Math 4022 Assignment 1

Due: Friday, 6 Sept, at 23:59 on GradeScope

Fully justify all your answers in complete sentences. Feel free to use any result proved or presented in class, but be sure to cite it appropriately. Questions 1-3 will be graded out of 4 each (for descriptions of what each point value corresponds to, see the syllabus). Each part of Question 4 will be graded out of 2.

(Question 1) Let G be a connected graph, and suppose P and Q are both maximum-length paths in G. Prove that $V(P) \cap V(Q) \neq \emptyset$. (Note: your solution should not include any pictures, though you are encouraged to draw pictures in your notebook to build your intuition.) Let G be a connected graph, and let P and Q be maximum-length paths in G. We will prove that $V(P) \cap V(Q) \neq \emptyset$.

Proof. Assume, for contradiction, that $V(P) \cap V(Q) = \emptyset$. Since G is connected, there exists a path R connecting a vertex $p \in P$ to a vertex $q \in Q$ with length r.

Let a and b be the lengths of the subpaths of P from its start to p and from p to the end of P, respectively. Thus, $|P| = a + b$. Similarly, let c and d be the lengths of the subpaths of Q from its start to q and from q to the end of Q, respectively. Thus, $|Q| = c + d$.

Consider the following paths: - A: from the start of P to p, then R, then to the end of Q, with length $a + r + d$. - B: from the end of P to p, then R, then to the start of Q, with length $b + r + c$. - C: from the start of P to p, then R, then to the start of Q, with length $a + r + c$.

The sum of the lengths of A, B, and C is:

$$(a + r + d) + (b + r + c) + (a + r + c) = 2a + b + 2c + d + 3r$$

The sum of the lengths of P and Q is:

$$(a + b) + (c + d) = a + b + c + d$$

The difference between these sums is:

$$2a + b + 2c + d + 3r - (a + b + c + d) = a + c + 3r$$

Since $a + c + 3r > 0$, the average length of A, B, and C is:

$$\frac{2a + b + 2c + d + 3r}{3}$$

which is greater than the average length of P and Q:

$$\frac{a + b + c + d}{2}$$

Thus, at least one of A, B, or C must be longer than both P and Q, contradicting the assumption that P and Q are maximum-length paths. Hence, $V(P) \cap V(Q) \neq \emptyset$.

$\square$

(Question 2) Let $k \in \mathbb{N}$ with $k \geq 2$, and let G be the graph with vertex set $V(G) = \{0,1\}^k$ (that is, $V(G)$ is the set of k-tuples with elements in $\{0,1\}$) where for each pair $u, v \in V(G)$, $uv \in E(G)$ if and only if u and v differ in exactly two positions. How many components does G have? Prove your answer is correct. (Note: it might be helpful to draw out some small examples (for $k = 2, 3, 4...$) and try to spot a pattern. As usual, you should not include these drawings in your solution.)

Proof. To determine the number of components in G, consider the following:

The vertex set $V(G)$ is $\{0,1\}^k$, the set of all k-tuples of 0s and 1s. Two vertices u and v are adjacent if and only if they differ in exactly two positions.

To analyze connectivity, consider paths that involve changing exactly two positions. Any path between two vertices u and v in G can be constructed by changing exactly two differing positions incrementally.

Select a specific vertex $v_0 \in V(G)$, for instance, $v_0 = (0, 0, \ldots, 0)$. Consider vertices that differ from v_0 in exactly i positions, where i ranges from 0 to k. These vertices form a connected subgraph. Each subgraph of vertices differing from v_0 in exactly i positions is connected because any two such vertices can be connected through a series of edges involving changes in exactly two positions.

Since there are $k+1$ possible numbers of differing positions (from 0 to k), there are $k+1$ distinct connected subgraphs. Thus, the number of connected components in G is $k+1$. $\square$

(Question 3) Show the following two statements are equivalent.

1. For every pair of vertices u, v in a graph G, there exists a (u, v)-walk.
2. For every partition of $V(G)$ into two sets A and B, there exists an edge from A to B. (Recall: given a set X, a partition $X_1, X_2, \ldots, X_n$ is a collection of nonempty subsets of X such that for all $i, j \in \{1, 2, \ldots, n\}$, $X_i \cap X_j \neq \emptyset \to i = j$ and $X_1 \cup ... \cup X_n = X$ (that is: the sets $X_1, \ldots, X_n$ are pairwise disjoint, and their union is equal to X).)

Proof. $\to$

Assume Statement 1 is true, i.e., for every pair of vertices u, v in G, there exists a (u, v)-walk.

Consider any partition of $V(G)$ into two disjoint sets A and B, such that $A \cup B = V(G)$ and $A \cap B = \emptyset$. We need to show that there is an edge from A to B, meaning there exists at least one edge connecting a vertex in A to a vertex in B.

Suppose, for contradiction, that no edge connects A to B. This would imply that all edges are either entirely within A or entirely within B.

Since G is connected (as assumed in Statement 1), for any $u \in A$ and $v \in B$, there exists a (u, v)-walk in G.

However, if there are no edges between A and B, then any (u, v)-walk from $u \in A$ to $v \in B$ would have to pass entirely through A or B. This contradicts the assumption that G is connected, because it implies that A and B are not connected by any edges.

Thus, there must be at least one edge from A to B.

$\leftarrow$

Assume Statement 2 is true, i.e., for every partition of $V(G)$ into two sets A and B, there exists an edge from A to B.

We need to show that for every pair of vertices u, v in G, there exists a (u, v)-walk.

Consider any pair of vertices u, v in G. We will show that there is a walk from u to v.

If u and v are in the same set of a partition, then we can partition $V(G)$ into two sets $A = \{u\} \cup (V(G) \setminus \{u\})$ and $B = \{v\} \cup (V(G) \setminus \{v\})$. By Statement 2, there exists an edge from A to B, and since u and v are in different sets, there must be an edge from u to v, implying a direct walk.

If u and v are in different sets, then Statement 2 guarantees an edge connecting u and v directly or indirectly through vertices in the sets. Since G is connected by assumption, there exists a path or walk between any pair of vertices in G.

Therefore, Statement 2 implies that every pair of vertices u and v is connected by a walk, thus proving Statement 1.

$\square$

(Question 4) Recall that, given a graph G, $\delta(G) := \min_{v \in V(G)} \deg(v)$. Recall also that the girth of a graph is the length of a shortest cycle in the graph (where if the graph has no cycles, we define the girth to be infinite). In what follows, let G be a graph.

(a) Show that G contains a path of length at least $\delta(G)$.

Proof:

Let $\delta(G) = d$. Consider a vertex $v \in V(G)$ with $\deg(v) = d$.

Starting from v, follow d edges to d distinct neighbors. This constructs a path of length d, as each neighbor can be connected directly.

Hence, G contains a path of length at least d, i.e., $\delta(G)$.

(b) Show that if $\delta(G) \geq 2$, then G contains a cycle of length at least $\delta(G) + 1$.

Proof:

Let $\delta(G) = d \geq 2$. Consider a vertex v with $\deg(v) \geq d$.

In the subgraph induced by the neighbors of v, each vertex has degree $\geq d - 1$. A $(d-1)$-regular graph with $d \geq 2$ contains a cycle of length $\geq d + 1$.

Thus, G contains a cycle of length at least $d + 1$, i.e., $\delta(G) + 1$.

(c) Show that if $\delta(G) \geq k \geq 2$ and $g(G) \geq 4$, then G contains a cycle of length at least $2k$.

Proof:

Let $\delta(G) = k \geq 2$ and $g(G) \geq 4$.

Consider a vertex v with $\deg(v) \geq k$. Each neighbor of v has $\geq k - 1$ neighbors. The subgraph induced by $N(v)$ is $(k-1)$-regular.

For $g(G) \geq 4$, no edge connects two neighbors of v. Thus, the subgraph induced by $N(v)$ has a cycle of length $\geq 2k$.

Therefore, G contains a cycle of length at least $2k$.