NONLINEAR EQUATIONS

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Outline

1. A little more motivation (if needed)
2. More on the Navier-Stokes equations
   - Simplification to potential flow
   - Turbulence modeling
3. Numerical methods for the Navier-Stokes equations
   - Derivation of the weak formulation
   - Finite element method for the space discretization and finite difference method for the time discretization
   - Inf-sup condition
4. Treatment of the convective term
   - Explicit, semi-implicit, and implicit schemes
   - Newton's method
   - Fixed point methods
5. The coupled fluid-structure interaction problem
   - Derivation of the weak formulation
   - Algorithms for coupled problems

Introduction

We will use fluid-structure interaction problems as an example of multiphysics problems.

When do we have fluid-structure interaction?

Whenever there is a significant **exchange of energy** between a moving fluid and a solid structure.

Examples:

- Civil engineering: the effect of the wind on civil structures such as bridges, skyscrapers, suspended cables etc. (seen yesterday)
- Aeronautics: the action of the air on aeronautic structures (aeroelasticity)
- Biomechanics: blood flow in the circulatory system.
- Mechanical engineering: sloshing of a fluid in a tank.

FSI in aeronautics: flutter

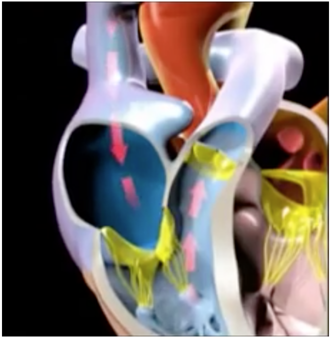
PLAY

A Piper PA-30 Twin Comanche was used at the NASA Dryden Flight Research Center as testbed for general aviation flight control research. This movie shows a tail flutter test done in 1966.

Glider wings instabilities

PLAY

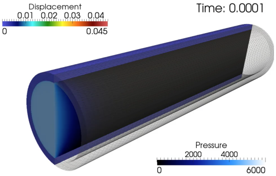
FSI in biomechanics



PLAY

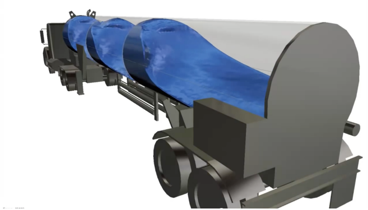
The fluid-structure interaction in the cardiovascular system is the mechanism that generates pressure waves traveling from the heart to peripheral vessels and guarantees a constant pressure at capillary level.

Numerical simulation of blood flow in an elastic pipe.



PLAY

FSI in mechanical engineering: sloshing



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Slosh refers to the movement of liquid inside another object which is also undergoing motion. The liquid must have a free surface to constitute a slosh dynamics problem. The slosh may affect the dynamics of the object containing the liquid.

THE NAVIER-STOKES EQUATIONS

The incompressible Navier-Stokes equations

The Navier-Stokes equations for an incompressible, homogeneous, and Newtonian fluid are given by:

$$
\begin{aligned}
&\rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}, p) = \mathbf{f} \\
&\nabla \cdot \mathbf{u} = 0 \\
&\boldsymbol{\sigma}_f(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\epsilon\left(\mathbf{u}\right) \\
&\rho_f = \rho_0 \text{ constant}
\end{aligned}
$$

Possible boundary conditions:

- imposed velocity profile: $\mathbf{u} = \mathbf{g}$ on Γ_D
- imposed traction (pressure and/or shear): $\boldsymbol{\sigma}_f \mathbf{n} = \mathbf{d}$ on Γ_N

Bernoulli law

We consider now an inviscid, incompressible fluid subject to conservative forces $\mathbf{f} = \nabla V$.

Kelvin theorem: If the flow is initially irrotational, it will stay so for all times.

Assume $\nabla \times \mathbf{u} = \mathbf{0}$. Then, for a simply connected domain Ω_f, there exists a potential ϕ such that $\mathbf{u} = \nabla\phi$. We have:

- Continuity eq.: $\nabla \cdot \mathbf{u} = 0 \implies \Delta\phi = 0$
- Momentum eq.: $\rho_f \left(\frac{\partial \nabla\phi}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p = \nabla V$

  Using vector identity $\mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u}\times\cancelto{0}{(\nabla\times\mathbf{u})}$, we get

$$\nabla\left(\rho_f\frac{\partial\phi}{\partial t} + \rho_f\frac{1}{2}|\mathbf{u}|^2 + p - V\right) = \mathbf{0}$$

Bernoulli Law

$$\rho_f\frac{\partial\phi}{\partial t} + \rho_f\frac{1}{2}|\mathbf{u}|^2 + p - V = \text{constant}$$

Potential flows

The velocity and pressure of a potential fluid flow are given by:

$$\Delta\phi = 0$$
$$\mathbf{u} = \nabla\phi$$
$$p = c + V - \rho_f\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\mathbf{u}|^2\right)$$

where c is the constant that you have in Bernoulli Law:

$$\rho_f\frac{\partial\phi}{\partial t} + \rho_f\frac{1}{2}|\mathbf{u}|^2 + p - V = c$$

Turbulence

The Navier-Stokes equations become unstable when the transport term $(\mathbf{u} \cdot \nabla \mathbf{u})$ becomes dominant with respect to the viscous term $(2\mu\epsilon(\mathbf{u}))$, i.e. when $Re \gg 1$.

In such a situation, energy is transferred from large eddies to smaller ones up to a characteristic scale, so called Kolmogorov scale, where they are dissipated by viscous forces.

$$\text{Kolmogorov scale:} \qquad \eta = Re^{-3/4} L$$

A direct numerical simulation (DNS) aims at simulating all relevant scales up to the Kolmogorov scale. Therefore, the mesh size has to be $h \approx \eta$.

$$\text{3D simulations:} \qquad \#Dofs \sim \left(\frac{L}{h}\right)^3 \sim Re^{9/4}$$

The computational cost required by DNS becomes unaffordable for nowadays computers for Re greater than a few thousands.

RANS vs LES

Alternative to DNS: model the effects of the small scales (not directly solved) to the medium and large scales.
The Navier-Stokes equations can be averaged:

- in time $\rightarrow$ Reynolds-Averaged Navier-Stokes (RANS) equations
- in space $\rightarrow$ Large Eddy Simulation (LES) techniques

The idea is to split the fluid motion in: $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$ and $p = \overline{p} + p'$, where

- $\overline{\mathbf{u}}$, $\overline{p}$: mean velocity and mean pressure
- $\mathbf{u}'$, p': fluctuations around the mean

By applying the average operator $\overline{(\cdot)}$ to the Navier-Stokes equations for Newtonian incompressible fluids we obtain:

$$\frac{\partial(\rho_f \overline{\mathbf{u}})}{\partial t} + \overline{\nabla \cdot (\rho_f \mathbf{u} \otimes \mathbf{u})} - 2\mu\epsilon\left(\overline{\mathbf{u}}\right) + \nabla \overline{p} = \overline{f}$$
$$\nabla \cdot \overline{\mathbf{u}} = 0$$

Problem: $\overline{\nabla \cdot (\rho_f \mathbf{u} \otimes \mathbf{u})} \neq \nabla \cdot (\rho_f \overline{\mathbf{u}} \otimes \overline{\mathbf{u}})$
We need further equations to close the system.

Turbulence models

Reynolds stress tensor

$$\boldsymbol{\tau}_R = \rho_f(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}} - \overline{\mathbf{u} \otimes \mathbf{u}})$$

We have:

$$\begin{aligned}\overline{\nabla \cdot (\rho_f \mathbf{u} \otimes \mathbf{u})} &= \underbrace{\overline{\nabla \cdot (\rho_f \mathbf{u} \otimes \mathbf{u})} - \nabla \cdot (\rho_f \overline{\mathbf{u}} \otimes \overline{\mathbf{u}})}_{= -\nabla \cdot \boldsymbol{\tau}_R} + \underbrace{\nabla \cdot (\rho_f \overline{\mathbf{u}} \otimes \overline{\mathbf{u}})}_{= \rho_f \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}}} \\ &= -\nabla \cdot \boldsymbol{\tau}_R + \rho_f \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}}\end{aligned}$$

Averaged equations

$$\rho_f \left(\frac{\partial \overline{\mathbf{u}}}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{\mathbf{u}} \right) - \nabla \cdot (2\mu\epsilon(\overline{\mathbf{u}}) + \boldsymbol{\tau}_R) + \nabla \overline{p} = \overline{\mathbf{f}}$$

$$\nabla \cdot \overline{\mathbf{u}} = 0$$

Turbulence models aim at modeling $\boldsymbol{\tau}_R$ as a function of $\nabla \overline{\mathbf{u}}$.

FINITE ELEMENT APPROXIMATION OF THE NAVIER-STOKES EQUATIONS

Finite element approximation of the Navier-Stokes equations

We will consider flows in laminar regimes (no turbulence models):

$$
\begin{aligned}
&\rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}, p) = \mathbf{f} && \text{in } \Omega_f, t > 0, \\
&\nabla \cdot \mathbf{u} = 0 && \text{in } \Omega_f, t > 0 \\
&\mathbf{u} = \mathbf{g} && \text{on } \Gamma_D, t > 0 \\
&\boldsymbol{\sigma}_f \mathbf{n} = \mathbf{d} && \text{on } \Gamma_N, t > 0 \\
&\mathbf{u} = \mathbf{u}_0 && \text{in } \Omega_f, t = 0,
\end{aligned}
$$

with $\boldsymbol{\sigma}_f(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\epsilon(\mathbf{u})$.

Weak formulation

Let us define the following functional spaces:

$$\mathbf{V} = [H^1(\Omega_f)]^d, \quad \mathbf{V}_0 = [H^1_{\Gamma_D}(\Omega_f)]^d \equiv \{\mathbf{v} \in \mathbf{V},\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\},$$
$$Q = L^2(\Omega_f)$$

Remark: in $\Gamma_D = \partial\Omega_f$, the pressure defined only up to a constant. The pressure space should be $Q = L^2(\Omega_f)\backslash\mathbb{R}$.

We multiply the continuity equation by a test function $q \in Q$ and the momentum equation by a test function $\mathbf{v} \in \mathbf{V}_0$ and integrate over the domain.

Integration by parts of the stress term:

$$\int_{\Omega_f} -(\nabla \cdot \boldsymbol{\sigma}_f) \cdot \mathbf{v} = \int_{\Omega_f} \boldsymbol{\sigma}_f : \nabla \mathbf{v} - \int_{\partial\Omega_f} (\boldsymbol{\sigma}_f \mathbf{n}) \cdot \mathbf{v}$$

$$= \int_{\Omega_f} 2\mu\epsilon(\mathbf{u}) : \nabla\mathbf{v} - \int_{\Omega_f} \underbrace{p\mathbf{I} : \nabla\mathbf{v}}_{= p\nabla\cdot\mathbf{v}} - \cancelto{0}{\int_{\Gamma_D} (\boldsymbol{\sigma}_f \mathbf{n}) \cdot \mathbf{v}} - \int_{\Gamma_N} \underbrace{(\boldsymbol{\sigma}_f \mathbf{n})}_{= \mathbf{d}} \cdot \mathbf{v}$$

Weak formulation

The weak formulation of the Navier-Stokes equations is:
Find $(\mathbf{u}(t), p(t)) \in \mathbf{V} \times Q$, $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{u}(t) = \mathbf{g}(t)$ on Γ_D such that

$$\int_{\Omega_f} \rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) \cdot \mathbf{v} + \int_{\Omega_f} 2\mu\epsilon(\mathbf{u}) : \nabla \mathbf{v} - \int_{\Omega_f} p \nabla \cdot \mathbf{v} = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{d} \cdot \mathbf{v}$$

$$\int_{\Omega_f} q \nabla \cdot \mathbf{u} = 0$$

$$\forall (\mathbf{v}, q) \in \mathbf{V}_0 \times Q$$

Weak solutions exist for all time ([Leray, 1934], [Hopf, 1951]).
Uniqueness is guaranteed in 2D, it is still an open problem in 3D.

Energy estimate

Assume zero forcing terms ($\mathbf{f} = \mathbf{g} = \mathbf{d} = \mathbf{0}$) and take $\mathbf{v} = \mathbf{u}$ in the weak formulation

$$\int_{\Omega_f} \rho_f \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{u} = \int_{\Omega_f} \rho_f \frac{1}{2} \frac{\partial |\mathbf{u}|^2}{\partial t} = \frac{d}{dt} \underbrace{\int_{\Omega_f} \frac{1}{2} \rho_f |\mathbf{u}|^2}_{\text{kinetic energy } E_k}$$

$$\int_{\Omega_f} \rho_f (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} = \int_{\Omega_f} \rho_f \mathbf{u} \cdot \frac{1}{2} \nabla |\mathbf{u}|^2 = \cancelto{0}{-\int_{\Omega_f} (\nabla \cdot \mathbf{u}) |\mathbf{u}|^2} + \underbrace{\int_{\Gamma_N} \frac{1}{2} \rho_f |\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{n}}_{\text{flux of } E_k \text{ through } \Gamma_N}$$

$$\int_{\Omega_f} 2\mu \epsilon(\mathbf{u}) : \nabla \mathbf{u} = \underbrace{\int_{\Omega_f} 2\mu \epsilon(\mathbf{u}) : \epsilon(\mathbf{u})}_{\text{dissipation rate}} > 0$$

$\int_{\Omega_f} p \nabla \cdot \mathbf{u} = 0$ by incompressibility constraint

Energy estimate

Putting everything together

$$\frac{\partial E_k}{\partial t} + \int_{\Omega_f} 2\mu\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{u}) = -\int_{\Gamma_N} E_k \mathbf{u}\cdot\mathbf{n}$$

Fully Dirichlet problem $(\Gamma_D = \partial\Omega_f)$
In this case we have

$$\frac{\partial E_k}{\partial t} + \int_{\Omega_f} 2\mu\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{u}) = 0 \quad \Rightarrow \quad E_k(T) \leq E_k(0)$$

i.e., the kinetic energy decreases due to viscous dissipation.

Mixed Dirichlet/Neumann boundary conditions $(|\Gamma_N| > 0)$
In this case, the term $-\int_{\Gamma_N} E_k \mathbf{u}\cdot\mathbf{n}$ is positive on the part of the boundary where $\mathbf{u}\cdot\mathbf{n} < 0$ (pumps energy in the system) and negative on the part of the boundary where $\mathbf{u}\cdot\mathbf{n} > 0$ (takes energy outside the domain).
It is a good practice, therefore, to impose Neumann boundary conditions only on outflow boundaries so that the system has a descreasing energy.

Finite element approximation

Introduce finite dimensional spaces of finite element type

$$\mathbf{V}_h \subset \mathbf{V}, \quad \mathbf{V}_{h0} = \mathbf{V}_h \cap \mathbf{V}_0, \quad Q_h \subset Q$$

Observe that pressure functions do not need to be continuous. Moreover let us denote by $\mathbf{u}_{h0} \in \mathbf{V}_h$ and $\mathbf{g}_h \in \mathbf{V}_h(\Gamma_D)$ suitable approximations of the initial datum and Dirichlet boundary datum.

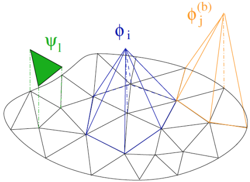
The Finite Element approximation (continuous in time) is:
Find $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{V}_h \times Q_h$, $\mathbf{u}_h(0) = \mathbf{u}_{h0}$, $\mathbf{u}_h(t) = \mathbf{g}_h(t)$ on Γ_D such that

$$\int_{\Omega_f} \rho_f \left(\frac{\partial \mathbf{u}_h}{\partial t} + \mathbf{u}_h \cdot \nabla \mathbf{u}_h \right) \cdot \mathbf{v} + \int_{\Omega_f} 2\mu\epsilon(\mathbf{u}_h) : \nabla \mathbf{v} - \int_{\Omega_f} p_h \nabla \cdot \mathbf{v} = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{d} \cdot \mathbf{v}$$

$$\int_{\Omega_f} q \nabla \cdot \mathbf{u}_h = 0$$

$$\forall (\mathbf{v}, q) \in \mathbf{V}_{h0} \times Q_h$$

Algebraic formulation



- $\{\phi_i\}_{i=1}^{N_u}$: basis of $\mathbf{V}_{h0}$
- $\{\phi_j^b\}_{j=1}^{N_u^b}$: basis of $\mathbf{V}_h \backslash \mathbf{V}_{h0}$ (shape functions corresponding to boundary nodes)
- $\{\psi_l\}_{l=1}^{N_p}$: basis of Q_h

Expand the solution $(\mathbf{u}_h(t), p_h(t))$ on the finite element basis

$$\mathbf{u}_h(\mathbf{x}, t) = \sum_{i=1}^{N_u} u_i(t)\phi_i(\mathbf{x}) + \sum_{j=1}^{N_u^b} g_j(t)\phi_j^b(\mathbf{x})$$

$$p_h(\mathbf{x}, t) = \sum_{l=1}^{N_p} p_i(t)\psi_l(\mathbf{x})$$

Vectors of unknown dofs (nodal values if Lagrange basis functions are used)

$$\mathbf{U}(t) = [\mathbf{u}_1(t), \ldots, \mathbf{u}_{N_u}(t)]^T, \quad \mathbf{P}(t) = [p_1(t), \ldots, p_{N_p}(t)]^T$$

Algebraic system

Plugging the expansions of $\mathbf{u}_h$ and p_h in the finite element formulation and testing with shape functions ϕ_i and ψ_l we get the following system of nonlinear ODEs for $t > 0$:

$$
\begin{aligned}
&M\frac{d\mathbf{U}}{dt} + A\mathbf{U} + N(\mathbf{U})\mathbf{U} + B^T\mathbf{P} = \mathbf{F}_u(\mathbf{U}),\\
&B\mathbf{U} = \mathbf{F}_p,
\end{aligned}
$$

where

- $M_{ij} = \int_{\Omega_f} \rho_f \phi_j \cdot \phi_i$ mass matrix
- $A_{ij} = \int_{\Omega_f} 2\mu\epsilon\left(\phi_j\right) : \nabla\phi_i$ stiffness matrix
- $N(\mathbf{U})_{ij} = \int_{\Omega_f} \mathbf{u}_h \cdot \nabla\phi_j \cdot \phi_i$ convective term matrix: it depends on $\mathbf{U}$
- $B_{ij} = -\int_{\Omega_f} \psi_l \nabla \cdot \phi_i$ divergence matrix

Algebraic system

Plugging the expansions of $\mathbf{u}_h$ and p_h in the finite element formulation and testing with shape functions ϕ_i and ψ_l we get the following system of nonlinear ODEs for $t > 0$:

$$
\begin{aligned}
& M\frac{d\mathbf{U}}{dt} + A\mathbf{U} + N(\mathbf{U})\mathbf{U} + B^T\mathbf{P} = \mathbf{F}_u(\mathbf{U}), \\
& B\mathbf{U} = \mathbf{F}_p,
\end{aligned}
$$

where

- $(\mathbf{F}_u(\mathbf{U}))_i = \int_{\Omega_f} \mathbf{f}\cdot\phi_i + \int_{\Gamma_N} \mathbf{d}\cdot\phi_i - \int_{\Omega_f} \rho_f \left(\frac{\partial \mathbf{g}_h}{\partial t} + \mathbf{u}_h \cdot \nabla \mathbf{g}_h\right)\cdot\phi_i$ $- \int_{\Omega_f} 2\mu\epsilon(\mathbf{g}_h) : \nabla\phi_i$
- $(\mathbf{F}_p)_l = \int_{\Omega_f} \psi_l \nabla\cdot\mathbf{g}_h$

Spurious pressure modes

Just like for the Stokes problem, a necessary and sufficient condition to avoid the presence of spurious pressure modes is that the finite elements spaces $\mathbf{V}_h \times Q_h$ satisfy

$$\text{inf-sup condition}: \quad \inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\left| \int_{\Omega_f} q_h \nabla \cdot \mathbf{v}_h \right|}{||\mathbf{v}_h||_{H^1} ||q_h||_{L^2}} \geq \beta_h > 0$$

Observe that the inf-sup condition is satisfied by the continuous spaces $\mathbf{V} \times Q$ but not necessarily by the finite element spaces $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset Q$.

$\Longrightarrow$ one needs to use inf-sup stable (or compatible) finite element spaces.

Temporal discretization: recall

Divide the time interval of interest $[0, T]$ into subintervals of length Δt and let $t^n = n\Delta t$ and $\mathbf{u}^n \approx \mathbf{u}(t^n)$.

- A first order scheme: Implicit Euler

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t^n} \approx [\mathbf{u}]_t^n = \frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t}$$

- A second order scheme: Backward differentiation formula (BDF) 2

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{t^n} \approx [\mathbf{u}]_t^n = \frac{3\mathbf{u}^n - 4\mathbf{u}^{n-1} + \mathbf{u}^{n-2}}{2\Delta t}$$

What do we do with the convective term?

TREATMENT OF THE CONVECTIVE TERM

Explicit, semi-implicit, implicit

At time t^n, we have:

$$\begin{aligned}
&[\mathbf{u}]_t^n + \mathbf{u}^? \cdot \nabla \mathbf{u}^? - \nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}^n, p^n) = \mathbf{f}^n && \text{in } \Omega_f, n = 1, 2, ...,\\
&\nabla \cdot \mathbf{u}^n = 0 && \text{in } \Omega_f, n = 1, 2, ...,\\
&\mathbf{u}^n = \mathbf{g} \text{ on } \Gamma_D, \quad \boldsymbol{\sigma}_f^n \cdot \mathbf{n} = \mathbf{d} \text{ on } \Gamma_N, \quad \mathbf{u}^0 = \mathbf{u}_0
\end{aligned}$$

The convective term can be treated in different ways:

- explicit : $\mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^{n-1}$ $\rightarrow$ the problem becomes linear but there is a stability condition on Δt:

$$\Delta t \leq \frac{h}{||\mathbf{u}_h||_{L^\infty}};$$

- semi-implicit : $\mathbf{u}^* \cdot \nabla \mathbf{u}^n$ $\rightarrow$ $\mathbf{u}^*$ is known so the problem becomes linear but the system matrix is not symmetric;
- implicit : $\mathbf{u}^n \cdot \nabla \mathbf{u}^n$ $\rightarrow$ no stability condition but the system is nonlinear.

Semi-implicit schemes

- First order: implicit Euler and convective term

$$\mathbf{u} \cdot \nabla \mathbf{u}\big|_{t^n} \approx \mathbf{u}^* = \mathbf{u}^{n-1} \nabla \mathbf{u}^n$$

This leads to a linear system to be solved at every time step:

$$\begin{bmatrix} \frac{1}{\Delta t} M + A + N(\mathbf{U}^{n-1}) & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}^n \\ \mathbf{P}^n \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u^n + \frac{1}{\Delta t} M \mathbf{U}^{n-1} \\ \mathbf{F}_p^n \end{bmatrix}$$

- Second order: BDF2 and convective term

$$\mathbf{u} \cdot \nabla \mathbf{u}\big|_{t^n} \approx \mathbf{u}^* = (2\mathbf{u}^{n-1} - \mathbf{u}^{n-2}) \nabla \mathbf{u}^n$$

Resulting linear system to be solved at every time step:

$$\begin{bmatrix} \frac{3}{2\Delta t} M + A + N(2\mathbf{U}^{n-1} - \mathbf{U}^{n-2}) & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}^n \\ \mathbf{P}^n \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u^n + \frac{M}{2\Delta t}(4\mathbf{U}^{n-1} - \mathbf{U}^{n-2}) \\ \mathbf{F}_p^n \end{bmatrix}$$

Semi-implicit schemes: linear system

At every time step, we have to solve a linear system in the form:

$$\begin{bmatrix} C & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}^n \\ \mathbf{P}^n \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{F}_p \end{bmatrix}$$

where $C = \frac{\alpha}{\Delta t} M + A + N(\mathbf{U}^*)$ with α and $\mathbf{U}^*$ that depend on the selected time marching scheme.

In 3D problems we have 3 velocity components + the pressure: this sparse linear system can be huge. Moreover, depending on the choice of physical and discretization parameters it can be ill-conditioned.

Implicit discretization

For simplicity, let us consider the steady Navier-Stokes equations:

$$\begin{aligned} &\mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}, p) = \mathbf{f} && \text{in } \Omega_f, \\ &\nabla \cdot \mathbf{u} = 0 && \text{in } \Omega_f, \\ &\mathbf{u} = \mathbf{g} \text{ on } \Gamma_D, \quad \boldsymbol{\sigma}_f \cdot \mathbf{n} = \mathbf{d} \text{ on } \Gamma_N, \quad \mathbf{u}^0 = \mathbf{u}_0 \end{aligned}$$

The corresponding linear system is:

$$\begin{bmatrix} A + N(\mathbf{U}) & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{F}_p \end{bmatrix}$$

This is a nonlinear system. To linearize it, we can use

- Newton's method.
- Fixed point methods.

Newton's method: recall

We look for $\mathbf{u} \in \mathbb{R}^n$ such that:

$$\mathcal{F}(\mathbf{u}) = \mathbf{0}, \quad \mathcal{F} : \mathbb{R}^n \times \mathbb{R}^n$$

Algorithm:
Given $\mathbf{u}^0 \in \mathbb{R}^n$, $\forall k = 1, 2, \ldots$ find $\delta\mathbf{u} \in \mathbb{R}^n$ such that

$$J(\mathbf{u}^k)\delta\mathbf{u} = -\mathcal{F}(\mathbf{u}^k)$$

and set:

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \delta\mathbf{u}.$$

Here, J is the Jacobian matrix and $J(\mathbf{u}^k)\delta\mathbf{u} = \nabla\mathcal{F}|_{\mathbf{u}^k} \cdot \delta\mathbf{u}$ represents a directional derivative of $\mathcal{F}$ in the direction $\delta\mathbf{u}$, evaluated at $\mathbf{u}^k$.

If $J(\mathbf{u})$ is not singular, $\exists\, \mathcal{U}(\mathbf{u})$ such that $\forall \mathbf{u}^0 \in \mathcal{U}(\mathbf{u})$ the series $\{\mathbf{u}^k\}$ converges to $\mathbf{u}$ quadratically: $||\mathbf{u} - \mathbf{u}^{k+1}|| \leq C||\mathbf{u} - \mathbf{u}^k||^2$.

Note: $\forall k$ we have to solve a linear system.

Newton's method applied to the Navier-Stokes problem

We can rewrite the steady Navier-Stokes equations as:

$$\mathcal{N}(\mathbf{u}, p) - \mathbf{F} = \mathbf{0}$$

with

$$\mathcal{N}(\mathbf{u}, p) = \begin{bmatrix} -\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p \\ \nabla\cdot\mathbf{u} \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

Thus, given $(\mathbf{u}^0, p^0)$, $\forall k = 1, 2, \ldots$ find $(\delta\mathbf{u}, \delta p) \in \mathbf{V} \times Q$ such that:

$$\begin{aligned} D\mathcal{N}_{\mathbf{u}^k, p^k}(\delta\mathbf{u}, \delta p) &= \mathbf{F} - \mathcal{N}(\mathbf{u}^k, p^k) \\ \mathbf{u}^{k+1} &= \mathbf{u}^k + \delta\mathbf{u} \\ p^{k+1} &= p^k + \delta p \end{aligned}$$

Newton's method applied to the Navier-Stokes problem

$D\mathcal{N}_{\mathbf{u}^k,p^k}(\delta\mathbf{u},\delta p)$ is the directional derivative of the $\mathcal{N}$ operator in the direction $(\delta\mathbf{u},\delta p)$ computed at $(\mathbf{u}^k,p^k)$. This is the Gateaux derivative:

$$
\begin{aligned}
D\mathcal{N}_{\mathbf{u}^k,p^k}(\delta\mathbf{u},\delta p) &= \lim_{\epsilon\to 0}\frac{\mathcal{N}(\mathbf{u}^k+\epsilon\delta\mathbf{u},p^k+\epsilon\delta p)-\mathcal{N}(\mathbf{u}^k,p^k)}{\epsilon}\\
&= \begin{bmatrix} -\nu\Delta\delta\mathbf{u}+\delta\mathbf{u}\cdot\nabla\mathbf{u}^k+\mathbf{u}^k\cdot\nabla\delta\mathbf{u}+\nabla\delta p\\ \nabla\cdot\delta\mathbf{u}\end{bmatrix}
\end{aligned}
$$

So, we have:

$$
-\nu\Delta\delta\mathbf{u}+\mathbf{u}^k\cdot\nabla\delta\mathbf{u}+\delta\mathbf{u}\cdot\nabla\mathbf{u}^k+\nabla\delta p=\mathbf{f}+\nu\Delta\mathbf{u}^k-\mathbf{u}^k\cdot\nabla\mathbf{u}^k-\nabla p^k
$$

$$
\nabla\cdot\delta\mathbf{u}=-\nabla\cdot\mathbf{u}^k
$$

which in weak form reads

$$
\begin{aligned}
&\int_{\Omega_f}\nu\nabla\delta\mathbf{u}:\nabla\mathbf{v}+\int_{\Omega_f}\left(\mathbf{u}^k\cdot\nabla\delta\mathbf{u}\right)\cdot\mathbf{v}+\int_{\Omega_f}\left(\delta\mathbf{u}\cdot\nabla\mathbf{u}^k\right)\cdot\mathbf{v}-\int_{\Omega_f}\delta p\nabla\cdot\mathbf{v}\\
&\quad=\int_{\Omega_f}\mathbf{f}\cdot\mathbf{v}+\int_{\Gamma_N}\mathbf{d}\cdot\mathbf{v}-\int_{\Omega_f}\nu\nabla\mathbf{u}^k:\nabla\mathbf{v}-\int_{\Omega_f}\left(\mathbf{u}^k\cdot\nabla\mathbf{u}^k\right)\cdot\mathbf{v}+\int_{\Omega_f}p^k\nabla\cdot\mathbf{v}\\
&\int_{\Omega_f}q\nabla\cdot\delta\mathbf{u}=-\int_{\Omega_f}q\nabla\cdot\mathbf{u}^k\\
&\forall(\mathbf{v},q)\in\mathbf{V}_{h0}\times Q_h
\end{aligned}
$$

Algebraic system

Plugging the expansions of $\mathbf{u}_h$ and p_h in the finite element formulation and testing with shape functions we get the following system of linear ODEs:

$$A\delta\mathbf{U} + N(\mathbf{U}^k)\delta\mathbf{U} + L(\mathbf{U}^k)\delta\mathbf{U} + B^T\delta\mathbf{P} = \mathbf{F}_u(\mathbf{U}^k) - A\mathbf{U}^k - N(\mathbf{U}^k)\mathbf{U}^k - B^T\mathbf{P}^k,$$
$$B\delta\mathbf{U} = \mathbf{F}_p - B\mathbf{U}^k,$$

where we have already seen all the matrices except

- $L(\mathbf{U}^k)_{ij} = \int_{\Omega_f} \phi_j \cdot \nabla\mathbf{u}_h^k \cdot \phi_i$

Notice that at every iteration we have to solve a linear system in the form:

$$\begin{bmatrix} C & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \delta\mathbf{U} \\ \delta\mathbf{P} \end{bmatrix} = \begin{bmatrix} RHS_u \\ RHS_p \end{bmatrix}$$

where $C = A + N(\mathbf{U}^k) + L(\mathbf{U}^k)$.

Convergence of Newton's method

Once $\delta\mathbf{U}$ and $\delta\mathbf{P}$ are computed, we update velocity and pressure:

$$\mathbf{U}^{k+1} = \mathbf{U}^k + \delta\mathbf{U}$$
$$\mathbf{P}^{k+1} = \mathbf{P}^k + \delta\mathbf{P}$$

If

- the operator $D\mathcal{N}_{\mathbf{u}^k,p^k}(\delta\mathbf{u},\delta p)$ is not singular $\rightarrow$ problem

$$D\mathcal{N}_{\mathbf{u}^k,p^k}(\delta\mathbf{u},\delta p) = \begin{bmatrix}\mathbf{F}\\ 0\end{bmatrix}$$

  is well-posed $\forall \mathbf{f} \in [H^{-1}\Omega_f]^d$;
- $\mathbf{u}^0$ is close enough to $\mathbf{u}$;

then the series $\{\mathbf{u}^k\}$ is converging quadratically:
$||\mathbf{u}^k - \mathbf{u}|| \leq ||\mathbf{u}^{k-1} - \mathbf{u}||^2$.

Implicit discretization

Let us go back to the steady Navier-Stokes equations:

$$\begin{aligned} &\mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}, p) = \mathbf{f} && \text{in } \Omega_f, \\ &\nabla \cdot \mathbf{u} = 0 && \text{in } \Omega_f, \\ &\mathbf{u} = \mathbf{g} \text{ on } \Gamma_D, \quad \boldsymbol{\sigma}_f \cdot \mathbf{n} = \mathbf{d} \text{ on } \Gamma_N, \quad \mathbf{u}^0 = \mathbf{u}_0 \end{aligned}$$

Another possibility is to linearize the problem with a fixed point method.

Algorithm:
Given $(\mathbf{u}^0, p^0)$, $\forall k = 1, 2, \ldots$ find $(\mathbf{u}^k, p^k) \in \mathbf{V} \times Q$ such that:

$$\begin{aligned} &\mathbf{u}^{k-1} \cdot \nabla \mathbf{u}^k - \nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}^k, p^k) = \mathbf{f} && \text{in } \Omega_f, \\ &\nabla \cdot \mathbf{u}^k = 0 && \text{in } \Omega_f, \\ &\mathbf{u} = \mathbf{g} \text{ on } \Gamma_D, \quad \boldsymbol{\sigma}_f \cdot \mathbf{n} = \mathbf{d} \text{ on } \Gamma_N, \quad \mathbf{u}^0 = \mathbf{u}_0 \end{aligned}$$

until a stopping criterion is satisfied.

Fixed point method

At each iteration of the fixed point method, we have to solve a linear system:

$$\begin{bmatrix} A+N(\mathbf{U}^{k-1}) & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}^k \\ \mathbf{P}^k \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{F}_p \end{bmatrix}$$

Possible stopping criterion: $||\mathbf{u}^k - \mathbf{u}^{k-1}||_{L^2} \leq tol$

The convergence of this fixed point method is only linear:

$$\begin{aligned} ||\mathbf{u} - \mathbf{u}^k|| &\leq \rho||\mathbf{u} - \mathbf{u}^{k-1}|| \\ &\leq \rho^k||\mathbf{u} - \mathbf{u}^0|| \end{aligned}$$

One last thing

The standard Galerkin approximation of the incompressible Navier-Stokes equations leads to instabilities when the convective term is dominant, i.e. when $Re \gg 1$.

A stabilization method is needed. A few options consists in adding extra terms to the variational formulation of the problem:

- Streamline Upwind Petrov-Galerkin (SUPG)
- Galerkin Least Squares (GALS)
- Variational Multiscale (VMS)

Some references

V. Girault and P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*. Springer-Verlag, New York, (1986).

M. Gunzburger, *Finite element method for viscous incompressible flows: a guide to theory, practice and algorithms*. Computer Science and Scientific Computing, Academic Press, San Diego, 1989.

S.C. Brenner, L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 2008.

A. Quarteroni, A. Valli. *Numerical Approximation of Partial Differential Equations*. Springer-Verlag, Berlin Heidelberg, 1994.

THE COUPLED PROBLEM

Models and numerical techniques

There are several models available and several numerical techniques, depending on the application.

We can define two categories of problems:

1. Small deformation regimes
    - The focus is on small vibrations of the structure
    - The fluid domain can be kept fixed
    - Often a simplified fluid dynamics may be used (potential flow; linearization around a steady state flow)

   Examples: aeroelasticity, acoustic vibrations etc.
2. Moderate/large deformation regimes
    - The focus is on both fluid and structure
    - Lot of energy exchange between fluid and structure
    - Moderate deformation: typically an Arbitrary Lagrangian Eulerian (ALE) formulation is used
    - Large deformation: extended ALE methods, Fictitious Domain method, Immersed Boundary method, etc.

   Examples: pulsatile flow in elastic pipes, particles moving in a fluid etc.

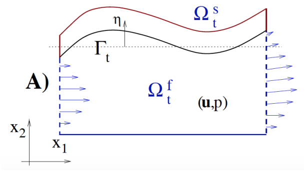
Eulerian vs Lagrangian

We consider now an incompressible fluid interacting with an elastic structure.

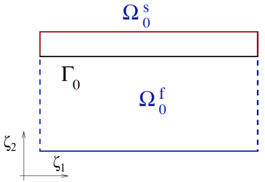
We have seen that

- Fluid equations are typically written in Eulerian form
- Structure equations are typically written in Lagrangian form on the reference domain Ω_s^0

WARNING: The fluid equations may be defined in a moving domain $\Omega_f(t)$



Current config.



Reference config.

Fluid-structure coupling conditions

At the common interface $\Gamma(t) = \Gamma_t$, we impose:

- Continuity of velocity (kinematic condition)
- Continuity of the normal stress (dynamic condition)

Note: the continuity of stresses on Γ_t involves the physical stresses, i.e. the Cauchy stress tensors.

Continuity of the velocity

$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial \boldsymbol{\eta}}{\partial t}(\boldsymbol{\xi}, t), \quad \text{with } \mathbf{x} = \mathcal{L}_t(\boldsymbol{\xi})$$

Continuity of the normal stress

$$\boldsymbol{\sigma}_f(\mathbf{u}, p)\mathbf{n}_f = -\boldsymbol{\sigma}_s(\boldsymbol{\eta})\mathbf{n}_s$$

Remember: $\boldsymbol{\sigma}_s(\boldsymbol{\eta}) = \frac{1}{J(\boldsymbol{\eta})}\boldsymbol{\sigma}_s^0(\boldsymbol{\eta})\mathbf{F}^T(\boldsymbol{\eta})$

The coupled fluid-structure problem

- Fluid problem:

$$
\begin{aligned}
&\frac{\partial(\rho_f \mathbf{u})}{\partial t} + \nabla \cdot (\rho_f \mathbf{u} \otimes \mathbf{u}) = \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}_f, && \text{in } \Omega_f(\boldsymbol{\eta}, t),\\
&\nabla \cdot \mathbf{u} = 0, && \text{in } \Omega_f(\boldsymbol{\eta}, t),
\end{aligned}
$$

- Structure problem:

$$
\rho_s^0 \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} - \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{\sigma}_s^0 = \mathbf{f}^0, \qquad \text{in } \Omega_s^0,
$$

- Coupling conditions:

$$
\begin{aligned}
&\mathbf{u}(\mathbf{x}, t) = \frac{\partial \boldsymbol{\eta}}{\partial t}(\boldsymbol{\xi}, t), \text{ with } \mathbf{x} = \mathcal{L}_t(\boldsymbol{\xi}) && \text{on } \Gamma_t,\\
&\boldsymbol{\sigma}_f(\mathbf{u}, p)\mathbf{n}_f = -\boldsymbol{\sigma}_s(\boldsymbol{\eta})\mathbf{n}_s, && \text{on } \Gamma_t.
\end{aligned}
$$

Normal convention: $\mathbf{n}_f = -\mathbf{n}_s$

Sources of nonlinearity

The coupled fluid-structure problem is highly nonlinear:

- Convective term $\nabla \cdot (\mathbf{u} \otimes \mathbf{u})$ in the Navier-Stokes equations.
- Nonlinearity in the structure model when using nonlinear elasticity.
- The fluid domain itself may bean unknown and, if so, the structure displacement deforms the fluid domain in a nonlinear way This is called geometric nonlinearity.

Global weak formulation

Fluid problem:

$$\int_{\Omega_f^t} \rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \right) \cdot \mathbf{v} + \int_{\Omega_f^t} \boldsymbol{\sigma}_f : \nabla \mathbf{v} + \int_{\Omega_f} q \nabla \cdot \mathbf{u}$$

$$= \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_t} \boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}$$

Structure problem

$$\int_{\Omega_s^0} \rho_s^0 \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} \cdot \phi + \int_{\Omega_s^0} \boldsymbol{\sigma}_s^0 : \nabla_{\boldsymbol{\xi}} \phi = \int_{\Omega_s^0} \mathbf{f}^0 \cdot \phi + \int_{\Gamma_0} \boldsymbol{\sigma}_s^0 \mathbf{n}_s^0 \cdot \phi$$

Global weak formulation

If we take matching test functions at the interface: $\mathbf{v} \circ \mathcal{L}_t(\xi) = \phi(\xi)$ and and thanks to the continuity of stresses, the interface terms cancel.

Fluid-structure functional space

$$V_{FS} = \{(\mathbf{v}, q, \phi) \ : \ \mathbf{v} \circ \mathcal{L}_t = \phi \text{ on } \Gamma_0\}$$

Find $(\mathbf{u}, p, \boldsymbol{\eta})$ such that $\mathbf{u} \circ \mathcal{L}_t = \frac{\partial \eta}{\partial t}$ on Γ_0 and

$$\int_{\Omega_f^t} \rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \right) \cdot \mathbf{v} + \int_{\Omega_f^t} \boldsymbol{\sigma}_f : \nabla \mathbf{v} + \int_{\Omega_f^t} q \nabla \cdot \mathbf{u}$$
$$+ \int_{\Omega_s^0} \rho_s^0 \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} \cdot \phi + \int_{\Omega_s^0} \boldsymbol{\sigma}_s^0 : \nabla_{\boldsymbol{\xi}} \phi = \int_{\Omega_f^t} \mathbf{f} \cdot \mathbf{v} + \int_{\Omega_s^0} \mathbf{f}^0 \cdot \phi$$

$\forall (\mathbf{v}, q, \phi) \in V_{FS}$.

Energy inequality

Taking as test functions $(\mathbf{v}, q, \phi) = (\mathbf{u}, p, \dot{\eta})$, we can derive energy inequality for hyperelastic materials

Fluid-structure energy (kinetic + elastic)

$$\mathcal{E}_{FS}(t) = \frac{\rho_f}{2}||\mathbf{u}(t)||^2_{L(\Omega_f^t)} + \frac{\rho_s^0}{2}||\ddot{\boldsymbol{\eta}}(t)||^2_{L(\Omega_s^0)} + W(\boldsymbol{\eta}(t))$$

Then, for an isolated system (no external forces)

$$\mathcal{E}_{FS}(T) + \underbrace{2\mu \int_0^T \int_{\Omega_f^t} \epsilon(\mathbf{u}) : \epsilon(\mathbf{u}) \; d\mathbf{x} \; dt}_{\text{energy dissipated by the fluid viscosity}} \leq \mathcal{E}_{FS}(0)$$

Energy inequality

Key points in deriving an energy inequality:

- Perfect balance of work at the interface

$$\int_{\Gamma_t} \boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{u} = -\int_{\Gamma_0} \boldsymbol{\sigma}_s^0 \mathbf{n}_s^0 \cdot \dot{\boldsymbol{\eta}}$$

- No kinetic flux through the interface

$$\int_{\Omega_f^t} \rho_f \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{u} = \frac{\rho_f}{2} \frac{d}{dt} \int_{\Omega_f^t} |\mathbf{u}|^2 - \frac{\rho_f}{2} \int_{\Gamma_t} |\mathbf{u}^2| \mathbf{w} \cdot \mathbf{n}_f$$

$$\int_{\Omega_f^t} \rho_f \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \cdot \mathbf{u} = \frac{\rho_f}{2} \int_{\Gamma_t} |\mathbf{u}^2| \mathbf{u} \cdot \mathbf{n}_f$$

where $\mathbf{w}$ the velocity at which the interface moves. Since $\mathbf{w} = \mathbf{u} = \dot{\boldsymbol{\eta}}$ on Γ_t, the kinetic flux $\frac{\rho_f}{2} \int_{\Gamma_t} |\mathbf{u}^2| (\mathbf{u} - \mathbf{w}) \cdot \mathbf{n}_f = 0$.

Recovering the fluid and structure subproblems

Take as a test function $(\mathbf{v}, q, \mathbf{0})$. We get fluid equations with Dirichlet boundary conditions:

$$\int_{\Omega_f^t} \rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \right) \cdot \mathbf{v} + \int_{\Omega_f^t} \boldsymbol{\sigma}_f : \nabla \mathbf{v} + \int_{\Omega_f^t} q \nabla \cdot \mathbf{u} = \int_{\Omega_f^t} \mathbf{f} \cdot \mathbf{v},$$

$$\mathbf{u} = \frac{\partial \boldsymbol{\eta}}{\partial t} \circ (\mathcal{L}_t)^{-1} \quad \text{on } \Gamma_0$$

Take as a test function $(\mathrm{Ext}(\boldsymbol{\phi}), q, \boldsymbol{\phi})$, where $\mathrm{Ext}(\boldsymbol{\phi})$ is a suitable extension of $\boldsymbol{\phi}$ in the fluid domain:
$\mathrm{Ext}(\boldsymbol{\phi}) \in [H^1(\Omega_f^t)]^d$, $\mathrm{Ext}(\boldsymbol{\phi}) \circ \mathcal{L}_t(\boldsymbol{\xi}) = \boldsymbol{\phi}(\boldsymbol{\xi})$ on Γ_0.
We get Structure equation with Neumann boundary conditions

$$\int_{\Omega_s^0} \rho_s^0 \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} \cdot \boldsymbol{\phi} + \int_{\Omega_s^0} \boldsymbol{\sigma}_s^0 : \nabla_{\boldsymbol{\xi}} \boldsymbol{\phi} = \int_{\Omega_s^0} \mathbf{f}^0 \cdot \boldsymbol{\phi} + \langle \mathcal{R}_f(\mathbf{u}, p), \mathrm{Ext}(\boldsymbol{\phi}) \rangle$$

where $\langle \mathcal{R}_f(\mathbf{u}, p), \mathrm{Ext}(\boldsymbol{\phi}) \rangle$ is the residual of the fluid momentum equation.

Recovering the fluid and structure subproblems

Observe that the term $\langle \mathcal{R}_f(\mathbf{u}, p), \mathrm{Ext}(\phi) \rangle$ represents the stress of the fluid on the structure.

$$
\begin{aligned}
&\langle \mathcal{R}_f(\mathbf{u}, p), \mathrm{Ext}(\phi) \rangle \\
&= \int_{\Omega_f^t} \left[\mathbf{f} - \rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \right) \right] \cdot \mathrm{Ext}(\phi) - \boldsymbol{\sigma}_f : \nabla \mathrm{Ext}(\phi) \\
&= \underbrace{\int_{\Omega_f^t} \left[\mathbf{f} - \rho_f \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \right) + \nabla \cdot \boldsymbol{\sigma}_f \right] \cdot \mathrm{Ext}(\phi)}_{0} - \int_{\Gamma_t} (\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathrm{Ext}(\phi) \\
&= - \int_{\Gamma_t} (\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \phi
\end{aligned}
$$

Therefore, the structure problem is endowed with Neumann boundary conditions at the interface:

$$
\frac{1}{J} \boldsymbol{\sigma}_s^0 \mathbf{F}^T \mathbf{n}^s = -\boldsymbol{\sigma}_f \mathbf{n}_f
$$

Other splittings are possible.

Some references

P. Le Tallec and J. Mouro. *Fluid structure interaction with large structural displacements*. Comput. Methods Appl. Mech. Engrg., 190:3039-3067, 2001.

F. Nobile. *Numerical approximation of fluid-structure interaction problems with application to haemodynamics*. PhD Thesis n. 2458, Ecole Polytechnique Federale de Lausanne.

Q. Du, M.D. Gunzburger, L.S. Hou, J. Lee. *Semidiscrete finite element approximations of a linear fluid-structure interaction problem*. SIAM J. Numer. Anal. 42(1):1-29, 2004.

Cardiovascular Mathematics. Modeling and simulation of the circulatory system (Chapter 9). L. Formaggia, A. Quarteroni, A. Veneziani (Eds.) Springer, 2009.

ALGORITHMS FOR FLUID-STRUCTURE INTERACTION

(and other coupled problems)

FSI problem

- Fluid problem: $Fl(\mathbf{w}^n, \mathbf{u}^n, p^n)$ in Ω_f^n
- Structure problem: $St(\boldsymbol{\eta}^n, \dot{\boldsymbol{\eta}}^n)$ in Ω_s^0
- Coupling conditions:

  kinematic condition: $\mathbf{u}^n = \dot{\boldsymbol{\eta}}^n$ on Γ_0

  dynamic condition: $\boldsymbol{\sigma}_f(\mathbf{u}^n, p^n)\mathbf{n}_f = -\boldsymbol{\sigma}_s(\boldsymbol{\eta}^n)\mathbf{n}_s$ on Γ_t

We have a fully coupled nonlinear problem in the unknowns $(\mathbf{u}^n, p^n, \boldsymbol{\eta}^n, \dot{\boldsymbol{\eta}}^n)$.

Monolithic vs partitioned

- A monolithic method tries to solve the non-linear problem all at once. We assemble and solve a "huge" linear system in the unknowns $(\mathbf{u}^n, p^n, \boldsymbol{\eta}^n, \dot{\boldsymbol{\eta}}^n)$. A monolithic method requires an ad-hoc fluid-structure solver.
- A partitioned method solves iteratively the fluid and structure subproblems (never form the full matrix). This allows one to couple two existing codes, one solving the fluid equations and the other solving the structure problem.
- There are other options in between: for instance one can "formally" write the monolithic problem and then use a block preconditioner to solve it.

Weakly coupled vs strongly coupled

A FSI algorithm is weakly coupled if the coupling conditions are not "exactly" satisfied at every time step. Otherwise it is called strongly coupled.

Weakly coupled methods may have serious stability problems in certain applications. Dangerous situations appear for incompressible fluids when the structure mass is comparable to the fluid mass.

	weakly coupled	strongly coupled
partitioned	can be unstable*	can be computationally expensive†
monolithic	does not exists	require you to implement it

* for the non-perfect balance of energy transfer at the interface (added mass effect)

† Many strategies have been proposed: fixed point algorithms with acceleration techniques, Domain Decomposition strategies, exact Newton methods, quasi-Newton methods, etc.

A simple weakly coupled partitioned FSI algorithm

Given the solution $(\mathbf{u}^{n-1}, p^{n-1}, \boldsymbol{\eta}^{n-1}, \dot{\boldsymbol{\eta}}^{n-1})$, at the time t^{n-1}

1. Extrapolate the displacement of the interface, e.g.: $\widetilde{\dot{\boldsymbol{\eta}}}^n = \dot{\boldsymbol{\eta}}^{n-1}$ on Γ
2. Solve the fluid problem with Dirichlet boundary conditions:

$$\begin{aligned} &Fl(\mathbf{u}^n, \mathbf{u}^n, p^n) = 0 && \text{in } \Omega_f \\ &\mathbf{u}^n = \widetilde{\dot{\boldsymbol{\eta}}}^n && \text{on } \Gamma \end{aligned}$$

3. Solve the structure problem with Neumann boundary conditions:

$$\begin{aligned} &St(\boldsymbol{\eta}^n, \dot{\boldsymbol{\eta}}^n) = 0 && \text{in } \Omega_s^0 \\ &\boldsymbol{\sigma}_s(\boldsymbol{\eta}^n)\mathbf{n}_s = -\boldsymbol{\sigma}_f(\mathbf{u}^n, p^n)\mathbf{n}_f && \text{on } \Gamma^n \end{aligned}$$

4. Go to next time step.

This algorithm is a classical Domain Decomposition algorithm called Dirichlet-Neumann.

A fully implicit partitioned FSI algorithm

The weakly coupled partitioned FSI algorithm does not balance the energy transfer at the interface.

In order to have a perfect energy balance, we need to satisfy "exactly" at each time step the kinematic and dynamic coupling conditions.

Possible remedy: subiterate at each time step (a relaxation step might be needed)

- if $||\boldsymbol{\eta}^n - \widetilde{\boldsymbol{\eta}}^n|| > tol$, set

$$\widetilde{\boldsymbol{\eta}}^n \leftarrow \omega \boldsymbol{\eta}^n + (1-\omega)\widetilde{\boldsymbol{\eta}}^n$$

  and repeat steps 2. to 4.
- This algorithm is stable, in general.
- However, whenever the weakly coupled scheme is unstable, this algorithm may need a very small relaxation parameter ω and the convergence may be very slow.

Monolithic approach: global week formulation

We recall the global weak formulation for the fluid-structure problem:
Find $(\mathbf{u}, p, \boldsymbol{\eta})$ such that $\mathbf{u} \circ \mathcal{L}_t = \frac{\partial \boldsymbol{\eta}}{\partial t}$ on Γ_0 and

$$\int_{\Omega_f^t} \rho_f \left(\left. \frac{\partial \mathbf{u}}{\partial t} \right|_{\boldsymbol{\xi}} + \mathbf{u} \cdot \nabla \mathbf{u} \right) \cdot \mathbf{v} + \int_{\Omega_f^t} \boldsymbol{\sigma}_f : \nabla \mathbf{v} + \int_{\Omega_f^t} q \nabla \cdot \mathbf{u}$$

$$+ \int_{\Omega_s^0} \rho_s^0 \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} \cdot \boldsymbol{\phi} + \int_{\Omega_s^0} \boldsymbol{\sigma}_s^0 : \nabla_{\boldsymbol{\xi}} \boldsymbol{\phi} = \int_{\Omega_f^t} \mathbf{f} \cdot \mathbf{v} + \int_{\Omega_s^0} \mathbf{f}^0 \cdot \boldsymbol{\phi}$$

$\forall (\mathbf{v}, q, \boldsymbol{\phi}) \in V_{FS}$ with $V_{FS} = \{(\mathbf{v}, q, \boldsymbol{\phi}) \ : \ \mathbf{v} \circ \mathcal{L}_t = \boldsymbol{\phi} \text{ on } \Gamma_0\}$

Monolithic approach: time discretization

E.g.: Implicit Euler scheme for the fluid and the Midpoint scheme for the structure.

Find $(\mathbf{u}^n, p^n, \boldsymbol{\eta}^n, \dot{\boldsymbol{\eta}}^n)$ such that

$$\int_{\Omega_f} \rho_f \left(\frac{\mathbf{u}^n - \widetilde{\mathbf{u}}^{n-1}}{\Delta t} + \mathbf{u}^n \cdot \nabla \mathbf{u} \right) \cdot \mathbf{v} + \boldsymbol{\sigma}_f : \nabla \mathbf{v} + q \nabla \cdot \mathbf{u}^n$$

$$+ \int_{\Omega_s^0} \rho_s^0 \frac{\dot{\boldsymbol{\eta}}^n - \dot{\boldsymbol{\eta}}^{n-1}}{\Delta t} \cdot \boldsymbol{\phi} + \boldsymbol{\sigma}_s^0 \left(\frac{\boldsymbol{\eta}^n + \boldsymbol{\eta}^{n-1}}{2} \right) : \nabla_{\boldsymbol{\xi}} \boldsymbol{\phi}$$

$$= \int_{\Omega_f} \mathbf{f}^n \cdot \mathbf{v} + \int_{\Omega_s^0} \mathbf{f}^0 \cdot \boldsymbol{\phi}$$

$\forall (\mathbf{v}, q, \boldsymbol{\phi}) \in V_{FS}$, $\boldsymbol{\psi} \in V_{\mathcal{A}}$. Moreover:

$$\dot{\boldsymbol{\eta}}^n = \frac{2}{\Delta t}(\boldsymbol{\eta}^n - \boldsymbol{\eta}^{n-1}) - \dot{\boldsymbol{\eta}}^{n-1} \quad \text{and} \quad \mathbf{u}^n = \dot{\boldsymbol{\eta}}^n$$

The monolithic fluid-structure system can be linearized with a fixed point method or with Newton's method, similarly to what done for the Navier-Stokes problem.

Some references

C. Farhat, M. Lesoinne, P. LeTallec. *Load and motion transfer algorithms for fluid/structure interaction problems with non-matching discrete interfaces: momentum and energy conservation, optimal discretization and application to aeroelasticity*. Comput. Methods Appl. Mech. Engrg. 157(1-2):95-114, 1998.

S. Deparis, M. Discacciati, G. Fourestey, A. Quarteroni. *Fluid-structure algorithm based on Steklov-Poincare operators*. Comput. Methods Appl. Mech. Engrg., 195, 5797-5812, 2006.

U. Kutter, W.A. Wall. *Fixed-point fluid-structure interaction solvers with dynamic relaxation*. Comput. Mech. 43:61-72, 2009.

M. Heil. *An efficient solver for the fully coupled solution of large-displacement fluid-structure interaction problems*. Comput. Methods Appl. Mech. Engrg. 193(1-2):1-23, 2004.

M.A. Fernandez, M. Moubachir. *A Newton method using exact Jacobians for solving fluid-structure coupling*. Computers & Structures, 83(2-3):127-142, 2005.