MIXED FINITE ELEMENT METHODS

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Outline of this session

1. Theory

2. Implementation

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   1. Revisiting Poisson Equation
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    3. Playtime!

Revisiting Poisson Equation

Let $\Omega \subset \mathbb{R}^d$ be a polygonal domain.

$$\begin{aligned} -\nabla\cdot(\mathbf{K}(\mathbf{x})\nabla p) &= f \quad \text{in } \Omega, \\ p &= g \quad \text{on } \partial\Omega, \end{aligned}$$

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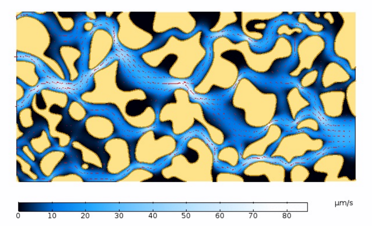


Mixed Formulation

Introduce $\mathbf{u} = -\mathbf{K}\nabla p$,

$$\begin{aligned} K^{-1}\mathbf{u} + \nabla p &= \quad 0 \quad \text{in } \Omega, \\ -\nabla\cdot\mathbf{u} &= -f \quad \text{in } \Omega, \\ p &= g \quad \text{on } \partial\Omega. \end{aligned}$$

Mixed Weak Formulation

Left multiply the above system by suitable test functions $\mathbf{v}, q$ and integrating over Ω gives us:

$$\begin{aligned} (\mathbf{v}, K^{-1}\mathbf{u})_\Omega + (\mathbf{v}, \nabla p)_\Omega &= 0, \\ -(q, \nabla\cdot\mathbf{u})_\Omega &= -(q, f)_\Omega. \end{aligned}$$

Integration by parts of $(\mathbf{v}, \nabla p)_\Omega$ yields:

$$\begin{aligned} (\mathbf{v}, K^{-1}\mathbf{u})_\Omega - (\nabla\cdot\mathbf{v}, p)_\Omega &= -(g, \mathbf{v}\cdot\mathbf{n})_{\partial\Omega}, \\ -(q, \nabla\cdot\mathbf{u})_\Omega \qquad\qquad &= -(f, q)_\Omega. \end{aligned}$$

Mixed Weak Formulation

Introduce the following spaces:

$$\mathbf{V} = \mathbf{H}(\mathrm{div}, \Omega) := \left\{ \mathbf{v} \in L^2(\Omega)^d \;\middle|\; \nabla\cdot\mathbf{v} \in L^2(\Omega) \right\}, \quad Q = L^2(\Omega)/\mathbb{R}.$$

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \ell^{(1)}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= \ell^{(2)}(q) \quad \forall q \in Q. \end{aligned}$$

where

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, K^{-1}\mathbf{u})_\Omega, \quad b(\mathbf{v}, p) = -(\nabla\cdot\mathbf{v}, p)_\Omega,$$
$$\ell^{(1)}(\mathbf{v}) = -(g, \mathbf{v}\cdot\mathbf{n})_{\partial\Omega}, \quad \ell^{(2)}(q) = -(q, f)_\Omega.$$

Operator Form

For each of the bilinear forms, we associate the following operators:

$$
\begin{aligned}
M : \mathbf{V} \to \mathbf{V}^* \quad & \langle Mu, v \rangle = a(\mathbf{u}, \mathbf{v}), \\
B : Q \to \mathbf{V}^* \quad & \langle Bp, v \rangle = b(\mathbf{v}, p) \\
B^T : \mathbf{V} \to Q^* \quad & \langle B^T u, q \rangle = b(\mathbf{u}, q),
\end{aligned}
$$

here $\langle \cdot, \cdot \rangle$ is the canonical pairing between the dual space and the corresponding space.

$$
\begin{bmatrix} M & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \ell^{(1)} \\ \ell^{(2)} \end{bmatrix}
$$

Solvability of the Operator Form

Write the above saddle-point system as

$$\begin{aligned} M\mathbf{u} + Bp &= \ell^{(1)} \\ B^T\mathbf{u} &= \ell^{(2)} \end{aligned}$$

Suppose the operator M is invertible. Substituting $\mathbf{u} = M^{-1}\ell^{(1)} - Bp$ in the second equation gives us

$$\underbrace{-B^T M^{-1} B}_{S} \, p = \ell^{(2)} - B^T M^{-1} \ell^{(1)}.$$

Definition (Schur Complement)

The Schur complement operator $S : Q \to Q^*$ of the system as

$$S = -B^T M^{-1} B$$

Solvability of the Operator Form

Theorem

The solution $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *to the saddle-point system can be obtained by solving*

$$\begin{aligned} Sp &= \ell^{(2)} - B^T M^{-1} \ell^{(1)}, \\ M\mathbf{u} &= \ell^{(1)} - Bp. \end{aligned}$$

Well Posedness

Theorem

Assume that:

- $a(\cdot,\cdot) : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ *is elliptic on* $Ker(B^T)$ *i.e.,* $\exists\ \alpha > 0$ *st.*

$$a(\mathbf{v}_0, \mathbf{v}_0) \geq \alpha \|\mathbf{v}_0\|^2_{\mathbf{V}} \quad \forall \mathbf{v}_0 \in Ker(B^T).$$

- $b(\cdot,\cdot) : \mathbf{V} \times Q \to \mathbb{R}$ *satisfies the inf-sup condition:*

$$\inf_{q \in Q \setminus Ker(B)} \sup_{v \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\| \ \|q\|} \geq \beta > 0.$$

Then, for any given $\ell^{(1)} \in \mathbf{V}^*$ *and* $\ell^{(2)} \in Q^*$, *the above saddle-point system admits a solution* $(\mathbf{u}, p) \in \mathbf{V} \times Q$.

Note: $\mathbf{u} \in \mathbf{V}$ is uniquely determined and $p \in Q$ is unique up to an element of $Ker(B)$.

Finite Element Approximation

Let $\{\mathbb{T}_h\}_h$ be a uniformly shape regular family of partitions of Ω into rectangular cells.
Introduce finite dimensional spaces

$$\mathbf{V}_h \subset \mathbf{V} \quad Q_h \subset Q.$$

Not all choices of subspaces inherit the inf-sup condition!

$$B^T|_{\mathbf{V}_h} \neq B_h^T$$

where $B_h^T : \mathbf{V}_h \to Q_h^* \quad \langle B_h^T \mathbf{u}_h, q_h \rangle = b(\mathbf{u}_h, q_h).$

RT Element

Definition

The **Raviart Thomas element** of degree $k \geq 0$ on the reference cell/ hypercube $\hat{T} = [-1, 1]^d$ consists of th polynomial space

$$RT_{[k]}(\hat{T}) = \mathbb{Q}_k(\hat{T})^d + \mathbf{x}\, \mathbb{Q}_k(\hat{T}).$$

Its nodal functionals are

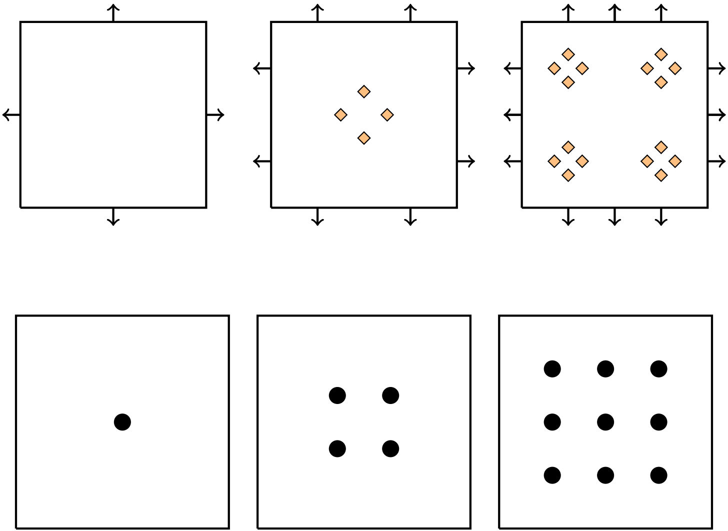
$$\mathcal{N}_{1,i,j}(\mathbf{v}) = \int_{F_i} \mathbf{v} \cdot \mathbf{n} q_j \, ds, \quad q_j \in \mathbb{Q}_k(F_i), \quad F_i \subset \hat{T},$$

$$\mathcal{N}_{2,i}(\mathbf{v}) = \int_{\hat{T}} \mathbf{v} \cdot \mathbf{w}_i \, dx, \quad \mathbf{w}_i \in \mathbb{Q}_{k-1,k,\cdots,k} \times \mathbb{Q}_{k,k-1,\cdots,k}.$$

There holds,

$$\dim RT_{[k]} = d(k+1)^{d-1}(k+2), \quad \text{and} \quad \nabla \cdot RT_{[k]} = \mathbb{Q}_k.$$

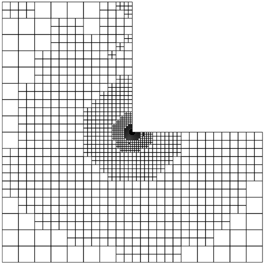
$RT_{[k]} \times \mathbb{Q}_k$



Numerical Results for nonsmooth Stokes Problem

$$\mathbf{u}(r,\theta) := r^{\lambda} \begin{pmatrix} (1+\lambda)\sin(\theta)\Psi(\theta) + \cos(\theta)\Psi'(\theta) \\ \sin(\theta)\Psi'(\theta) - (1+\lambda)\cos(\theta)\Psi(\theta) \end{pmatrix},$$

$$p(r,\theta) := -r^{\lambda-1}\frac{(1+\lambda)^2\Psi'(\theta) + \Psi'''(\theta)}{(1-\lambda)},$$



where,

$$\begin{aligned} \Psi(\theta) &= \sin((1+\lambda)\theta)\cos(\lambda\omega)/(1+\lambda) \; - \cos((1+\lambda)\theta) \\ &\quad - \sin((1-\lambda)\theta)\cos(\lambda\omega)/(1-\lambda) \; + \; \cos((1-\lambda)\theta), \\ \omega &= \frac{3\pi}{2}, \quad \lambda \approx 0.544. \end{aligned}$$

- The singularity for the velocity is of the form r^{λ}.
- However, adaptive refinement enables to retrieve the optimal rate of N/d, N is polynomial degree, $d = 2$ dimension of the space as opposed to the rate λ.

Convergence History