International seminar
on Automorphic forms
13/04/2021

# The algebraic geometry of Fourier coefficients of Poincaré series

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Summary:

- Relate Fourier coefficients of Poincaré series (of positive and negative weight) with periods of algebraic varieties.
- Algebraicity / transcendence questions.

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Basic notation:

upper half-plane

- $\mathbb{H} = \{ \tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0 \}$
- $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}$

Hecke's congruence subgroup

acting on $\mathbb{H}$ via $\gamma \tau = \dfrac{a\tau + b}{c\tau + d}$.

**Definition** A weakly holomorphic modular form of weight $k$ and level $N$ is a holomorphic fct. $f : \mathbb{H} \to \mathbb{C}$ s.t.

1) $\forall \gamma \in \Gamma$, $\quad (cz+d)^{-k} f(\gamma z) = f(z)$

2) $\forall \gamma \in SL_2(\mathbb{Z})$, $\exists \rho \in \mathbb{R}$ s.t.

$$(cz+d)^{-k} f(\gamma z) = O\left(e^{\rho \, \mathrm{Im} z}\right) \quad \text{as } \mathrm{Im} z \to +\infty$$

meromorphic at the cusps

$\mathbb{C}$-vector space : $M^!_k(\Gamma_0(N))$

**Example** (Poincaré series)

$k \geq 4$, $m \in \mathbb{Z}$, $N \geq 1$

$$\boxed{P_{m,k,N}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{e^{2\pi i m \gamma z}}{(cz+d)^k}} \in M^!_k(\Gamma_0(N))$$

$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}$

- $k = 2$ ("Hecke's trick")
- $m = 0$ $\rightsquigarrow$ Eisenstein series

Fourier coefficients of $f \in M_k^!(\Gamma_0(N))$:

$a_n(f) \in \mathbb{C}$ s.t.

$$f(z) = \sum_{n \geq -\infty} a_n(f)\, q^n, \qquad q = e^{2\pi i z}$$

## Example

$m > 0, \quad k \geq 2, \quad N \geq 1$

$$P_{m,k,N}(z) = q^m + \sum_{n \geq 1} \left( 2\pi (-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c \geq 1 \\ N \mid c}} \frac{K(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right) q^n$$

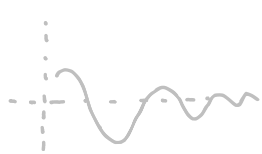
$$P_{-m,k,N}(z) = q^{-m} + \sum_{n \geq 1} \left( 2\pi (-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c \geq 1 \\ N \mid c}} \frac{K(-m,n;c)}{c} I_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right) q^n$$

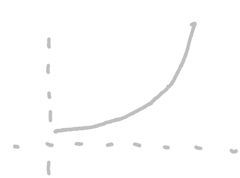
where

Kloosterman sum

- $K(a,b;c) = \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^\times} e^{\frac{2\pi i}{c}(ax + bx^{-1})} \in \mathbb{R} \cap \overline{\mathbb{Q}}$

- $J_{k-1}$, $I_{k-1}$ Bessel functions





What can we say about $a_n(P_{\pm m,k,N})$ ?

- Growth in $n$ : well understood
- Arithmetic nature : ? ? ?

Some examples:

doesn't look algebraic!

1) $P_{1,12,1} = 2{,}84028\ldots\,\Delta$

where $\Delta(\tau) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1} \underbrace{\tau(n)}_{\in\mathbb{Z}} q^n$

Lehmer's conjecture : $\tau(n)\neq 0 \quad \forall\, n\geq 1$

2) $P_{-1,2,1} = -q\dfrac{dj}{dq} \quad \in \mathbb{Z}[\![q]\!]$

where $j(\tau) = \dfrac{1}{q} + 744 + 196884\,q + \cdots$

Corollary (Petersson, Rademacher) : $a_n(j) \sim \dfrac{e^{2\pi\sqrt{n}}}{\sqrt{2}\; n^{3/4}}$

3) $P_{m,4,8} \equiv 0 \;, \quad m = 2, 4, 6, 8, \ldots$

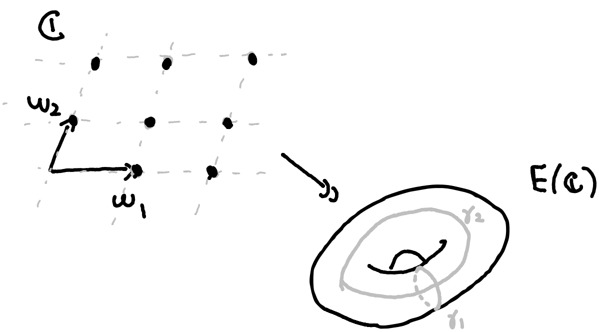
4) $P_{-1,4,9} = \dfrac{1}{q} + 2q^2 - 49q^5 + 48q^8 + \cdots \in \mathbb{Z}[\![q]\!]$

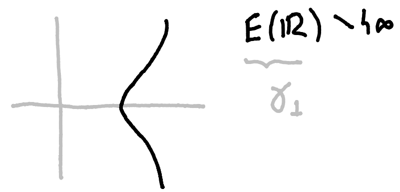
(Bruinier - Ono - Rhoades '08, Candelori '14)

Slogan: Fourier coefficients of Poincaré series are given by single-valued periods of modular motives!

Consider the elliptic curve

$$E \,:\, y^2 + y = x^3 - x^2 - 10x - 20$$




| Periods | Quasi-periods |
|---|---|
| $\omega_1 = \int_{\gamma_1} \frac{dx}{2y+1}$ | $\eta_1 = \int_{\gamma_1} x \frac{dx}{2y+1}$ |
| $= 1.269\ldots$ | $= -2.214\ldots$ |
| $\omega_2 = \int_{\gamma_2} \frac{dx}{2y+1}$ | $\eta_2 = \int_{\gamma_2} x \frac{dx}{2y+1}$ |
| $= 0.634\ldots + i\, 1.458\ldots$ | $= -1.107\ldots + i\, 2.405\ldots$ |

One can numerically verify that:

$$a_1(P_{1,2,11}) = 1.696\ldots = -\frac{2\pi i}{\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2}$$

$$a_1(P_{-1,2,11}) = -0.952\ldots = \frac{\bar{\omega}_1\eta_2 - \bar{\omega}_2\eta_1}{\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2} - 1$$

Reason : $E = X_0(11)$ (over $\mathbb{C}$, compact. of $\Gamma_0(11)\backslash\mathbb{H}$)

and $S = \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}^{-1} \begin{pmatrix} \bar{\omega}_1 & \bar{\eta}_1 \\ \bar{\omega}_2 & \bar{\eta}_2 \end{pmatrix} = \frac{1}{2\pi i}\begin{pmatrix} \frac{\bar{\omega}_1\eta_2 - \bar{\omega}_2\eta_1}{\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2} & \frac{\bar{\eta}_1\eta_2 - \eta_1\bar{\eta}_2}{\omega_1\bar{\eta}_2 - \omega_2\bar{\eta}_1} \end{pmatrix}$

is the single-valued period matrix of $H^1(X_0(11))$.

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Periods (Kontsevich - Zagier)

$X_{/\mathbb{Q}}$ smooth affine $\rightsquigarrow$ motives $H^n(X)$

$$\text{comp}: \; H^n_{dR}(X)\otimes_{\mathbb{Q}}\mathbb{C} \xrightarrow{\sim} H^n_B(X)\otimes_{\mathbb{Q}}\mathbb{C}$$

$$[\omega] \mapsto \left([\sigma] \mapsto \int_\sigma \omega\right)$$

Period matrix :

$$P = \begin{pmatrix} \int_{\sigma_1}\omega_1 & \cdots & \int_{\sigma_1}\omega_r \\ \vdots & & \vdots \\ \int_{\sigma_r}\omega_1 & \cdots & \int_{\sigma_r}\omega_r \end{pmatrix} \in GL_r(\mathbb{C})$$

Single-valued periods (Brown-Dupont '18):

(combinations of integrals of the form $\int_\sigma \omega \wedge \bar{\eta}$, archimedean analogs of $p$-adic periods)

Complex conj. $X(\mathbb{C}) \to X(\mathbb{C}) \rightsquigarrow F_\infty : H^n_B(X) \to H^n_B(X)$

induces via comp

$$\mathfrak{s} : H^n_{dR}(X) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} H^n_{dR}(X) \otimes_{\mathbb{Q}} \mathbb{R}$$

Single-valued period matrix:

$$S = P^{-1} \bar{P} = P^{-1} F_\infty P \in GL_r(\mathbb{R})$$

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Modular motives: $k \geq 2$ even, $N \geq 1$

weakly holomorphic cusp forms ($a_0 = 0$) with $a_n \in \mathbb{Q}$ $\forall n$

$$\begin{tikzcd}
 & M(k, N) \arrow[dl, dashed, "dR"'] \arrow[dr, dashed, "B"] & \\
\dfrac{S^!_k(\Gamma_0(N))_{\mathbb{Q}}}{D^{k-1} M^!_{-k}(\Gamma_0(N))_{\mathbb{Q}}} \otimes_{\mathbb{Q}} \mathbb{C} \arrow[rr, "\text{comp}", "\sim"'] & & H^1_{par}(\Gamma_0(N), V_{k-2}) \otimes_{\mathbb{Q}} \mathbb{C}
\end{tikzcd}$$

"Eichler-Shimura"

subquotient of $H^{k-1}(\underbrace{\mathcal{E} \times_{Y_0(N)} \times \cdots \times_{Y_0(N)} \mathcal{E}}_{k-2})$

- $M(2, N) = H^1(X_0(N))$
- Periods are periods and "quasi-periods" of modular forms (cf. Brown - Hain '18)
- Single-valued periods ?

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<u>Theorem</u> (F. '20) Let $S = (s_{ij})_{1\leq i,j\leq r} \in GL_r(\mathbb{R})$ be a single-valued period matrix of $M(k, N)$. Then:

$$\mathbb{Q}(s_{ij} : 1 \leq i,j \leq r) = \mathbb{Q}(a_n(P_{m,k,N}) : m,n \in \mathbb{Z})$$

Easy corollaries :

- $\mathbb{Q}(a_n(P_{m,k,N}) : m,n \in \mathbb{Z})$ is finitely generated.
- If $M(k, N) = 0$, then $a_n(P_{m,k,N}) \in \mathbb{Q}$ $\forall m,n$

  e.g. $M(2,1) = H^1(X_0(1)) = H^1(\mathbb{P}^1) = 0$

  $$P_{-1,2,1} = \frac{1}{q} - 196884\, q + 42987520\, q^2 + \dots$$

$\Longleftrightarrow$ $S_k(\Gamma_0(N)) = 0$

Assume $M(k,N)$ has rank 2 ($\Leftrightarrow \dim S_k(\Gamma_0(N)) = 1$)

Let $f \in S_k(\Gamma_0(N))_{\mathbb{Q}}$ and $g \in S_k^!(\Gamma_0(N))_{\mathbb{Q}}$ induce a basis of $M(k,N)_{dR} = S_k^!(\Gamma_0(N))_{\mathbb{Q}} / D^{k-1} M^!_{2-k}(\Gamma_0(N))$ and $S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ be the corresponding single-valued period matrix.

**Theorem** (F. '20) For every $m \geq 1$, there is $h_m \in M^!_{2-k}(\Gamma_0(N))_{\mathbb{Q}}$ s.t., for every $n \geq 1$,

$$a_n(P_{m,k,N}) = -\frac{(k-2)!}{m^{k-1}} a_m(f) a_n(f) \frac{1}{s_{21}}$$

$$a_n(P_{-m,k,N}) = \frac{(k-2)!}{m^{k-1}} a_m(f) a_n(f) \frac{s_{11}}{s_{21}} + r_{m,n}$$

where $r_{m,n} = \frac{(k-2)!}{m^{k-1}} a_m(f) a_n(g) + n^{k-1} a_n(h_m)$.

Note: $s_{21} \doteq (f,f)_{Pet}$, $s_{11}/s_{21} \doteq C_f^+(1)$

**Corollary** If $f$ has CM, then $a_n(P_{-m,k,N}) \in \mathbb{Q}$.

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix} \Rightarrow \frac{s_{11}}{s_{21}} = \frac{b}{\bar{a} - a}$$

e.g. $(k,N) = (4,9)$ CM by $\mathbb{Q}(\sqrt{-3}) \rightsquigarrow a_n(P_{-m,4,9}) \in \mathbb{Q}$.

# How to prove?

- Explicit description of

$$s : M(k,N)_{dR} \otimes \mathbb{R} \longrightarrow M(k,N)_{dR} \otimes \mathbb{R}$$

via harmonic Maass forms

$\xi_{2-k}(F) = \frac{1}{(\mathrm{Im}\,\tau)^{2-k}}\overline{\frac{\partial F}{\partial \tau}}$ $\qquad\qquad$ $D = q\frac{d}{dq}$

$$\begin{tikzcd}
 & \mathcal{H}^!_{2-k}(\Gamma_0(N)) \arrow[dl, "\frac{1}{(4\pi)^{k-1}}\xi_{2-k}"'] \arrow[dr, "\frac{1}{(k-2)!}D^{k-1}"] & \\
M^!_k(\Gamma_0(N)) \arrow[d] \arrow[rr, phantom, "\circlearrowright"] & & M^!_k(\Gamma_0(N)) \arrow[d] \\
\frac{M^!_k(\Gamma_0(N))}{D^{k-1}M^!_{2-k}(\Gamma_0(N))} \arrow[rr, "s \otimes c_{dR}"'] & & \frac{M^!_k(\Gamma_0(N))}{D^{k-1}M^!_{2-k}(\Gamma_0(N))}
\end{tikzcd}$$

$\left(\mathbb{Q} \oplus M(k,N)_{dR}\right) \otimes \mathbb{C}$ $\qquad\qquad$ "flipping operator"

- Bringmann - Ono '07

$$\Rightarrow \quad s([P_{m,k,N}]) = -[P_{-m,k,N}]$$

- "Linear algebra" ...