SYMMETRIC GROUPS CONTINUED..

Cosets

Author
Tom Jeong

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Contents

1 symmetric group

last time: we talked about inversions. n_σ
$sgn : S_n \to \{1, -1\}$
$A_n = ker(sgn)$
$|A_n| = \frac{n!}{2}$

example: $A_2 = \{id\}$

Proposition 1.1 (2.9.17).
let $n \geq 2$ then

1. a transposition $\tau = (ij) \in S_n$ is an odd permutation
2. the sign of a k cycle $(x_1 x_2 \ldots x_k)$ is $(-1)^{k-1}$

Proof.
1. let $(xy) \in S_n$ be a transposition. Then $\exists \sigma \in S_n$ s.t. $\sigma(1) = x$ and $\sigma(2) = y$. so $\sigma(12)\sigma^{-1} = (xy)$. (lemma 2.9.8)
so $sgn(xy) = sgn(\sigma(12)\sigma^{-1}) = sgn(\sigma)sgn(12)sgn(\sigma^{-1}) = sgn(12) = -1$
2. $(x_1 x_2 \ldots x_k) = (x_1 x_k)(x_1 x_{k-1}) \ldots (x_1 x_3)(x_1 x_2)$
so $sgn(x_1 x_2 \ldots x_k) = (-1)^{k-1}$ □

ex. $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 6 & 1 & 7 & 2 & 5 \end{bmatrix} = (13624)(57)$

$$sgn(\sigma) = (-1)^4(-1)^1 = -1$$

Q: Why are symmetric groups important?

Theorem 1.2 (Cayley's Theorem).

Every finite group G is isomorphic to a subgrouop of S_n or $S_{|G|}$ for some n. where $n = |G|$.

Proof.
Let S_G be the group of permutations on the set G.
Define a map $f : G \to S_G$ by $f(x) = \phi_x$ where $\phi_x : G \to G$ is defined by $\phi_x(g) = xg$.
inverse of ϕ_x is $\phi_{x^{-1}}$
similarly $\phi_x^{-1}\phi_x(g) = g$.
ϕ_x is a bijection, since ϕ_x^{-1} is its left and right inverse

1. f is a homomorphism: $f(x) \cdot f(y) = \phi_x \cdot \phi_y = \phi_x \circ \phi_y = \phi_{xy} = f(xy)$
2. f is injective: $f(x) = f(y) \implies \phi_x = \phi_y \implies \phi_x(e) = \phi_y(e) \implies x = y$

So $f : G \to S_G$ is an bijective homomorphism. ie an isomorphism. (co domain is the image so it is surjective by definition)

$\square$

ex. $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to a subgroup of S_n.

$$f : \mathbb{Z}/n\mathbb{Z}$$

$f(k) = \phi_k$ where $\phi_k(x) = k = x$
$\mathbb{Z}/n\mathbb{Z} \cong < (1234 \ldots n) > \subseteq S_n$
Ex2. $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ where $|G| = 4$
$f : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to S_4$
$f((0,0)) = id$
$f((1,0)) = (12)$
$f((0,1)) = (34)$
$f((1,1)) = (12)(34)$

Actions of groups

Recall: consider the symmetric group S_n and the set $M_n = \{1, 2, \ldots, n\}$. we kow

1. $e(i) = i \forall i \in M_n$
2. $(\tau\sigma)(i) = \tau(\sigma(i))$

This is a special case of more general phenomenon.

Definition 1.1 (2.10.1)**.**
Let G be a group and S set. We say that G acts (from the left) on S if there is a map

$$\alpha : G \times S \to S$$

such that

1. $e \cdot s = s \forall s \in S$
2. $(gh) \cdot s = g \cdot (h \cdot s) \forall g, h \in G, s \in S$

example S_n acts on M_n notation: we write $S_n \circlearrowright M_n$ "acts on "

Idea: if G acts on S that is like saying that at least some of the symmetries of S are described by G.
EX. recall $D_3 = \{$symmetris of equilateral triangle $\}$ and Let $S = \{$Points on an equilateral triangle$\}$ we can say that $D_3 \circlearrowright S$

$< s_1 > \cong \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z} \circlearrowright S$ in the following way:

$$\begin{aligned}\alpha : \mathbb{Z}/2\mathbb{Z} \times S &\to S\\ \alpha([n], p) &\to s_1^n(p)\\ \alpha([0], p) &\to p\\ \alpha([1], p) &\to s_1(p)\end{aligned}$$

Definition 1.2. Let G act on a set S and let $X \subseteq S$ and $s \in S$

1. fix $s \in S$ then $G \cdot s = Gs = \{g \cdot s | g \in G\}$ is called the orbit of s under the action of G
2. The set of orbits $\{Gs | S \in S\}$ is denoted S/G
3. Fix $g \in G$ let $g \cdot X = gX = \{gx | x \in X\}$ Then, $G_x = \{g \in G | g \cdot X = X\}$ is called the stabilizer of x under the action of G.
   if $X = \{x\}$., we deote G_x by G_x
4. We say $s \in S$ is a fixed Point for the action of G on S if $g \cdot s = s \forall g \in G$. The set of fixed points of G is denoted S^G

remark: Can define an equivalence relation on S. given $s, t \in S$ we say $s \sim t$ if $\exists g \in G$ such that $g \cdot s = t$. We will see that the orbit is the equivalence class of s. and the set of orbits of S is the partition of S induced by the equivalence relation $\sim$ ex: $S_n \circlearrowright M_n$
by $\alpha : S_n \times M_n \to M_n$
and $\alpha(\sigma, i) = \sigma(i)$

the orbit $S_n \cdot i = \{\sigma(i) | \sigma \in S_n\} \subseteq M_n = M_n$ since $\forall j \in M_n$ $\exists \sigma \in S_n$ such that $\sigma(i) = j$

ex. fix $\sigma \in S_n$ and let $H =< \sigma >= \{\sigma^k | k \in \mathbb{Z}\}$
Then $H \circlearrowright M_n$
$\alpha : H \times M_n \to M_n$
$\alpha(\sigma^k, i) = \sigma^k(i)$
$M_n/H \leftrightarrow$ dijoint cycles of σ
e.g. $\sigma = (123)(56) \in S_6$
$H =< \sigma >$ $H \cdot 1 = \{1, 2, 3\} = H \cdot 2 = H \cdot 3$
$H \cdot 4 = \{4\}$
$H \cdot 5 = \{5, 6\} = H \cdot 6$

$M_6/H = \{\{1, 2, 3\}, \{4\}, \{5, 6\}\}$
sets of orbits
$H_{M_6} = \{h \in H | h \cdot M_6 = M_6\} = H$
$H_{\{H\}} = \{\sigma^2, \sigma^4, e\}$
$M_6^H = \{4\}$