Epidemics in Networks
SIS: Equilibrium Size and Dynamics

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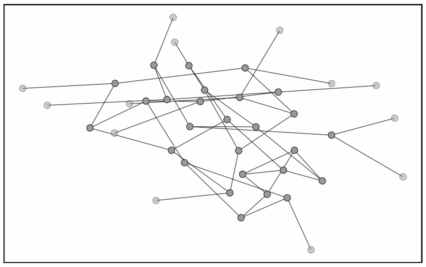
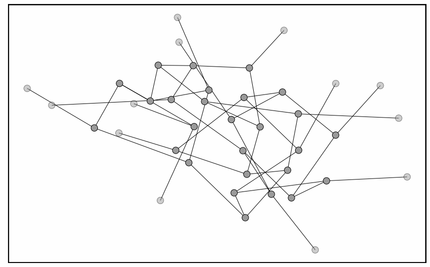
Introduction

 $\mapsto$ 

- For technical reasons it is difficult to handle SIS on configuration model networks.
- The standard solution is to turn to annealed networks.
- There are some subtle but important differences between epidemics in the two cases.

History

- The model was introduced in 1980 [1] and used widely in mathematical biology [2, 3]: "Social heterogeneity"
- In 2001 it was used to study powerlaw degree distributions [4]: "Degree-based mean field"
- Many people use it as if it were correct for configuration model networks.
- The major result of [4] is altered if the network is static [5].

What's the problem?

- If a high degree node is infected in a static network, it will infect many partners.
- When it recovers, these partners are likely to reinfect it.
- A local island of infection forms.
- These islands occassionally go extinct, and occassionally they spread disease to another high degree node, starting a new island.

SIS dynamics

- Because partnerships are constantly changing, the probability a partner is infected is independent of the history of a node.
- Let $\langle I \rangle$ denote the probability a partner is infected.
- Then
$$\langle I \rangle = \frac{\sum_k kP(k)I_k}{\sum_k kP(k)} = \frac{\sum_k kP(k)I_k}{\langle K \rangle}$$
Where I_k is the probability a degree k individual is infected.
- We get

$$\dot{S}_k = -\beta k S_k \langle I \rangle + \gamma I_k$$
$$\dot{I}_k = \beta k S_k \langle I \rangle - \gamma I_k$$

SIS equilibrium

At equilibrium infection balances recovery.

$$\begin{aligned}
\beta k S_k \langle I \rangle &= \gamma I_k \\
\beta k(1 - I_k) \langle I \rangle &= \gamma I_k \\
\beta k \langle I \rangle &= (\gamma + \beta k \langle I \rangle) I_k \\
\Rightarrow \quad I_k &= \frac{\beta k \langle I \rangle}{\gamma + \beta k \langle I \rangle}
\end{aligned}$$

where

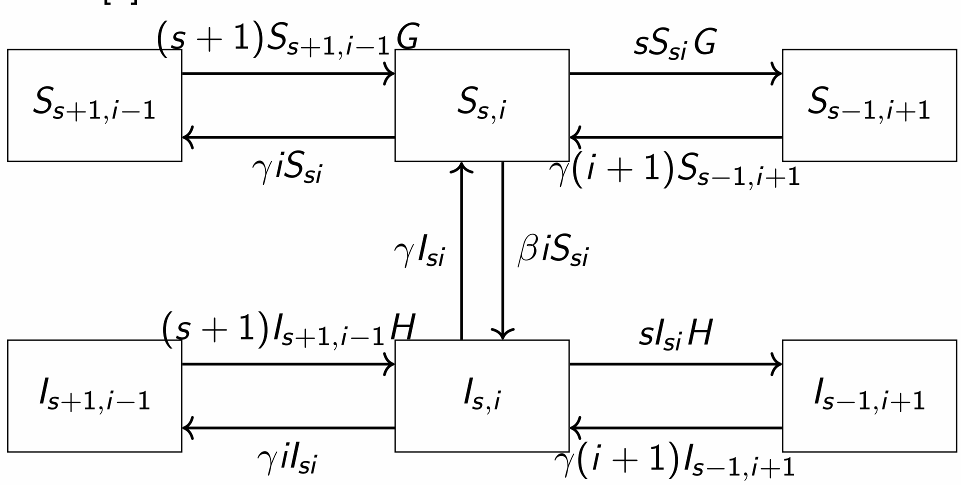
$$\langle I \rangle = \frac{\sum_k k P(k) I_k}{\langle K \rangle}$$

Other models

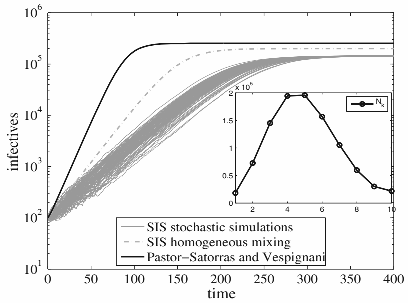
I am aware of two other systems of equations that appear to do a reasonable job on configuration model networks

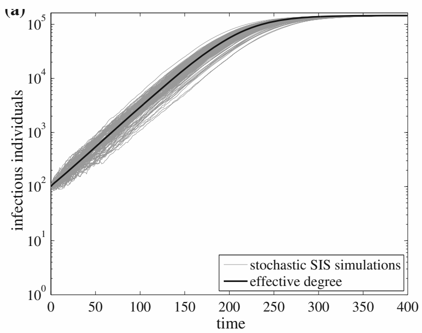
Effective degree model

From [6]



$$G = \frac{\sum_{k=1}^{M}\sum_{j+l=k} j\beta l S_{jl}}{\sum_{k=1}^{M}\sum_{j+l=k} jS_{jl}} \qquad H = \frac{\sum_{k=1}^{M}\sum_{j+l=k} \beta l^2 S_{jl}}{\sum_{k=1}^{M}\sum_{j+l=k} jI_{jl}}$$



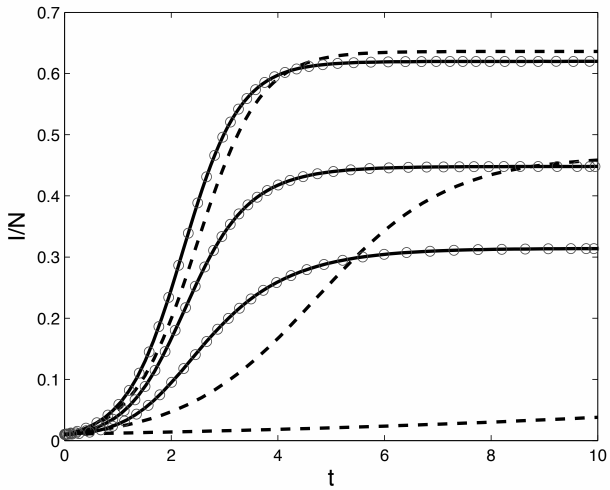


An alternate model by [7] has just 5 equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}[S] = \gamma[I] - \beta[SI]$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[I] = \beta[SI] - \gamma[I]$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[SI] = \gamma([II] - [SI]) + \beta[SI]([SS] - [SI])Q - \beta[SI]$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[SS] = 2\gamma[SI] - 2\beta[SI][SS]Q$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[II] = -2\gamma[II] + 2\beta[SI]^2 Q + 2\beta[SI]$$

where

$$Q = \frac{1}{[S]n_S}\left(\frac{\langle K^2\rangle (\langle K^2\rangle - n_S\langle K\rangle) + \langle K^3\rangle (n_S - \langle K\rangle)}{n_S(\langle K^2\rangle - \langle K\rangle^2)} - 1\right)$$
$$n_S = \frac{[SI] + [SS]}{[S]}$$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