SYMMETRIC GROUPS CONTINUED..

Cosets

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Contents

1 symmetric group

Lemma 1.1 (2.9.8)**.**
suppose $\tau = (i, i_2, \ldots, i_k)$ is a k-cycle and $\sigma \in S_n$ then, $\sigma\tau\sigma^{-1} = (\sigma(i), \sigma(i_2), \ldots, \sigma(i_k))$. where $\sigma(i_j) \to \sigma(i_{j+1})$

ex.

$$(134) = (12)(234)(12)$$

Proof. let $j = \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)\}$, both the LHS and RHS of the lemma formula take on the same values for $j \in J$ and $j \notin J$ to itself. Hence LSH are RHS are equal. ☐

Another way to see that every permutation is a product of simple transpositions: (idea of bubble sort) ex:

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{bmatrix}$$

use bubble sort: each time a neighboring pair is out of order, swap the two, and iterate the process.

$$52143 \to^{(12)} 25143$$

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{bmatrix} (12) = \sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{bmatrix}$$

$$25143 \to^{(23)} 21543 \to^{(34)} 21354 \to^{(45)} 21345 \ldots$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{bmatrix} (12)(23)(12)(34)(45)(34) = e$$

so equivelently $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{bmatrix} = (12)(23)(12)(34)(45)(34)$

Q: what is the smallest number of simple transpositions needed to express $\sigma \in S_n$?

Definition 1.1 (2.9.10)**.**
let $\sigma \in S_n$ A pair of indices (i, j) where $1 \le i < j \le n$ is called <u>inversion of σ</u> if $\sigma(i) > \sigma(j)$.
Let $I_\sigma = \{(i, j) | 1 \le i < j \le n \text{ and } \sigma(i) > \sigma(j)\}$ denote the set of inversions of σ and $n_\sigma = |I_\sigma|$ denote the number of inversions of σ.

ex: $\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 2 & 1 \end{bmatrix}$ then we have $I_\sigma = \{(12), (13), (23)\}$

Proposition 1.2 (2.9.12)**.**

1. the permutation $\sigma \in S_n$ is the identity iff $n_\sigma = 0$
2. if $\sigma \neq id$ then $\exists i \in \{1, 2 \ldots, n-1\}$ such that $\sigma(i) > \sigma(i+1)$

Proof. 1. $\to$: true since identity has no inversions
$\leftarrow$ if $\sigma \neq id$ then $\exists$ smallest $i \in M_n$ such that $\sigma(i) > j$ then $(i, \sigma^{-1}(i))$ is an inversion of σ, hence $n_\sigma \neq 0$

2. if $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$ then $n_\sigma = 0$ so $\sigma = id$

$\square$

Lemma 1.3 (2.9.13)**.**
Let $s_i \in S_n$ be the imple transposition $(i, i+1)$ then $n_{\sigma s_i} = \begin{cases} n_\sigma + 1 & \text{if } \sigma(i) < \sigma(i+1) \\ n_\sigma - 1 & \text{if } \sigma(i) > \sigma(i+1) \end{cases}$

Proof. suppose $\sigma(i) < \sigma(i+1)$ then $(i, i+1)$ is not an inversion of σ but is an inversion of σs_i so $n_{\sigma s_i} = n_\sigma + 1$

$\square$

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$$\phi : I_\sigma \to I_{\sigma s_i} \backslash \{(i, i+1)\}$$

$$(k, l) \to (s_i(k), s_i(l))$$

is a bijection and so

$$n_{\sigma s_i} = n_\sigma + 1$$

Proof. proof of claim: If $(kl) \in I_\sigma$ then $s_i(k) > s_i(l)$ since theo nly way for $s_i(k) < s_i(l)$ is if $k = i$ and $l = i+1$ but $(kl) \neq (i, i+1)$ Hence $(s_i(k), s_i(l)) \in I_{\sigma s_i} \backslash \{(i, i+1)\}$. Similarly if $(s_i(k), s_i(l)) \in I_{\sigma s_i} \backslash \{(i, i+1)\}$ then $s_i(k) > s_i(l)$ so $(kl) \in I_\sigma$ $\square$

Proposition 1.4 (2.9.14)**.**
let $\sigma \in S_n$ THen

1. σ is a product of n_σ simple transpositions

2. n_σ is the minimal number of simple transpositions needed to write σ as a product of simple transpositions.

Proof. 1. Induction on n_σ
Base Case: if $n_\sigma = 0$, then by proposition 2.9.12 $\sigma = id$ so σ is the empty product of zero simple transpositions.
Inductive step: if $n_\sigma ! = 0$ then by prop 2.9.12 there exists $i \in \{1, 2, \ldots, n-1\}$ such that $\sigma(i) > \sigma(i+1)$ then by lemma 2.9.13 $n_{\sigma s_i} = n_\sigma - 1$ so by induction $\sigma = s_i \sigma s_i$ is a product of $n_\sigma - 1$ simple transpositions.

By the induction hypothesis σs_i can be written as a product of $n_{\sigma s_i} = n_\sigma - 1$ simple transpositions, then $\sigma s_i s_i = \sigma$ is a product of n_σ simple transpositions.

2. let $l(\sigma) =$ the min number of simple transpositions needed to express σ. By part1, $l(\sigma) \leq n_\sigma$. Suppose $l(\sigma) = k < n_\sigma$ then $\sigma = s_{i_1} s_{i_2} \ldots s_{i_k}$ is a product

inductive step: assume $l(\sigma) > 0$. then we can find a simple transposition s_i such that $l(\sigma s_i) = l(\sigma) - 1$. Therefore, $l(\sigma s_i) = n_{\sigma s_i}$ by induction hypothesis and $l(\sigma) \geq n_\sigma$ by lemme 2.9. 13

$$l(\sigma) - 1 = l(\sigma s_i) = n\sigma s_i = n_\sigma \pm 1$$

□

Definition 1.2 (2.9.15)**.**
the signs of $\sigma \in S_n$ is $sgn(\sigma) = (-1)^{n_\sigma}$

Proposition 1.5 (2.9.16)**.**
the sign of a permutation

$$sgn : S_n \to \{\pm 1\}$$

is a group homomorphism where composition on $\{\pm 1\}$ is multiplication s

Proof. by lemme 2.9.13, $n_{\sigma s_i} = n_\sigma \pm 1$ so $sgn(\sigma s_i) = (-1)^{n_\sigma \pm 1} = (-1)^{n_\sigma}(-1)^{\pm 1} = sgn(\sigma) sgn(s_i)$ since any $\tau \in S_n$ can be written as a product of simple transpositions it follows that $sgn(\tau) = sgn(\sigma_1) sgn(\sigma_2) \ldots sgn(\sigma_k)$ □

Definition 1.3. the alternating group is

$$A_n = ker(sgn) = \{\sigma \in S_n | sgn(\sigma) = 1\}$$

Note: A_n is a normal subgroup of S_n by the isomorphism theorem, $S_n/A_n \cong \{\pm 1\}$