$
2. symmetric if xRy implies yRx for all $x, y \in S$
3. antisymmetric if xRy and yRx implies $x = y$ for all $x, y \in S$
4. transitive if xRy and yRz implies xRz for all $x, y, z \in S$

(sometimes we note $\sim$ for the relation R)
going back to examples: "is congruent to" is reflexive, symmetric, and transitive

2 equivalence and partial order relations

> **Definition 2.1.** two very important types of relations are:
>
> 1. equivalence relations
>   (a) reflextive
>   (b) symmetric
>   (c) transitive
> 2. partial order relations
>   (a) reflexive
>   (b) antisymmetric
>   (c) transitive

e.g. $R = \{(a, b) | b - a \in \mathbb{N}\} \subseteq \mathbb{Z} \times \mathbb{Z}$
is R an equivalence relation? a partial ordering ?

1. reflexive: aRa? yes since $a - a = 0 \in \mathbb{N}$
2. antisymmetric: aRb and bRa implies $a = b$? if aRb then $b - a \in \mathbb{N}$ and if bRa then $a - b \in \mathbb{N}$ so a and b are the same

3. transitive: aRb and bRc implies aRc? if aRb then $b-a \in \mathbb{N}$ and if bRc then $c-b \in \mathbb{N}$ so $c-a=(c-b)+(b-a) \in \mathbb{N}$

Exercise: check that aRb iff $a \leq b$

e.g. fix $n \in \mathbb{Z}^+$ $R = \{(a,b)|a-b \in n\mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{Z}$

Exercise this is an equivalece relation

aRb if $a-b=nk$ for some $k \in \mathbb{Z}$

we call this relation congruence module n

notation:if R is an equivalence relation, we often denote it by $\sim$

Definition 2.2 (equivalence class)**.** If $\sim$ is an equivalene relation on S we can fix and element $x \in S$ and consider all elements of S that is equivalent to x:

$$[x] = \{s \in S | s \sim x\} \subseteq S$$

and x is a representative of this equivalence class

e.g. $S = \{a,b,c,d,e\}$ and $\sim = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(b,a),(c,d),(d,c),(d,e),(e,d),(c,e),(e,c)\}$
$[a] = \{a,b\} = [b]$ and $[c] = \{c,d,e\} = [d] = [e]$

Q: are equivalence classes always disjoint?

A: yes, if $[x] \cap [y] \neq \emptyset$ then $[x] = [y]$

Lemma 2.1. Let $\sim$ be an equivalene relation on S and $x, y \in S$. Then $[x] = [y]$ iff $x \sim y$

proof of lemma 2.1. iff

$\to$: supose $[x] = [y]$ then $x \in [x]$, since $\sim$ is reflexive and so $x \in [y]$ Thus, $x \sim y$

$\leftarrow$: Let $s \in [x]$ since $x \sim x$ and $x \sim y$ (by assumption) then $s \sim y$ since it is transitive. hence $s \in [y]$ and so $[x] \subseteq [y]$

similarly, if $s \in [y]$, then $s \sim y$ and $x \sim y$ so since $\sim$ is symmetric and trasnstiive $s \sim x$ and so $xin[x]$ hence $[y] \subseteq [x]$ $\square$

Corollary:

$$[x] \cap [y] = \emptyset \text{ if } [x] \neq [y]$$

Proof. suppose $[x] \cap [y] \neq \emptyset$ then there exists $s \in [x] \cap [y]$ then $s \sim x$ and $s \sim y$ by lemma 2.1 $x \sim y$ and so $[x] = [y]$ $\square$

Definition 2.3. a partition on a set S is a collection $(S_i)_{i \in I}$ of nonempty subsets of S such that

1. $S = \bigcup_{i \in I} S_i$
2. $S_i \cap S_j = \emptyset$ for all $i, j \in I$ with $i \neq j$

e.g. $S = \{a, b, c, d, e\}$ and $S_1 = \{a, b\}$, $S_2 = \{c, d, e\}$ is a partition of S

Theorem 2.2. Let S be a set with an equivalece relation $\sim$: Then the set of equivalence classes $S/\sim = \{[x] | x \in S\}$ is a partition of S
conversely, if $(S_i)_{i \in I}$ is a partition of S then there exists an equivalence relation $\sim$ on S such that $S/\sim = (S_i)_{i \in I}$

Proof. $\rightarrow$ we will first show that the set of equivalence classes is a partition, we already showed $[x] \cap [y] = \emptyset$ if $[x] \neq [y]$ so we need to show that $S = \bigcup_{x \in S} [x]$
let $s \in S$ then $s \in [s]$ so $s \in \bigcup_{x \in S} [x]$ and so $S \subseteq \bigcup_{x \in S} [x]$
For each $x \in S, [x] \subseteq S$ so $\bigcup_{x \in S} [x] \subseteq S$ and so $S = \bigcup_{x \in S} [x]$

$\leftarrow$ now suppose $(S_i)_{i \in I}$ is a partition of S. Define $x \sim y$ iff $x, y \in S_i$ for some i. We have $\sim$ is reflexive since each $x \in S_i$ for some $i \in I$. As $\bigcup_{i \in I} S_i = S$.
$\sim$ is symmetric since containm ent in a set have an order
$\sim$ is transitive since if $x, y \in S_i$ and $y, z \in S_j$ then $x, y, z \in S_i \cap S_j = \emptyset$ so $x \sim z$ □

e.g. $S = \mathbb{Z}$, $\sim$ is $\equiv \mod n$
$S/\sim = \{[0], [1], \ldots, [n-1]\}$ is a partition of $\mathbb{Z}$

$$[0] = \{\ldots, -2n, -n, 0, n, 2n, \ldots\} \tag{1}$$
$$[1] = \{\ldots, -2n+1, -n+1, 1, n+1, 2n+1, \ldots\} \tag{2}$$
$$\vdots \tag{3}$$
$$[n-1] = \{\ldots, -2n+n-1, -n+n-1, n-1, 2n+n-1, \ldots\} \tag{4}$$

3 Partial orders

<u>notation:</u> when a relation R is a partial order, we often denote it by $\leq$
e.g. $\leq$ is a partial order on $mathbbR$
Let S be a set and $\mathbb{P}(s)$ its power set. $\leq$ is a partial order on $\mathbb{P}(S)$
ex: "Divides" is a partial order on $\mathbb{N}$ (but not on $\mathbb{Z}$)

Definition 3.1 (minimal and first element)**.** let $\leq$ be ap artial order on a set S.
An element $s \in S$ is <u>minimal</u> if $x \leq s \rightarrow x = s \forall x \in S$.
An element $t \in S$ is called a <u>first</u> element if $t \leq x \forall x \in S$ .. (Warning: not every minimal element is a first element)

e.g. $S = \{2, 3, 4, 5, 6 \ldots\}$ divides is a partial order on S
2 is a minimal element. 3 is a minimal element . 5 is a minimal element. every prime number is a minimal element But they are not first elements

Proposition 3.1.

1. A first element is unique
2. a first element is a minimal element
3. a minimal element needs not to be a first element***