On universal vector extensions of elliptic curves and elliptic KZB equations

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IMECC - Unicamp

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Joint work with **Nils Matthes**.

- *Towards algebraic iterated integrals for elliptic curves via the universal vectorial extension.*
  RIMS Kôkyurôku, No. 2160 (2020).
- *A note on the Gauss-Manin connection for abelian schemes.*
  Rend. Sem. Mat. Univ. Padova 152 (2024).
- *Elliptic KZB connections via universal vector extensions*.
  To appear in Algebra & Number Theory.

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- Related via elliptic functions

$$z \mapsto (\wp_\tau(z), \wp'_\tau(z))$$

  Coefficients $g_2(\tau), g_3(\tau)$ are modular forms.

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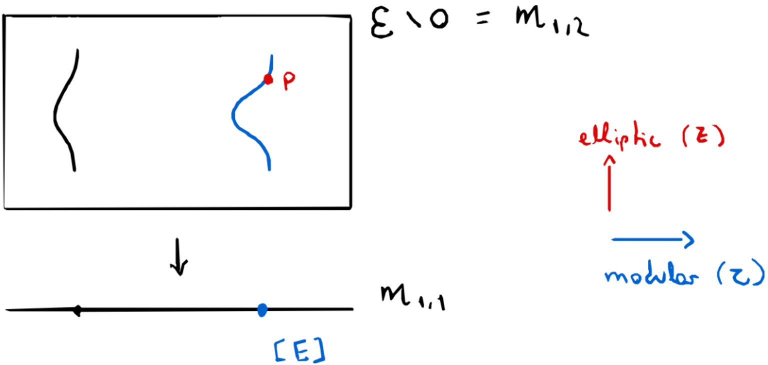
$$\mathcal{E} \setminus O = \mathcal{M}_{1,2} \to \mathcal{M}_{1,1}$$

- Uniformisation:

$$\mathcal{E}^{an} = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2\backslash\!\backslash\mathbb{H} \times \mathbb{C}$$

A point in $\mathcal{M}_{1,2}^{an}$ is determined by two coordinates $(\tau, z) \in \mathbb{H} \times \mathbb{C}$, where τ is 'modular' (horizontal) and z is 'elliptic' (vertical).



universal family elliptic curves punctured at the identity

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- Explicit formula:

$$\nabla_{KZB} = d - dz \otimes ad_a F_\tau(z, ad_a)b - \frac{d\tau}{2\pi i} \otimes (ad_a F'_\tau(z, ad_a)b + D_\tau)$$

where $F'_\tau(z,x) = \frac{\partial}{\partial x} F_\tau(z,x) + \frac{1}{x^2}$, and

$$D_\tau = b\frac{\partial}{\partial a} + \frac{1}{2}\sum_{n\geq 2}(2n-1)G_{2n}(\tau)\sum_{\substack{j+k=2n-1\\ j,k>0}}[(-ad_a)^j b, (ad_a)^k b]\frac{\partial}{\partial b}.$$

and

$$F_\tau(z,x) = \frac{\theta'_\tau(0)\theta_\tau(z+x)}{\theta_\tau(z)\theta_\tau(x)}$$

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- $(\mathcal{V}, \nabla)$ is unipotent if there is a filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$$

  such that $\nabla \mathcal{V}_i \subset \Omega^1_{X/k} \otimes \mathcal{V}_{i-1}$ and $(\mathcal{V}_i/\mathcal{V}_{i-1}, \nabla) \cong (\mathcal{O}_X, d)$.

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Problem: classify (and describe) unipotent integrable vector bundles with connection on X.

Example

Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and

$$\mathcal{V} = \mathcal{O}_X^{\otimes 3}, \qquad \nabla = d - \begin{pmatrix} 0 & \frac{dz}{z} & 0 \\ 0 & 0 & \frac{dz}{1-z} \\ 0 & 0 & 0 \end{pmatrix}$$

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Fundamental solution of $\nabla = 0$:

$$\begin{pmatrix} 1 & log(z) & -Li_2(z) \\ 0 & 1 & log(1-z) \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$Li_2(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^2} = \int_0^z \int_0^y \frac{dx}{x-1} \frac{dy}{y}.$$

Theorem ('well-known...')

Every unipotent vector bundle with integrable connection on $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ *is canonically isomorphic to*

$$(\mathcal{O}_X \otimes V, d - \frac{dz}{z} \otimes X_0 - \frac{dz}{1-z} \otimes X_1)$$

where V *is a finite-dimensional vector space and* $X_0, X_1 \in End(V)$ *are simultaneously strictly upper triangularizable.*

Key properties:

$$\begin{cases} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k \\ H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = Ext^1(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) = 0. \end{cases}$$

"Deligne's good conditions"

KZ connection on $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$:

$$\mathcal{V}_{KZ} = \mathcal{O}_X \hat{\otimes} k\langle\langle x_0, x_1 \rangle\rangle, \qquad \nabla_{KZ} = d - \frac{dz}{z} \otimes x_0 - \frac{dz}{1-z} \otimes x_1$$

Universal property: given $b \in X$, for every unipotent $(\mathcal{V}, \nabla)$ with $v \in \mathcal{V}_b$, there is a unique $(\mathcal{V}_{KZ}, \nabla_{KZ}) \to (\mathcal{V}, \nabla)$ such that $1 \mapsto v$.

Questions:

- Something similar for $X = E \setminus O$?
- What about moduli $\mathcal{M}_{1,1}$? Extend to $\mathcal{M}_{1,2} = \mathcal{E} \setminus O$?

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- $(\mathcal{V}_E, \nabla_E)$ over $E \setminus O$ such that

$$p^*\mathcal{V}_E \cong \mathcal{O}\hat{\otimes}\mathbb{C}\langle\langle a, b\rangle\rangle$$

and

$$\begin{aligned} p^*\nabla_E &= d - dz \otimes ad_a F_\tau(z, ad_a)b \\ &= d - dz \otimes b - \frac{\theta'_\tau(z)}{\theta_\tau(z)} dz \otimes [a, b] - \cdots \end{aligned}$$

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- Note: $H^1(E, \mathcal{O}_E) \neq 0$. Deligne's good conditions are *not* satisfied.

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- If $char(k) = 0$, then

$$\begin{cases} H^0(E^\natural, \mathcal{O}_{E^\natural}) \cong k \\ H^1(E^\natural, \mathcal{O}_{E^\natural}) = 0 \end{cases}$$

Deligne's good conditions are satisfied!

Theorem (F.–Matthes)

If $char(k) = 0$, *then*

$$\Gamma(E^\natural, \Omega^1(log \pi^{-1} O)) = \underbrace{k\nu \oplus k\omega^{(0)}}_{\Gamma(E^\natural, \Omega^1)} \oplus k\omega^{(1)} \oplus \cdots$$

where $\omega^{(n)}$ *are uniquely determined by* $\omega^{(0)} \in \pi^* \Gamma(E, \Omega^1)$ *and*

- $d\omega^{(n)} = \nu \wedge \omega^{(n-1)}$
- $\omega^{(n)} \wedge \omega^{(0)} = 0$
- $Res(\omega^{(n)}) = t^n/(n-1)!$

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The pullback of $(\mathcal{V}_E, \nabla_E)$ *to* $E^\natural$ *is given by*

$$\pi^*\mathcal{V}_E \cong \mathcal{O}\hat{\otimes} k\langle\langle a, b\rangle\rangle, \qquad \pi^*\nabla_E = d - \nu \otimes a - \sum_{n\geq 0} \omega^{(n)} \otimes ad_a^n b.$$

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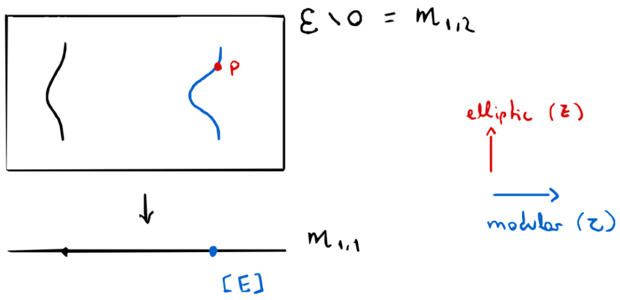
Note: if $k = \mathbb{C}$ and $E^{an} \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, then

$$E^{\natural,an} \cong \mathbb{C}^2/L_\tau, \qquad L_\tau = \{(m + n\tau, n) : m, n \in \mathbb{Z}\}.$$

With coordinates (z, w) on $\mathbb{C}^2$:

$$\nu = dw, \qquad \omega^{(0)} = dz, \qquad \omega^{(1)} = \left(\frac{\theta'_\tau(z)}{\theta_\tau(z)} + 2\pi i w\right) dz, \qquad \ldots$$

What about the universal KZB?



$$\nabla_{KZB} = d - dz \otimes ad_a F_\tau(z, ad_a) b - \frac{d\tau}{2\pi i} \otimes (ad_a F'_\tau(z, ad_a) b + D_\tau)$$

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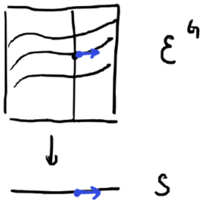
- *Canonical* horizontal foliation (isomonodromy of $(\mathcal{L}, \nabla)$)

$$0 \longrightarrow T_{\mathcal{E}^\natural/S} \longrightarrow T_{\mathcal{E}^\natural} \longrightarrow f^* T_S \longrightarrow 0$$



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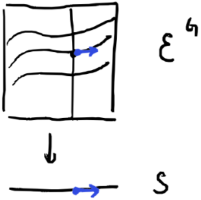

- **Laumon** '96, **Coleman** '98: If $char(S) = 0$,

$$f_* \mathcal{O}_{\mathcal{E}^\natural} \cong \mathcal{O}_S, \qquad R^1 f_* \mathcal{O}_{\mathcal{E}^\natural} = 0.$$

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- Our construction relies on a theorem which provides canonical lifts of relative logarithmic differential forms

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1_S \arrow[r] & f_*\Omega^1_{\mathcal{E}^\natural}(\log \pi^{-1} O) \arrow[r] & f_*\Omega^1_{\mathcal{E}^\natural/S} \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$

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- Example:

$$
\begin{aligned}
\widetilde{\omega}^{(1)} = & \left(\frac{\theta'_\tau(z)}{\theta_\tau(z)} + 2\pi i w \right) dz \\
& + \frac{1}{2\pi i} \left(\frac{1}{2} \frac{\theta''_\tau(z)}{\theta_\tau(z)} - \frac{1}{6} \frac{\theta'''_\tau(0)}{\theta'_\tau(0)} - \frac{(2\pi i w)^2}{2} \right) d\tau.
\end{aligned}
$$

Thank you!