Norwegian University of
Science and Technology

RANDOMNESS 3

TTM4205 – Lecture 4

Tjerand Silde

31.08.2023

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Reference Group

I am looking for (at least) three students to form a reference group in this course, preferably students from different programs. We will meet three times during the semester, and your feedback is extremely valuable.

Send me an email and/or talk to me in the break :)

Open PhD Position

Figure: https://www.jobbnorge.no/en/available-jobs/job/246480/phd-candidate-in-cryptography-engineering

Uniped Observation

I am completing a course in University Pedagogy (Uniped) this year, and next week, on Tuesday September 5th, I have so-called *collegial coaching*. This means that a few other lecturers from different departments at NTNU will be observing my lecture and they will provide feedback to me afterwards. They are **not** observing you.

Contents

Primality Testing

How do we check if a number is prime?

Deterministic Methods

- Brute Force
- Sieving methods
- Wilson's Theorem?

Brute Force Testing

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This is infeasible to compute! 2^{128} is considered impossible.

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It still requires exponential work to check all possibilities!

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But it is possible to use similar techniques to speed it up.

Randomized Methods

- Monte Carlo algorithms
- The Miller-Rabin method

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Some commonly used algorithms: Soloway-Strassen, Fermat (warning: Carmichael numbers) and Miller-Rabin.

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If this is false, then p is composite. However: the above fact is true for roughly $\frac{1}{4}$ composite numbers with respect to a.

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If we sample λ random values a, the Miller-Rabin primality testing algorithm has $\frac{1}{4}^{\lambda}$ chance of being wrong every time.

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2. Check p is divisible by any prime number in the list.
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4. If all checks succeeds, then output: *probably prime*.

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Sometimes it is a mix between fixed a's and freshly sampled a's, still giving the adversary a good chance to fool the test.

Primality Testing in OpenSSL

Prime and Prejudice: Primality Testing Under Adversarial Conditions

Martin R. Albrecht[1], Jake Massimo[1], Kenneth G. Paterson[1], and Juraj Somorovsky[2]

[1] Royal Holloway, University of London
[2] Ruhr University Bochum, Germany
martin.albrecht@rhul.ac.uk, jake.massimo.2015@rhul.ac.uk, kenny.paterson@rhul.ac.uk,
juraj.somorovsky@rub.de

Figure: https://eprint.iacr.org/2018/749.pdf

The Need for Secure Primality Testing

Safety in Numbers: On the Need for Robust Diffie-Hellman Parameter Validation

Steven Galbraith[1], Jake Massimo[2], and Kenneth G. Paterson[2]

[1] University of Auckland
[2] Royal Holloway, University of London
s.galbraith@auckland.ac.nz, jake.massimo.2015@rhul.ac.uk, kenny.paterson@rhul.ac.uk

Figure: https://eprint.iacr.org/2019/032.pdf

Secure Primality Testing API

A Performant, Misuse-Resistant API for Primality Testing

Jake Massimo[1] and Kenneth G. Paterson[2]

[1] Information Security Group,
Royal Holloway, University of London
jake.massimo.2015@rhul.ac.uk
[2] Department of Computer Science,
ETH Zurich
kenny.paterson@inf.ethz.ch

Figure: https://eprint.iacr.org/2020/065.pdf

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How do we factor large bi-primes?

Deterministic Methods

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- Pollard's Rho algorithm find largest prime factor in $\sqrt[4]{n}$

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Randomized Methods

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Then we *might* find a factor of n by computing the greatest common divisor between n and $a - b$ and $a + b$.

Number Field Sieve

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Factoring as a service: In 2015 it was possible to factor 512 bit RSA keys in less than four hours.

Factoring as a Service

Factoring as a Service

Luke Valenta, Shaanan Cohney, Alex Liao,
Joshua Fried, Satya Bodduluri, Nadia Heninger

University of Pennsylvania

Figure: https://eprint.iacr.org/2015/1000.pdf

State of the Art

The state of the art in integer factoring and breaking public key cryptography

Fabrice Boudot[1], Pierrick Gaudry[2], Aurore Guillevic[2], Nadia Heninger[3], Emmanuel Thomé[2], and Paul Zimmermann[2]

[1]Université de Limoges, XLIM, UMR 7252, F-87000 Limoges, France
[2]Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France
[3]University of California, San Diego, USA

Figure: `https://hal.science/hal-03691141/document`

RSA Failures in Practice

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- Same seed when sampling primes
- Same seed + added entropy between sampling
- Low entropy RNG or PRNG from known algorithm
- Related primes from known algorithm

PKE in the Wild

RSA, DH and DSA in the Wild*

Nadia Heninger

University of California, San Diego, USA

Figure: https://eprint.iacr.org/2022/048.pdf

Fermat in the Wild

Fermat Factorization in the Wild

Hanno Böck

January 8, 2023

Figure: https://eprint.iacr.org/2023/026.pdf

Shared Prime Factors

Ron was wrong, Whit is right

Arjen K. Lenstra[1], James P. Hughes[2],
Maxime Augier[1], Joppe W. Bos[1], Thorsten Kleinjung[1], and Christophe Wachter[1]

[1] EPFL IC LACAL, Station 14, CH-1015 Lausanne, Switzerland
[2] Self, Palo Alto, CA, USA

Figure: `https://eprint.iacr.org/2012/064.pdf`

Shared Prime Factors

Mining Your Ps and Qs: Detection of Widespread Weak Keys in Network Devices

Nadia Heninger[†*] Zakir Durumeric[‡*] Eric Wustrow[‡] J. Alex Halderman[‡]

[†] *University of California, San Diego*
nadiah@cs.ucsd.edu

[‡] *The University of Michigan*
{zakir, ewust, jhalderm}@umich.edu

Figure: Check out the blog post, paper and slides: 1) `https://freedom-to-tinker.com/2012/02/15/new-research-theres-no-need-panic-over-factorable-keys-just-mind-your-ps-and-qs`, 2) `https://factorable.net/weakkeys12.extended.pdf`, 3) `https://crypto.stanford.edu/RealWorldCrypto/slides/nadia.pdf`

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De-Randomized Crypto

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- HMAC with key for deterministic MAC
- Hedging techniques (next slide)

Shared Prime Factors

**Hedged Public-Key Encryption:
How to Protect Against Bad Randomness**

Mihir Bellare* Zvika Brakerski† Moni Naor‡ Thomas Ristenpart§
Gil Segev¶ Hovav Shacham‖ Scott Yilek**

April 21, 2012

Figure: https://www.cs.utexas.edu/~hovav/dist/hedge.pdf

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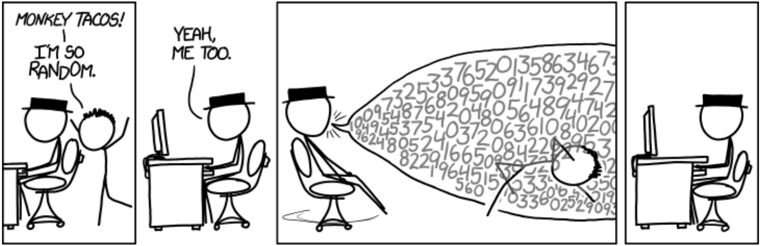
I am so random



Random Number Generation

Check the quality of the built-in RNG that you rely on:

- How does it collect randomness?
- Is the RNG seeded / pre-seeded?
- How much entropy does it provide?
- Does it warn you about issues?
- Is it cryptographically secure?
- (Linux's */dev/random* vs */dev/urandom*)

Faulty Voting Randomness

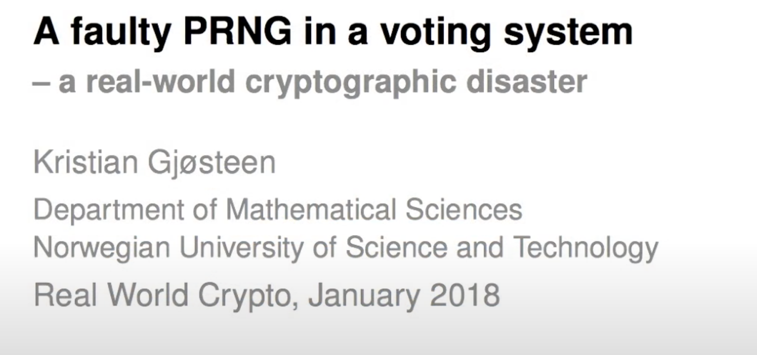


Figure: `https://youtu.be/xq_6ey2JGAE?feature=shared`

Pseudo-Random Number Generation

Check the quality of the built-in PRNG that you rely on:

- Does it rely on a proper RNG as seed? Is it pre-seeded?
- Is the PRNG cryptographically secure? NIST-approved?
- Verify the output: Do values repeat? Correct bit-size?
- Which library/version is used? Known vulnerabilities?

Some good resources are available at `https://github.com/veorq/cryptocoding#use-strong-randomness`.

NIST Standard

National Institute of
Standards and Technology
Technology Administration
U.S. Department of Commerce

Special Publication 800-22
Revision 1a

A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications

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Figure: https://csrc.nist.gov/pubs/sp/800/22/r1/upd1/final

Choice of Primitives

Check the cryptographic primitive that you rely on:

- Does it rely on a proper PRNG? Is it pre-seeded?
- Is it the newest/most secure primitive? NIST-approved?
- Verify the output: Do values repeat? Correct bit-size?
- Which library/version is used? Known vulnerabilities?
- Are there de-randomized algorithms available instead?

Rolling Your Own Crypto

Figure: https://securitycryptographywhatever.buzzsprout.com/1822302/8953842-the-great-roll-your-own-crypto-debate-with-filippo-valsorda

Questions?