SYSTEM OF EQUATIONS

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Outline

1. A little bit motivation (if needed)

2. Introduction of two fluid models
   - Derivation of the incompressible Navier-Stokes equations
   - Non-dimensionalization of the Navier-Stokes equations and derivation of the Stokes problem

3. Numerical methods for the Stokes problem
   - Derivation of the weak formulation
   - Finite element method for the space discretization and finite difference method for the time discretization
   - Inf-sup condition

4. Introduction to structural dynamics
   - General elastodynamics
   - A linear elasticity model

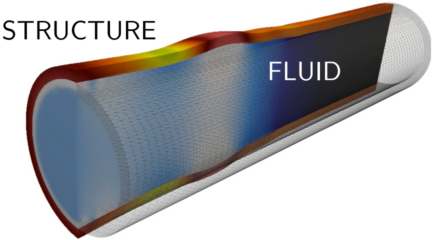
5. Numerical methods for the linear elasticity model
   - Derivation of the weak formulation
   - Finite element method for the space discretization and finite difference method for the time discretization

A little bit motivation

Tomorrow we will talk briefly about multiphysics problems, i.e. coupled systems involving more than one simultaneously occurring physical fields.

Example of muliphysics problems:

- Magnetohydrodynamics
- Thermomechanics
- Fluid-structure interaction (FSI)



FSI in civil engineering: flutter

PLAY

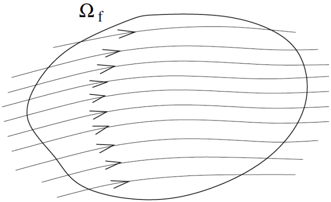
The first Tacoma Narrows Bridge opened on July 1, 1940. At the time of its construction the bridge was the third-longest suspension bridge in the world. Its main span collapsed four months later on November 7, 1940, due to aeroelastic flutter caused by a 67 kilometers per hour wind.

I will show more motivating examples tomorrow.

THE NAVIER-STOKES EQUATIONS AND STOKES FLOW

Derivation of the Navier-Stokes equations

Eulerian frame of reference: We consider a domain Ω_f occupied by a fluid moving at velocity $\mathbf{u}(\mathbf{x}, t)$, $\mathbf{x} \in \Omega_f$, $t > 0$.



Trajectory of fluid particles:

$$\mathbf{x}(t) : \quad \dot{\mathbf{x}}(t) = \mathbf{u}(\mathbf{x}, t)$$

Given a scalar function $f(\mathbf{x}, t)$, the material derivative is its time variation along the particle trajectory:

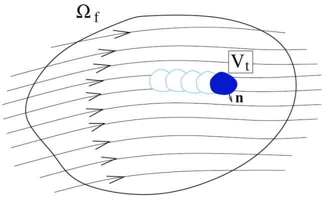
$$\frac{Df}{dt}(\mathbf{x}, t) = \frac{D}{dt} f(\mathbf{x}(t), t)$$

By applying chain rule, we have:

$$\frac{Df}{dt} = \frac{d}{dt} f(\mathbf{x}(t), t) = \frac{\partial f}{\partial t} + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t} = \frac{\partial f}{\partial t} + (\mathbf{u} \cdot \nabla) f$$

Derivation of the Navier-Stokes equations

Consider an arbitrary volume V_t transported by the fluid and a scalar $f(\mathbf{x}, t)$ defined in Ω_f.



Reynolds transport theorem

$$\frac{d}{dt}\int_{V_t} f \; d\mathbf{x} = \int_{V_t} \frac{\partial f}{\partial t} d\mathbf{x} + \int_{\partial V_t} f\mathbf{u}\cdot\mathbf{n} \; dA$$
$$= \int_{V_t} \left(\frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u})\right) d\mathbf{x}$$

Mass conservation

Principle of mass conservation: given an arbitrary volume V_t moving with the fluid, the mass contained in it will not change in time.

We denote by $\rho_f(\mathbf{x}, t)$ the fluid density. We have

$$\frac{d}{dt}\int_{V_t} \rho_f \, d\mathbf{x} = 0 \quad \xRightarrow{\text{by Reynolds th}} \quad \int_{V_t} \left(\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \mathbf{u}) \right) d\mathbf{x} = 0$$

Given that volume V_t is arbitrary, we conclude:

Mass conservation equation (or continuity equation)

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \mathbf{u}) = 0, \quad \text{in } \Omega_f, \; t > 0.$$

Incompressibility assumption

The Mach number is the ratio between the fluid velocity and the speed of sound in that fluid. The incompressibility assumption is acceptable for Mach $\ll$1. Many fluids at low velocity can be considered incompressible.

A flow is incompressible if any arbitrary subdomain V_t transported by the flow does not change its volume.

$$\frac{d}{dt}\int_{V_t} 1 d\mathbf{x} = 0 \quad \xRightarrow{\text{by Reynolds th}} \quad \nabla \cdot \mathbf{u} = 0$$

The continuity equation becomes:

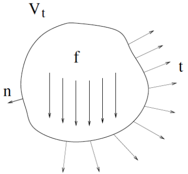
$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \mathbf{u}) = \frac{\partial \rho_f}{\partial t} + \cancelto{0}{\rho_f \nabla \cdot \mathbf{u}} + \mathbf{u} \cdot \nabla \rho_f = \frac{D\rho_f}{Dt} = 0$$

i.e. the density remains constant along particle trajectories.

Note: under the incompressibility assumption, a fluid initially homogeneous ($\rho_f|_{t=0} = \rho_0$ constant) remains homogeneous at all times, that is $\rho_f(\mathbf{x}, t) = \rho_0$, $\forall \mathbf{x} \in \Omega_f$, $t > 0$.

Momentum conservation

Principle of momentum conservation (Newton's second law): the variation of the momentum equals the forces applied to the system



$$\frac{d}{dt}\int_{V_t}\rho_f \mathbf{u}d\mathbf{x} = \int_{V_t}\mathbf{f}d\mathbf{x} + \int_{\partial V_t}\mathbf{t}dA$$

Here:

- $\mathbf{f}$ are the external volume forces per unit volume (e.g., gravity$\cdot\rho_f$)
- $\mathbf{t}$ are the internal surface tractions due to the interaction with the surrounding fluid particles.

Cauchy postulate: $\mathbf{t} = \mathbf{t}(\mathbf{x}, \mathbf{n}, t)$ depends only on the outward unit normal $\mathbf{n}$ to V_t at $\mathbf{x}$ and not on the particular volume V_t under consideration.

Momentum conservation

Cauchy theorem: $\mathbf{t}$ is necessarily a linear function of $\mathbf{n}$, i.e. there exists a tensor $\boldsymbol{\sigma}_f$, called Cauchy stress tensor, such that $\mathbf{t} = \mathbf{n}^T \boldsymbol{\sigma}_f$.

For the principle of conservation of angular momentum, equilibrium requires that the summation of momentum with respect to an arbitrary point is zero: $\boldsymbol{\sigma}_f$ is symmetric. Thus, $\mathbf{t} = \mathbf{n}^T \boldsymbol{\sigma}_f = \boldsymbol{\sigma}_f \mathbf{n}$.

By applying Reynolds transport theorem, divergence theorem, and arbitrariness of V_t, we get

Momentum conservation equation (first form)

$$\frac{\partial(\rho_f \mathbf{u})}{\partial t} + \nabla \cdot (\rho_f \mathbf{u} \otimes \mathbf{u}) = \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}_f, \qquad \text{in } \Omega_f, \; t > 0.$$

Momentum conservation

Notice that:

$$\frac{\partial(\rho_f \mathbf{u})}{\partial t} + \nabla\cdot(\rho_f \mathbf{u}\otimes\mathbf{u}) = \frac{\partial \rho_f}{\partial t}\mathbf{u} + \rho_f \frac{\partial \mathbf{u}}{\partial t} + \rho_f \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}\nabla\cdot(\rho_f\mathbf{u})$$

$$= \mathbf{u}\cancelto{0}{\left(\frac{\partial \rho_f}{\partial t} + \nabla\cdot(\rho_f\mathbf{u})\right)} + \rho_f\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right)$$

Thus, we get a second form for the momentum conservation equation:

Momentum conservation equation (second form)

$$\rho_f\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \mathbf{f} + \nabla\cdot\boldsymbol{\sigma}_f, \quad \text{in } \Omega_f,\ t > 0.$$

or

$$\rho_f\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \nabla\cdot\boldsymbol{\sigma}_f = \mathbf{f}, \quad \text{in } \Omega_f,\ t > 0.$$

Constitutive relations

To characterize a particular fluid, the Cauchy stress tensor has to be related to the fluid motion.

Hypotheses of Stokes:

- For a fluid at rest, $\boldsymbol{\sigma}_f = -p\mathbf{I}$, where p is the fluid pressure
- $\boldsymbol{\sigma}_f$ is a continuous function of $\nabla\mathbf{u}$: $\boldsymbol{\sigma}_f(\nabla\mathbf{u})$
- $\boldsymbol{\sigma}_f$ is isotropic (no preferred direction)
  $\Rightarrow Q^T\boldsymbol{\sigma}_f(\nabla\mathbf{u})Q = \boldsymbol{\sigma}_f(Q^T(\nabla\mathbf{u})Q)$ for all orthogonal matrices Q

It follows that $\boldsymbol{\sigma}_f$ has to be a function of the strain rate tensor $\epsilon(\mathbf{u}) = \frac{\nabla\mathbf{u}+(\nabla\mathbf{u})^T}{2}$.

A fluid that satisfies the hypotheses of Stokes and has a linear relation $\boldsymbol{\sigma}_f = \boldsymbol{\sigma}_f(\epsilon(\mathbf{u}))$ is called Newtonian fluid. Then, a Newtonian fluid has:

$$\boldsymbol{\sigma}_f(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\epsilon(\mathbf{u}) + \lambda(\nabla\cdot\mathbf{u})\mathbf{I}$$

where μ is the dynamic viscosity and λ is the second coefficient of viscosity.

The incompressible Navier-Stokes equations

The Navier-Stokes equations for an incompressible, homogeneous, and Newtonian fluid are given by:

$$
\begin{aligned}
&\rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}, p) = \mathbf{f} \\
&\nabla \cdot \mathbf{u} = 0 \\
&\boldsymbol{\sigma}_f(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\epsilon(\mathbf{u}) \\
&\rho_f = \rho_0 \text{ constant}
\end{aligned}
$$

Possible boundary conditions:

- imposed velocity profile: $\mathbf{u} = \mathbf{g}$ on Γ_D
- imposed traction (pressure and/or shear): $\boldsymbol{\sigma}_f \mathbf{n} = \mathbf{d}$ on Γ_N

Note

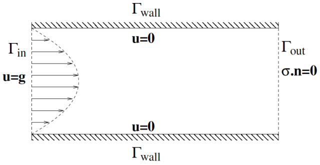
In the continuous formulation, $\nabla \cdot \nabla \mathbf{u}^T = \nabla(\nabla \cdot \mathbf{u}) = 0$, due to the continuity equation. Therefore, the Navier-Stokes equations are often formulated as:

$$\rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p - \mu \Delta \mathbf{u} = \mathbf{f},$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\rho_f = \rho_0 \text{ constant}$$

However, the contribution of the term $\nabla \mathbf{u}^T$ does not vanish when the Navier-Stokes problem is formulated in its weak form, as done in the finite element approximation.

An example of boundary conditions

Let's consider the flow in a channel:



Typically, the following boundary conditions are assigned:

- $\mathbf{u} = \mathbf{g}$ on Γ_{in} $\rightarrow$ inflow velocity profile
- $\mathbf{u} = \mathbf{0}$ on Γ_{wall} $\rightarrow$ adherence to the wall
- $\boldsymbol{\sigma}_f \mathbf{n} = \mathbf{0}$ on Γ_{out} $\rightarrow$ stress free condition

Non-dimensionalization of the Navier-Stokes equations

Let:

- U be a characteristic speed $\rightarrow \mathbf{u}^* = \mathbf{u}/U$
- L be a characteristic length $\rightarrow \mathbf{x}^* = \mathbf{x}/L$
- L/U be a characteristic time $\rightarrow t^* = t/(L/U)$
- $\rho_f U^2$ be a characteristic pressure $\rightarrow p^* = p/(\rho_f U^2)$

Let us assume no forcing term $(\mathbf{f} = \mathbf{0})$. The momentum balance equation can be rewritten as:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho_f} \nabla p - \nu \Delta \mathbf{u} = \mathbf{0},$$

where $\nu = \frac{\mu}{\rho_f}$ is the fluid kinematic viscosity.

Next, we switch to non-dimensional variables.

Non-dimensionalization of the Navier-Stokes equations

$$\frac{U^2}{L}\frac{\partial \mathbf{u}^*}{\partial t^*} + \frac{U^2}{L}\mathbf{u}^* \cdot \nabla^* \mathbf{u}^* + \frac{U^2}{L}\nabla^* p^* - \nu \frac{U}{L^2}\Delta^* \mathbf{u}^* = \mathbf{0} \rightarrow$$
$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* + \nabla p^* - \frac{\nu}{UL}\Delta^* \mathbf{u}^* = \mathbf{0}$$

The ratio

$$\frac{UL}{\nu} = Re$$

is called Reynolds number and it measures the relative importance of inertial effects versus viscous effects.

So, we can write:

$$Re\ \frac{\partial \mathbf{u}^*}{\partial t^*} + Re\ \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* + Re\ \nabla p^* - \Delta^* \mathbf{u}^* = \mathbf{0}$$

Stokes flow

When $Re \to 0$ (creeping flow), the advective inertial forces are small compared with viscous forces. Thus, one can consider a simplified linear problem:

Stokes problem

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}, p) &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

plus boundary conditions. Or, as an alternative

Time-dependent Stokes problem

$$\begin{aligned} \rho_f \frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}, p) &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

plus boundary conditions.

FINITE ELEMENT APPROXIMATION OF THE STOKES EQUATIONS

Finite element approximation of the Stokes equations

Let us consider the time-dependent Stokes problem:

$$
\begin{aligned}
&\rho_f \frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}, p) = \mathbf{f} && \text{in } \Omega_f, t > 0, \\
&\nabla \cdot \mathbf{u} = 0 && \text{in } \Omega_f, t > 0 \\
&\mathbf{u} = \mathbf{g} && \text{on } \Gamma_D, t > 0 \\
&\boldsymbol{\sigma}_f \mathbf{n} = \mathbf{d} && \text{on } \Gamma_N, t > 0 \\
&\mathbf{u} = \mathbf{u}_0 && \text{in } \Omega_f, t = 0,
\end{aligned}
$$

with $\boldsymbol{\sigma}_f(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\epsilon(\mathbf{u})$.

Weak formulation

Let us define the following functional spaces:

$$\mathbf{V} = [H^1(\Omega_f)]^d, \quad \mathbf{V}_0 = [H^1_{\Gamma_D}(\Omega_f)]^d \equiv \{\mathbf{v} \in \mathbf{V},\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\},$$
$$Q = L^2(\Omega_f)$$

Remark: if $\Gamma_D = \partial\Omega_f$, the pressure is defined only up to a constant. The pressure space should be $Q = L^2(\Omega_f)\backslash\mathbb{R}$.

We multiply the continuity equation by a test function $q \in Q$ and the momentum equation by a test function $\mathbf{v} \in \mathbf{V}_0$ and integrate over the domain.

Integration by parts of the stress term:

$$\int_{\Omega_f} -(\nabla \cdot \boldsymbol{\sigma}_f) \cdot \mathbf{v} = \int_{\Omega_f} \boldsymbol{\sigma}_f : \nabla \mathbf{v} - \int_{\partial\Omega_f} (\boldsymbol{\sigma}_f \mathbf{n}) \cdot \mathbf{v}$$

$$= \int_{\Omega_f} 2\mu\epsilon(\mathbf{u}) : \nabla \mathbf{v} - \int_{\Omega_f} \underbrace{p\mathbf{I} : \nabla \mathbf{v}}_{= p\nabla \cdot \mathbf{v}} - \cancelto{0}{\int_{\Gamma_D} (\boldsymbol{\sigma}_f \mathbf{n}) \cdot \mathbf{v}} - \int_{\Gamma_N} \underbrace{(\boldsymbol{\sigma}_f \mathbf{n})}_{= \mathbf{d}} \cdot \mathbf{v}$$

Weak formulation

The weak formulation of the Stokes problem is:
Find $(\mathbf{u}(t), p(t)) \in \mathbf{V} \times Q$, $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{u}(t) = \mathbf{g}(t)$ on Γ_D such that

$$\int_{\Omega_f} \rho_f \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} + \int_{\Omega_f} 2\mu \boldsymbol{\epsilon}(\mathbf{u}) : \nabla \mathbf{v} - \int_{\Omega_f} p \nabla \cdot \mathbf{v} = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{d} \cdot \mathbf{v}$$

$$\int_{\Omega_f} q \nabla \cdot \mathbf{u} = 0$$

$$\forall (\mathbf{v}, q) \in \mathbf{V}_0 \times Q$$

Finite element approximation

Introduce finite dimensional spaces of finite element type

$$\mathbf{V}_h \subset \mathbf{V}, \quad \mathbf{V}_{h0} = \mathbf{V}_h \cap \mathbf{V}_0, \quad Q_h \subset Q$$

Observe that pressure functions do not need to be continuous. Moreover let us denote by $\mathbf{u}_{h0} \in \mathbf{V}_h$ and $\mathbf{g}_h \in \mathbf{V}_h(\Gamma_D)$ suitable approximations of the initial velocity and Dirichlet boundary data.

The Finite Element approximation (continuous in time) is:
Find $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{V}_h \times Q_h$, $\mathbf{u}_h(0) = \mathbf{u}_{h0}$, $\mathbf{u}_h(t) = \mathbf{g}_h(t)$ on Γ_D such that

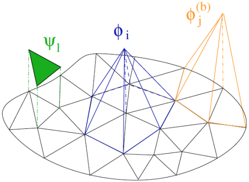
$$\int_{\Omega_f} \rho_f \frac{\partial \mathbf{u}_h}{\partial t} \cdot \mathbf{v} + \int_{\Omega_f} 2\mu\epsilon\left(\mathbf{u}_h\right) : \nabla \mathbf{v} - \int_{\Omega_f} p_h \nabla \cdot \mathbf{v} = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{d} \cdot \mathbf{v}$$

$$\int_{\Omega_f} q \nabla \cdot \mathbf{u}_h = 0$$

$$\forall (\mathbf{v}, q) \in \mathbf{V}_{h0} \times Q_h$$

Algebraic formulation

- $\{\phi_i\}_{i=1}^{N_u}$: basis of $\mathbf{V}_{h0}$
- $\{\phi_j^b\}_{j=1}^{N_u^b}$: basis of $\mathbf{V}_h \backslash \mathbf{V}_{h0}$ (shape functions corresponding to boundary nodes)
- $\{\psi_l\}_{l=1}^{N_p}$: basis of Q_h



Expand the solution $(\mathbf{u}_h(t), p_h(t))$ on the finite element basis

$$\mathbf{u}_h(\mathbf{x}, t) = \sum_{i=1}^{N_u} u_i(t)\phi_i(\mathbf{x}) + \sum_{j=1}^{N_u^b} g_j(t)\phi_j^b(\mathbf{x})$$

$$p_h(\mathbf{x}, t) = \sum_{l=1}^{N_p} p_i(t)\psi_l(\mathbf{x})$$

Vectors of unknown degrees of freedom (nodal values if Lagrange basis functions are used)

$$\mathbf{U}(t) = [\mathbf{u}_1(t), \ldots, \mathbf{u}_{N_u}(t)]^T, \quad \mathbf{P}(t) = [p_1(t), \ldots, p_{N_p}(t)]^T$$

Algebraic system

Plugging the expansions of $\mathbf{u}_h$ and p_h in the finite element formulation and testing with shape functions ϕ_i and ψ_l we get the following system of ODEs for $t > 0$:

$$M\frac{d\mathbf{U}}{dt} + A\mathbf{U} + B^T\mathbf{P} = \mathbf{F}_u,$$
$$B\mathbf{U} = \mathbf{F}_p,$$

where

- $M_{ij} = \int_{\Omega_f} \rho_f \boldsymbol{\phi}_j \cdot \boldsymbol{\phi}_i$ mass matrix
- $A_{ij} = \int_{\Omega_f} 2\mu\boldsymbol{\epsilon}\left(\boldsymbol{\phi}_j\right) : \nabla\boldsymbol{\phi}_i$ stiffness matrix
- $B_{ij} = -\int_{\Omega_f} \psi_l \nabla \cdot \boldsymbol{\phi}_i$ divergence matrix
- $(\mathbf{F}_u)_i = \int_{\Omega_f} \mathbf{f} \cdot \boldsymbol{\phi}_i + \int_{\Gamma_N} \mathbf{d} \cdot \boldsymbol{\phi}_i - \int_{\Omega_f} \rho_f \frac{\partial \mathbf{g}_h}{\partial t} \cdot \boldsymbol{\phi}_i - \int_{\Omega_f} 2\mu\boldsymbol{\epsilon}(\mathbf{g}_h) : \nabla\boldsymbol{\phi}_i$
- $(\mathbf{F}_p)_l = \int_{\Omega_f} \psi_l \nabla \cdot \mathbf{g}_h$

Spurious pressure modes

If these exists a pressure p_h^* such that

$$-\int_{\Omega_f} p_h^* \nabla \cdot \mathbf{v} = 0,$$

it follows that if $(\mathbf{u}_h, p_h)$ is a solution of the discretized Stokes problem, then also $(\mathbf{u}_h, p_h + p_h^*)$ is a solution to the system and we lose uniqueness of the pressure. Such a pressure p_h^* is called spurious mode.

A necessary and sufficient condition to avoid the presence of spurious pressure modes is that the finite elements spaces $\mathbf{V}_h \times Q_h$ satisfy

inf-sup condition : $\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \dfrac{\left| \int_{\Omega_f} q_h \nabla \cdot \mathbf{v}_h \right|}{||\mathbf{v}_h||_{H^1} ||q_h||_{L^2}} \geq \beta_h > 0$

Observe that the inf-sup condition is satisfied by the continuous spaces $\mathbf{V} \times Q$ but not necessarily by the finite element spaces $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset Q$.

Spurious pressure modes

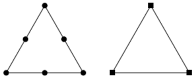
At the algebraic level, the inf-sup condition is equivalent to the request that $\mathrm{Ker}(B^T) = \emptyset$.

Spaces that satisfy the inf-sup condition are called compatible.

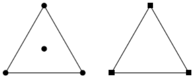
Assume the spaces $(\mathbf{V}_h, Q_h)$ are compatible with β_h independent of h. Then, the optimality of the Galerkin projection holds:

$$||\mathbf{u}_h - \mathbf{u}_{ex}||_{H^1} + ||p_h - p_{ex}||_{L^2} \leq C \left(\inf_{\mathbf{v}_h \in \mathbf{V}_h} ||\mathbf{v}_h - \mathbf{v}_{ex}||_{H^1} + \inf_{q_h \in Q_h} ||q_h - q_{ex}||_{L^2} \right)$$

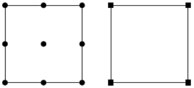
Examples of spaces that satisfy the inf-sup condition



$\mathbb{P}_2 \,/\, \mathbb{P}_1$



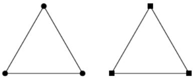
$\mathbb{P}_1^{bubble} \,/\, \mathbb{P}_1$



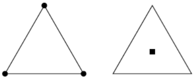
$\mathbb{Q}_2 \,/\, \mathbb{Q}_1$

Spurious pressure modes

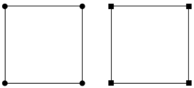
Examples of spaces that do not satisfy the inf-sup condition



$\mathbb{P}_1 \,/\, \mathbb{P}_1$



$\mathbb{P}_1 \,/\, \mathbb{P}_0$

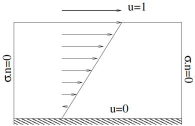


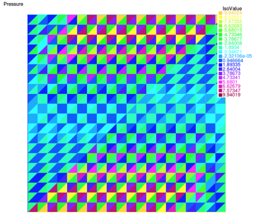
$\mathbb{Q}_1 \,/\, \mathbb{Q}_1$

Example: Couette flow

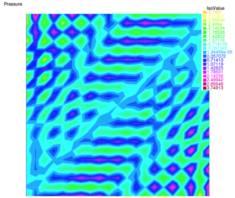
$u_x = y$

$u_y = p = 0$





Pressure with $\mathbb{P}_1/\mathbb{P}_0$



Pressure with $\mathbb{P}_1/\mathbb{P}_1$

Temporal discretization

Divide the time interval of interest $[0, T]$ into subintervals of length Δt and let $t^n = n\Delta t$ and $\mathbf{u}^n \approx \mathbf{u}(t^n)$.

- A first order scheme: Implicit Euler

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t^n} \approx [\mathbf{u}]_t^n = \frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t}$$

- A second order scheme: Backward differentiation formula of order 2 (BDF2)

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t^n} \approx [\mathbf{u}]_t^n = \frac{3\mathbf{u}^n - 4\mathbf{u}^{n-1} + \mathbf{u}^{n-2}}{2\Delta t}$$

Temporal discretization

The time discrete problem reads:

$$\rho_f [\mathbf{u}]_t^n - \nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}^n, p^n) = \mathbf{f}^n \qquad \text{in } \Omega_f, n = 1, 2, ...,$$

$$\nabla \cdot \mathbf{u}^n = 0 \qquad \text{in } \Omega_f, n = 1, 2, ...,$$

$$\mathbf{u}^n = \mathbf{g} \text{ on } \Gamma_D, \quad \boldsymbol{\sigma}_f^n \cdot \mathbf{n} = \mathbf{d} \text{ on } \Gamma_N, \quad \mathbf{u}^0 = \mathbf{u}_0$$

which leads to a linear system to solving at every time step:

- Implicit Euler

$$\begin{bmatrix} \frac{1}{\Delta t} M + A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}^n \\ \mathbf{P}^n \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u^n + \frac{1}{\Delta t} M \mathbf{U}^{n-1} \\ \mathbf{F}_p^n \end{bmatrix}$$

- BDF2

$$\begin{bmatrix} \frac{3}{2\Delta t} M + A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}^n \\ \mathbf{P}^n \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u^n + \frac{M}{2\Delta t}(4\mathbf{U}^{n-1} - \mathbf{U}^{n-2}) \\ \mathbf{F}_p^n \end{bmatrix}$$

with a direct or iterative method.

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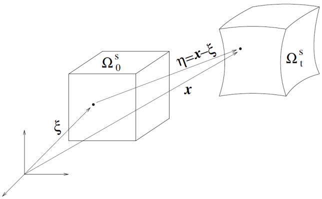
ELASTICITY MODELS

Structural models: kinematics

Dynamics of structures is more conveniently described in Lagrangian coordinates, i.e. equations are written in the reference configuration Ω_s^0 (e.g., the initial configuration).

Lagrangian map: $\mathbf{x} = \mathbf{x}(\xi, t) = \mathcal{L}_t(\xi)$



- ξ: coordinate of a material point in reference configuration
- $\mathbf{x}$: coordinate of a material point in deformed configuration
- $\eta(\xi, t) = \mathbf{x} - \xi$ displacement of a material point

Kinematics:

velocity: $$\mathbf{u}(\xi, t) = \frac{\partial \mathbf{x}}{\partial t}(\xi, t) = \dot{\mathbf{x}}(\xi, t) = \dot{\eta}(\xi, t)$$

acceleration: $$\mathbf{a}(\xi, t) = \frac{\partial^2 \mathbf{x}}{\partial t^2}(\xi, t) = \ddot{\mathbf{x}}(\xi, t) = \ddot{\eta}(\xi, t)$$

Eulerian vs Lagrangian

Lagrangian map (mostly used for solids)
Kinematics is described in the reference configuration

- position $\mathbf{x} = \mathbf{x}(\xi, t)$
- velocity $\mathbf{u} = \mathbf{u}(\xi, t) = \dot{\mathbf{x}}(\xi, t)$
- acceleration $\mathbf{a} = \mathbf{a}(\xi, t) = \ddot{\mathbf{x}}(\xi, t)$

Eulerian map (mostly used for fluids)
Kinematics is described in the current configuration

- position (trajectories) $\mathbf{x} = \mathbf{x}(\xi, t)$: $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t)$, $\mathbf{x}(\xi, 0) = \xi$
- velocity $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$
- acceleration $\mathbf{a} = \mathbf{a}(\mathbf{x}, t) = \frac{D\mathbf{u}}{Dt}(\mathbf{x}, t) = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u}$

Measures of strain

Let us introduce the deformation gradient tensor

$$\mathbf{F} = \nabla_{\boldsymbol{\xi}} \mathbf{x} = \mathbf{I} + \nabla_{\boldsymbol{\xi}} \boldsymbol{\eta}.$$

Given an infinitesimal material line segment $d\boldsymbol{\xi}$ in Ω_s^0, it will be transformed through the Lagrangian map into $d\mathbf{x} = \mathbf{F} d\boldsymbol{\xi}$.

- Right Cauchy-Green tensor: $\mathbf{C} = \mathbf{F}^T\mathbf{F}$
  Measures the change in length of the infinitesimal material line segment due to the motion

$$\frac{||d\mathbf{x}||^2}{||d\boldsymbol{\xi}||^2} = \frac{d\mathbf{x}^T d\mathbf{x}}{||d\boldsymbol{\xi}||^2} = \frac{d\boldsymbol{\xi}^T \mathbf{F}^T \mathbf{F} d\boldsymbol{\xi}}{||d\boldsymbol{\xi}||^2} = \frac{d\boldsymbol{\xi}^T}{||d\boldsymbol{\xi}||} \mathbf{C} \frac{d\boldsymbol{\xi}}{||d\boldsymbol{\xi}||} = \mathbf{v}^T \mathbf{C} \mathbf{v}$$

where $\mathbf{v} = d\boldsymbol{\xi}/||d\boldsymbol{\xi}||$ is a unit vector.

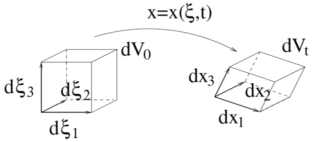
Measures of strain

- Green strain tensor: $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$
  measures the relative elongation of $d\boldsymbol{\xi}$

$$\frac{||d\mathbf{x}||^2 - ||d\boldsymbol{\xi}||^2}{||d\boldsymbol{\xi}||^2} = \frac{d\boldsymbol{\xi}^T \mathbf{C} d\boldsymbol{\xi} - d\boldsymbol{\xi}^T d\boldsymbol{\xi}}{||d\boldsymbol{\xi}||^2} = \mathbf{v}^T(\mathbf{C} - \mathbf{I})\mathbf{v} = 2\mathbf{v}^T\mathbf{E}\mathbf{v}$$

  where $\mathbf{v} = d\boldsymbol{\xi}/||d\boldsymbol{\xi}||$ is a unit vector.

Deformation of volume elements



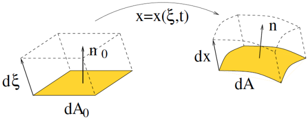
Consider an infinitesimal volume $dV_0 = \det(d\boldsymbol{\xi}_1, d\boldsymbol{\xi}_2, d\boldsymbol{\xi}_3)$

After deformation:

$$\begin{aligned} dV_t &= \det(d\mathbf{x}_1, d\mathbf{x}_2, d\mathbf{x}_3) \\ &= \det(\mathbf{F}d\boldsymbol{\xi}_1, \mathbf{F}d\boldsymbol{\xi}_2, \mathbf{F}d\boldsymbol{\xi}_3) \\ &= \det(\mathbf{F}) \det(d\boldsymbol{\xi}_1, d\boldsymbol{\xi}_2, d\boldsymbol{\xi}_3) \\ &= \det(\mathbf{F}) dV_0 \end{aligned}$$

Thus, $J = \det(\mathbf{F}) = \frac{dV_t}{dV_0}$ measures the change of volumes.

Deformation of surface elements



Consider an infinitesimal surface element dA_0 with normal $\mathbf{n}_0$ and an infinitesimal vector $d\boldsymbol{\xi}$.
Infinitesimal volume:
$dV_0 = d\boldsymbol{\xi}^T \mathbf{n}_0 dA_0$

After deformation

$$dV_t = d\mathbf{x}^T \mathbf{n} dA = J dV_0 = d\boldsymbol{\xi}^T (J \mathbf{n}_0 dA_0) = d\mathbf{x}^T (J \mathbf{F}^{-T} \mathbf{n}_0 dA_0)$$

Since $d\mathbf{x}$ is arbitrary, we obtain

Nanson's formula

$$\mathbf{n} dA = J \mathbf{F}^{-T} \mathbf{n}_0 dA_0$$

Equations of motion: conservation of mass

The equations have to be cast into the reference configuration Ω_0^s.

Conservation of mass:

$$\frac{d}{dt}\int_{V_t} \rho_s d\mathbf{x} = 0$$

where $\rho_s = \rho_s(\mathbf{x}, t)$ is the density in the current configuration.

Let ρ_s^0 be the density in the reference configuration, e.g, the density at the initial time. Then $\forall t > 0$, we have

$$\int_{V_t} \rho_s dV_t = \int_{V_0} \rho_s^0 dV_0 \quad \Rightarrow \rho_s = J^{-1}\rho_s^0$$

Mass conservation equation

$$\frac{d\rho_s^0}{dt} = 0, \quad \text{in } \Omega_s^0, \ t > 0.$$

Equations of motion: conservation of momentum

Conservation of momentum:

$$\frac{d}{dt}\int_{V_t} \rho_s \mathbf{u}\ d\mathbf{x} = \int_{V_t} \mathbf{f}\ d\mathbf{x} + \int_{\partial V_t} \mathbf{t}\ dA$$

Switching to the reference configuration:

$$\frac{d}{dt}\int_{V_t} \rho_s \mathbf{u}\ d\mathbf{x} = \frac{d}{dt}\int_{V_0} \underbrace{J\rho_s}_{\rho_s^0} \mathbf{u}\ d\boldsymbol{\xi} = \int_{V_0} \rho_s^0 \mathbf{a}\ d\boldsymbol{\xi} = \int_{V_0} \rho_s^0 \ddot{\boldsymbol{\eta}}\ d\boldsymbol{\xi}$$

$$\int_{V_t} \mathbf{f}\ d\mathbf{x} = \int_{V_0} \underbrace{J\mathbf{f}(\mathbf{x}(\boldsymbol{\xi}, t))}_{\mathbf{f}^0(\boldsymbol{\xi})}\ d\boldsymbol{\xi} = \int_{V_0} \mathbf{f}^0(\boldsymbol{\xi})\ d\boldsymbol{\xi}$$

$$\int_{\partial V_t} \mathbf{t}\ dA = \int_{\partial V_t} \underbrace{\boldsymbol{\sigma}_s \mathbf{n}}_{\mathbf{t}}\ dA = \int_{\partial V_0} \underbrace{J\boldsymbol{\sigma}_s \mathbf{F}^{-T}}_{\boldsymbol{\sigma}_s^0} \mathbf{n}_0\ dA = \int_{V_0} \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\sigma}_s^0\ d\boldsymbol{\xi}$$

Momentum conservation equation

$$\rho_s^0 \ddot{\boldsymbol{\eta}} - \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\sigma}_s^0 = \mathbf{f}^0, \quad \text{in } \Omega_s^0,\ t > 0.$$

Nominal stress tensor

$$\boldsymbol{\sigma}_s^0 = J\boldsymbol{\sigma}_s\mathbf{F}^{-T}$$

We have introduced $\boldsymbol{\sigma}_s^0$, which is called nominal stress tensor or first Piola-Kirchhoff stress tensor. It has the property that:

$$\boldsymbol{\sigma}_s^0\mathbf{n}_0 dA = \boldsymbol{\sigma}_s\mathbf{n} dA$$

It represents the "stress tensor" in the reference configuration.

Problem: $\boldsymbol{\sigma}_s^0$ is not symmetric because $\mathbf{F}$ is not symmetric.
Remember: the Cauchy stress tensor $\boldsymbol{\sigma}_s$ is symmetric.

One can define the second Piola-Kirchhoff stress tensor $\mathbf{S} = \mathbf{F}^{-1}\boldsymbol{\sigma}_s^0 = J\mathbf{F}^{-1}\boldsymbol{\sigma}_s\mathbf{F}^{-T}$, which is symmetric but does not have a direct physical interpretation.

Constitutive laws

Unlike fluids, an elastic body presents internal stresses as a consequence of deformation gradients (instead of gradients of deformation rates).

Cauchy elastic material:

- $\boldsymbol{\sigma}_s^0$ is a continuous function of $\mathbf{F}$: $\boldsymbol{\sigma}_s^0 = \boldsymbol{\sigma}_s^0(\mathbf{F})$
- Frame indifference:

$$\boldsymbol{\sigma}_s^0(Q\mathbf{F}) = \boldsymbol{\sigma}_s^0(\mathbf{F})Q^T$$

  for all Q orthogonal. This implies that $\boldsymbol{\sigma}_s^0$ can be expressed as a function of $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ only: $\boldsymbol{\sigma}_s^0 = \boldsymbol{\sigma}_s^0(\mathbf{C})$
- Incompressible material: add one constraint $\det \mathbf{F} = 1$ and a corresponding Lagrange multiplier p (pressure)

$$\boldsymbol{\sigma}_s = \tilde{\boldsymbol{\sigma}}_s(\boldsymbol{\eta}) - p\mathbf{I}, \quad \boldsymbol{\sigma}_s^0 = \tilde{\boldsymbol{\sigma}}_s^0(\boldsymbol{\eta}) - pJ\mathbf{F}^{-T}$$

Constitutive laws: Green elastic materials

For a Green elastic material (also called hyperelastic) the stress tensor is derived from a strain energy function W

$$\boldsymbol{\sigma}_s^0 = \frac{\partial W}{\partial \mathbf{F}}$$

- Frame indifference: $W(Q\mathbf{F}) = W(\mathbf{F})$, for all Q orthogonal. This implies that W can be expressed as a function of $\mathbf{C}$

$$W = W(\mathbf{C}), \quad \text{thus } \boldsymbol{\sigma}_s^0 = \frac{\partial W}{\partial \mathbf{F}} = \frac{\partial W}{\partial \mathbf{C}} \frac{\partial \mathbf{C}}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}$$

- Isotropic material: $W(\mathbf{F}Q) = W(\mathbf{F})$, for all Q orthogonal. This relation, together with the frame indifference implies that W depends only on the eigenvalues of $\mathbf{C}$.

Examples of constitutive laws

Example of incompressible material ($J = 1$)
Neo-Hookean: $W(\mathbf{C}) = \frac{\mu}{2}(\mathrm{tr}(\mathbf{C}) - 3)$

$$\boldsymbol{\sigma}_s^0 = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}} - pJ\mathbf{F}^{-T}$$

Example of compressible material
Saint-Venant Kirchhoff: $W(\mathbf{C}) = \frac{\lambda}{2}\mathrm{tr}(\mathbf{E})^2 + \mu\mathrm{tr}(\mathbf{E}^2)$, with $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$
and λ, μ Lamé constants

$$\boldsymbol{\sigma}_s^0 = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}} = \mathbf{F}\frac{\partial W}{\partial \mathbf{E}} = \mathbf{F}(\lambda\mathrm{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E})$$

Equations of motion: hyperelastic materials

Compressible materials

$$\rho_s^0 \ddot{\boldsymbol{\eta}} - \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\sigma}_s^0(\boldsymbol{\eta}) = \mathbf{f}^0, \quad \boldsymbol{\sigma}_s^0 = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}$$

Incompressible materials

$$\rho_s^0 \ddot{\boldsymbol{\eta}} - \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\sigma}_s^0(\boldsymbol{\eta}, p) = \mathbf{f}^0, \quad \boldsymbol{\sigma}_s^0 = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}} - pJ\mathbf{F}^T$$

$$J = 1$$

Possible boundary conditions:

- imposed displacement: $\boldsymbol{\eta} = \mathbf{g}$ on Γ_D
- imposed nominal stress: $\boldsymbol{\sigma}_s^0 \mathbf{n}_0 = \mathbf{d}$ on Γ_N

Linear Saint-Venant Kirchhoff model

Recall that for this hyperelastic material we have

$$\boldsymbol{\sigma}_s^0 = \mathbf{F}(\lambda \mathrm{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E})$$

where λ, μ are the Lamé constants. Since $\mathbf{F} = \mathbf{I} + \nabla_{\boldsymbol{\xi}}\boldsymbol{\eta}$, we have

$$\begin{aligned}\mathbf{E} &= \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}((\mathbf{I} + \nabla_{\boldsymbol{\xi}}\boldsymbol{\eta})^T(\mathbf{I} + \nabla_{\boldsymbol{\xi}}\boldsymbol{\eta}) - \mathbf{I})) \\ &= \frac{1}{2}(\cancel{\mathbf{I}} + \nabla_{\boldsymbol{\xi}}\boldsymbol{\eta} + \nabla_{\boldsymbol{\xi}}\boldsymbol{\eta}^T + \nabla_{\boldsymbol{\xi}}\boldsymbol{\eta}^T\nabla_{\boldsymbol{\xi}}\boldsymbol{\eta} - \cancel{\mathbf{I}}),\end{aligned}$$

which under the hypothesis of small deformations can be approximated as

$$\mathbf{E} \approx \frac{1}{2}(\nabla_{\boldsymbol{\xi}}\boldsymbol{\eta} + \nabla_{\boldsymbol{\xi}}\boldsymbol{\eta}^T) = \epsilon(\boldsymbol{\eta}),$$

Thus, under the same hypothesis

$$\boldsymbol{\sigma}_s^0 = (\mathbf{I} + \nabla_{\boldsymbol{\xi}}\boldsymbol{\eta})(\lambda \mathrm{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}) \approx \lambda(\nabla \cdot \boldsymbol{\eta})\mathbf{I} + 2\mu\epsilon(\boldsymbol{\eta}).$$

FINITE ELEMENT APPROXIMATION OF THE EQUATIONS FOR COMPRESSIBLE STRUCTURES

Weak formulation

Let $\mathbf{V}$ be a suitable functional space and let $\mathbf{V}_0 = \{\mathbf{v} \in \mathbf{V},\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}$.

We multiply the momentum equation by a test function $\phi \in \mathbf{V}_0$ and integrate over the domain.

Integration by parts of the stress term:

$$\int_{\Omega_s^0} -(\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\sigma}_s^0) \cdot \phi \, d\boldsymbol{\xi} = \int_{\Omega_s^0} \boldsymbol{\sigma}_s^0 : \nabla_{\boldsymbol{\xi}} \phi \, d\boldsymbol{\xi} - \int_{\partial \Omega_s^0} (\boldsymbol{\sigma}_s^0 \cdot \mathbf{n}_0) \cdot \phi \, dA_0$$

$$= \int_{\Omega_s^0} \boldsymbol{\sigma}_s^0 : \nabla_{\boldsymbol{\xi}} \phi \, d\boldsymbol{\xi} - \cancelto{0}{\int_{\Gamma_D} (\boldsymbol{\sigma}_s^0 \cdot \mathbf{n}_0) \cdot \phi \, dA_0} - \int_{\Gamma_N} \underbrace{(\boldsymbol{\sigma}_s^0 \cdot \mathbf{n}_0)}_{= \mathbf{d}} \cdot \phi \, dA_0$$

The weak formulation of the structure equations is: Find $\boldsymbol{\eta}(t) \in \mathbf{V}$, $\boldsymbol{\eta}(0) = \boldsymbol{\eta}_0$, $\boldsymbol{\eta}(t) = \mathbf{g}(t)$ on Γ_D such that

$$\int_{\Omega_s^0} \rho_s^0 \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} \cdot \phi \, d\boldsymbol{\xi} + \int_{\Omega_s^0} \boldsymbol{\sigma}_s^0 : \nabla_{\boldsymbol{\xi}} \phi \, d\boldsymbol{\xi} = \int_{\Omega_s^0} \mathbf{f}^0 \cdot \phi \, d\boldsymbol{\xi} + \int_{\Gamma_N} \mathbf{d} \cdot \phi \, dA_0, \ \forall \phi \in \mathbf{V}_0$$

Particular case: linear model

For the linear Saint-Venant Kirchhoff model, the stress term becomes:

$$\int_{\Omega_s^0} \boldsymbol{\sigma}_s^0 : \nabla_{\boldsymbol{\xi}} \phi \, d\boldsymbol{\xi} = \int_{\Omega_s^0} 2\mu\epsilon(\mathbf{u}) : \nabla_{\boldsymbol{\xi}} \phi \, d\boldsymbol{\xi} + \int_{\Omega_s^0} \underbrace{\lambda(\nabla \cdot \boldsymbol{\eta})\mathbf{I} : \nabla_{\boldsymbol{\xi}} \phi}_{=\lambda(\nabla \cdot \boldsymbol{\eta})(\nabla_{\boldsymbol{\xi}} \cdot \phi)} \, d\boldsymbol{\xi}$$

$$= \int_{\Omega_s^0} 2\mu\epsilon(\boldsymbol{\eta}) : \nabla_{\boldsymbol{\xi}} \phi \, d\boldsymbol{\xi} + \int_{\Omega_s^0} \lambda(\nabla \cdot \boldsymbol{\eta})(\nabla_{\boldsymbol{\xi}} \cdot \phi) \, d\boldsymbol{\xi}$$

Then, the weak formulation of the structure equations is: Find $\boldsymbol{\eta}(t) \in \mathbf{V}$, $\boldsymbol{\eta}(0) = \boldsymbol{\eta}_0$, $\boldsymbol{\eta}(t) = \mathbf{g}(t)$ on Γ_D such that

$$\int_{\Omega_s^0} \rho_s^0 \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} \cdot \phi \, d\boldsymbol{\xi} + \int_{\Omega_s^0} 2\mu\epsilon(\boldsymbol{\eta}) : \nabla_{\boldsymbol{\xi}} \phi \, d\boldsymbol{\xi} + \int_{\Omega_s^0} \lambda(\nabla \cdot \boldsymbol{\eta})(\nabla_{\boldsymbol{\xi}} \cdot \phi) \, d\boldsymbol{\xi}$$

$$= \int_{\Omega_s^0} \mathbf{f}^0 \cdot \phi \, d\boldsymbol{\xi} + \int_{\Gamma_N} \mathbf{d} \cdot \phi \, dA_0, \ \forall \phi \in \mathbf{V}_0$$

Energy estimate

Consider a hyperelastic material with $\boldsymbol{\sigma}_s^0 = \frac{\partial W}{\partial \mathbf{F}}$ and assume zero forcing terms ($\mathbf{f}^0 = \mathbf{g} = \mathbf{d} = \mathbf{0}$). Take $\phi = \dot{\boldsymbol{\eta}}$ in the weak formulation

$$\int_{\Omega_s^0} \rho_s^0 \ddot{\boldsymbol{\eta}} \cdot \dot{\boldsymbol{\eta}} \, d\boldsymbol{\xi} = \frac{\rho_s^0}{2} \frac{d}{dt} \int_{\Omega_s^0} (\dot{\boldsymbol{\eta}})^2 \, d\boldsymbol{\xi} = \frac{\rho_s}{2} \frac{d}{dt} ||\dot{\boldsymbol{\eta}}||_{L^2}^2$$

$$\int_{\Omega_s^0} \boldsymbol{\sigma}_s^0 : \nabla_{\boldsymbol{\xi}} \dot{\boldsymbol{\eta}} \, d\boldsymbol{\xi} = \int_{\Omega_s^0} \frac{\partial W}{\partial \mathbf{F}} : \dot{\mathbf{F}} \, d\boldsymbol{\xi} = \int_{\Omega_s^0} \frac{\partial W}{\partial t} \, d\boldsymbol{\xi}$$

After integration in time over the interval $[0, T]$, we obtain:

energy conservation

$$\frac{\rho_s^0}{2} ||\dot{\boldsymbol{\eta}}(T)||_{L^2}^2 + \int_{\Omega_s^0} W(\boldsymbol{\eta}(T)) \, d\boldsymbol{\xi} = \frac{\rho_s^0}{2} ||\dot{\boldsymbol{\eta}}_0||_{L^2}^2 + \int_{\Omega_s^0} W(\boldsymbol{\eta}_0) \, d\boldsymbol{\xi}$$

Finite element approximation

Consider a finite element space $\mathbf{V}_h \subset \mathbf{V}$ defined on a suitable triangulation $\mathcal{T}_h$ of Ω_s^0.

Moreover let us denote by $\boldsymbol{\eta}_{h0} \in \mathbf{V}_h$ and $\mathbf{g}_h \in \mathbf{V}_h(\Gamma_D)$ suitable approximations of the initial datum and Dirichlet boundary datum.

The Finite Element approximation (continuous in time) is:
Find $\boldsymbol{\eta}_h(t) \in \mathbf{V}_h$, $\boldsymbol{\eta}_h(0) = \boldsymbol{\eta}_{h0}$, $\boldsymbol{\eta}_h(t) = \mathbf{g}_h(t)$ on Γ_D such that

$$\int_{\Omega_s^0} \rho_s^0 \frac{\partial^2 \boldsymbol{\eta}_h}{\partial t^2} \cdot \phi \, d\boldsymbol{\xi} + \int_{\Omega_s^0} \boldsymbol{\sigma}_s^0(\boldsymbol{\eta}_h) : \nabla_{\boldsymbol{\xi}} \phi \, d\boldsymbol{\xi} = \int_{\Omega_s^0} \mathbf{f}^0 \cdot \phi \, d\boldsymbol{\xi} + \int_{\Gamma_N} \mathbf{d} \cdot \phi \, dA_0$$

for all $\phi \in \mathbf{V}_{h0}$.

Algebraic formulation: linear case

Introduce a basis $\{\phi_i\}_{i=1}^{N_s}$ of $\mathbf{V}_{h0}$, a basis $\{\phi_j^b\}_{j=1}^{N_s^b}$ for $\mathbf{V}_h \backslash \mathbf{V}_{h0}$ and expand $\boldsymbol{\eta}_h$:

$$\boldsymbol{\eta}_h(\boldsymbol{\xi}, t) = \sum_{i=1}^{N_s} \eta_i(t)\phi_i(\boldsymbol{\xi}) + \sum_{j=1}^{N_s^b} g_j(t)\phi_j^b(\boldsymbol{\xi})$$

Vectors of unknown dofs: $\mathbf{D}(t) = [\boldsymbol{\eta}_1(t), \ldots, \boldsymbol{\eta}_{N_s}(t)]^T$.

Plugging the expansions of $\boldsymbol{\eta}_h$ into the finite element formulation and testing with shape functions ϕ_i, we get the following system of linear ODEs for $t > 0$:

$$M_s\ddot{\mathbf{D}} + K_s\mathbf{D} = \mathbf{F}_s$$

CAREFUL: if the structural model is nonlinear, then $K_s\mathbf{D}$ has to be replaced with the nonlinear term $K_s(\mathbf{D})$.

Algebraic system: linear case

$$M_s\ddot{\mathbf{D}} + K_s\mathbf{D} = \mathbf{F}_s$$

where

- $(M_s)_{ij} = \int_{\Omega_s^0} \rho_s^0 \phi_j \cdot \phi_i$ mass matrix
- $(K_s(\mathbf{D}))_{ij} = \int_{\Omega_s^0} 2\mu\epsilon(\phi_j) : \nabla_{\xi}\phi_i \, d\xi + \int_{\Omega_s^0} \lambda(\nabla \cdot \phi_j)(\nabla_{\xi} \cdot \phi_i) \, d\xi$ stiffness matrix
- $(\mathbf{F}_s)_i = \int_{\Omega_s^0} \mathbf{f}^0 \cdot \phi_i + \int_{\Gamma_N} \mathbf{d} \cdot \phi_i - \int_{\Omega_s^0} \rho_s^0 \ddot{\mathbf{g}}_h \cdot \phi_i - \int_{\Omega_s^0} \boldsymbol{\sigma}_s^0(\mathbf{g}_h) : \nabla_{\xi}\phi_i$

CAREFUL: if the structural model is nonlinear, the stiffness matrix depends on $\mathbf{D}$

- $(K_s(\mathbf{D}))_{ij} = \int_{\Omega_s^0} \boldsymbol{\sigma}_s^0(\eta_h) : \nabla_{\xi}\phi_i$

Temporal discretization: Midpoint scheme

A popular scheme is the Midpoint scheme.

The equation is collocated at $t^{n-1/2}$. Central finite differences are used to approximate both the velocity and the acceleration. For the linear case, we have:

$$
\begin{aligned}
M_s \frac{\dot{\mathbf{D}}^n - \dot{\mathbf{D}}^{n-1}}{\Delta t} + K_s \frac{\mathbf{D}^n + \mathbf{D}^{n-1}}{2} = \mathbf{F}_s^{n-1/2}, \\
\frac{\mathbf{D}^n - \mathbf{D}^{n-1}}{\Delta t} = \frac{\dot{\mathbf{D}}^n + \dot{\mathbf{D}}^{n-1}}{2}
\end{aligned}
$$

- Second order accurate.
- No numerical dissipation for linear problems.

We can eliminate the velocity $\dot{\mathbf{D}}^n$ in the first equation and solve the problem in displacement only, i.e. at each time step we have to solve a linear system in the displacement vector $\mathbf{D}^n$.

Temporal discretization: Newmark scheme

Another popular scheme is the following Newmark scheme.

Let: $\mathbf{D}^n \approx \mathbf{D}(t^n)$, $\dot{\mathbf{D}}^n \approx \dot{\mathbf{D}}(t^n)$, $\ddot{\mathbf{D}}^n \approx \ddot{\mathbf{D}}(t^n)$. Define two parameters $\gamma \in [0,1]$ and $\beta \in [0, \frac{1}{2}]$.
For the linear case, we have:

$$M_s \ddot{\mathbf{D}}^n + K_s \mathbf{D}^n = \mathbf{F}_s^n, \qquad \text{eq. collocated at time } t^n$$

$$\dot{\mathbf{D}}^n = \dot{\mathbf{D}}^{n-1} + \Delta t(\gamma \ddot{\mathbf{D}}^n + (1-\gamma)\ddot{\mathbf{D}}^{n-1}), \qquad \text{Taylor exp. for } \dot{\mathbf{D}}$$

$$\mathbf{D}^n = \mathbf{D}^{n-1} + \Delta t \dot{\mathbf{D}}^{n-1} + \frac{\Delta t^2}{2}(2\beta \ddot{\mathbf{D}}^n + (1-2\beta)\ddot{\mathbf{D}}^{n-1}), \qquad \text{Taylor exp. for } \mathbf{D}$$

- Leads to a linear system in $(\mathbf{D}^n, \dot{\mathbf{D}}^n, \ddot{\mathbf{D}}^n)$ at each time step
- For linear second order systems it is unconditionally stable for $\beta \geq \frac{1}{4}$ and second order accurate for $\gamma = \frac{1}{2}$

Temporal discretization: Newmark scheme

We can use the expansion for $\mathbf{D}^n$ and $\dot{\mathbf{D}}^n$ to express the acceleration $\ddot{\mathbf{D}}^n$ in terms of the displacement $\mathbf{D}^n$. We get:

$$\ddot{\mathbf{D}}^n = \frac{1}{\beta \Delta t^2}\mathbf{D}^n - \zeta^{n-1}, \text{ with } \zeta^{n-1} = \frac{1}{\beta \Delta t^2}(\mathbf{D}^{n-1} + \Delta t \dot{\mathbf{D}}^{n-1}) + \frac{1-2\beta}{2\beta}\ddot{\mathbf{D}}^{n-1}$$

Now, the algebraic system can be written in the displacement only:

$$\frac{1}{\beta \Delta t^2} M_s \mathbf{D}^n + K_s \mathbf{D}^n = \mathbf{F}_s^n + M_s \zeta^{n-1}$$

At each time step we have to solve a linear system in the displacement vector $\mathbf{D}^n$.

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