Algebraicity of elliptic KZB equations

Tiago J. Fonseca

Unicamp, FAPESP

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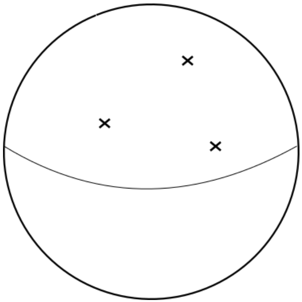
Genus 0



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  When $X/\mathbb{Q}$, the above isomorphism gives rise to *periods*. Ex.: $\mathbb{P}^1 \setminus \{0,\infty\} = \mathit{Spec}\ \mathbb{Q}[z,z^{-1}]$, with period

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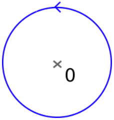

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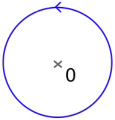

- What about homotopy groups?

- Line integrals cannot detect non-abelian phenomena in $\pi_1(X)$:

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- **Chen**: consider 'iterated line integrals'. Let $c : [0,1] \to X$ be a path and $\omega_i \in A^1(X)$. Define

$$\int_c \omega_1 \cdots \omega_r := \int_{0 \leq t_1 \leq \cdots \leq t_r \leq 1} f_1(t_1) \cdots f_r(t_r) dt_1 \cdots dt_r.$$

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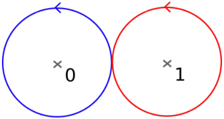
  where $c^*\omega_i = f_i(t_i)dt_i$.
- Example: $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$$\int_{aba^{-1}b^{-1}} \frac{dz}{z} \frac{dz}{z-1} = (2\pi i)^2.$$



- Let $x \in X$ and $Ch(P_x X, \mathbb{R})$ be the space of iterated integrals

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- Let I be the augmentation ideal of $\mathbb{Z}[\pi_1(X, x)]$. **Chen**'s theorem:

$$\varinjlim_n Hom(\mathbb{Z}[\pi_1(X, x)]/I^{n+1}, \mathbb{R}) \cong H^0 Ch(P_x X, \mathbb{R}),$$

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- Remarks:
  - The LHS is dual to the **Mal'cev** completion of $\pi_1(X,x)$.
  - Can replace $\mathbb{R}$ by $\mathbb{C}$.
  - There is also a version with two base points x, y.

- If $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$,

$$H^0 Ch(P_x X, \mathbb{C}) \cong \mathbb{Q}\langle \omega_0, \omega_1 \rangle \otimes \mathbb{C}, \qquad \omega_0 = \frac{dz}{z}, \quad \omega_1 = \frac{dz}{1-z}$$

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- Multiple zeta value (MZV):

$$\begin{aligned}\zeta(n_1, \ldots, n_r) &= \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} \\ &\overset{\textbf{Kontsevich}}{=} \int_0^1 \omega_1 \underbrace{\omega_0 \cdots \omega_0}_{n_1-1} \cdots \omega_1 \underbrace{\omega_0 \cdots \omega_0}_{n_r-1}\end{aligned}$$

Relations are explained by the integral formula. Example: $\zeta(3) = \zeta(1, 2)$ comes from change of variables $z \mapsto 1 - z$.

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- **Brown**'s theorem: every MZV is a $\mathbb{Q}$-linear combination of MZVs of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$.

- Polylogarithm:

$$Li_n(z) = \int_0^z \omega_1 \underbrace{\omega_0 \cdots \omega_0}_{n-1} = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.$$

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- More generally, if w is a word in x_0, x_1, define

$$Li_w(z)$$

  in such a way that $Li_{n_1,\ldots,n_r}(z) = Li_{x_0^{n_1-1}x_1 \cdots x_0^{n_r-1}x_1}(z)$.

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- KZ connection (**Knizhnik–Zamolodchikov**):

$$\nabla_{KZ} = d - \omega_0 \otimes x_0 - \omega_1 \otimes x_1$$

on the trivial infinite-rank vector bundle $\mathcal{V}_{KZ} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ with fibres $\mathbb{C}\langle\langle x_0, x_1 \rangle\rangle$.

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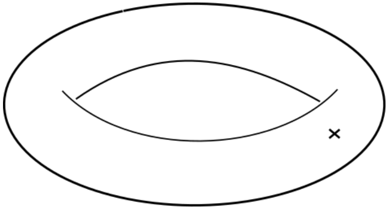
  such that $(\mathcal{V}_i, \nabla_i)/(\mathcal{V}_{i-1}, \nabla_{i-1})$ is trivial for every i.
- KZ is so nice (constant vector bundle, forms with log singularities) because

$$H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C}, \qquad H^1(\mathbb{P}^1, \mathcal{O}) = 0$$

  Every unipotent vector bundle over $\mathbb{P}^1$ is *canonically* trivial.

Genus 1



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- **Bloch**'s elliptic dilogarithm:

$$D_E(z) = \sum_{m\in\mathbb{Z}} \mathcal{L}_2(q^m z)$$

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  where $\mathcal{L}_2(z) = -2iIm(Li_2(z)) + 2\log|z|\log(1 - \bar{z})$.
- There are general theories of elliptic MPLs as iterated integrals on elliptic curves (e.g. **Brown–Levin**), but the underlying algebraic geometry is not clear.

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9.2. **The formula.** The connection is defined by a 1-form

$$\omega \in \Omega^1(\mathbb{C} \times \mathfrak{h}, \log \Lambda) \hat{\otimes} \operatorname{End} \mathfrak{p}.$$

via the formula

$$\nabla f = df + \omega f$$

where $f : \mathbb{C} \times \mathfrak{h} \to \mathfrak{p}$ is a (locally defined) section of (9.1). Specifically,

$$\omega = \frac{1}{2\pi i} d\tau \otimes \mathbf{a} \frac{\partial}{\partial \mathbf{t}} + \psi + \nu$$

where

$$\psi = \sum_{m \geq 1} \left(\frac{(2\pi i)^{2m+1}}{(2m)!} G_{2m+2}(\tau) d\tau \otimes \sum_{\substack{j+k=2m+1 \\ j,k>0}} (-1)^j [\operatorname{ad}_{\mathbf{t}}^j(\mathbf{a}), \operatorname{ad}_{\mathbf{t}}^k(\mathbf{a})] \frac{\partial}{\partial \mathbf{a}} \right)$$

and

$$\nu = \mathbf{t} F(\xi, \mathbf{t}, \tau) \cdot \mathbf{a}\, d\xi + \frac{1}{2\pi i} \left(\frac{1}{\mathbf{t}} + \mathbf{t} \frac{\partial F}{\partial \mathbf{t}}(\xi, \mathbf{t}, \tau) \right) \cdot \mathbf{a}\, d\tau.$$

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- Problem (cf. **Luo**): is this algebraic? Defined over $\mathbb{Q}$?

- Two main difficulties for a purely algebraic theory:
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  2. presence of moduli.
- Joint work in progress with **Nils Matthes**: work over the $\mathbb{A}^1$-bundle

$$\pi : E^\natural \to E$$

such that $H^0(E^\natural, \mathcal{O}) = \mathbb{C}$ and $H^1(E^\natural, \mathcal{O}) = 0$ (cf. **Deligne**, **Enriquez–Etingof**).

- Let $\mathcal{V}$ be the vector bundle on $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ given by

$$(m + n\tau) \cdot (v_1, v_2, z) = (v_1 + nv_2, v_2, z + m + n\tau)$$

$$\begin{array}{ccc} \mathbb{C}^2 \times \mathbb{C} & & \mathcal{V} \\ \downarrow & \overset{\text{mod } \mathbb{Z}+\tau\mathbb{Z}}{\rightsquigarrow} & \downarrow \\ \mathbb{C} & & E \end{array}$$

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- A splitting

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{V} \arrow[r] & \mathcal{O} \arrow[l, bend right] \arrow[r] & 0 \end{tikzcd}$$

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- No such holomorphic r. Real-analytic: $r(z) = Im(z)/Im(\tau)$ (cf. **Brown–Levin**).

- How to algebraize? Consider $\mathbb{C}^2$ with coordinates (z, r), and lift the action of $\mathbb{Z} + \mathbb{Z}\tau$ by

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- The quotient $\mathbb{C}^2/(\mathbb{Z} + \mathbb{Z}\tau)$ has a natural structure of algebraic variety such that the projection on the first coordinate

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is algebraic! In fact,

$$\mathbb{C}^2/(\mathbb{Z} + \mathbb{Z}\tau) \cong H^1(E, \mathbb{C})/H^1(E, \mathbb{Z}) \cong H^1(E, \mathbb{C}^\times)$$

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- Define $E^\natural$ as the moduli of *algebraic* flat line bundles on E and $\pi : E^\natural \to E \cong Pic^0(E)$ by $[(\mathcal{L}, \nabla)] \mapsto [\mathcal{L}]$.

Theorem (F.–Matthes; cf. Enriquez–Etingof)

Set $D = \pi^{-1}(O)$. There is a canonical decomposition

$$\Gamma(E^\natural, \Omega^1(\log D)) = \Gamma(E^\natural, \Omega^1) \oplus K^{(1)} \oplus K^{(2)} \oplus \cdots$$

where $K^{(n)}$ are 1-dimensional subspaces uniquely determined by:

1. $dK^{(n)} = \Gamma(E^\natural, \Omega^1) \wedge K^{(n-1)}$, *where* $K^{(0)} := \pi^*\Gamma(E, \Omega^1)$,
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3. $Res_D(K^{(n)})$ *has degree* $n-1$.

We can find $\nu, \omega^{(0)}, \omega^{(1)}, \omega^{(2)}, \ldots$ such that

$$\Gamma(E^\natural, \Omega^1) = \mathbb{C}\nu \oplus \mathbb{C}\omega^{(0)}, \qquad K^{(n)} = \mathbb{C}\omega^{(n)}$$

and $d\omega^{(n)} = \nu \wedge \omega^{(n-1)}$, $\omega^{(n)} \wedge \omega^{(0)} = 0$, $Res_D(\omega^{(n)}) = t^{n-1}/(n-1)!$.

Let $\mathcal{V}_{KZB}$ be the trivial vector bundle over $E^\natural \setminus D$ with fibres $\mathbb{C}\langle\langle a, b\rangle\rangle$, and

$$\nabla_{KZB}^{vert} = d - \nu \otimes a - \sum_{n\geq 0} \omega^{(n)} \otimes ad_a^n b$$

By the last theorem, $(\mathcal{V}_{KZB}, \nabla_{KZB}^{vert})$ is flat. Fix $b \in E^\natural \setminus D$.

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Theorem (F.–Matthes)

For every unipotent flat connection $(\mathcal{V}, \nabla)$ on $E^\natural \setminus D$ with $v \in \mathcal{V}_b$, there is a unique horizontal map

$$\varphi : (\mathcal{V}_{KZB}, \nabla^{vert}_{KZB}) \to (\mathcal{V}, \nabla), \qquad \varphi(1) = v.$$

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- Can be descended to E.
- If $E/\mathbb{Q}$, $(\mathcal{V}_{KZB}, \nabla_{KZB}^{vert})$ is also defined over $\mathbb{Q}$.

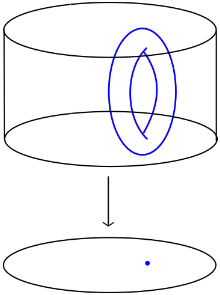
- Universal vector extension behaves well in families: given $E \to S$, get $f : E^\natural \to S$.
- All of the above results generalize to families. Get

$$\nabla_{KZB}^{vert} : \mathcal{V}_{KZB} \to \Omega^1_{E^\natural/S}(\log D) \otimes \mathcal{V}_{KZB}$$

- Universal KZB

$$\nabla_{KZB} : \mathcal{V}_{KZB} \to \Omega^1_{E^\natural}(\log D) \otimes \mathcal{V}_{KZB}$$

  is obtained as an "isomonodromic deformation" of ∇_{KZB}^{vert}. Proof relies on canonical lifts of relative forms (next slide).
- If $S/\mathbb{Q}$, then $(\mathcal{V}_{KZB}, \nabla_{KZB})$ is also defined over $\mathbb{Q}$.



Theorem (F.–Matthes)

The sequence

$$0 \longrightarrow \Omega^1_S \longrightarrow f_*\Omega^1_{E^\natural}(\log D) \longrightarrow f_*\Omega^1_{E^\natural/S}(\log D) \longrightarrow 0$$

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- Canonical lift to $f_*\Omega^1_E(\log D)$:

$$\widetilde{\omega}^{(1)} = \left(\frac{\theta'_\tau(z)}{\theta_\tau(z)} + 2\pi i\, r\right) dz + \frac{1}{2\pi i}\left(\frac{1}{2}\frac{\theta''_\tau(z)}{\theta_\tau(z)} - \frac{1}{6}\frac{\theta'''_\tau(0)}{\theta'_\tau(0)} - \frac{(2\pi i\, r)^2}{2}\right) d\tau$$

Thank you!