Global solvability of real non-singular closed one-forms on compact manifolds

Vinícius Novelli
(joint with Gabriel Araújo, Paulo Dattori da Silva and Bruno de Lessa Victor)

Instituto de Ciências Matemáticas e de Computação - ICMC/USP
Seminário de Geometria e Topologia - Unicamp

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Introduction

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- Recall that $\mathscr{F}$ is a decomposition $\{L_\alpha\}$ of Ω into immersed, smooth, connected n-dimensional submanifolds L_α $(n := N - d)$ called *leaves* which is locally trivial in the following sense:
- For every point $p \in \Omega$, there is a coordinate system $x = (x_1, \ldots, x_N) : U \to \mathbb{R}^n \times \mathbb{R}^m$ defined in a neighborhood U of p such that, for each leaf L_α, every connected component of $U \cap L_\alpha$ is of the form

$$\{q \in U;\, x_{n+1}(q) = \ldots = x_N(q) = \text{constant}\}.$$

Introduction

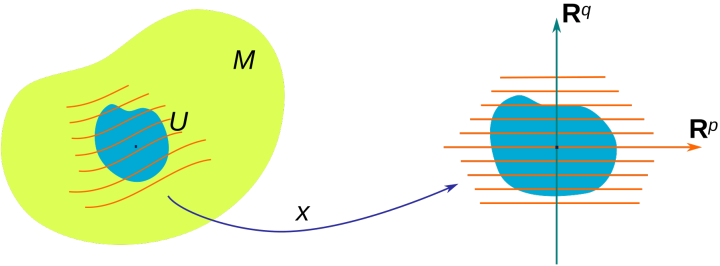

Figure: Local triviality

Introduction

- The data of this foliation is equivalent, by Frobenius' theorem, to a smooth, integrable[1] subbundle $\mathscr{V} \subset T\Omega$ of the tangent bundle of Ω, whose fibers are the tangent space of the leaf through that point.

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- We can think of a local basis of sections of $\mathscr{V}$ as a *system of first-order PDEs* defined in the manifold, and we can study its properties from a PDE point of view.

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- We can think of a local basis of sections of $\mathscr{V}$ as a *system of first-order PDEs* defined in the manifold, and we can study its properties from a PDE point of view.
- Indeed, $\mathscr{V}$ (or, rather, its complexification $\mathbb{C} \otimes \mathscr{V}$) is a particular type of *involutive structure*, as introduced by Treves, which is an involutive subbundle of the complexified tangent bundle $\mathbb{C}T\Omega$. This formalism includes elliptic systems of (complex) linear PDE, CR structures, complex structures, etc.

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- Let

$$\mathfrak{N}^q_{\mathscr{V}}(\Omega) \subset F^q(\Omega)$$

denote the space of q-forms ω such that $\omega(X_1, \ldots, X_q) = 0$ whenever $X_1, \ldots, X_q$ are sections of $\mathscr{V}$ (if $q = 0$, set $\mathfrak{N}^0_{\mathscr{V}}(\Omega) := 0$).

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- Note that $\mathfrak{N}^q_{\mathscr{V}}(\Omega)$ is a closed subspace of $F^q(\Omega)$, $0 \leqslant q \leqslant N$.

Introduction

- We consider the quotient

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  which is a Fréchet space for the quotient topology.
- This construction can be localized to open subsets $U \subset \Omega$ and, moreover, for every q, there is a smooth vector bundle $\Lambda^q_{\mathscr{V}} \to \Omega$ such that $\Lambda^q_{\mathscr{V}}(U)$ is the space of smooth sections of $\Lambda^q_{\mathscr{V}} \to \Omega$ over U.

Introduction

- The involutivity of the bundle $\mathscr{V}$ implies easily that the exterior derivative d satisfies

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  which form a differential complex.
- This complex is the exterior derivative "along the leaves", in some sense. Our goal is to study its properties, focusing on questions of exactness (or global solvability).

Introduction

- One can form the associated cohomology spaces

$$H^q_{\mathscr{V}}(\Omega) := \frac{\ker \mathrm{d}'_q}{\operatorname{ran} \mathrm{d}'_{q-1}}, \ q \geqslant 1,$$

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- These spaces are well-known in foliation theory literature, usually called the *foliated cohomology* spaces of $\mathscr{V}$.

Introduction

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- Very often it is *not* true that $H^q_{\mathscr{V}}(\Omega)$ vanishes, or even is finite-dimensional (although understanding when that happens is certainly interesting).
- We are interested in a weaker property, namely that the quotient topology on $H^q_{\mathscr{V}}(\Omega)$ is separated (or Hausdorff).

Definition

We say $\mathscr{V}$ is (strongly) *globally solvable in degree* q if $H^q_{\mathscr{V}}(\Omega)$ is a Hausdorff space for the quotient topology, or equivalently, if $\mathrm{d}'_{q-1} : \Lambda^{q-1}_{\mathscr{V}}(\Omega) \to \Lambda^q_{\mathscr{V}}(\Omega)$ has closed range.

Introduction

- This condition is very natural from the PDE point of view, since it is equivalent (by functional-analytic nonsense) to the property that one can find a global solution $u \in \Lambda^{q-1}_{\mathscr{V}}(\Omega)$ to

$$\mathrm{d}'_{q-1} u = f,$$

provided $f \in \Lambda^{q}_{\mathscr{V}}(\Omega)$ satisfies obvious compatibility conditions (namely, being in the annihilator of the kernel of the transpose operator ${}^t\mathrm{d}'_{q-1}$).

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  provided $f \in \Lambda^{q}_{\mathscr{V}}(\Omega)$ satisfies obvious compatibility conditions (namely, being in the annihilator of the kernel of the transpose operator ${}^{t}\mathrm{d}'_{q-1}$).
- One can also consider an even weaker property, allowing distributional (or current) solutions $u \in \mathscr{D}'^{q-1}_{\mathscr{V}}(\Omega)$. We refer to this property as *weak* global solvability in degree q.

Introduction

- So the main question we are concerned with is the following:

Question

Given a smooth, non-singular foliation $\mathscr{V}$, is it possible to characterize when $H^q_{\mathscr{V}}(\Omega)$ is Hausdorff for $0 \leqslant q \leqslant n$?

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Question

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- Note that, if $\mathscr{V}$ is the trivial foliation by points, then d' reduces to the usual exterior derivative, which is always globally solvable in every degree (compactness is not necessary and the result is a simple consequence of de Rham's theorem).

Examples

- This question is wide open (and maybe hopeless) in general, so we will restrict attention to very simple, codimension 1 foliations: those defined by closed, real, non-singular one-forms $\zeta \in F^1(\Omega)$. Given such a form, we let

$$\mathscr{V}_p = \ker \zeta_p \subset T_p\Omega, \;\; p \in \Omega,$$

which defines an involutive smooth subbundle $\mathscr{V} \subset T\Omega$.

Example - Greenfield-Wallach

- Let us illustrate how $H^q_{\mathscr{V}}(\Omega)$ can fail to be Hausdorff in a very simple example, considered by Greenfield-Wallach[2].

[2] Greenfield, Wallach, "Global hypoellipticity and Liouville numbers", Proc. Amer. Math. Soc., 1972.

Example - Greenfield-Wallach

- Let us illustrate how $H^q_{\mathscr{V}}(\Omega)$ can fail to be Hausdorff in a very simple example, considered by Greenfield-Wallach[2].
- Let $\Omega = \mathbb{T}^2 = S^1 \times S^1$ be the 2-torus, with standard coordinates $(t, x) \in S^1 \times S^1$, and consider the closed, non-singular one-form

$$\zeta = \mathrm{d}x + a \cdot \mathrm{d}t,$$

where $a \in \mathbb{R}^\times$ is a non-zero real number.

[2] Greenfield, Wallach, "Global hypoellipticity and Liouville numbers", Proc. Amer. Math. Soc., 1972.

Example - Greenfield-Wallach

- The foliation induced on $\mathbb{T}^2$ is given by the integral curves of the vector field $X = -a\partial_x + \partial_t$. This is the standard linear flow on the torus and the leaves are circles (periodic) if a is rational, and lines (dense) if a is irrational.

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- We can concretely represent the space $\Lambda^1_{\mathscr{V}}(\mathbb{T}^2)$ by

$$\Lambda^1_{\mathscr{V}}(\mathbb{T}^2) = \left\{ f(t,x)\mathrm{d}t;\ f \in C^\infty(\mathbb{T}^2) \right\},$$

and the associated differential is given by

$$\begin{aligned} \mathrm{d}'_0 : C^\infty(\mathbb{T}^2) &\to \Lambda^1_{\mathscr{V}}(\mathbb{T}^2) \\ u &\mapsto (\partial_t u - a\partial_x u)\,\mathrm{d}t. \end{aligned}$$

Example - Greenfield-Wallach

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$$(Lu)(t,x) = 2\pi i \sum_{\xi\in\mathbb{Z}^2} (\xi_1 - a\xi_2) u_{\xi_1,\xi_2} e^{2\pi i(t\xi_1 + x\xi_2)}.$$

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- We must distinguish cases depending on the arithmetic nature of a:

Example - Greenfield-Wallach

- Assume $a \in \mathbb{Q}$ and let $a = p/q$, with p, q coprime and $q > 0$. Let

$$X := \{f \in C^\infty(\mathbb{T}^2);\ f_{\xi_1,\xi_2} = 0 \text{ for all } (\xi_1, \xi_2) \in \mathbb{Z} \cdot (p, q)\}.$$

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Clearly, $\operatorname{ran} L \subset X$. Moreover, given $f \in X$, we let

$$u(t, x) := \frac{1}{2\pi i} \sum_{\xi \in \mathbb{Z}^2 \setminus \left(\mathbb{Z} \cdot (p,q)\right)} \frac{f_{\xi_1,\xi_2}}{\xi_1 - a\xi_2} e^{2\pi i (t\xi_1 + x\xi_2)}.$$

Since $|\xi_1 - a\xi_2| \geqslant q^{-1}$ for all $\xi \in \mathbb{Z}^2 \setminus (\mathbb{Z} \times (p, q))$, this defines a smooth function such that $Lu = f$. We conclude that $\operatorname{ran} L = X$, and therefore, $\mathscr{V}$ is globally solvable in degree 1.

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- Assume now that $a \notin \mathbb{Q}$. Let $X = \{f \in C^\infty(\mathbb{T}^2);\ f_{0,0} = 0\}$. Given $f \in X$, the obvious candidate for u is the same:

$$u(t,x) = \frac{1}{2\pi i} \sum_{(\xi_1,\xi_2) \neq (0,0)} \frac{f_{\xi_1,\xi_2}}{\xi_1 - a\xi_2} e^{i(t\xi_1 + x\xi_2)}.$$

All we have to check is if u is smooth. This depends on decay inequalities for $(\xi_1 - a\xi_2)^{-1}$, i.e., how well is a *approximable* by rational numbers.

Example - Greenfield-Wallach

Definition

Let $a \in \mathbb{R}\backslash\mathbb{Q}$. We say a is a *Liouville number* if, for every $n \in \mathbb{N}$, there exists a pair of integers $(p, q) \in \mathbb{Z}^2$ with $q > 1$ such that

$$\left|a - \frac{p}{q}\right| < \frac{1}{q^n}$$

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- More generally, one has the notion of *Liouville vector*:

Definition

Let $a = (a_1, \ldots, a_r) \in \mathbb{R}^r\backslash\mathbb{Q}^r$. We say a is a *Liouville vector* if, for every $n \in \mathbb{N}$, there are integers $(p_1, \ldots, p_r) \in \mathbb{Z}^r$ and $q > 1$ such that

$$\max_{j=1,\ldots,r} \left| a_j - \frac{p_j}{q} \right| < \frac{1}{q^n}.$$

Example - Greenfield-Wallach

- It follows easily that, if $a \in \mathbb{R} \backslash \mathbb{Q}$ is *not* a Liouville number, then $u \in C^\infty(\mathbb{T}^2)$ and $\mathscr{V}$ is globally solvable in degree 1.

Example - Greenfield-Wallach

- It follows easily that, if $a \in \mathbb{R} \backslash \mathbb{Q}$ is *not* a Liouville number, then $u \in C^\infty(\mathbb{T}^2)$ and $\mathscr{V}$ is globally solvable in degree 1.
- Conversely, one can also show (using functional-analytic methods, for example) that if $L : C^\infty(\mathbb{T}^2) \to C^\infty(\mathbb{T}^2)$ has closed range, then a cannot be a Liouville number. The conclusion, therefore, is the following:

Theorem

The structure $\mathscr{V}$ is globally solvable in degree 1 if and only if a is either rational or a non-Liouville irrational.

Examples - Tube type structures

- A more general class of structures of this type are the *tube-type* structures: let $\Omega = M \times S^1$, where M is a compact manifold. Denote the variables by $(t, x) \in M \times S^1$ and let

$$\zeta := \mathrm{d}x - \omega,$$

where $\omega \in F^1(M)$ is a closed, real-valued 1-form in M.

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where $\omega \in F^1(M)$ is a closed, real-valued 1-form in M.

- In this case, $\Lambda^q_{\mathscr{V}}(\Omega)$ can be identified with the space of q-forms which are locally of the form

$$f = \sum_{|J|=q} f_J(t, x) \mathrm{d}t_J,$$

where $(t_1, \ldots, t_n)$ is a coordinate system over M.

Examples - Tube type structures

- The differential $\mathrm{d}'_q : \Lambda^q_{\mathscr{V}}(\Omega) \to \Lambda^{q+1}_{\mathscr{V}}(\Omega)$ is then given by

$$\mathrm{d}'_q f = \mathrm{d}_t f + \omega \wedge (\partial_x f),$$

and (weak) global solvability in the last degree ($q = n - 1$) was characterized by Bergamasco, Cordaro and Petronilho.[3].

[3]Bergamasco, Cordaro, Petronilho, "Global solvability for certain classes of underdetermined systems of vector fields", Math. Z., 1996.

Examples

- From now on, we shall consider a foliation induced by a closed, real 1-form ζ on Ω. We let $N = \dim \Omega = n + 1$.

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Examples

- From now on, we shall consider a foliation induced by a closed, real 1-form ζ on Ω. We let $N = \dim \Omega = n + 1$.
- The study of this type of structure was started by Meziani[4], where he was concerned with the property of *global hypoellipticity*:

(GH): If $u \in \mathscr{D}'(\Omega)$ is such that $\mathrm{d}_0' u \in F^1(\Omega)$, then $u \in C^\infty(\Omega)$.

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The group of periods

- As we saw in the Greenfield-Wallach example, global solvability was determined by arithmetic properties of $-a = \int_{\{c\}\times S^1} (\mathrm{d}t - a\mathrm{d}x)$.

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The group of periods

- As we saw in the Greenfield-Wallach example, global solvability was determined by arithmetic properties of $-a = \int_{\{c\}\times S^1} (\mathrm{d}t - a\mathrm{d}x)$.
- The analogue object in the general case is the *group of periods* of ζ:

$$\mathscr{P}(\zeta) = \left\{ \int_\sigma \zeta; \ \ [\zeta] \in H_1(\Omega, \mathbb{Z}) \right\} = \left\{ \int_\sigma \zeta; \ \ [\sigma] \in \pi_1(\Omega, p_0) \right\},$$

which is a finitely generated abelian subgroup of $(\mathbb{R}, +)$ (since Ω is compact, $H_1(\Omega, \mathbb{Z})$ is a finitely generated abelian group).

The group of periods

- We say the form ζ is *rational* is $\mathscr{P}(\zeta)$ has a single generator, i.e.,

$$\mathscr{P}(\zeta) = c \cdot \mathbb{Z}, \ c \in \mathbb{R}^{\times},$$

(if $c = 1$, we say the form is *integral*) and it is *irrational* otherwise.

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- From previous experience with classes of forms ζ, one expects to have global solvability in the rational case. In the irrational case, this property should hold if and only if some diophantine condition on ζ (or $\mathscr{P}(\zeta)$) is satisfied.

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- We are able to confirm the first expectation, and the second one in a restricted case.

The rational case – statement

Theorem

Assume the form ζ is rational. Then, there is a compact hypersurface $M_1 \subset \Omega$ such that, for every $0 \leqslant q \leqslant n$, there is a vector bundle $\mathbb{H}^q_{\mathrm{dR}}(M_1) \to S^1$, with fiber given by the finite-dimensional vector space $H^q_{\mathrm{dR}}(M_1)$, such that there is a topological isomorphism

$$H^q_{\mathscr{V}}(\Omega) \simeq C^\infty(S^1, \mathbb{H}^q_{\mathrm{dR}}(M_1)),$$

where $C^\infty(S^1, \mathbb{H}^q_{\mathrm{dR}}(M_1))$ denotes the space of smooth sections of the bundle $\mathbb{H}^q_{\mathrm{dR}}(M_1) \to S^1$, endowed with the standard C^∞ Fréchet topology.

The rational case – statement

Corollary

If the form ζ is rational, d'_q is (strongly) globally solvable for every $0 \leqslant q \leqslant n$.

The rational case – ideas of the proof

- The condition of solvability is invariant under multiplication of ζ by a non-zero constant, so we can assume $\mathscr{P}(\zeta) = 2\pi\mathbb{Z}$.

The rational case – ideas of the proof

- The condition of solvability is invariant under multiplication of ζ by a non-zero constant, so we can assume $\mathscr{P}(\zeta) = 2\pi\mathbb{Z}$.
- We fix a smooth vector field X on Ω that is transversal for ζ, i.e., $\zeta(X) = 1$. We denote the flow of X at time t by Φ_t^X.

The rational case – ideas of the proof

- The flow Φ_t^X has the following property: let $L \subset \Omega$ be a leaf of $\mathscr{F}$. Then,

$$\mathscr{P}(\zeta) = \left\{ t \in \mathbb{R};\ \Phi_t^X(L) = L \right\}.$$

The rational case – ideas of the proof

- Let $\Pi : \widetilde{\Omega} \to \Omega$ be the universal covering, with $\Pi^*\zeta = \mathrm{d}\psi$, $\psi \in C^\infty(\widetilde{\Omega}; \mathbb{R})$. By a well-known result, the exponential $e^{i\psi}$ is of the form $\Pi \circ G$, for some function $G : \Omega \to \mathbb{C}$. Since it has absolute value 1, we can think of it as a function $g : \Omega \to S^1$. It has the property that

$$g^*\mathrm{d}\theta = \zeta.$$

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The rational case – ideas of the proof

- The leaves of the foliation $\mathscr{F}$ are given by the fibers $M_z := g^{-1}(z)$, $z \in S^1$, which are all compact.
- The key idea is that g is a *locally trivial fibration*:

1. Let $U_z := S^1 \backslash \{z\}$, for $z = \pm 1$ and fix angle coordinates $\theta_z : U_z \to \mathbb{R}$. Let $\Omega_z := g^{-1}(U_z)$.

The rational case – ideas of the proof

- The leaves of the foliation $\mathscr{F}$ are given by the fibers $M_z := g^{-1}(z)$, $z \in S^1$, which are all compact.
- The key idea is that g is a *locally trivial fibration*:

1. Let $U_z := S^1 \backslash \{z\}$, for $z = \pm 1$ and fix angle coordinates $\theta_z : U_z \to \mathbb{R}$. Let $\Omega_z := g^{-1}(U_z)$.
2. One verifies that the maps

$$\begin{aligned} \phi_z : \Omega_z &\to U_z \times M_1 \\ x &\mapsto \left(g(x), \Phi^X_{\theta_z\left(\frac{1}{g(x)}\right)}(x) \right) \end{aligned}$$

are diffeomorphisms that satisfy

$$\phi_z \circ \phi_{-z}^{-1}(w, y) = \left(w, \Phi^X_{2\pi k}(y) \right),$$

where $w \in U_1 \cap U_{-1}$, $y \in M_1$ and k is such that $\theta_z - \theta_{-z} = 2\pi k$ on $U_1 \cap U_{-1}$.

The rational case – ideas of the proof

- The diffeomorphism ϕ_z induces an isomorphism

$$(\phi_z)_* : H^q_{\mathscr{V}}(\Omega_z) \to H^q_{\mathscr{W}}(U_z \times M_1),$$

where $\mathscr{W}$ is the product involutive structure, i.e., if $(x,t) \in U_z \times M_1$, the induced $\mathrm{d}'_{\mathscr{W}}$ operator is just d_t (partial de Rham in the leaf directions).

The rational case – ideas of the proof

- The cohomology of the partial de Rham complex is easy to compute:

Proposition

The partial de Rham $\mathrm{d}_t : \Lambda^q_{\mathscr{W}}(U_z \times M_1) \to \Lambda^{q+1}_{\mathscr{W}}(U_z \times M_1)$ *has closed range in every degree* $q = 0, 1, \ldots, n$ *and, moreover, there is a topological isomorphism*

$$H^q_{\mathscr{W}}(U_z \times M_1) \simeq C^\infty(U_z; H^q_{\mathrm{dR}}(M_1)).$$

The rational case – ideas of the proof

- Then, we define the following map:

$$\begin{aligned} T : H^q_{\mathscr{V}}(\Omega) &\to C^\infty(U_{-1}, H^q_{\mathrm{dR}}(M_1)) \times C^\infty(U_1; H^q_{\mathrm{dR}}(M_1)) \\ [\omega] &\mapsto \left((\phi_{-1})_* [\omega\big|_{U_{-1}}], (\phi_1)_* [\omega\big|_{U_1}] \right). \end{aligned}$$

The rational case – ideas of the proof

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[\omega] &\mapsto \left((\phi_{-1})_*[\omega\big|_{U_{-1}}], (\phi_1)_*[\omega\big|_{U_1}]\right).
\end{aligned}
$$

- In general, such a map would not be injective (as one knows from the Mayer Vietoris long exact sequence), but in this case, injectivity holds. Indeed, let $\{\rho_{-1}, \rho_1\}$ be a partition of unity subordinated to the open cover $\{U_{-1}, U_1\}$ of S^1.

The rational case – ideas of the proof

- Then, given $f \in \Lambda^q_{\mathscr{V}}(\Omega)$ such that $f\big|_{\Omega_z} = \mathrm{d}'_{q-1} v_z$, for $v_z \in \Lambda^{q-1}_{\mathscr{V}}(\Omega_z)$, we set

$$v := (g^* \rho_{-1}) v_{-1} + (g^* \rho_1) v_1 \in \Lambda^q_{\mathscr{V}}(\Omega),$$

which satisfies

$$\mathrm{d}'_{q-1} v = f.$$

The rational case – ideas of the proof

- We now describe the range of T. Consider the map

$$\tau : C^\infty(U_{-1} \cap U_1; H^q_{\mathrm{dR}}(M_1)) \longrightarrow C^\infty(U_{-1} \cap U_1; H^q_{\mathrm{dR}}(M_1))$$

given by

$$\tau := (\phi_1 \circ \phi_{-1}^{-1})_* : \sigma(\cdot) \longmapsto (\Phi^X_{2\pi k})_* \sigma(\cdot),$$

where $\Phi_{2\pi}$ is Poincaré's first return map on the leaf M_1 for the transversal vector field X.

The rational case – ideas of the proof

- One can then form the (closed) subspace $\mathscr{R}$ of compatible pairs

$$(\sigma_{-1}, \sigma_1) \in C^\infty(U_{-1}; H^q_{\mathrm{dR}}(M_1)) \times C^\infty(U_1; H^q_{\mathrm{dR}}(M_1))$$

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  such that $\tau\sigma_{-1} = \sigma_1$.
- It is clear that $\operatorname{ran} T = \mathscr{R}$ and one can explicitely (using the partition of unity) write down the continuous inverse.

The irrational case

- In the irrational case, one should consider appropriate diophantine conditions on ζ. We say that ζ is *Liouville* if there exist a sequence of integral 1-forms $\{\zeta_\nu\}_{\nu \in \mathbb{N}}$ and a sequence of integers $\{q_\nu\}_{\nu \in \mathbb{N}}$, with $q_\nu \geqslant 2$, such that

$$\left\{ q_\nu^\nu (\zeta - q_\nu^{-1} \zeta_\nu);\ \nu \in \mathbb{N} \right\} \quad \text{is a bounded set in } F^1(\Omega).$$

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- This condition is equivalent to the following: replacing ζ with a multiple, we can assume that $1 \in \mathscr{P}(\zeta)$. Then, if we write the generators of $\mathscr{P}(\zeta)$ as $(1, a_1, \ldots, a_r)$, ζ is Liouville if and only if $(a_1, \ldots, a_r)$ is a Liouville vector.

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- We shall study a class of structures endowed with symmetries[5]. We assume that Ω is a *principal* S^1*-bundle*:

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2. The quotient is smooth: there is a smooth manifold M and a smooth quotient map $p : \Omega \to M$ such that the fibers $p^{-1}(x)$ are the orbits of the S^1 action.

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2. The quotient is smooth: there is a smooth manifold M and a smooth quotient map $p : \Omega \to M$ such that the fibers $p^{-1}(x)$ are the orbits of the S^1 action.
3. For every point in M there is a neighborhood $U \subset M$ and a diffeomorphism $\chi : p^{-1}(U) \to U \times S^1$ of the form $\chi(x) = (\pi(x), S_\chi(x))$, where S_χ is S^1-equivariant.

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The symmetric case

- The presence of the S^1 action introduces the *fundamental vector field*:

$$X_x = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} x \cdot e^{it} = (A_x)_* \left(\frac{\partial}{\partial \theta}\Big|_1 \right), \ x \in \Omega,$$

where $A_x : S^1 \to x \cdot S^1 \subset \Omega$ is the action map. The flow of this vector field is just the action:

$$\Phi_t^X(x) = x \cdot e^{it}, \ x \in \Omega, \ t \in \mathbb{R}.$$

It is easy to check that X is invariant by rotations, meaning that $(R_z)_* X\big|_x = X\big|_{x \cdot z}$, where $R_z : \Omega \to \Omega$ is the diffeomorphism $x \mapsto x \cdot z$, $z \in S^1$.

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2. $T_x\Omega = \mathscr{V}_x \oplus \ker \mathrm{d}p\big|_x$, $x \in \Omega$.

- In this case, there is a 1-form $\zeta \in F^1(\Omega)$ such that $\ker \zeta_x = \mathscr{V}_x$, is rotation invariant ($R_z^* \zeta = \zeta$) and such that $\zeta(X) = 1$. Such a ζ is called a *connection form*. If ζ is closed, we say the connection is *flat*.

The symmetric case

Theorem

Let Ω be a compact, connected, orientable circle bundle, endowed with a flat connection $\mathscr{V} \subset T\Omega$. Let ζ be the connection form associated to $\mathscr{V}$. Then, the following are equivalent:

1. *Weak global solvability holds in degree n.*
2. *Strong global solvability holds in degree n.*
3. *Strong global solvability holds in degree 1.*
4. *ζ is either rational or irrational non-Liouville.*

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- The proof splits in the following cases: first, one shows (1) and (4) are equivalent. Then, (1) and (2) are equivalent (by general functional-analytic arguments). Then, one shows (3) and (4) are equivalent.

The symmetric case - (1) $\implies$ (4)

- To prove that $(1) \implies (4)$, the idea is to violate an *a priori* inequality which is implied by weak global solvability.

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- It is enough to construct sequences $(f_\nu)_{\nu\in\mathbb{N}}$ in $\Lambda^n_{\mathscr{V}}(\Omega)$ and $(v_\nu)_{\nu\in\mathbb{N}} \in C^\infty(\Omega)$ with the following properties:

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  1. For all $\nu \in \mathbb{N}$,
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  2. There is a non-zero constant c such that for every $\nu \in \mathbb{N}$,
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  1. For all $\nu \in \mathbb{N}$,
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  2. There is a non-zero constant c such that for every $\nu \in \mathbb{N}$,
  $$\int_\Omega (v_\nu \cdot f_\nu) \wedge \zeta = c.$$
  3. For every $k, l \in \mathbb{Z}_+$,
  $$\|f_n\|_{W^k} \cdot \|\mathrm{d}_0' v_\nu\|_{W^l} \to 0$$
  as $\nu \to \infty$.

The symmetric case - (1) $\implies$ (4)

- Assume ζ is irrational Liouville and let $(\zeta_\nu)_{\nu\in\mathbb{N}}$ be a sequence of integral forms and $(q_\nu)_{n\in\mathbb{N}}$ a sequence of integers, $q_\nu \geqslant 2$, such that, for every $l \in \mathbb{Z}_+$,

$$\|\zeta - q_\nu^{-1}\zeta_\nu\|_{W^l} \leqslant C_l q_\nu^{-\nu}, \ \nu \in \mathbb{N}.$$

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$$\|\zeta - q_\nu^{-1}\zeta_\nu\|_{W^l} \leqslant C_l q_\nu^{-\nu}, \ \nu \in \mathbb{N}.$$

- The first thing we can do is apply an *averaging operator* on ζ_ν: if $\eta \in F^1(\Omega)$, we let

$$(\mathscr{A}\eta)_p(v) = \frac{1}{2\pi}\int_{S^1} \left(R_\theta^*\eta\right)_p \cdot v \, \mathrm{d}\theta,$$

which defines a continuous linear operator $\mathscr{A} : F^1(\Omega) \to F^1(\Omega)$.

The symmetric case - (1) $\implies$ (4)

- A simple calculation shows that $\mathscr{A}$ preserves integral forms, so replacing ζ_ν by $\mathscr{A}\zeta_\nu$ allows us to assume that every ζ_ν is rotation-invariant.

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- Let $g_\nu : \Omega \to S^1$ be the fibration such that $g_\nu^* \mathrm{d}\theta = 2\pi\zeta_\nu$ and let $G_\nu = i \circ g_\nu$, where $i : S^1 \to \mathbb{C}$ is the inclusion. The function G_ν satisfies

$$\mathrm{d}G_\nu = 2\pi i G_\nu \zeta_\nu.$$

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$$\mathrm{d}G_\nu = 2\pi i G_\nu \zeta_\nu.$$

- We set $v_\nu := G_\nu^{-1}$. One shows that $\|\mathrm{d}_0' v_\nu\|_{W^l}$ decays like $q_\nu^{A-\nu}$ for some constant $A > 0$.

The symmetric case - (1) $\implies$ (4)

- Fix a Riemannian metric on Ω, let $\mathrm{d}V$ denote the volume form and set
$$f_\nu := G_\nu \cdot \iota_X \mathrm{d}V$$
as a class in $\Lambda^n_{\mathscr{V}}(\Omega)$.

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$$f_\nu := G_\nu \cdot \iota_X \mathrm{d}V$$
as a class in $\Lambda^n_{\mathscr{V}}(\Omega)$.
- Similarly, one can show that $\|f_\nu\|_{W^k}$ is bounded by a fixed power of q_ν, so condition $3)$ is satisfied.

- To show the second item, observe that

$$\int_\Omega (v_\nu \cdot f_\nu) \wedge \zeta = \int_\Omega \iota_X \mathrm{d}V \wedge \zeta.$$

However, we have

$$\iota_X \mathrm{d}V \wedge \zeta = (-1)^{n+1} \mathrm{d}V \wedge (\iota_X \zeta) = (-1)^{n+1} \mathrm{d}V,$$

so

$$\int_\Omega (v_\nu \cdot f_\nu) \wedge \zeta = (-1)^{n+1} \operatorname{vol}(\Omega),$$

which is a non-zero constant.

The symmetric case - (1) $\implies$ (4)

- It remains to prove the first item, i.e.,

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- Notice that

$$\int_\Omega f_\nu \wedge \zeta = \int_\Omega G_\nu \cdot \iota_X(\mathrm{d}V) \wedge \zeta = (-1)^{n+1} \int_\Omega G_\nu \mathrm{d}V,$$

so we have to prove that G_ν has zero average for every ν.

The symmetric case - (1) $\implies$ (4)

- The key point (where the extra symmetries are important) is that there are non-zero integers r_ν such that $2\pi r_\nu^{-1}\zeta_\nu$ are connection forms for all ζ. In particular, the same vector field X is transversal for all of them. Therefore,

$$\begin{aligned}\int_\Omega G_\nu \mathrm{d}V &= (-1)^{n+1}\int_\Omega G_\nu i_X(\mathrm{d}V)\wedge\left(2\pi r_\nu^{-1}\zeta_\nu\right)\\ &= (-1)^{n+1} r_\nu^{-1}\int_\Omega G_\nu i_X(\mathrm{d}V)\wedge\left(g_\nu^*\mathrm{d}\theta\right)\\ &= (-1)^{n+1} r_\nu^{-1}\int_{S^1}\left(\int_{g_\nu^{-1}(z)} G_\nu \iota_X(\mathrm{d}V)\right)\mathrm{d}\theta,\end{aligned}$$

where one applies integration along the fibers of the submersion g_ν.

The symmetric case - (1) $\implies$ (4)

- Then,

$$\int_{\Omega} G_\nu \mathrm{d}V = (-1)^{n+1} r_\nu^{-1} \int_{S^1} z \left(\int_{g_\nu^{-1}(z)} \iota_X(\mathrm{d}V) \right) \mathrm{d}\theta.$$

The final ingredient is to choose an appropriate (translation-invariant) Riemannian metric on Ω such that

$$z \in S^1 \mapsto \int_{g_\nu^{-1}(z)} \iota_X(\mathrm{d}V)$$

is constant, which can be achieved using the connection $\mathscr{V}$.

The symmetric case - (4) $\implies$ (1)

- To show that (4) $\implies$ (1), we exploit the property of global hypoellipticity via the following standard result:

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Proposition

If $\mathrm{d}'_0 : C^\infty(\Omega) \to \Lambda^1_{\mathscr{V}}(\Omega)$ *is globally hypoelliptic, then* d'_{n-1} *is weakly globally solvable.*

The symmetric case - (4) $\implies$ (1)

- The idea is to use Meziani's[6] result on global hypoellipticity.
- His result, however, requires an additional hypothesis which does not hold in general, but it *does* hold in our case of principal S^1-bundles[7].

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[6] Meziani, "Hypoellipticity of nonsingular closed 1-forms on compact manifolds", Comm. Partial Differential Equations, 2002

[7] We can choose an arbitrarily good rational approximation η of ζ with the property that, in its suspension construction $\Omega = K \times \mathbb{R}/\sim_\phi$, a power of ϕ coincides with the identity over K.

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- The idea is to use Meziani's[6] result on global hypoellipticity.
- His result, however, requires an additional hypothesis which does not hold in general, but it *does* hold in our case of principal S^1-bundles[7].
- We conclude that, if ζ is irrational and non-Liouville, then d'_0 is globally hypoelliptic and, therefore, d'_{n-1} is weakly globally solvable, as desired.

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The symmetric case

- As we mentioned, equivalence of (1) and (2) is a formal argument. To show that (3) is equivalent to (4), we use a completely different method, which we illustrate in the general case.

The general case - speculations

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- Fix a point $p_0 \in \Omega$ and consider

$$H = \left\{ [\gamma] \in \pi_1(\Omega, p_0);\ \int_\gamma \zeta = 0 \right\},$$

which is a normal subgroup of the fundamental group of Ω at p_0 and contains the commutator subgroup $[\pi_1(\Omega, p_0), \pi_1(\Omega, p_0)]$.

The general case - speculations

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- The surjective map $\pi_1(\Omega, p_0) \to \mathscr{P}(\zeta)$ given by $[\gamma] \mapsto \int_{[\gamma]} \zeta$ induces a group isomorphism

$$\mathscr{P}(\zeta) \simeq \pi_1(\Omega, p_0)/H.$$

The general case - speculations

- We can then consider the Galois covering

$$\pi : \widehat{\Omega} \to \Omega$$

associated with the subgroup $H \subset \pi_1(\Omega, p_0)$, which we call the *minimal covering*.

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- Fixing a point $\widehat{p_0} \in p^{-1}(p_0)$, we know from the general theory that $\widehat{\Omega}$ is a connected $(n+1)$-dimensional smooth manifold such that $\pi_*\Big(\pi_1(\widehat{\Omega}, \widehat{p_0})\Big) = H$ and we can define a smooth function $f : \widehat{\Omega} \to \mathbb{R}$ by

$$f(q) = \int_{\pi(\gamma)} \zeta = \int_{\gamma} \pi^* \zeta, \; q \in \widehat{\Omega},$$

where $\gamma : [0,1] \to \widehat{\Omega}$ is a smooth path on $\widehat{\Omega}$ satisfying $\gamma(0) = \widehat{p_0}$ and $\gamma(1) = q$. It satisfies $\mathrm{d}f = \pi^* \zeta$.

The general case - speculations

- f is a submersion, and the foliation induced by it coincides with the pullback foliation $\widehat{\mathscr{F}} = \pi^* \mathscr{F}$. The leaves $\widehat{L}$ of $\pi^* \widehat{\mathscr{F}}$ are given by the connected components of the pre-images $\pi^{-1} L$, and $\pi\big|_{\widehat{L}} \to L$ is a diffeomorfism, where L is a leaf of $\mathscr{F}$.

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- FIx X a transversal vector field for ζ and consider the vector field $\widehat{X}$ on $\widehat{\Omega}$ which satisfies

$$(\mathrm{d}\pi)_q \widehat{X}_q = X_{\pi(q)}, \ \ q \in \widehat{\Omega}.$$

Let $\psi : \mathbb{R} \times \Omega \to \Omega$ denote the flow of X and $\widehat{\psi} : \mathbb{R} \times \widehat{\Omega} \to \widehat{\Omega}$ denote the flow of $\widehat{X}$.

The general case - speculations

- One can check that the map

$$h : \mathbb{R} \times \widehat{L} \to \widehat{\Omega}$$
$$(t, q) \mapsto \widehat{\psi}_t(q)$$

is a diffeomorphism, where $\widehat{L}$ is any fixed leaf of $\widehat{F}$. Composing with π, we obtain a new covering map

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- The foliation induced on $\mathbb{R} \times \widehat{L}$ via this map is the standard one, where the leafs are $\{t\} \times \widehat{L}$. In particular, we can think of it as an involutive structure (denoted by $\widehat{\mathscr{V}}$) where $\mathrm{d}'_{\widehat{\mathscr{V}}}$ is the partial de Rham $\mathrm{d}_{\widehat{L}}$ on the $\widehat{L}$ directions.

The general case - speculations

- Let $\mathrm{Deck}(g)$ denote the group of automorphisms of the covering g, which are the diffeomorphisms $T : \mathbb{R} \times \widehat{L} \to \mathbb{R} \times \widehat{L}$ that satisfy $g = g \circ T$. It follows from standard covering space theory that $\mathrm{Deck}(g)$ acts freely and properly on $\mathbb{R} \times \widehat{L}$, and $\mathbb{R} \times \widehat{L} / \mathrm{Deck}(g) \simeq \Omega$.

The general case - speculations

- We can explicitely compute the maps in $\mathrm{Deck}(g)$:

Proposition

The group of deck transformations of g consists of maps $T_\alpha : \mathbb{R} \times \widehat{L} \to \mathbb{R} \times \widehat{L}$ of the following form:

$$T_\alpha(t,q) = \left(t+\alpha, \left(\pi\big|_{\widehat{L}}\right)^{-1} \circ \psi_{-\alpha} \circ \left(\pi\big|_{\widehat{L}}\right)(q)\right), \ (t,q) \in \mathbb{R} \times \widehat{L},$$

where $\alpha \in \mathscr{P}(\zeta)$.

The general case - speculations

- Let $\mathscr{S}$ denote the sheaf of smooth solutions of d'_0 on Ω. Since the complex d' is locally exact, it follows from standard arguments of sheaf theory that

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- The pullback sheaf $\widehat{\mathscr{S}} = g^* \mathscr{S}$ is the sheaf of smooth solutions of $\mathrm{d}_{\widehat{L}}$ on $\mathbb{R} \times \widehat{L}$. Since the group of periods $\mathscr{P}(\zeta)$ acts on $\mathbb{R} \times \widehat{L}$, it also acts on the sheaf $\widehat{\mathscr{S}}$.

The general case - speculations

- There is a spectral sequence (the Cartan-Leray spectral sequence) such that

$$E_2^{pq} = H^p\left(\mathscr{P}(\zeta); H^q(\mathbb{R}\times\widehat{L};\widehat{\mathscr{S}})\right) \implies H^{p+q}_{\mathscr{V}}(\Omega),$$

  where $H^p(\mathscr{P}(\zeta); E)$ denotes the group cohomology of the $\mathscr{P}(\zeta)$-module E.
- Since $\mathscr{P}(\zeta)$ is a free group, the group cohomology is only non-trivial in degrees 0 and 1. This implies the spectral sequence collapses and we obtain an (algebraic) isomorphism

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- Both spaces on the right hand side have natural topologies.

The general case - speculations

- The H^0 term (which corresponds to invariant *cohomology classes*) is always a Hausdorff space, however, the H^1 term is Hausdorff if and only if the map

$$C^\infty(\mathbb{R}; H^{q-1}_{\mathrm{dR}}(\widehat{L})) \to \prod_{\alpha \in \mathscr{P}(\zeta)} C^\infty(\mathbb{R}; H^{q-1}_{\mathrm{dR}}(\widehat{L}))$$

$$f \mapsto \big(T_\alpha^* f - f\big)_{\alpha \in \mathscr{P}(\zeta)}$$

has closed range.

The general case - speculations

- For example, if $q = 1$, the H^1 term is Hausdorff if and only if, for every sequence $\{f_n\} \subset C^\infty(\mathbb{R})$ such that

$$f_n(x + \alpha) - f_n(x) \to g_\alpha(x)$$

for every $\alpha \in \mathscr{P}(\zeta)$, then there is $f \in C^\infty(\mathbb{R})$ such that $g_\alpha(x) = f(x + \alpha) - f(x)$ for every $\alpha \in \mathscr{P}(\zeta)$.

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- If we assume that our initial structure is a principal S^1-bundle, then the same constructions apply, but $\mathbb{R}$ is replaced by S^1. This is precisely equivalent to $\mathscr{P}(\zeta)$ being generated by a irrational non-Liouville vector, i.e., ζ being irrational and non-Liouville. This is the structure that allows one to show (3) and (4) are equivalent in this case.

The general case - speculations

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- Notice that, since any pair of transversal vector fields are homotopic, the action of $\psi_{-\alpha}$ on the cohomology of the leaves is *independent* of the transversal vector field.
- It is tempting to think that this condition should be relevant for (strong) global solvability in general.

Thank you very much!