Advanced Numerical Concepts: Discontinuous Galerkin Method and Adaptive Mesh Refinement

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Outline of this session

1. Discontinuous Galkerin (DG) Method

2. Adaptivity

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    1. Overview of Existing Schemes
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   1. Overview of Existing Schemes
   2. Why DG?
   3. DG Formulation
2. Adaptivity

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$$h_i \frac{d\bar{u}_i}{dt} + F_{i+1/2} - F_{i-1/2} = 0$$

where $F_{i+1/2} = F(u_{i+1/2}),\; u_{i+1/2} = \frac{1}{2}(\bar{u}_{i+1} + \bar{u}_i)$.

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Good on Complex Geometry, local conservative property. But bad for adaptivity.

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Good on Complex Geometry, High-order accuracy. Performs poorly for certain pdes.

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2. Local conservative representation for the PDE as seen for FVM.

DG in 1D

We consider a partition of Ω into intervals. Then our approximation locally assumes the form

$$u_h(x,t) = \sum_{i=1}^{N} u_i(x_i,t)\phi_i(x), \quad x \in T.$$

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The choice of flux F^* here is crucial to guarantee other desirable properties of the scheme!

Numerical Example: 1D Wave Equation

$$\partial_t u - 2\pi \partial_x u = 0, \quad u(x,0) = \sin\left(kx\right), \ k = \frac{2\pi}{\lambda}, \quad \Omega = [0, 2\pi].$$

$p \backslash K$	2	4	8	16	32	64	Order
1	–	4.0e-1	9.1e-2	2.3e-02	5.7e-03	1.4e-03	2.0
2	2.0e-01	4.3e-02	6.3e-03	8.0e-04	1.0e-04	1.3e-05	3.0
4	3.3e-03	3.1e-04	9.9e-06	3.2e-07	1.0e-08	3.3e-10	5.0
8	2.1e-07	2.5e-09	4.8e-12	2.2e-13	5.0e-13	6.6e-13	$\approx$ 9.0

Table: Global L^2 errors with varying polynomial degree p and using K elements.

The error behaves like

$$\|u - u_h\| \leq Ch^{p+1}$$

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The scheme is very flexible!

What is DG?

1. The standard Galerkin Method is also referred to as Continuous Galkerin Method (CG).
2. The DG Method is a variant of the CG Method by relaxing these continuity requirements.
3. DG methods have more degrees of freedom to solve than CG methods.

What we gain by relaxing continuity...

1. Assembly of stiffness matrix is easier to implement. Easy to parallelize.
2. Adaptivity is more flexible
3. Locally conservative property
4. Thanks to the natural hierarchy, it is suitable for multilevel methods integrated into the solvers.
5. Supports higher order local approximations that can vary nonuniformly over the mesh.

DG for Elliptic Problems

Let $\Omega \subset \mathbb{R}^d$ be a bounded, open polygonal domain with Lipschitz continuous boundary Γ.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= u_D \quad \text{on } \Gamma_D, \\ \mathbf{n} \cdot \nabla u &= u_N \quad \text{on } \Gamma_N, \end{aligned}$$

where $\Gamma = \Gamma_D \cup \Gamma_N$ and $\mathbf{n}$ denotes the outward normal to Ω.

DG Method

Let $\mathbb{T}_h$: uniform partition of Ω into cells. Introduce our DG space:

$$\mathrm{V}_h := \{\mathrm{v}_h : \bar{\Omega} \to \mathbb{R} \mid \mathrm{v}_h|_T \in \mathcal{P}_k(T), T \in \mathbb{T}_h\}$$

Multiplying by $v \in \mathrm{V}_h$ integrating over each $T \in \mathbb{T}_h$ and summing over all possible cells we arrive at:

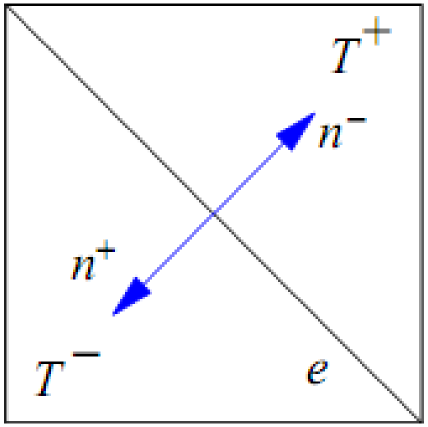
$$-\sum_{T \in \mathbb{T}_h} \int_T \Delta u v \; dx = \sum_{T \in \mathbb{T}_h} \int_T fv \; dx$$

$$\sum_{T \in \mathbb{T}_h} \left(\int_T \nabla u \cdot \nabla v \; dx - \int_{\partial T} \nabla u \cdot \mathbf{n} v \; ds \right) = \sum_{T \in \mathbb{T}_h} \int_T fv \; dx$$

DG Method

$$\int_{\partial T} \nabla u \cdot \mathbf{n} v \; ds = \int_{e} \nabla u^{+} \cdot \mathbf{n}^{+} v + \nabla u^{-} \cdot \mathbf{n}^{-} v \; ds$$

$$= \int_{e} \nabla u^{+} \cdot \mathbf{n}^{+} v + \nabla u^{-} \cdot \mathbf{n}^{-} v \; ds$$



$$\sum_{T\in\mathbb{T}_h}\int_{\partial T}\nabla u\cdot\mathbf{n}v\;ds=\sum_{e\in\mathbf{E}_h^i}\int_e\left(\nabla u^+\cdot\mathbf{n}^+v+\nabla u^-\cdot\mathbf{n}^-v\right)\;ds+$$

$$\sum_{e\in\Gamma_D}\int_e\nabla u\cdot\mathbf{n}v\;ds+\sum_{e\in\Gamma_N}\int_e g_Nv\;ds.$$

Resulting in

$$\sum_{T\in\mathbb{T}_h}\int_T\nabla u\cdot\nabla v\;dx-\sum_{e\in\mathbf{E}_h^i}\int_e\left(\nabla u^+\cdot\mathbf{n}^+v+\nabla u^-\cdot\mathbf{n}^-v\right)\;ds$$

$$+\sum_{e\in\Gamma_D}\int_e\nabla u\cdot\mathbf{n}v\;ds=\sum_{T\in\mathbb{T}_h}\int_T fv\;dx+\sum_{e\in\Gamma_N}\int_e g_Nv\;ds$$

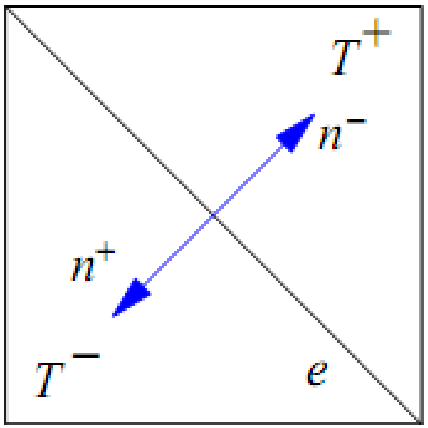
DG Method

$$\nabla u^+ \cdot \mathbf{n}^+ v + \nabla u^- \cdot \mathbf{n}^- v = \{\frac{\partial u}{\partial \mathbf{n}}\}_e [v]_e$$

where,

$$\left[v\right]_e = v^+ - v^- \qquad \text{(jump of v)}$$

$$\{\frac{\partial v}{\partial \mathbf{n}}\}_e = \frac{1}{2}(\nabla v^- \cdot \mathbf{n}^- + \nabla v^+ \cdot \mathbf{n}^+) \qquad \text{(average of v)}$$



Penalty Method

Find $u_h \in V_h$, such that

$$a_h^{IP}(u_h, v_h) = \ell(v_h)\ dx \quad \forall v_h \in V_h$$

where

$$\ell(v_h) = \int_\Omega f v_h\ dx + \int_{\Gamma_N} g_N v\ ds - \int_{\Gamma_D} g_D \frac{\partial v_h}{\partial \mathbf{n}}\ ds + \frac{\alpha}{|e|}\int_{\Gamma_D} g_D v_h\ ds$$

$$\begin{aligned} a_h^{IP}(u_h, v_h) := &\sum_{T \in \mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h\ dx - \sum_{e \in \mathcal{E}_h} \int_e \{\frac{\partial u_h}{\partial \mathbf{n}}\}_e \cdot [v_h]_e\ ds \\ &- \sum_{e \in \mathcal{E}_h} \int_e [u_h]_e \cdot \{\frac{\partial v_h}{\partial \mathbf{n}}\}_e\ ds + \sum_{e \in \mathcal{E}_h} \frac{\alpha}{|e|} \int_e [u_h]_e \cdot [v_h]_e\ ds \end{aligned}$$

where $\alpha > 0$ is a suitably chosen penalty parameter.

Well Posedness

Lemma (Coercivity and Continuity)

For penalty parameter $\alpha > \alpha_0 \geq 1$, we have

$$a_h^{IP}(u,v) \leq C_{cont} |||u||| \, |||u|||,$$
$$C_{coer} |||u|||^2 \leq a_h^{IP}(u,u).$$

where

$$|||v|||^2 = \sum_{T \in \mathbb{T}_h} \|\nabla v\|_{0,T}^2 + \sum_{e \in \mathbf{E}_h} \frac{\alpha}{|e|} \|[\mathrm{u}_h]\|_{0,e}^2.$$

A priori Estimates

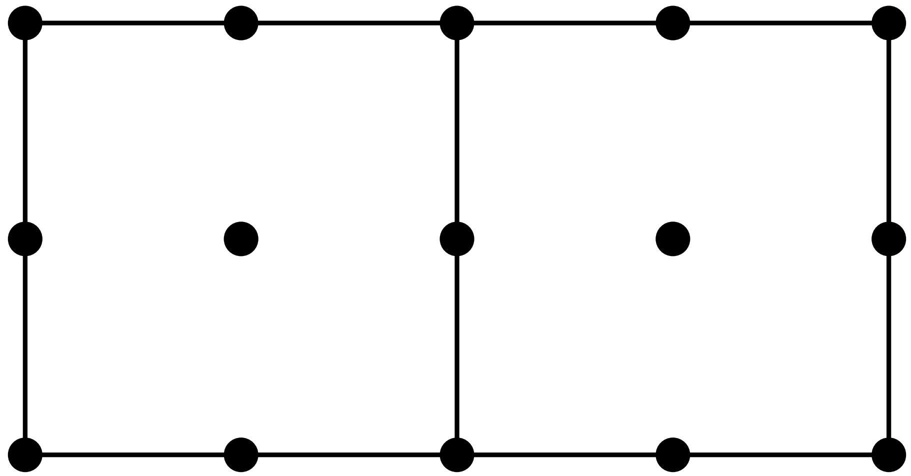
Theorem

For $u \in H^s(\mathbb{T}_h),\ s > 3/2$ and for penalty parameter large enough, we can find a constant $C > 0$ independent of h such that the following optimal a priori estimates hold:

$$|||u - u_h|||^2 \leq Ch^{\min\{k+1,s\}-1}|||v|||^2_{H^s}$$

deal.II Implementation

```
FE_Q<dim> fe_cg;
```



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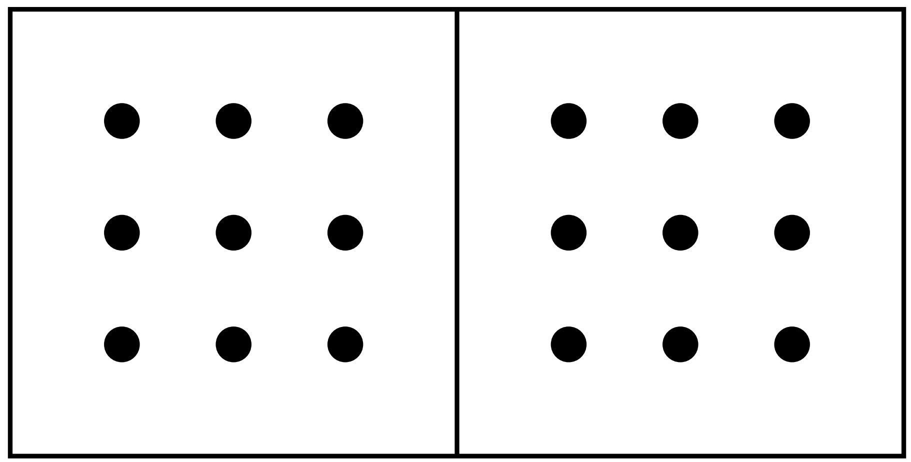

How does deal.II do it?

For `assemble_system()`, we want to code:

$$a_h^{IP}(u_h, v_h) := \sum_{T\in\mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h \; dx - \sum_{e\in\mathcal{E}_h} \int_e \{\frac{\partial u_h}{\partial \mathbf{n}}\}_e [v_h]_e \; ds$$

$$- \sum_{e\in\mathcal{E}_h} \int_e [u_h]_e \{\frac{\partial v_h}{\partial \mathbf{n}}\}_e \; ds + \sum_{e\in\mathcal{E}_h} \frac{\alpha}{|e|} \int_e [u_h]_e [v_h]_e \; ds$$

$$\ell(v_h) = \int_\Omega f v_h \; dx + \int_{\Gamma_N} g_N v \; ds - \int_{\Gamma_D} g_D \frac{\partial v_h}{\partial \mathbf{n}} \; ds + \frac{\alpha}{|e|} \int_{\Gamma_D} g_D v_h \; ds$$

1. `LocalIntegrators::Laplace::cell(dinfo,info)`
2. `LocalIntegrators::Laplace::boundary(dinfo,info)`
3. `LocalIntegrators::Laplace::face(dinfo1,dinfo2,info1,info2)`

`MeshWorker::integration_loop(···)` assembles all the local contributions into the global matrix.
Look into `step-39.cc` for more details!

ADAPTIVE FINITE ELEMENT METHODS

Need for adaptivity

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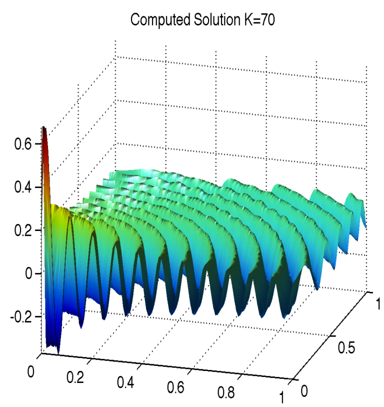

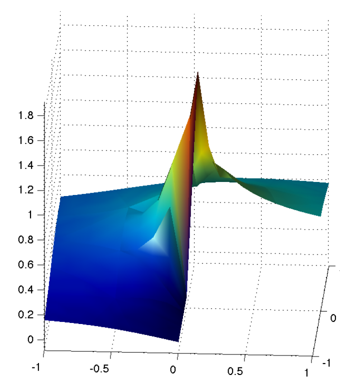

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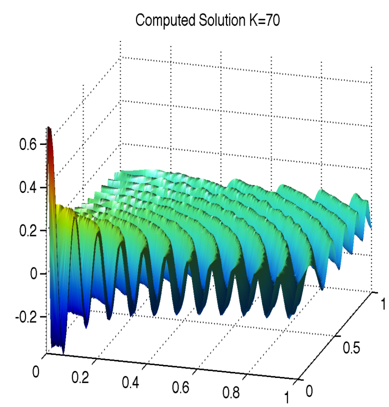
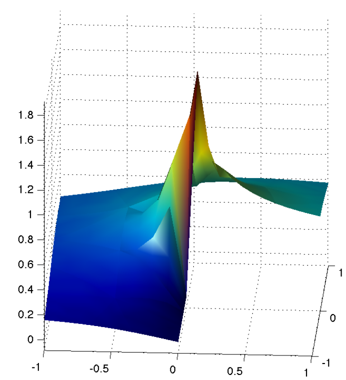




Devise an algorithm Solve-Estimate-Mark-Refine that efficiently identifies regions where the solution requires better resolution.

Adaptive Algorithm

The implementation of the adaptive algorithm is done according to the cycle:

SOLVE $\Longrightarrow$ ESTIMATE $\Longrightarrow$ MARK $\Longrightarrow$ REFINE

- SOLVE amounts to seeking a solution $u_h \in V_h$ st.

$$a_h(u_h, v_h) = \ell(v_h) \quad v_h \in V_h \quad \text{holds.}$$

- ESTIMATE the computation of η_h based on

$$\mathcal{R}(v) = \ell(v) - a_h(u_h, v)$$

- MARK deals with the selection of the elements and edges for refinement based on $0 < \theta < 1$

$$\theta \; \eta_h \;\lesssim \eta_{\mathcal{M}}$$

- REFINEMENT realizes the subdivision of cells.

Desirable Properties of the estimator

The estimator η_h consists of

$$\eta_h = \sum_{T \in \mathbb{T}_h} \eta_T + \sum_{e \in \mathbf{E}_h} \eta_e$$

E.g: For Poisson Equation, the residual based estimator has cell and edge terms like:

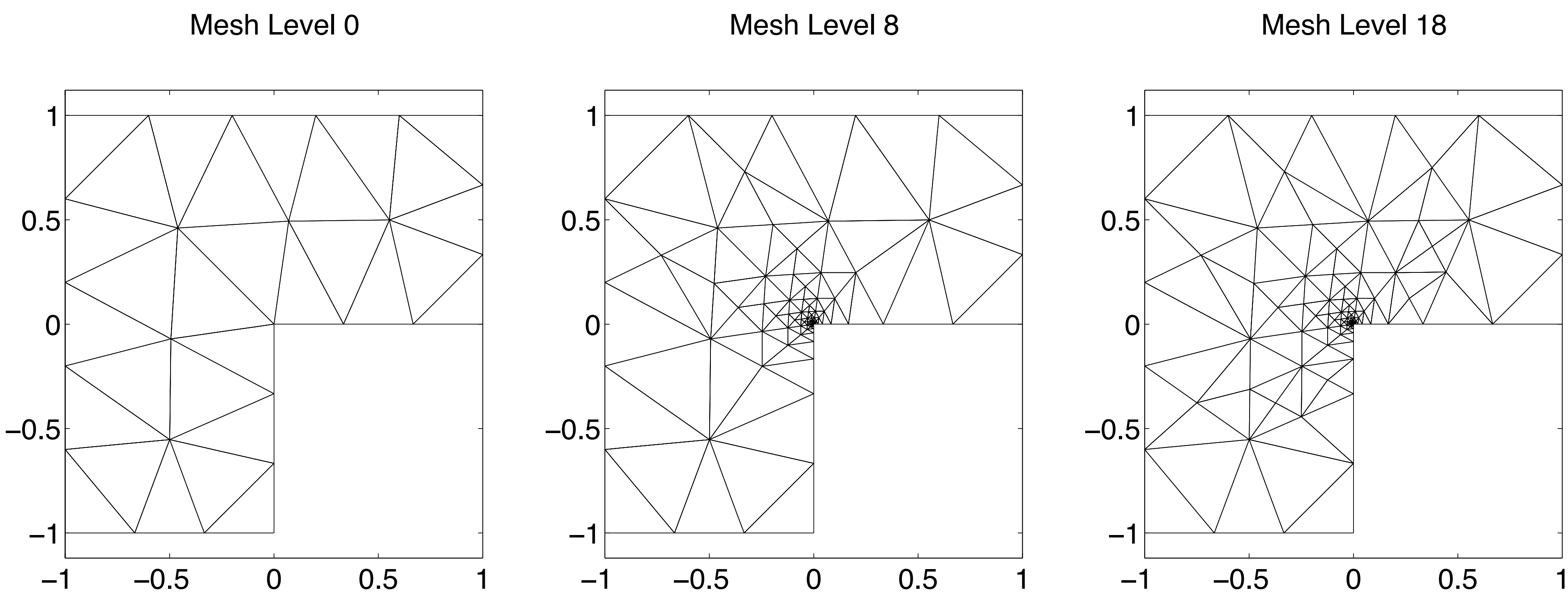
$$\eta_T = h_T ||f - (-\Delta u_h)||_{0,T}, \quad \eta_e = h_e^{1/2} \|[\partial_n u_h]\|_{0,e}$$

- Cheaply computable estimator
- η_h must be reliable, i.e,
$$\|u - u_h\| \leq C_{rel} \eta_h.$$
- η_h must be efficient, i.e,
$$C_{eff} \eta_h \leq \|u - u_h\|.$$

Notorious L-Shaped Domain

$$
\begin{aligned}
-\Delta u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma_D := \{0\} \times [-1,0] \cup [0,1] \times \{0\}, \\
\mathbf{n} \cdot \nabla u &= u_N \quad \text{on } \Gamma_N := \Gamma \setminus \Gamma_D,
\end{aligned}
$$

where the data is chosen according to $u(r,\phi) = grad(r^{\frac{2}{3}} \sin(\frac{2}{3}\phi))$.



$\theta = 0.1, k = 4$

Moving Peak

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in} \quad I \times \Omega,$$
$$u = 0 \quad \text{on} \quad I \times \partial\Omega,$$
$$u(0,.) = u_0(.) \quad \text{on} \quad \Omega$$

where $I = [0,1]$ and we choose data according to

$$u(\mathbf{x}, t) = g(t) \exp^{\beta((x_1-0.5)^2+(x_2-0.5)^2)}$$

- $\beta = 25$,
- $g(t) = 1 - \exp^{-\gamma(t-0.5)^2}$, $\gamma = 10$.

Numerical results: Adaptive algorithm

Moving peak

(Loading Video...)