Algebraic formulae for differential equations of polylogarithms: genus 0, 1, and beyond?

Tiago J. Fonseca

IMECC - Unicamp

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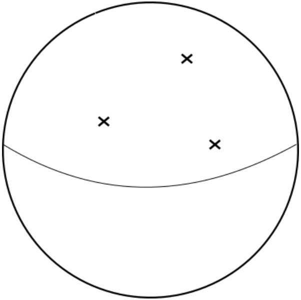
Plan of the talk:

- Overview on classical (multiple) polylogarithms and their differential equations. Higher genus analogues?
- These are period functions associated to motivic fundamental groups, but our focus is on their differential equations.
- Why the genus 1 case is so much more difficult than genus 0?
- Joint work with **Nils Matthes** on genus 1.
- Some thoughts on higher genus.
- Slides:

`https://tjfonseca.github.io/YMSC-2024.pdf`

Genus 0



- Polylogarithms:
$$Li_k(z) = \sum_{n>0} \frac{z^n}{n^k}.$$
Example: $Li_1(z) = -\log(1-z)$.
- Multiple polylogarithms:
$$Li_{k_1,\ldots,k_r}(z) = \sum_{n_1>\cdots>n_r>0} \frac{z^{n_1}}{n_1^{k_1}\cdots n_r^{k_r}}.$$
- Multivalued analytic continuation to $\mathbb{P}^1 \setminus \{0,1,\infty\}$.
- Arithmetic phenomena when $z \in \overline{\mathbb{Q}}$.

Example (Special values of Dedekind zeta functions)

Zagier's conjecture:

$$\zeta_F^*(1-m) \sim \det(\mathcal{L}_m(\xi_j^\sigma)),$$

where $\mathcal{L}_m(z)$ are 'single-valued polylogarithms', e.g.

$$\mathcal{L}_2(z) = -2i\mathrm{Im}(Li_2(z)) + 2\log|z|\log(1-\overline{z})$$

($m = 2, 3, 4$ proved by **Zagier**, **Goncharov**, **Goncharov-Rudenko**.)

Example (Multiple zeta values)

$$\zeta(k_1, \ldots, k_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

Theorem (**Brown** '11): $\zeta(k_1, \ldots, k_r)$ with $k_i \in \{2, 3\}$ span the $\mathbb{Q}$-vector space of MZVs.

MPLs are solutions of differential equations on $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$:

Example

Define a connection $\nabla : \mathcal{O}_X^{\oplus 3} \to \Omega_X^1 \otimes \mathcal{O}_X^{\oplus 3}$ by

$$\nabla = d - A_0 \frac{dz}{z} - A_1 \frac{dz}{1-z}, \qquad A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Fundamental system of solutions:

$$\begin{pmatrix} 1 & \log z & -Li_2(z) \\ 0 & 1 & -Li_1(z) \\ 0 & 0 & 1 \end{pmatrix}$$

Expression as iterated integral:

$$Li_2(z) = \int_0^z Li_1(y) \frac{dy}{y} = \int_0^z \int_0^y \frac{dx}{x-1} \frac{dy}{y}$$

A flat vector bundle $(\mathcal{V}, \nabla)$ is unipotent if it can be obtained by successive extensions of $(\mathcal{O}_X, d)$.

Theorem

Let k be a field of characteristic zero. Every unipotent flat vector bundle on $X = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ is canonically isomorphic to some

$$(\mathcal{O}_X \otimes V, d - \frac{dz}{z} \otimes A_0 - \frac{dz}{1-z} \otimes A_1)$$

with $A_0, A_1 \in End_k(V)$ "simultaneously strictly upper-triangular".

Proof: Consider the canonical extension $\overline{\nabla} : \overline{\mathcal{V}} \to \Omega^1_{\mathbb{P}^1_k}(\log D) \otimes \overline{\mathcal{V}}$ and apply

$$\left.\begin{array}{l} H^0(\mathbb{P}^1_k, \mathcal{O}) = k \\ H^1(\mathbb{P}^1_k, \mathcal{O}) = Ext^1(\mathcal{O}, \mathcal{O}) = 0 \end{array}\right\} \text{Deligne's good conditions.}$$

KZ connection $(\mathcal{V}_{KZ}, \nabla_{KZ})$ (**Knizhnik–Zamolodchikov**)

- Consider all unipotent flat connections on $X = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ "at the same time".
- $\mathcal{V}_{KZ}$ is the trivial (pro-)vector bundle $\mathcal{O}_X \hat{\otimes} k\langle\langle x_0, x_1 \rangle\rangle$ and

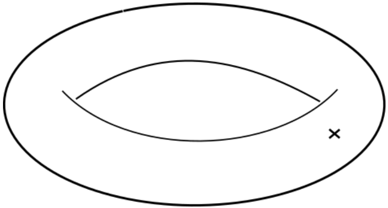
$$\nabla_{KZ} = d - \frac{dz}{z} \otimes x_0 - \frac{dz}{1-z} \otimes x_1,$$

  where x_i act by left multiplication.
- Universal property: given $b \in X(k)$ (or a fibre functor),

  for all flat unipotent $(\mathcal{V}, \nabla)$ with $v \in \mathcal{V}(b)$,
  there is a unique $(\mathcal{V}_{KZ}, \nabla_{KZ}) \to (\mathcal{V}, \nabla)$ with $1 \mapsto v$.

- Motivic interpretation: $\mathcal{V}_{KZ}$ is dual to the $\mathcal{O}_X$-algebra corresponding to the "universal torsor" $\pi_1^{dR}(X; b, \cdot)$.

Genus 1



Elliptic versions of polylogarithms (90's)?

- Let E be a complex elliptic curve with identity O.
- Analogues of $Li_k(z)$ defined over $E \setminus O$?
- **Bloch**'s elliptic dilogarithm: write $E = \mathbb{C}^\times / q^{\mathbb{Z}}$ and define

$$D_E(x) = \sum_{m=-\infty}^{\infty} \mathcal{L}_2(q^m x).$$

- When E is defined over $\mathbb{Q}$, computes $L(E, 2)$ (**Bloch**, **Beilinson**, **Goncharov–Levin**, ...).
- Elliptic polylogarithmic sheaves (**Beilinson–Levin**).
- Some analytic formulae for EPLs (**Levin**).

What about multiple polylogarithms?

- Should be related to horizontal sections of unipotent flat vector bundles on $E \setminus O$.
- **Levin–Racinet** ('07): let $(\mathcal{V}_E, \nabla_E)$ be the pro-flat vector bundle satisfying the universal property: given $b \in E \setminus O$ (or a fibre functor) there is $1_E \in \mathcal{V}_E(b)$ such that

  for all flat unipotent $(\mathcal{V}, \nabla)$ with $v \in \mathcal{V}(b)$,

  there is a unique $(\mathcal{V}_E, \nabla_E) \to (\mathcal{V}, \nabla)$ with $1_E \mapsto v$.

  Similar motivic interpretation in terms of $\pi_1^{dR}(E \setminus O, b, \cdot)$.
- KZB connection (**Knizhnik–Zamolodchikov–Bernard**).
- How to compute it? Problem: $H^1(E, \mathcal{O}) \neq 0$ (Deligne's good conditions are not satisfied).

Levin–Racinet:

- Write $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for some $\tau \in \mathbb{H}$ and let $\pi : \mathbb{C} \to E$ be the uniformisation.
- Consider the Kronecker function

$$F_\tau(z, x) = \frac{\theta'_\tau(0)\theta_\tau(z + x)}{\theta_\tau(z)\theta_\tau(x)}.$$

- Then $\pi^*\mathcal{V}_E \cong \mathcal{O}\hat{\otimes}\mathbb{C}\langle\langle a, b\rangle\rangle$ and

$$\pi^*\nabla_E = d - dz \otimes ad_a F_\tau(z, ad_a)b$$

  where $ad_a c = ac - ca$.
- Problems:
  1. Not algebraic. Field of definition?
  2. What are the differential forms analogous to $\frac{dz}{z}$ and $\frac{dz}{1-z}$ in genus 0?

Given a Weierstrass equation

$$E : y^2 = 4x^3 - g_2 x - g_3, \qquad g_2, g_3 \in k.$$

What's wrong with the naïve KZB

$$\nabla_n = d - \frac{dx}{y} \otimes a - x\frac{dx}{y} \otimes b \ \ ?$$

We do have $(\mathcal{V}_E, \nabla_E) \cong (\mathcal{O}\hat{\otimes} k\langle\langle a, b\rangle\rangle, \nabla_n)$ over $E \setminus O$ but:

- Higher order poles (bad for MHS and 'regularisation').
- Non-canonicity: any basis of $H^1_{dR}(E \setminus O)$ would do.
- Iterated integrals?
- Trouble to put in families.

Brown–Levin MEPLs ('11):

- $F_\tau(z,x)$ do not descend to E: $F_\tau(z+\tau,x) = e^{-2\pi i x} F_\tau(z,x)$.
- Let $r(z) = Im(z)/Im(\tau)$, so that

$$r(z+\tau) = r(z) + 1.$$

- Consider the real-analytic 1-forms

$$\nu_{BL} = 2\pi i\, dr, \qquad \omega_{BL}^{(n)}, \quad n \geq 0$$

where

$$e^{2\pi i\, r\, x} F_\tau(z,x) dz = \sum_{n\geq 0} \omega_{BL}^{(n)} x^{n-1}.$$

Example: $\omega_{BL}^{(0)} = dz$, $\omega_{BL}^{(1)} = d\log\theta_\tau(z) + 2\pi i\, r\, dz$, ...

- MEPLs are iterated integrals of $\nu_{BL}, \omega_{BL}^{(n)}$. Agrees with q-averaging MPLs, $q = e^{2\pi i \tau}$.

How to algebraise?

- Use the universal vector extension $E^\natural \to E$ (cf. **Deligne**, **Enriquez–Etingof**).
- Consider $\mathbb{C}^2$ with coordinates (z, r), and lift the action of $\mathbb{Z} + \tau\mathbb{Z}$ by

$$(m + n\tau) \cdot (z, r) = (z + m + n\tau, r + n).$$

- The quotient $E^\natural = \mathbb{C}^2/(\mathbb{Z} + \tau\mathbb{Z})$ has a natural structure of algebraic variety (commutative algebraic group): moduli space of flat line bundles on E.
- **Laumon**, **Coleman**: if E is defined over a field k of characteristic zero,

$$H^0(E^\natural, \mathcal{O}) = k \text{ and } H^1(E^\natural, \mathcal{O}) = 0.$$

Theorem (F.–Matthes)

Let $\pi : E^\natural \to E$ be the projection and set $D = \pi^{-1}O$. Then:

$$\Gamma(E^\natural, \Omega^1(\log D)) = k\nu \oplus k\omega^{(0)} \oplus k\omega^{(1)} \oplus k\omega^{(2)} \oplus \cdots$$

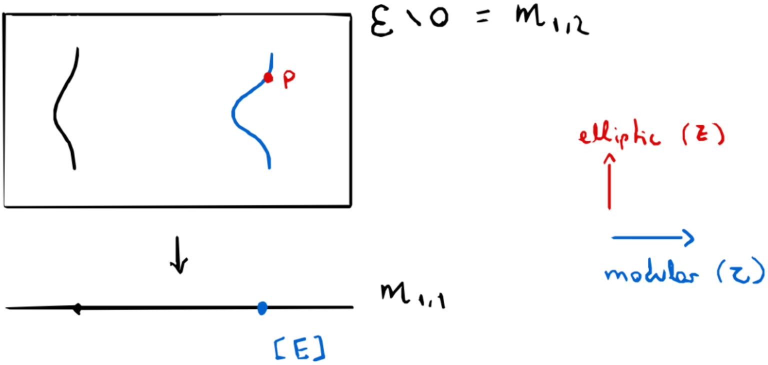
where $\Gamma(E^\natural, \Omega^1) = k\nu \oplus k\omega^{(0)}$, $\pi^*\Gamma(E, \Omega^1) = k\omega^{(0)}$, and

1. $d\omega^{(n)} = \nu \wedge \omega^{(n-1)}$,
2. $\omega^{(n)} \wedge \omega^{(0)} = 0$,
3. $Res_D(\omega^{(n)}) = t^{n-1}/(n-1)!$, *where t is the coordinate on D induced by ν.*

The $\omega^{(n)}$ are uniquely determined by the choice of ν. Moreover, the pullback of the KZB connection is

$$\pi^*\mathcal{V}_E \cong \mathcal{O}\hat{\otimes} k\langle\langle a, b\rangle\rangle, \qquad \pi^*\nabla_E = d - \nu \otimes a - \sum_{n\geq 0} \omega^{(n)} \otimes ad_a^n b.$$

New problem: elliptic KZB in families?



universal family elliptic curves punctured at the identity

- **Calaque–Enriquez–Etingof** '09: universal elliptic KZB

$$\nabla_{KZB} = d - dz \otimes ad_a F_\tau(z, ad_a) b - \frac{d\tau}{2\pi i} \otimes (ad_a F'_\tau(z, ad_a) b + D_\tau)$$

where $F'_\tau(z,x) = \frac{\partial}{\partial x} F_\tau(z,x) + \frac{1}{x^2}$, and

$$D_\tau = b\frac{\partial}{\partial a} + \frac{1}{2}\sum_{n\geq 2}(2n-1)G_{2n}(\tau)\sum_{\substack{j+k=2n-1\\ j,k>0}}[(-ad_a)^j b, (ad_a)^k b]\frac{\partial}{\partial b}.$$

"Isomonodromic deformation of KZB on a fibre of $\mathcal{E} \to \mathcal{M}_{1,1}$."

- **Calaque–Gonzalez** '20: higher levels (puncture at torsion points)
- **Luo** '19: algebraicity over $\mathbb{Q}$ in level 1.

F.–Matthes '23:

- Given $E \to S$, get $f : E^\natural \to S$.
- "Vertical KZB" $\nabla_{/S}$ is an S-connection defined on $f^*\mathcal{H}^\vee$, where

$$\mathcal{H} = H^0(B(f_*\Omega^\bullet_{E^\natural/S}(\log D)))$$

Locally, $\mathcal{H}^\vee \cong \mathcal{O}_S\langle\langle a, b\rangle\rangle$ and

$$\nabla_{/S} = d - \nu \otimes a - \sum_{n\geq 0} \omega^{(n)} \otimes ad_a^n b$$

where $\nu, \omega^{(n)}$ are now relative 1-forms in $\Gamma(E^\natural, \Omega^1_{E^\natural/S}(\log D))$.

- "Absolute KZB" ∇ is a k-connection on $f^*\mathcal{H}^\vee$, locally

$$\nabla = f^*\delta^\vee - \widetilde{\nu} \otimes a - \sum_{n\geq 0} \widetilde{\omega}^{(n)} \otimes ad_a^n b$$

where $\widetilde{\nu}, \widetilde{\omega}^{(n)}$ are canonical lifts and $\delta : \mathcal{H} \to \mathcal{H}$ is a Gauss–Manin connection.

Theorem (F.–Matthes)

The sequence

$$0 \longrightarrow \Omega^1_{S/k} \longrightarrow f_*\Omega^1_{E^\natural/k}(\log D) \longrightarrow f_*\Omega^1_{E^\natural/S}(\log D) \longrightarrow 0$$

is exact and canonically split.

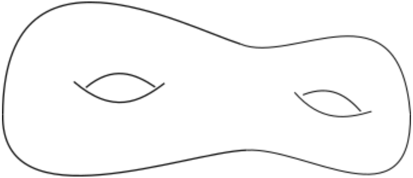
Example for $E^\natural \to \mathbb{H}$ with $E^\natural_\tau = \mathbb{C}^2/(\mathbb{Z} + \tau\mathbb{Z})$. Relative form:

$$\omega^{(1)} = \left(\frac{\theta'_\tau(z)}{\theta_\tau(z)} + 2\pi i\, r \right) dz$$

Canonical lift:

$$\widetilde{\omega}^{(1)} = \left(\frac{\theta'_\tau(z)}{\theta_\tau(z)} + 2\pi i\, r \right) dz + \frac{1}{2\pi i} \left(\frac{1}{2} \frac{\theta''_\tau(z)}{\theta_\tau(z)} - \frac{1}{6} \frac{\theta'''_\tau(0)}{\theta'_\tau(0)} - \frac{(2\pi i\, r)^2}{2} \right) d\tau$$

Higher genus?



- C curve of genus ≥ 1.
- Same problem: $H^1(C, \mathcal{O}) \neq 0$ (there are non-trivial unipotent vector bundles).
- Find good scheme $\pi : \widetilde{C} \to C$ over which unipotent vector bundles trivialize.
- **N. Dogra**'s remark: $E^\natural \cong \pi_1^{coh}(E; O, \cdot)$ is a Tannakian universal torsor for the category of unipotent vector bundles.
- $\widetilde{C} = \pi_1^{coh}(C; b, \cdot)$?

Thank you!