LECTURE 6 SEP 9

Cosets

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Contents

1 Normal Subgroup

Proposition 1.1. If N is a normal subgroup then $(xN)(yN) = (xy)N \forall x, y \in G$

Corollary 2.3.3 Let N be a normal subgroup of G then composition of left cosets makes G/N into a group weher $(xN)(yN) = (xy)N \forall xN, yN \in G/N$

Proof. Since cmposition in G is associative, $(g_1N)(g_2N)(g_3N) = g_1N(g_2Ng_3N) = g_1N(g_2g_3)N = (g_1g_2)g_3N = (g_1g_2)N(g_3N) = (g_1Ng_2N)(g_3N)$ The identity is $N = eN$ since $eNgN = (eg)N = gN$ and $gNeN = gN$. The inverse of gN is $g^{-1}N$ since $gNg^{-1}N = (gg^{-1})N = N$ and $g^{-1}NgN = (g^{-1}g)N = N$

$\square$

Definition 1.1. Let N be a normal subgroup of G. The group G/N (g mod n) is called the Quotient Group of G by N.

Let us look at some examples... $G = (\mathbb{Z}/6\mathbb{Z}, +)$ and $N = \{[0], [3]\}$ left cosets of N:

1. $[0] + N = \{[0], [3]\}$
2. $[1] + N = \{[1], [4]\}$
3. $[2] + N = \{[2], [5]\}$

Then $G/N = \mathbb{Z}/6\mathbb{Z}/\{[0], [3]\} = \{N, [1] + N, [2] + N\}$

Definition 1.2.
$(\mathbb{Z}/n\mathbb{Z})^* = \{[a] \in \mathbb{Z}/n\mathbb{Z} | gcd(a, n) = 1\}$

example: $(\mathbb{Z}/8\mathbb{Z})^* = \{[1], [3], [5], [7]\}$

$\cdot$	[1]	[3]	[5]	[7]
[1]	[1]	[3]	[5]	[7]
[3]	[3]	[1]	[7]	[5]
[5]	[5]	[7]	[1]	[3]
[7]	[7]	[5]	[3]	[1]

Table 1: Multiplication Table for $(\mathbb{Z}/8\mathbb{Z})^* = \{[1], [3], [5], [7]\}$

<u>NOTE:</u> If p is prime, $(\mathbb{Z}/p\mathbb{Z})^* = \{[0], [1], \dots, [p-1]\}$

2 Group homomorphism

Definition 2.1 (2.4.1)**.**
Let G and k be groups. A function $f : G \to K$ is called a group homomorphism if $f(ab) = f(a)f(b) \forall a, b \in G$

examples: $f : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$

$$x \rightarrowtail 2x$$

$$
\begin{aligned}
f(x+y) &= 2(x+y) \\
&= 2x + 2y \\
&= f(x) + f(y)
\end{aligned}
$$

$\uparrow$: Not an isomorphisim because it is not surjective. It is injective though.

example 2: $f : (\mathbb{R}, +) \to (\mathbb{R}^*, \cdot)$

$$
\begin{aligned}
x &\rightarrowtail e^x \\
f(x+y) &= e^{x+y} \\
&= e^x e^y \\
&= f(x)f(y)
\end{aligned}
$$

$\uparrow$: Not an isomorphisim because it is not surjective (the image of e^x is always greater or equal to 0). It is injective though.

example3: determinant $f : (GL_2(\mathbb{R}), \cdot) \to (\mathbb{R}^*, \cdot)$

$$
\begin{aligned}
A &\rightarrowtail det(A) \\
f(AB) &= det(AB) \\
&= det(A)det(B) \\
&= f(A)f(B)
\end{aligned}
$$

Definition 2.2 (2.4.5)**.**
let $f : G \to K$ be a group homomorphism.

1. The kernel of f is the set is $ker(f) = \{g \in G | f(g) = e_K\}$
2. the image of f is the set $Im(f) = \{f(g) | g \in G\}$
3. if f is a bijection, then f is called an isomorphism and we say G and K are isomorphic and write $G \cong K$

example: recall $D_3 = \{e, r_1, r_2, s_1, s_2, s_3\}$ aka symmetries of a triangle and $S_3 = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3$ aka permutations of 3 elements. exercise: shwo that $D_3 \cong S_3$ by explicitly constructing an isomorphism $\phi : D_3 \to S_3$

$$
\phi(s_1) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}
$$

ex. Let $N \trianglelefteq G$ then the function $\Pi : G \to G/N$ given by $f(g) = gN$ is a surjective group homomorphism with kernel N

Proof. $\Pi(gh) = (gh)N = gNhN = \Pi(g)\Pi(h)$
$Ker(\Pi) = \{g \in G | \Pi(g) = N\} = \{g \in G | gN = N\} = N$

$Im(\pi) = \{gN | g \in G\} = G/N$

why is the identity of G/N N?

$$\Pi(e) = eN = N$$
$$\Pi(g) = gN = N \to g \in N$$

□

Proposition 2.1 (2.4.9)**.** Let $f : G \to K$ be a group homomorphism.

1. the image $f(G) \subseteq K$ is a subgroup kf K
2. The kernel $ker(f) \subseteq G$ and $ker(f) \trianglelefteq G$ it is a normal subgroup of G
3. f is injective iff $ker f = \{e_G\}$

Proof.

1. identity $f(g) = f(ge_G) = f(g)f(e_G) \to f(e_G) = e_K$
   Hence $e_K \in Im(f)$
   Closure if $k_1, k_2 \in Im(f)$ then $k_1 = f(g_1)$ and $k_2 = f(g_2)$ for some $g_1, g_2 \in G$ then $k_1k_2 = f(g_1)f(g_2) = f(g_1g_2) \in Im(f)$
   Inverse if $k \in Im(f)$ then $k = f(g)$ for some $g \in G$. And $e = kk^{-1} = f(g)f(g)^{-1}$ and $e_K = f(e_G) = f(gg^{-1}) = f(g)f(g)^{-1} = kk^{-1} \in Im(f)$ hence $(f(g))^{-1} = f(g^{-1}) \in Im(f)$
2. Id: since $f(e_G) = e_K$ then $e_G \in ker(f)$
   closure suppose $x, y \in ker(f)$ then $f(xy) = f(x)f(y) = e_Ke_K = e_K$ hence $xy \in ker(f)$
   inverses now if $x \in ker(f)$ by definition $e = f(x)$ then $f(x^{-1}) = f(x)^{-1} = e_K^{-1} = e_K$ hence $x^{-1} \in ker(f)$

□