From multiple polylogarithms to the universal vector extension of an elliptic curve

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- Joint work in progress with **Nils Matthes**.
- Some algebro-geometric aspects on an elliptic analogue of the theory of multiple polylogarithms.
- Algebraic de Rham fundamental group of punctured elliptic curves, over an arbitrary base.
- Classification of unipotent connections on punctured elliptic curves (Levin–Racinet, Brown–Levin, Hain, Enriquez–Etingof).
- Goal of the talk: make a case for the universal vector extension of an elliptic curve as the right framework to study these questions (Deligne).

- Polylogarithms:

$$Li_k(z) = \sum_{n>0} \frac{z^n}{n^k}.$$

Example: $Li_1(z) = -\log(1-z)$.

- Multiple polylogarithms:

$$Li_{k_1,\ldots,k_r}(z) = \sum_{n_1>\cdots>n_r>0} \frac{z^{n_1}}{n_1^{k_1}\cdots n_r^{k_r}}.$$

- Arithmetic phenomena when $z \in \overline{\mathbb{Q}}$.

Example (Special values of Dedekind zeta functions)

Zagier's conjecture:

$$\zeta_F^*(1-m) \sim \det(\mathcal{L}_m(\xi_j^\sigma)),$$

where $\mathcal{L}_m(z)$ are 'single-valued polylogarithms', e.g.

$$\mathcal{L}_2(z) = -2i\mathrm{Im}(Li_2(z)) + 2\log|z|\log(1-\overline{z})$$

($m = 2, 3, 4$ proved by Zagier, Goncharov, Goncharov-Rudenko.)

Example (Multiple zeta values)

$$\zeta(k_1, \ldots, k_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

Theorem (Brown '11): $\zeta(k_1, \ldots, k_r)$ with $k_i \in \{2, 3\}$ span the $\mathbb{Q}$-vector space of MZVs.

MPLs are solutions of differential equations which come from Algebraic Geometry.

Example

MPLs are iterated integrals of algebraic differential forms on $X = \mathbb{A}^1 \setminus \{0,1\} = \mathbb{P}^1 \setminus \{0,1,\infty\}$:

$$Li_2(z) = \int_0^z Li_1(y)\frac{dy}{y} = \int_0^z \int_0^y \frac{dx}{x-1}\frac{dy}{y}$$

Define a connection $\nabla : \mathcal{O}_X^{\oplus 3} \to \Omega_X^1 \otimes \mathcal{O}_X^{\oplus 3}$ by

$$\nabla = d + A_0 \frac{dt}{t} + A_1 \frac{dt}{t-1}, \qquad A_0 = \left(\begin{smallmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right),\ A_1 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$$

Fundamental system of solutions:

$$\begin{pmatrix} 1 & \log z & -Li_2(z) \\ 0 & 1 & -Li_1(z) \\ 0 & 0 & 1 \end{pmatrix}$$

Let X be a smooth variety over a field k.

- A connection on a vector bundle $\mathcal{V}$ on X is a morphism of sheaves $\nabla : \mathcal{V} \to \Omega^1_{X/k} \otimes \mathcal{V}$ satisfying

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

where f (resp. s) is a section of $\mathcal{O}_X$ (resp. $\mathcal{V}$). When $\mathcal{V} = \mathcal{O}_X \otimes V$, we can write

$$\nabla = d + A, \qquad A \in \Gamma(X, \Omega^1_{X/k}) \otimes End_k(V).$$

We say that ∇ is integrable (or flat) if $\nabla \circ \nabla = 0$, i.e.,

$$dA + A \wedge A = 0.$$

- We say that $(\mathcal{V}, \nabla)$ is unipotent if there exists a filtration

$$0 = (\mathcal{V}_0, \nabla_0) \subset (\mathcal{V}_1, \nabla_1) \subset \cdots \subset (\mathcal{V}_n, \nabla_n) = (\mathcal{V}, \nabla)$$

such that

$$(\mathcal{V}_i, \nabla_i)/(\mathcal{V}_{i-1}, \nabla_{i-1}) \cong (\mathcal{O}_X, d).$$

Theorem

Let k be a field of characteristic zero. Every unipotent vector bundle with integrable connection on $X = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ is canonically isomorphic to some

$$(\mathcal{O}_X \otimes V, d + \frac{dt}{t} \otimes A_0 + \frac{dt}{t-1} \otimes A_1),$$

where V is a finite dimensional k-vector space, and $A_0, A_1 \in End_k(V)$ are nilpotent.

Remark

- *Analytically, flat sections are given by MPLs.*
- *Related to KZ equations (Knizhnik–Zamolodchikov).*
- $\pi_1^{\mathrm{dR}}(X, x) \cong Spec(\bigoplus_{n \geq 0}(k\frac{dt}{t} \oplus k\frac{dt}{t-1})^{\otimes n})$ *canonically.*

Proof.

Let $(\mathcal{V}, \nabla)$ be a unipotent vector bundle with integrable connection on X. By unipotency, $(\mathcal{V}, \nabla)$ is regular singular at infinity and it extends canonically to a unipotent vector bundle with integrable logarithmic connection

$$\overline{\nabla} : \overline{\mathcal{V}} \to \Omega^1_{\mathbb{P}^1_k}(\log\{0, 1, \infty\}) \otimes \overline{\mathcal{V}}$$

Since $\overline{\mathcal{V}}$ is unipotent and

$$H^0(\mathbb{P}^1_k, \mathcal{O}) = k, \qquad H^1(\mathbb{P}^1_k, \mathcal{O}) = Ext^1(\mathcal{O}, \mathcal{O}) = 0$$

the canonical map $\mathcal{O} \otimes V \to \overline{\mathcal{V}}$ is an isomorphism, where $V = \Gamma(\mathbb{P}^1_k, \overline{\mathcal{V}})$. Thus,

$$\nabla = d + A,$$

where $A \in \Gamma(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k}(\log\{0, 1, \infty\})) \otimes End_k(V)$.To conclude, we remark that

$$\Gamma(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k}(\log\{0, 1, \infty\})) = k\frac{dt}{t} \oplus k\frac{dt}{t-1}. \qquad \square$$

Elliptic versions of multiple polylogarithms?

- We are looking for analogues of $Li_{k_1,\ldots,k_n}(z)$ defined over $E \setminus \{O\}$, where E is an elliptic curve and O is the origin.
- Elliptic dilogarithm (Bloch): write $E = \mathbb{C}^\times / q^{\mathbb{Z}}$ and define

$$D_E(x) = \sum_{m=-\infty}^{\infty} \mathcal{L}_2(qx).$$

- Computes $L(E_{/\mathbb{Q}}, 2)$ (Bloch, Beilinson, Goncharov–Levin).
- Elliptic polylogarithmic sheaves and the Eisenstein symbol (Beilinson, Levin, Deninger, etc.)

Brown–Levin's multiple elliptic polylogarithms ('13):

- Consider the Kronecker function

$$F_\tau(z,w) = \frac{\theta'_\tau(0)\theta_\tau(z+w)}{\theta_\tau(z)\theta_\tau(w)}.$$

- Let $r(z) = Im(z)/Im(\tau)$, and consider the 1-forms

$$\nu_{BL} = 2\pi i\, dr, \qquad \omega_{BL}^{(n)}, \quad n \geq 0$$

  where

$$e^{2\pi i\, r(z) w} F_\tau(z,w) = \sum_{n\geq 0} \omega_{BL}^{(n)} w^{n-1}.$$

  Example: $\omega_{BL}^{(0)} = dz$, $\omega_{BL}^{(1)} = d\log\theta_\tau(z) + 2\pi i\, r(z) dz$, ...

- MEPLs are iterated integrals of $\nu_{BL}, \omega_{BL}^{(n)}$ (agrees with q-averaging MPLs).

Where does $r(z) = Im(z)/Im(\tau)$ come from?

- Let $\mathcal{V}$ be the vector bundle on $X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ given by

$$(m + n\tau) \cdot (v_1, v_2, z) = (v_1 + nv_2, v_2, z + m + n\tau)$$

$$\begin{array}{ccc} \mathbb{C}^2 \times \mathbb{C} & & \mathcal{V} \\ \downarrow & \underset{\rightsquigarrow}{\text{mod } \mathbb{Z}+\tau\mathbb{Z}} & \downarrow \\ \mathbb{C} & & X \end{array}$$

- We have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{V} \to \mathcal{O}_X \to 0.$$

- A splitting corresponds to a function $r : X \to \mathbb{C}$ satisfying

$$f(z + m + n\tau) = f(z) + n.$$

- No such holomorphic f, but we can consider the real-analytic function r.

How to algebraize?

- Consider $\mathbb{C}^2$ with coordinates (z, r), and lift the action of $\mathbb{Z} + \mathbb{Z}\tau$ by
$$(m + n\tau) \cdot (z, r) = (z + m + n\tau, r + n).$$
- The quotient $\mathbb{C}^2/(\mathbb{Z} + \mathbb{Z}\tau)$ has a natural structure of algebraic variety such that the induced projection to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is algebraic!
- The universal vector extension of an elliptic curve $f : E \to S$ is a commutative group scheme $g : E^\natural \to S$ which sits into an exact sequence
$$0 \to \mathbb{V}(R^1 f_* \mathcal{O}_E) \to E^\natural \xrightarrow{\pi} E \to 0$$
and is universal for extensions of E by vector groups. (Rosenlicht, Serre, Grothendieck, Mazur–Messing, etc.)
- $g : E^\natural \to S$ is a smooth group scheme of rel. dimension 2 (not proper neither affine!).

Theorem (Laumon '96)

If S is of characteristic zero, then $g_*\mathcal{O}_{E^\natural} \cong \mathcal{O}_S$ and $R^n g_*\mathcal{O}_{E^\natural} = 0$ for $n \geq 1$.

- Use it to classify (relatively) unipotent vector bundles with integrable connection on $E^\natural \setminus D$, where $D = \pi^{-1}(O)$.
- Need to understand relative differential forms on $E^\natural$ with log poles along D.

Theorem (F.–Matthes '21)

There is a canonical decomposition

$$g_*\Omega^1_{E^\natural/S}(\log D) = g_*\Omega^1_{E^\natural/S} \oplus \bigoplus_{n\geq 1} \mathcal{K}^{(n)}$$

where $\mathcal{K}^{(n)}$ are rank 1 subbundles uniquely determined by

1. $d\mathcal{K}^{(n)} = g_*\Omega^1_{E^\natural/S} \wedge \mathcal{K}^{(n-1)}$, *where* $\mathcal{K}^{(0)} := f_*\Omega^1_{E/S}$,
2. $\mathcal{K}^{(n)} \wedge \mathcal{K}^{(0)} = 0$,
3. $Res_D(\mathcal{K}^{(n)})$ *has degree* $n-1$.

If $\nu, \omega^{(0)}$ trivializes $g_*\Omega^1_{E^\natural/S}$, with $\omega^{(0)}$ in $\mathcal{K}^{(0)}$, then there are unique trivializations $\omega^{(n)}$ of $\mathcal{K}^{(n)}$ such that

1. $d\omega^{(n)} = \nu \wedge \omega^{(n-1)}$,
2. $\omega^{(n)} \wedge \omega^{(0)} = 0$,
3. $Res_D(\omega^{(n)}) = \frac{t^{n-1}}{(n-1)!}$, where $t : D \xrightarrow{\sim} \mathbb{A}^1_S$ is induced by ν.

Theorem (F.–Matthes '21)

Every relatively unipotent vector bundle with integrable connection on $E^\natural \setminus D$ over S is canonically isomorphic to some

$$(g^*\mathcal{W}, d + A \otimes \nu + B_0 \otimes \omega^{(0)} + \sum_{n \geq 1} ad^n_A(B_0) \otimes \omega^{(n)})$$

where $\mathcal{W}$ is a vector bundle over S, and A, B_0 are nilpotent endomorphisms of $\mathcal{W}$.

Note: pullback by $\pi : E^\natural \to E$ gives a classification of relatively unipotent vector bundles with integrable connection on $E \setminus \{O\}$.

- Let $S = Spec(k)$. Using that $H^0(E^\natural, \Omega^1) \cong H^1_{dR}(E/k)$, get canonical isomorphism

$$\pi_1^{dR}(E \setminus \{0\}, x) \cong Spec(\bigoplus_{n \geq 0} H^1_{dR}(E/k)^{\otimes n}).$$

- Take $k = \mathbb{C}$, $E(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. Thus $E^\natural \cong \mathbb{C}^2/(\mathbb{Z} + \mathbb{Z}\tau)$ and we have a real-analytic section

$$\begin{tikzcd} E^\natural(\mathbb{C}) \arrow[d, "\pi"] \\ E(\mathbb{C}) \arrow[u, bend left, "\mathcal{S}"] \end{tikzcd} \qquad \mathcal{S}(z) = (z, Im(z)/Im(\tau))$$

Then

$$\mathcal{S}^*\nu = \nu_{BL}, \qquad \mathcal{S}^*\omega^{(n)} = \omega^{(n)}_{BL}, \quad n \geq 1.$$

- Real-analytic flat sections of unipotent vector bundles with integrable connection on $E(\mathbb{C}) \setminus \{O\}$ are given by Brown–Levin's MEPLs.

Comments on KZB:

- Last theorem is related to the 'universal elliptic KZB equation' (Knizhnik–Zamolodchikov-Bernard), which lives on $\mathcal{M}_{1,2}$.
- Calaque–Enriquez–Etingof, Levin–Racinet, Hain:

9.2. **The formula.** The connection is defined by a 1-form

$$\omega \in \Omega^1(\mathbb{C} \times \mathfrak{h}, \log \Lambda) \hat{\otimes} \operatorname{End} \mathfrak{p}.$$

via the formula

$$\nabla f = df + \omega f$$

where $f : \mathbb{C} \times \mathfrak{h} \to \mathfrak{p}$ is a (locally defined) section of (9.1). Specifically,

$$\omega = \frac{1}{2\pi i} d\tau \otimes \mathbf{a} \frac{\partial}{\partial \mathbf{t}} + \psi + \nu$$

where

$$\psi = \sum_{m \geq 1} \left(\frac{(2\pi i)^{2m+1}}{(2m)!} G_{2m+2}(\tau) d\tau \otimes \sum_{\substack{j+k=2m+1 \\ j,k>0}} (-1)^j [\operatorname{ad}_{\mathbf{t}}^j(\mathbf{a}), \operatorname{ad}_{\mathbf{t}}^k(\mathbf{a})] \frac{\partial}{\partial \mathbf{a}} \right)$$

and

$$\nu = \mathbf{t} F(\xi, \mathbf{t}, \tau) \cdot \mathbf{a}\, d\xi + \frac{1}{2\pi i} \left(\frac{1}{\mathbf{t}} + \mathbf{t} \frac{\partial F}{\partial \mathbf{t}}(\xi, \mathbf{t}, \tau) \right) \cdot \mathbf{a}\, d\tau.$$

- Algebraicity results (Hain, Luo) rely on explicit formulas and well-chosen $\mathbb{Q}$-structures.

- We can give a purely algebraic construction of KZB. Formulas reflect the geometry of the universal vector extension.
- Key structure: 'crystalline nature' of universal vector extensions.

Theorem (F.–Matthes '21)

Assume that S is a smooth k-scheme. Then

$$0 \to \Omega^1_{S/k} \to g_*\Omega^1_{E^\natural/k}(\log D) \to g_*\Omega^1_{E^\natural/S}(\log D) \to 0$$

has a canonical splitting.

- Example on the uniformization:

$$\tilde{\omega}^{(1)} = \left(\frac{\theta'_\tau(z)}{\theta_\tau(z)} + 2\pi i\, r\right) dz + \frac{1}{2\pi i}\left(\frac{1}{2}\frac{\theta''_\tau(z)}{\theta_\tau(z)} - \frac{1}{6}\frac{\theta'''_\tau(0)}{\theta'_\tau(0)} - \frac{(2\pi i\, r)^2}{2}\right) d\tau$$

Further commments and directions:

- Universal Mixed Elliptic Motives (Hain–Matsumoto).
- Case $E \setminus E[n]$. Analogous to theory of cyclotomic MZV and cyclotomic KZ equation. Algebraicity of level n KZB?
- Motivic theory (à la Brown) of elliptic multiple zeta values. Action of motivic Galois group. Explanation of modular/elliptic phenomena of MZVs.