STABILIZERS AND ORBITS

Cosets

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Contents

1 Stabilizers

From last class, $\sigma = (123)(56) \in S_6$
$H =< \sigma >= \{\sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5, e\}$ and
$H \circlearrowright M_6 = \{1, 2, 3, 4, 5, 6\}$
The sets of orbit $M_6/H = \{\{1, 2, 3\}, \{4, 5, 6\}\}$

The stablizer:

$$\begin{aligned} H_{\{1\}} &= \{e, \sigma^3\} \\ H_{\{4\}} &= H \\ H_{\{5\}} &= \{\sigma^2, \sigma^4, e\} \\ H_{\{5,6\}} &= \{h \in H | h \cdot \{5, 6\} = \{5, 6\}\} = H \end{aligned}$$

Ex: Let $H \leq G$
THen

$$\begin{aligned} \alpha : H \times G \to G \\ \alpha(h, g) = h \cdot g \end{aligned}$$

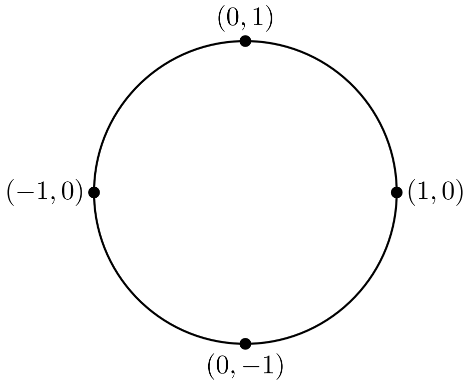
is an action $H \circlearrowright G$
the orbit $H \cdot g = \{hg | h \in H\}$ ie the right coset Hg the set of orbit is $H \backslash G$
ie the sest of right cosets fixed poitns $H_g = \{h \in H | hg = g\}$
ther is no fixed points since $H \neq \{e\}$

1.1 examples

ex. $GL_2(\mathbb{R}) \circlearrowright \mathbb{R}^2$ by

$$\begin{aligned} \alpha = GL_2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2 \\ \alpha(A, v) = A \cdot v \end{aligned}$$

$$\begin{aligned} A \cdot (0, 0) = (0, 0) \; \forall A \in GL_2(\mathbb{R}) \\ \text{If } \vec{v} \neq (0, 0), \\ GL_2(\mathbb{R}) \cdot \vec{v} = \mathbb{R}^2 \backslash \{(0, 0)\} \\ \text{Let } S' = \{\vec{v} \in \mathbb{R}^2 | ||\vec{v}|| = 1\} \end{aligned}$$



exercise: stabilizer of S' is $O_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) | A^T A = id\}$
exercise Let H be a subgroup of G not necessarily normal

Then Define

$$\alpha : G \times G/H \to G/H$$
$$\alpha(g, g'H) = (gg')H$$

Tere is only one orbit since for any $g_1 H$ and $g_2 H$,

$$\alpha(g_2 g_1^{-1}, g_1 H) = g_2 H$$

The stabilizer of $H \in G/H$ is $H \leq G$ since $\alpha(h, H) = hH = H$ for all $h \in H$

Proposition 1.1. Let $\alpha : G \times S \to S$ be a group action.

1. let $X \subseteq S$ Then the stabilizer G_X is a subsgroup of G
2. The set S is a union of G-ortbits.

$$S = \bigcup_{s \in S} G \cdot s$$

where $G \cdot s \neq G \cdot t \longrightarrow G \cdot s \cap G \cdot t = \emptyset$

3. Orbit-stabilizer lemma: Let $x \in S$ then

$$\tilde{f} : G/G_X \to G \cdot x$$

given by $\tilde{f}(gG_x) = g \cdot x$ is a well-defined bijection map between the set of left osets of G_X and the orbit of x

EX.

Example 1.2.

$$\sigma = (123)(56) \in S_6$$
$$H =< \sigma >$$
$$H \circlearrowleft M_6$$

set of orbits

$$M_6/H = \{\{1,2,3\},\{4\},\{5,6\}\}$$

with Stabilizer:

$$H_1 = \{e, \sigma^3\}$$
$$H_5 = \{\sigma^2, \sigma^4, e\}$$
$$H/H_5 = \{H_5, \sigma H_5\}$$
$$\text{Note that } \sigma H_5 = \{\sigma\sigma^2, \sigma\sigma^4, \sigma e\} = \{\sigma^3, \sigma^5, \sigma\} = H_1$$

with the bijection

$$\tilde{f} : H/H_5 \to \{5,6\}$$
$$\tilde{f}(H_5) = 5$$
$$\tilde{f}(\sigma H_5) = 6$$

Now since we have seen the sample, we will show the proof of the proposition.

Proof.

1. Prove that G_X is a subgroup $G_X = \{g \in G | gX = X\}$
   (Identity) $e \in G_X$ (by definitino of group action)
   (Closure) If $g, h \in G_X$ then $(gh) \cdot X = g(h \cdot X) = gX = X$
   (Inverse) If $g \in G_X$ then $g^{-1}X = g^{-1}(gX) = X$ since g is in the stabilizer

2. We will define an equivalence relation: define $\alpha : G \times S \to S$ be a group action. LEt $s, t \in S$ Define $s \sim t$ if $\exists g \in G$ such that $g \cdot s = t$

Lemma 1.3. This is an equivalence relation.

Proof.

(a) <u>reflexive</u>: $e \in G$ and $e \cdot s = s$ forall $s \in S$ so $s \sim s$

(b) <u>symmetric</u>: suppose $s \sim t$ then $\exists g \in G$ such that $g \cdot s = t$
then $g^{[} - 1] \cdot t = g^{-1}(gs) = e \cdot s = s$ so $t \sim s$

(c) <u>transitive</u>: suppose $s \sim t$ and $t \sim u$ then $\exists g, h \in G$ such that $g \cdot s = t$ and $h \cdot t = u$
$hg \cdot s = ht = u$ so $s \sim u$

So g orbits are eactly the equivalence classes of this relation. $\square$

3. wts: $\tilde{f} : G/G_X \to G \cdot x$ with $\tilde{f}(gG_X) = g \cdot x$ is a well-defined bijection.
Let $g_1, g_2 \in G$ suppose $g_2 G_x = g_2 G_X$. By Lemme 2.26

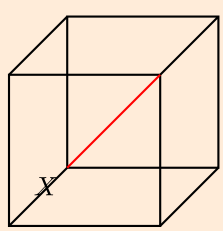
$$g_1 G_X = g_2 G_X \leftrightarrows g_1^{-1} g_2 \in G_X$$

we have that $g_1^{-1} g_2 \in G_x$
liff $g_1^{-1} g_2 \cdot x = x$
iff $g_1 \cdot x = g_2 \cdot x$ so $\tilde{f}$ is well-defined and injective
Let $s \in G \cdot x$ Then $s = g \cdot x$ for some $g \in G$.. Hence

$$\tilde{f}(gG_X) = g \cdot x$$

so $\tilde{f}$ is surjective

$\square$

Example 1.4. 

x should be on top righr coerner ngl Let $G = \{$ group of symmetries of a cube$\}$. WHat is the order of G?
Let x be a vertex, $|G \cdot x| = 8$ since the orbit of x is the number of vertices of the cube.
Stabilizer $|G_X| = 3$ which is the identity, rotation by $\frac{2\pi}{3}, \frac{4\pi}{3}$
about the red axis By LaGranges Theorem $|G| = |G/G_X| \cdot |G_X| = 8 \cdot 3 = 24$by Previous proposition $|G/G_X| = |G \cdot X| = 8$ SO we have $|G| = 24$

<u>Corollary</u> (2.10.7) Let $G \times S \to S$ be a group action where S is a finnite set

Then

$$|S| = |S^G| + \sum_x |G/G_X|$$

where S^G are the fixed points. where the summation is done by picking out an element x from eah orbit with more than one element.

Proof.
BY 2) of the previous proposition $|S| = \sum_{G\cdot X \in S/G} |G \cdot x| =$

$$|S^G| \text{ (orbits containing a single element) } + \sum_x |G/G_X| \text{ (by part 3 of proposition)}$$

□

Lemma 1.5 (Burnside's Lemma (2.10.8))**.**
Cauchy -Frobenius Lemma. Let $G \times S \to S$ be a group action where S is a finite set and G is a finite group. Then the number of orbits is equal to the average number of fixed points of elements of G.

$$|S/G| = \frac{\sum_{g\in G} |S^g|}{|G|}$$

and where $S^g = \{x \in S | gx = x\}$