

Intro to deterministic models

Joel C. Miller & Tom Hladish

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Pairwise Equations

Effective Degree Equations

References

Pairwise equations

We will develop a system of differential equations which predicts disease spread by tracking the numbers of partnerships involving individuals of various statuses.

These have been developed by [1, 2, 3, 4] among many others.

Transition rates

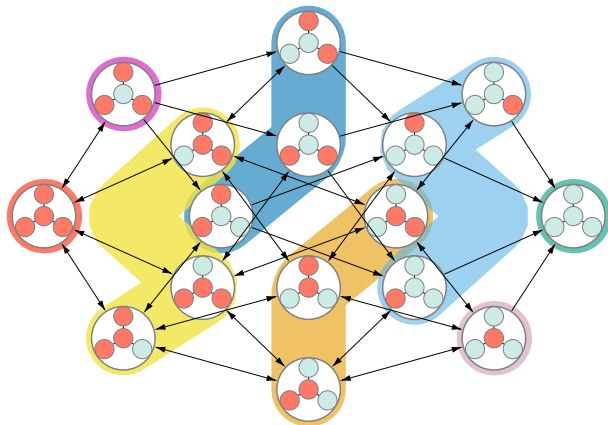
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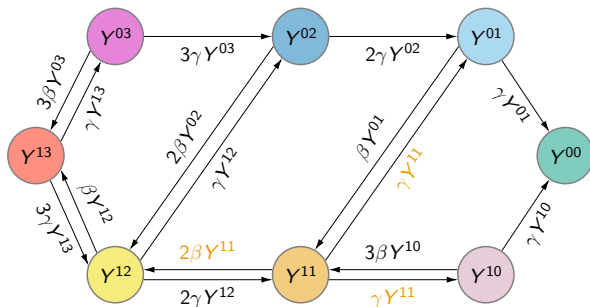
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Can you see the problem?

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Need for Closures

We need to find equations for $[SI]$. For the SIS case we have

$$[\dot{SI}] = \beta[SSI] - \beta[ISI] - \beta[SI] - \gamma[SI]$$

where $[SSI]$ denotes the number of (ordered) triples with statuses SSI and $[ISI]$ the number of (ordered) triples with statuses ISI.

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- ▶ The **third** term represents the susceptible node in an SI pair being infected by the infected node in the pair.

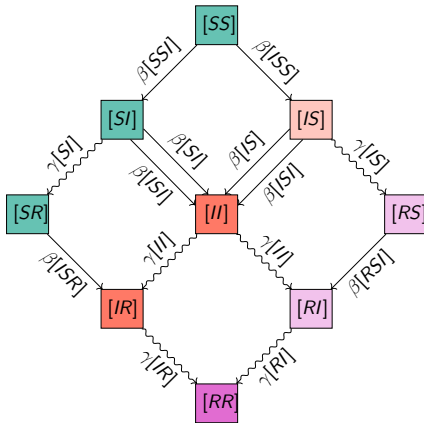
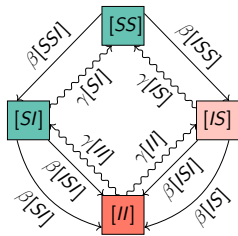
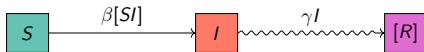
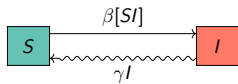
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- ▶ The **fourth** term represents the infected node in an SI pair recovering.



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The equations are equivalent to the Kermack-McKendrick equations

Appropriateness of $[SI] = \langle K \rangle SI$

What assumptions are we making when we set $[SI] = \langle K \rangle SI$?

- ▶ We're assuming that nodes are not preferentially infected by degree.
- ▶ We're assuming that neighbors of infected nodes are no more likely to be infected than any other node.

When are these assumptions appropriate?

- ▶ Same degree, annealed network. Partnerships have zero duration.
- ▶ Large very similar degrees, transmission probability per edge very low, and low clustering.
- ▶ As a general rule — if the disease will never transmit across the same partnership twice, we can use models that ignore partnership duration.

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 - ▶ So

$$\mathcal{R}_0 = \sum_k \frac{kP(k)}{\langle K \rangle} k \frac{\beta}{\gamma} = \frac{\beta}{\gamma} \frac{\langle K^2 \rangle}{\langle K \rangle}$$

Scale-free networks

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- ▶ But if $\langle K^2 \rangle$ is finite, then there is an epidemic threshold.

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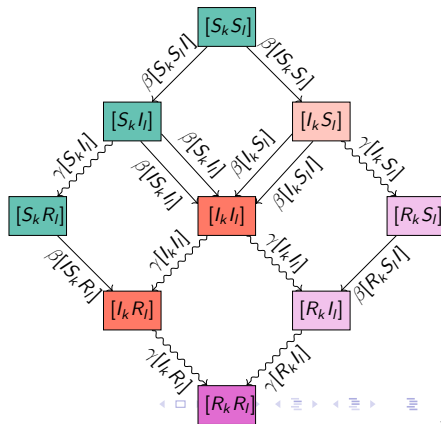
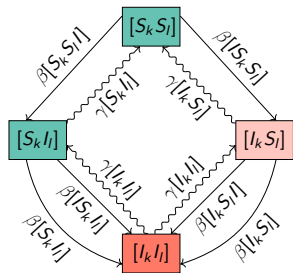
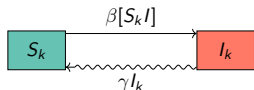
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- ▶ Why?
- ▶ High degree nodes get infected and infect their neighbors. Then they recover.
- ▶ So susceptible high degree nodes tend to have more infected neighbors.
- ▶ We expect to see islands of infection surrounding high degree nodes. Eventually the disease spreads to other high degree nodes. So it can persist even if the naive estimate has $\mathcal{R}_0 < 1$.

An improved heterogeneous degree model

Use S_k , I_k , and R_k to denote the proportion of the population that is in each state and has degree k : $\sum_k S_k + I_k + R_k = 1$.



Improved closure

We close this with:

$$[AS_k B] = \frac{k-1}{k} \frac{[AS_k][S_k B]}{S_k}$$

This assumes that statuses of two neighbors of a susceptible node are independent.

This is dubious when

- ▶ There is significant **clustering**.
- ▶ A node has transmitted to its neighbors and then recovered — **SIS**.

► SIS

$$[\dot{S}_k]_s = \gamma[I_k]_s - \beta[S_k I]_s,$$

$$[\dot{I}_k]_s = \beta[S_k I]_s - \gamma[I_k]_s,$$

$$[S_k \dot{I}_I]_s = \gamma([I_k I]_s - [S_k I]_s) + \beta([S_k S_I I]_s - [I S_k I]_s - [S_k I I]_s),$$

$$[S_k \dot{S}_I]_s = \gamma([S_k I]_s + [I_k S_I]_s) - \beta([S_k S_I I]_s + [I S_k S_I]_s)$$

$$[I_k \dot{I}_I]_s = \beta([S_k I]_s + [I_k S_I]_s) - 2\gamma[I_k I]_s + \beta([I S_k I]_s + [I_k S_I I]_s).$$

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$$[\dot{S}_k]_s = -\beta[S_k I]_s,$$

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$$[\dot{R}_k]_s = \gamma[I_k]_s,$$

$$[\dot{S}_k I]_s = -\gamma[S_k I]_s + \beta([S_k S I]_s - [I S_k I]_s - [S_k I]_s),$$

$$[\dot{S}_k S I]_s = -\beta([S_k S I]_s + [I S_k S I]_s).$$

Further simplifications are possible

We can further simplify the model if we assume that the probability a node's neighbor is susceptible is independent of the node's degree. (fails if there are degree correlations).

Pairwise Equations

Effective Degree Equations

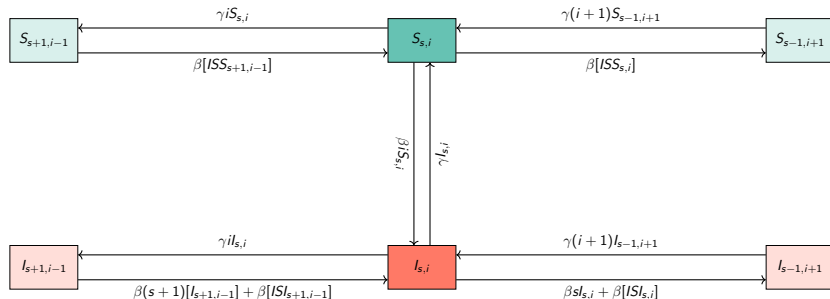
References

Effective Degree Equations aka Approximate Master Equations

Developed by [6, 7, 8, 9].

- ▶ The basic idea is that we can choose a different fundamental unit to study.
- ▶ Each node is the center of a “star”, and its risk of infection depends on how many infected neighbors it has.
- ▶ We use $S_{s,i}$ to denote the proportion of the population that is susceptible and has s susceptible and i infected neighbors. Similarly $I_{s,i}$.
- ▶ We can ignore recovered neighbors (hence “effective degree”)

SIS model



A term like $[ISS_{s+1,i-1}]$ represents $(1/N \text{ times})$ the number of paths that exist from an infected node to a susceptible node to a susceptible node having $s+1$ susceptible partners and $i-1$ infected partners.

We make the closures

$$[ISS_{s,i}] = \frac{[ISS]}{[SS]} sS_{s,i} \quad [ISl_{s,i}] = \frac{[ISl]}{[Sl]} sl_{s,i}$$

Which correspond to assuming that the susceptible neighbors of susceptible nodes are interchangeable and the susceptible neighbors of infected nodes are interchangeable.

SIS effective degree model

We get

$$\begin{aligned}\dot{S}_{s,i} = & -\tau i S_{s,i} + \gamma I_{s,i} + \gamma((i+1)S_{s-1,i+1} - iS_{s,i}) \\ & + \tau \frac{[ISS]}{[SS]}((s+1)S_{s+1,i-1} - sS_{s,i}),\end{aligned}$$

$$\begin{aligned}\dot{I}_{s,i} = & \tau i S_{s,i} - \gamma I_{s,i} + \gamma((i+1)I_{s-1,i+1} - iI_{s,i}) \\ & + \tau \left(\frac{[SI]}{[S]} + 1 \right) ((s+1)I_{s+1,i-1} - sI_{s,i}),\end{aligned}$$

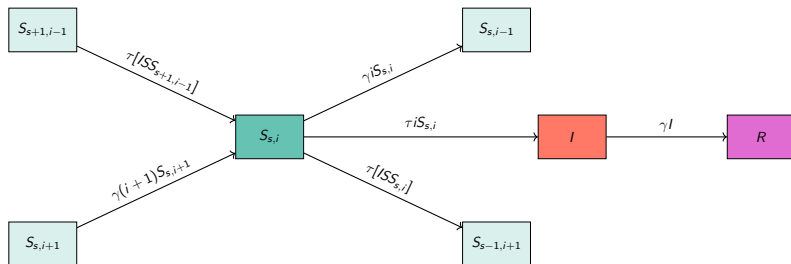
$$S = \sum_{s,i} S_{s,i}, \quad I = \sum_{s,i} I_{s,i},$$

with

$$[SS] = \sum_{s,i} s S_{s,i}, \quad [SI] = \sum_{s,i} i S_{s,i},$$

$$[ISS] = \sum_{s,i} i s S_{s,i}, \quad [ISI] = \sum_{s,i} i(i-1) S_{s,i}.$$

SIR effective degree model



SIR effective degree equations

$$\begin{aligned}\dot{S}_{s,i} &= -\tau i S_{s,i} + \gamma((i+1)S_{s,i+1} - iS_{s,i}) \\ &\quad + \tau \frac{[ISS]}{[SS]} [(s+1)S_{s+1,i-1} - sS_{s,i}] \\ \dot{R} &= \gamma I, \quad S = \sum_{s,i} S_{s,i}, \quad I = N - S - R,\end{aligned}$$

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References

References I

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