Elliptic KZB equations via the universal vector extension

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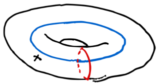
$\pi_1\Big($  $\Big) \cong F(\{a, b\})$

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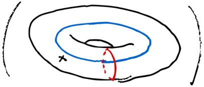
π_1  $\cong F(\{a, b\})$

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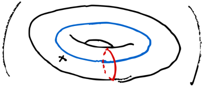
π_1  $\cong F(\{a, b\})$

- Make a case for the universal vector extension of an elliptic curve as the right framework to study the unipotent de Rham fundamental group (Deligne, Enriquez–Etingof).

- Applications to algebraicity and rationality problems for the universal KZB equations.

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- $U(X)$ category of unipotent flat vector bundles:

$$(\mathcal{V}, \nabla), \qquad \nabla : \mathcal{V} \to \Omega^1_{X/k} \otimes \mathcal{V}, \qquad \nabla^2 = 0$$

with

$$0 = (\mathcal{V}_0, \nabla_0) \subset (\mathcal{V}_1, \nabla_1) \subset \cdots \subset (\mathcal{V}_n, \nabla_n) = (\mathcal{V}, \nabla)$$

such that

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- $U(X)$ is Tannakian and every $x\in X(k)$ defines a fibre functor

$$\omega_x: U(X)\to Vect_k, \qquad (\mathcal{V},\nabla)\mapsto x^*\mathcal{V}$$

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$$\mathcal{O}(\pi_1^{dR}(X,x)) \cong T^c H^1_{dR}(X) = \bigoplus_{n\geq 0} H^1_{dR}(X)^{\otimes n}$$

where

$$H^1_{dR}(X) = H^0(\mathbb{P}^1_k, \Omega^1(\log\{0,1,\infty\})) = k\frac{dz}{z} \oplus k\frac{dz}{1-z}.$$

Nice properties: independence of base points, simple poles.

- Every unipotent flat vector bundle on $X = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ is canonically isomorphic to some

$$(\mathcal{O}_X \otimes V, d - \frac{dz}{z} \otimes A_0 - \frac{dz}{1-z} \otimes A_1)$$

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- Proof: consider the canonical extension

$$\overline{\nabla} : \overline{\mathcal{V}} \to \Omega^1_{\mathbb{P}^1_k}(\log\{0, 1, \infty\}) \otimes \overline{\mathcal{V}}$$

and apply

$$\left.\begin{array}{l} H^0(\mathbb{P}^1_k, \mathcal{O}) = k \\ H^1(\mathbb{P}^1_k, \mathcal{O}) = Ext^1(\mathcal{O}, \mathcal{O}) = 0 \end{array}\right\} \text{Deligne's good conditions}$$

Cette dernière hypothèse, très restrictive, est vérifiée si X est rationnelle.

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$$(m + n\tau) \cdot (v_1, v_2, z) = (v_1 + nv_2, v_2, z + m + n\tau)$$

$$\begin{array}{ccc} \mathbb{C}^2 \times \mathbb{C} & & \mathcal{V} \\ \downarrow & \overset{\text{mod } \mathbb{Z}+\tau\mathbb{Z}}{\rightsquigarrow} & \downarrow \\ \mathbb{C} & & E(\mathbb{C}) \end{array}$$

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- A splitting

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{V} \arrow[r] & \mathcal{O} \arrow[l, bend right] \arrow[r] & 0 \end{tikzcd}$$

corresponds to a function $r : E^{an} \to \mathbb{C}$ satisfying

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- No such holomorphic r.

- Real-analytic: $r(z) = Im(z)/Im(\tau)$ (cf. Brown–Levin)

3.5. **Massey products on $\mathcal{E}^{(n)}$.** We use the Eisenstein-Kronecker series F to write down some explicit one-forms on $\mathcal{E}^{(n)}$. First consider a single elliptic curve $\mathcal{E}^{\times}$ with coordinate ξ as above. Write $\xi = s + r\tau$, where $r, s \in \mathbb{R}$ and τ is fixed, and let $\omega = d\xi$ and $\nu = 2\pi i dr$. The classes $[\omega], [\nu]$ form a basis for $H^1(\mathcal{E}^{\times}; \mathbb{C})$.

Lemma 6. *The form $\Omega(\xi; \alpha) = \mathbf{e}(\alpha r) F(\xi; \alpha) d\xi$ is invariant under elliptic transformations $\xi \mapsto \xi + \tau$ and $\xi \mapsto \xi + 1$, and satisfies $d\,\Omega(\xi; \alpha) = \nu\alpha \wedge \Omega(\xi; \alpha)$.*

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- In fact,

$$\mathbb{C}^2/(\mathbb{Z}+\mathbb{Z}\tau) \cong H^1_{dR}(E^{an})/H^1(E^{an},\mathbb{Z}) \cong H^1(E^{an},\mathbb{C}^\times)$$

classifies holomorphic flat line bundles on E^{an}.

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  is universal for extensions of E by vector groups.
- Laumon, Coleman: if k has characteristic zero, then $H^0(E^\natural,\mathcal{O}) = k$ and $H^1(E^\natural,\mathcal{O}) = 0$.

Let $Z \subset E[N](k)$, and set $D := \pi^{-1}(Z) \subset E^\natural$.

Theorem (F.–Matthes; cf. Enriquez–Etingof)

There is a canonical decomposition

$$\Gamma(E^\natural, \Omega^1(\log D)) = \Gamma(E^\natural, \Omega^1) \oplus K^{(1)} \oplus K^{(2)} \oplus \cdots$$

where $K^{(n)}$ are 1-dimensional k-subspaces uniquely determined by:

1. $dK^{(n)} = \Gamma(E^\natural, \Omega^1) \wedge K^{(n-1)}$, *where* $K^{(0)} := \pi^*\Gamma(E, \Omega^1)$,
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We can find $\nu, \omega^{(0)}, \omega_P^{(1)}, \omega_P^{(2)}, \ldots$ such that

$$\Gamma(E^{\natural}, \Omega^1) = k\nu \oplus k\omega^{(0)}, \qquad K^{(n)} = \bigoplus_{P \in Z} k\omega_P^{(n)}$$

and $d\omega_P^{(n)} = \nu \wedge \omega_P^{(n-1)}$, $\omega_P^{(n)} \wedge \omega^{(0)} = 0$, $Res_D(\omega_P^{(n)}) = t_P^{n-1}/(n-1)!$.

Corollary

Every unipotent flat connection on $E^\natural \setminus D$ is isomorphic to

$$(\mathcal{O} \otimes V, d - \nu \otimes A - \omega^{(0)} \otimes B - \sum_{n \geq 1} \sum_{P \in Z} \omega_P^{(n)} \otimes ad_A^{n-1}(C_P))$$

with $A, B, C_P \in End_k(V)$ "simultaneously nilpotent", and $\sum_{P \in Z} C_P = [A, B]$.

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For every $x \in (E^\natural \setminus D)(k)$*, there are canonical isomorphism*

$$\begin{aligned} \mathcal{O}(\pi_1^{dR}(E^\natural \setminus D, x)) &\cong H^0(B(\Gamma(E^\natural, \Omega^\bullet(\log D))) \\ &\cong T^c\Gamma(E^\natural, \Omega^1(\log D))^{d=0} \end{aligned}$$

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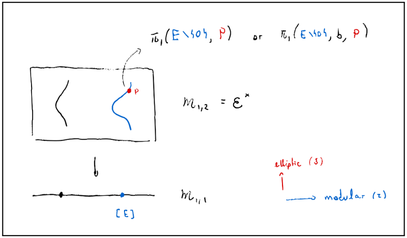
Note: if $Q = \pi(x)$, then $\pi_1^{dR}(E^\natural \setminus D, x) \cong \pi_1^{dR}(E \setminus Z, Q)$, and $\Gamma(E^\natural, \Omega^1(\log D))^{d=0} \cong H^1_{dR}(E \setminus Z)$.

Universal elliptic KZB equations

- Differential equations satisfied by multiple elliptic polylogarithms. Calaque–Enriquez–Etingof, Levin–Racinet, Hain, Luo.



9.2. **The formula.** The connection is defined by a 1-form

$$\omega \in \Omega^1(\mathbb{C} \times \mathfrak{h}, \log \Lambda) \hat{\otimes} \operatorname{End} \mathfrak{p}.$$

via the formula

$$\nabla f = df + \omega f$$

where $f : \mathbb{C} \times \mathfrak{h} \to \mathfrak{p}$ is a (locally defined) section of (9.1). Specifically,

$$\omega = \frac{1}{2\pi i} d\tau \otimes \mathbf{a} \frac{\partial}{\partial \mathbf{t}} + \psi + \nu$$

where

$$\psi = \sum_{m \geq 1} \left(\frac{(2\pi i)^{2m+1}}{(2m)!} G_{2m+2}(\tau) d\tau \otimes \sum_{\substack{j+k=2m+1 \\ j,k>0}} (-1)^j [\operatorname{ad}_{\mathbf{t}}^j(\mathbf{a}), \operatorname{ad}_{\mathbf{t}}^k(\mathbf{a})] \frac{\partial}{\partial \mathbf{a}} \right)$$

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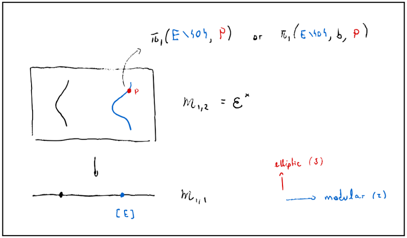

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- “Higher level” version: Calaque-Gonzales, Hopper.

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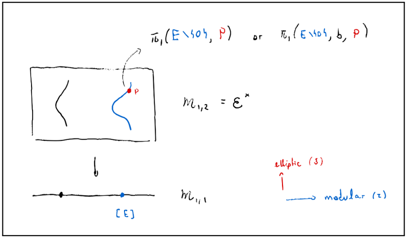

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- Universal vector extension behaves well in families: given $p : E \to S$, get $f : E^\natural \to S$.
- All of the above results generalize to families and are compatible with base change. "Relative fundamental Hopf algebra":

$$\mathcal{H} := H^0(B(f_*\Omega^\bullet_{E^\natural/S}(\log D)).$$

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- Elliptic KZB (vertical direction): integrable S-connection

$$\nabla : f^*\mathcal{H}^\vee \to \Omega^1_{E^\natural/S}(\log D)\hat{\otimes} f^*\mathcal{H}^\vee, \qquad \nabla = d - \Omega$$

$$\Omega = \nu \otimes a + \omega^{(0)} \otimes b + \sum_{n\geq 1}\sum_{P\in Z} \omega_P^{(n)} \otimes ad_a^{n-1}(c_P)$$

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  lifting ∇. "Isomonodromic deformation".
- Key: the universal vector extension is a crystal. Canonical splitting (with integrability condition) of

$$0 \longrightarrow f^*\Omega^1_{S/k} \longrightarrow \Omega^1_{E^\natural/k} \longrightarrow \Omega^1_{E^\natural/S} \longrightarrow 0$$

  or equivalently of

$$0 \longrightarrow \Omega^1_{S/k} \longrightarrow f_*\Omega^1_{E^\natural/k} \longrightarrow f_*\Omega^1_{E^\natural/S} \longrightarrow 0$$

Theorem (F.–Matthes)

The sequence

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$$\widetilde{\omega}^{(1)} = \left(\frac{\theta'_\tau(z)}{\theta_\tau(z)} + 2\pi i\, r\right) dz + \frac{1}{2\pi i}\left(\frac{1}{2}\frac{\theta''_\tau(z)}{\theta_\tau(z)} - \frac{1}{6}\frac{\theta'''_\tau(0)}{\theta'_\tau(0)} - \frac{(2\pi i\, r)^2}{2}\right) d\tau$$

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