Linear Equation is equation where n variables $x_1, x_2, ..., x_n$ can be expressed in the form $a_1x_1 + a_2x_2 + ... + a_nx_n = b$
where $a_1, a_2, ..., a_nx_n$ and b are constants, and $a's$ are not all zero.

System of linear equations or **linear system** is a finite test of linear equations.The variables are called **unknowns**

Solution of a linear system with n unknonws $x_1, x_2, ..., x_n$ is a sequance of n numbers $s_1, s_2, ...s_n$ for which the substitution $x_1 = s_1, x_2 = s_2, ..., x_n = s_n$ makes each equation a true statement.

Every system of linear equations has zero, one or infinitely many solutions. There are no other possibilities.

Consistent linear system is the one that has at least one solution. They are two equations in this system, one solution of infinitely many.

Inconsistent linear system is the one that has no solutions.

Matrix is rectangular array of numbers.

Size of A Matrix is described in terms of number of rows(horizontal lines) and columns(vertical lines). For example matrix with 3 rows and 2 columns have size 3 by 2 (written 3 x 2)

Augmented Matrix is a matrix obtained by appending the columns of two given matrices.
Given the matrices A and B, where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 5 & 2 & 2 \end{bmatrix} \quad , B = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

the augmented matrix $(A|B)$ is written as

$$(A|B) = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & 3 \\ 5 & 2 & 2 & 1 \end{bmatrix}$$

Augmented Matrix can be used to represent sytem equations. For example :

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Elementary Row Operations The basic method for solving a linear system is to perform appropiate algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simple system, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are as follows:

1. Multiply an equation through by a nonzero constant
2. Interchange two equations
3. Add a constant times of one equation to another

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operation on the rows of the augmented matrix

1. Multiply a row through by a nonzero constant
2. Interchange two rows
3. Add a constant times of one row to another

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first equation to all equation

Add -2 times the first row to all row

$$\begin{aligned} -x - y - 2z &= -9 \\ 2y - 7z &= -17 \\ x + 4y - 9z &= -18 \end{aligned}$$

$$\begin{bmatrix} -1 & -1 & -2 & -9 \\ 0 & 2 & -7 & -17 \\ 1 & 4 & -9 & -18 \end{bmatrix}$$

Add the first equation the third equation

Add the first row to the third row

$$\begin{aligned} -x - y - 2z &= -9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

$$\begin{bmatrix} -1 & -1 & -2 & -9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -1 of the third equation the second equation

Add -1 of the third row to the second row

$$\begin{aligned} -x - y - 2z &= -9 \\ -y + 4z &= 10 \\ 3y - 11z &= -27 \end{aligned}$$

$$\begin{bmatrix} -1 & -1 & -2 & -9 \\ 0 & -1 & +4 & 10 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -1 of the second equation the first equation

Add -1 of the second row to the first row

$$\begin{aligned} -x + 0y - 6z &= -19 \\ -y + 4z &= 10 \\ 3y - 11z &= -27 \end{aligned}$$

$$\begin{bmatrix} -1 & 0 & -6 & -19 \\ 0 & -1 & 4 & 10 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add 3 times of the second equation the third equation

Add 3 times of the second row to the third row

$$\begin{aligned} -x + 0y - 6z &= -19 \\ -y + 4z &= 10 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} -1 & 0 & -6 & -19 \\ 0 & -1 & 4 & 10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add 6 times of the third equation the first equation

Add 6 times of the third row to the first row

$$\begin{aligned} -x &= -1 \\ -y + 4z &= 10 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 4 & 10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -4 times of the third equation the second equation

Add -4 times of the third row to the second row

$$\begin{aligned} -x &= -1 \\ -y &= -2 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Multiply -1 times for each equation 1 and 2

Multiply -1 times for each row 1 and 2

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Echelon Forms and Reduced Echelon Forms

1. If row does not consists of entirely zeroes, then the first nonzero number in the row is a 1. This is called a *leading 1.*

2. It there are any rows that consist entirely of zeroes, the they are grouped together a the bottom of the matrix.

3. In any two succesive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right the the leading 1 in the higher row.

4. Each column that contains a leading 1 has zeroes everywhere else in the column.

1 to 3 is property of *row echelon form*, meanwhile 1 to 4 is property of *reduced row echelon form*.

Example of Reduced Echelon Form

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example of Echelon Form but not Reduced

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Solutions. If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is called **general solution** to the system.

Elimination Procedure is steps that can be used to reduce matrix to reduced echelon form

Gauss-Jordan Elimination.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

1. Locate the leftmost column that does not consist entirely zeros

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

2. Interchange the top row with another row, if necessary, to bring a nonzeor entry to the top of the column found in 1

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

3. If the entry that is now at the top of the column found in 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

4. Add a suitable multiples of the top row to the rows below so that all entries below the leading 1 becomes zeros

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

5. Now cover the top row in the matrix and begina again with step 1 applied to the submatrix that remains. Continue in this way untuk the entire matrix in row echelon form

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Homogeneous Linear Systems

A system of linear equations is said to be ***homogeneous*** if the constant terms all zero; that is, the system has the form

$$\begin{aligned} a_{11}x_1 \quad &+a_{12}x_2 \quad +\ldots + a_{1n}x_n = 0 \\ a_{21}x_1 \quad &+a_{22}x_2 \quad +\ldots + a_{2n}x_n = 0 \\ \ldots \\ a_{m1}x_1 \quad &+a_{m2}x_2 \quad +\ldots + a_{mn}x_n = 0 \end{aligned}$$

Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, x_n = 0$ as a solution. This solution is called

trivial solution; if there are other solutions, they are called ***nontrivial solutions***.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_6 = 0$
$x_3 = -2x_4$
$x_4 = t$
$x_3 = -2t$
$x_1 = -3v - 4t - 2w$
$x_2 = v$

Note that the trivial solution results when $v = t = w = 0$.

Leading Variables Are those whose columns in the *Reduced Row Echelon Form* contains leading 1.

Free Variables Are those whose columns in the *Reduced Row Echelon Form* do not contain leading 1.

x_1, x_3 and x_6 are all *Leading Variables*
x_2, x_4 and x_5 are all *Free Variables*

Free Variable Theorem for Homogeneous System. If a homogeneous has n unknowns, and if the *reduced row echelon form* of its *augmented matrix* has r nonzero rows, then the system has $n - r$ free variables.

Free Variables , Homogeneous System and Solutions. A *Homogeneous Linear System* with more unknowns than equations has infinitely many solutions.

Gaussian Elimination and Back-Substitution. For a larger system using *Gauss Jordan* requires lots of resource.

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian Elimination and Back-Substitution comes for help.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ x_3 + 2x_4 + 3x_6 &= 1 \\ x_6 &= \frac{1}{3} \end{aligned}$$

1. Solve the equations for the leading variables

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= \frac{1}{3} \end{aligned}$$

2. Beginning with the bottom equation and working upward, successively substitute each equation into all the equations above it.
Substituting $x_6 = \frac{1}{3}$ into the second equation yields

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \frac{1}{3} \end{aligned}$$

Substituting $x_3 = -2x_4$ into the first equation yields

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4x_6 & = \frac{1}{3} \end{aligned}$$

3. Assign arbitrary values to the free variables if any

matrix is a rectangular array of numbers.

column vector or **column matrix** is a matrix with only one column.

row vector or **row matrix** is a matrix with only one row.

cross product If A is an $m \times n$ matrix and B is $r \times n$ matrix, then the product **AB** is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, an then add up the resulting products.

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \\ a_{30} & a_{31} & a_{32} \end{bmatrix} B = \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \\ b_{20} & b_{21} \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{00} \times b_{00} + a_{01} \times b_{10} + a_{02} \times b_{20} & a_{00} \times b_{01} + a_{01} \times b_{11} + a_{02} \times b_{21} \\ a_{10} \times b_{00} + a_{11} \times b_{10} + a_{12} \times b_{20} & a_{10} \times b_{01} + a_{11} \times b_{11} + a_{12} \times b_{21} \\ a_{20} \times b_{00} + a_{21} \times b_{10} + a_{22} \times b_{20} & a_{20} \times b_{01} + a_{21} \times b_{11} + a_{22} \times b_{21} \\ a_{30} \times b_{00} + a_{31} \times b_{10} + a_{32} \times b_{20} & a_{30} \times b_{01} + a_{31} \times b_{11} + a_{32} \times b_{21} \end{bmatrix}$$

The definition of matrix multiplication requires that the number of columns of the first factor A be the same as the number of rows of the second factor B in order to form the product AB.

Partitioned Matrices A matrix can be subdivided or partitioned into smaller matrices by inserting horizontal and vertical rules between selected rows and columns.

Partitioning can be used to find particular rows or columns of a matrix product AB without computing the entire product.

Any individual column vector of AB can be obtained by partitioning B into column vectors.

$$AB = A \begin{bmatrix} b_0 & b_1 & ... & b_n \end{bmatrix}$$

Any individual row vector of AB can be obtained by partitioning A into row vectors.

$$AB = \begin{bmatrix} a_0 \\ a_1 \\ . \\ . \\ . \\ a_n \end{bmatrix} B$$

From previous AB with

$$b_{col_0} = \begin{bmatrix} b_{00} \\ b_{10} \\ b_{20} \end{bmatrix}$$

$$(AB)_{col_0} = Ab_{col_0} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \\ a_{30} & a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{00} \\ b_{10} \\ b_{20} \end{bmatrix}$$

and with

$$a_{row_0} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \end{bmatrix}$$

$$(AB)_{row_0} = a_{row_0}B = \begin{bmatrix} a_{00} & a_{01} & a_{02} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \\ b_{20} & b_{21} \end{bmatrix}$$

Linear combinations. if $A_1....A_m$ are matrices of the same size, and $c_1...c_m$ are scalars, the expression of the form

$$c_1A_1 + ... + c_mA_m$$

is called a *linear combination* of matrices $A_1...A_m$. The scalars $c_1...c_m$ are called the *coefficients* of the lienar combination.

If A is an $m \times n$ matrix, and if x is an $n \times 1$ column vector, then the product of Ax can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of x.

$$A = \begin{bmatrix} a_{00} & a_{01} & ... & a_{0n} \\ a_{10} & a_{11} & ... & a_{1n} \\ . & . & ... & . \\ . & . & ... & . \\ a_{m0} & a_{m1} & ... & a_{mn} \end{bmatrix} \; and \; x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$$

Transpose of any $m \times n$ matrix A (denoted by A^T)is an $n \times m$ matrix that results by interchanging the rows and columns of A, that is the first column of A^T is is the first row of A, the second column of A^T is the second row of A, and so forth.

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 3 & 1 \\ 2 & 5 & 6 \\ 7 & 6 & 2 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & 1 & 2 & 7 \\ 4 & 3 & 5 & 6 \\ 2 & 1 & 6 & 2 \end{bmatrix}$$

trace of any matrix A, denoted by $tr(A)$, is the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$tr(A) = a_{00} + a_{11} + a_{22}, tr(B) = -1 + 5 + 7 + 0 = 11$$

Square Matrix of Order n is a matrix with n rows and n columns.

Main Diagonal of a square matrix is entries with index a_{00}, a_{11}, a_{22}, ..., a_{nn}

Identity Matrix is a square matrix with 1's on the main diagonal and zero elsewheare.

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I_n is the identity matrix for the $n \times n$ matrix

I_n has the property for that every matrix A of size $n \times n$ it is true that $AI_n = A$ and $I_nA = A$

Theorem If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the indenitity matrix I_n

Zero Matrices are matrices with all elements are zero. Some example:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Some properties of zero matrices

1. $A + 0 = 0 + A = A$
2. $A - 0 = A$
3. $A - A = A + (-A) = 0$
4. $0A = 0$
5. If $cA = 0$, then $c = 0$ or $A = 0$

Inverse . If A is a square matrix, and if B is matrix of the same size and that $AB = BA = I$, then A is said to be ***invertible*** or ***nonsingular*** and B is called ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

$$\begin{aligned} XA &= B \\ XAA^{-1} &= BA^{-1} \\ XI &= BA^{-1} \\ X &= BA^{-1} \end{aligned}$$

A Real Life Example Bus and Train A group took a trip an a bus, at \$3

per child and \$3.2 per adult for a total of \$118.40
The took the train back at \$3.50 per child and \$3.60 per adult for a total of \$135.20
How many children, and how many adults?

$$
\begin{aligned}
\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix} &= \begin{bmatrix} 118.4 & 135.2 \end{bmatrix} \\
\begin{bmatrix} x_1 & x_2 \end{bmatrix} \quad I \quad &= \begin{bmatrix} 118.4 & 135.2 \end{bmatrix} \begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}^{-1}
\end{aligned}
$$

If A is a square matrix of size $n \times n$ and there any matrix B of the same size such that $AB = 0$ then A is not invertible.

Remember that a matrix is representation of several linear formula (linear map or linear function). I A is matrix , X is a set of unknown variable (Domain) and Y is a set of Codomain of AX. If for every unique combination of X there is a unique combination of Y then A is invertible and the set of function that can map Y to X is inverse of A.

A matrix is also said non invertible if can not reduced using elementary row operation to identity matrix.

Find Inverse of A Matrix using Elementary Row Operation

Inverse of any invertible matrix can be deduced by applying the same gauss jordan operation to get its *reduced row echelon form* ,which is *indentity matrix* if there is no row of zeros, to the *reduced row echelon form.*

Suppose three equations:

$$
\begin{aligned}
3x + 2y + 5z &= 12 \\
x - 3y + 2z &= -13 \\
5x - y + 4z &= 10
\end{aligned}
$$

$$
\begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{bmatrix} \quad R_1/3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$\begin{bmatrix} 1 & 2/3 & 5/3 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{bmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 5R_1 \end{matrix} \quad \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2/3 & 5/3 \\ 0 & -11/3 & 1/3 \\ 0 & -13/3 & -13/3 \end{bmatrix} \quad R_2 \times -3/11 \quad \begin{bmatrix} 1/3 & 0 & 0 \\ -1/3 & 1 & 0 \\ -5/3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2/3 & 5/3 \\ 0 & 1 & 1/11 \\ 0 & -13/3 & -13/3 \end{bmatrix} \quad R_3 \times 13/3R_2 \quad \begin{bmatrix} 1/3 & 0 & 0 \\ 1/11 & -3/11 & 0 \\ -5/3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2/3 & 5/3 \\ 0 & 1 & -1/11 \\ 0 & 0 & -156/33 \end{bmatrix} \quad R_3 \times -33/156 \quad \begin{bmatrix} 1/3 & 0 & 0 \\ 1/11 & -3/11 & 0 \\ -14/11 & -13/11 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2/3 & 5/3 \\ 0 & 1 & -1/11 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_1 - 5/3R_3 \\ R_2 + 1/11R_3 \end{matrix} \quad \begin{bmatrix} 1/3 & 0 & 0 \\ 1/11 & -3/11 & 0 \\ 14/52 & 13/52 & -11/52 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_1 - 2/3R_2 \quad \begin{bmatrix} -3/26 & -5/12 & 55/156 \\ 3/26 & -1/4 & -1/52 \\ 7/26 & 1/4 & -11/52 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -5/26 & -1/4 & 19/52 \\ 3/26 & -1/4 & -1/52 \\ 7/26 & 1/4 & -11/52 \end{bmatrix}$$

because

$$\begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \\ 10 \end{bmatrix}$$

so

$$\begin{bmatrix} -5/26 & -1/4 & 19/52 \\ 3/26 & -1/4 & -1/52 \\ 7/26 & 1/4 & -11/52 \end{bmatrix} \begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

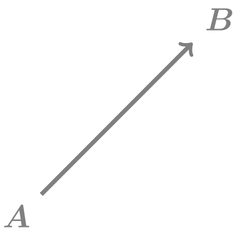
$$\begin{bmatrix} -5/26 & -1/4 & 19/52 \\ 3/26 & -1/4 & -1/52 \\ 7/26 & 1/4 & -11/52 \end{bmatrix} \begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5/26 & -1/4 & 19/52 \\ 3/26 & -1/4 & -1/52 \\ 7/26 & 1/4 & -11/52 \end{bmatrix} \begin{bmatrix} 12 \\ -13 \\ 10 \end{bmatrix}$$

then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4.59615384615385 \\ 4.44230769230769 \\ -2.13461538461538 \end{bmatrix}$$

and
$$\begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} 4.59615384615385 \\ 4.44230769230769 \\ -2.13461538461538 \end{bmatrix} = \begin{bmatrix} 12 \\ -13 \\ 10 \end{bmatrix}$$

Vector or also called **Eucledian Vector** is geometric object that has magnitude(length) and direction. It is frequently represented by a line segment with definite direction and denoted by $\overrightarrow{AB}$



norm of a vector is length of a vector. Using *Phytagorean theorem* it can be found that norm of a 2 space vector of x, and y is

$||v|| = \sqrt{x^2 + y^2}$

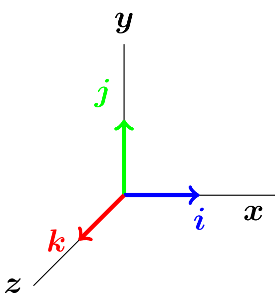
Using *Pythagorean Theorem* it is also can be defined Norm of any vector in n space.

unit vector is a vector of length 1. It often denoted by lower letter with circumflex $\hat{u}$

For any vector in any space it can be shown that

$$\hat{u} = \frac{u}{||u||}$$

standard basis (also called **natural basis**) is the set of *unit vectors* in the direction of the axes of *Cartesian Coordinate System*.



orthonormal basis is a set of unit vector that are orthogonal to each other.

Inner Product The standard *inner product* (also called *dot product*) of two n-vectors is defined as the scalar

$$a^T b = a_1 b_1 + a_2 b_2 + ... + a_n b_n$$

Geometric Definition of Dot Product

$$a.b = \|a\| \, \|b\| \, cos\theta \tag{1}$$

Equivalence of the definitions
if $e_1, ..., e_n$ are the standard basis vectors in $\mathbb{R}^n$, then we may write

$$a = \begin{bmatrix} a_1, & ..., & a_n \end{bmatrix} = \sum_i^n a_i e_i$$
$$b = \begin{bmatrix} b_1, & ..., & b_n \end{bmatrix} = \sum_i^n b_i e_i$$

The vectors e_i are an orthonormal basis, which means that they have unit length and are at right angels to each other. Hence since these vectors have unit length

$$e_i.e_i = 1$$

and for $i \neq$ j

$$e_i.e_j = 0$$

Thus we can say that :

$$e_i.e_j = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Also by the geometric definition, for any vector e_i and a vector a, we note

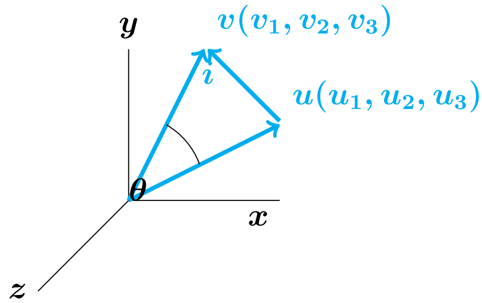
$$a.e_i = \|a\| \, \|e_i\| \cos\theta_i = \|a\| \cos\theta_i = a_i$$

where a_i is the component of vector a in the direction of e_i.

Applying the distributivity of geometric version of the dot product gives

$$a.b = a.\sum_i b_i e_i = \sum_i b_i(a.e_i) = \sum_i a_i b_i$$

Another way to prove this equivalence



From the law of cosine

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\,\|u\|\,\|v\|\,cos\theta$$

$$\|u\|\,\|v\|\,cos\theta = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \|v-u\|^2\right)$$

$$u.v = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \|v-u\|^2\right)$$

with

$$
\begin{aligned}
\|u\|^2 &= u_1^2 + u_2^2 + u_3^2 \\
\|v\|^2 &= v_1^2 + v_2^2 + v_3^2
\end{aligned}
$$

and

$$
\begin{aligned}
\|v-u\|^2 &= (v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2 \\
\|v-u\|^2 &= (v_1^2+u_1^2-2v_1u_1) + (v_2^2+u_2^2-2v_2u_2) + (v_3^2+u_3^2-2v_3u_3) \\
\|v-u\|^2 &= v_1^2+v_2^2+v_3^2+u_1^2+u_2^2+u_3^2-2v_1u_1-2v_2u_2-2v_3u_3 \\
\|v-u\|^2 &= v^2+u^2-2v_1u_1-2v_2u_2-2v_3u_3 \\
\|v-u\|^2 &= \|v\|^2+\|u\|^2-2v_1u_1-2v_2u_2-2v_3u_3
\end{aligned}
$$

since

$$\|v-u\|^2 = \|v-u\|^2$$

$$
\begin{aligned}
\|u\|^2+\|v\|^2-2\|u\|\|v\|cos\theta &= \|v\|^2+\|u\|^2-2v_1u_1-2v_2u_2-2v_3u_3 \\
-2\|u\|\|v\|cos\theta &= -2v_1u_1-2v_2u_2-2v_3u_3 \\
\|u\|\|v\|cos\theta &= v_1u_1+v_2u_2+v_3u_3
\end{aligned}
$$

orthogonality. Two vector **u** and **v** is said **orthogonal** or **perpendicular** if $\vec{u}.\vec{v} = 0$

normal of a given vector is another vector that perpendicular to previous.

A line that meet the equation

$$ax + by + c = 0$$

where :

$$
\begin{aligned}
a &\neq 0, \\
b &\neq 0 \\
c &\in \mathbb{Q}
\end{aligned}
$$

$$\forall c, d \in \mathbb{Q}$$
$$\vec{r} \perp ax + by + c => \vec{r} \perp ax + by + d$$

for $c = 0$

$$ax + by = 0$$

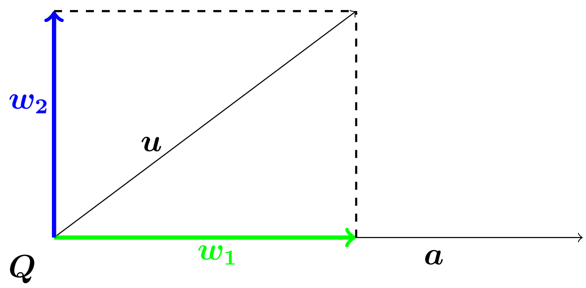
equals to

$$(a, b).(x, y) = 0$$

For

$$\forall p_n \neq 0 \wedge c, d \in \mathbb{Q}$$
$$\vec{r} \perp p_n x_n + p_{n-1} x_{n-1} + ... + p_1 x_1 + c => \vec{r} \perp p_n x_n + p_{n-1} xn - 1 + ... p_1 x_1 + d$$
$$(p_n, p_{n-1}, ..., p_1).(x_n, x_{n-1}, ..., x_1) = 0$$

Projection Theorem : If u and a are vectors in R^n, and if $a \neq 0$, then u can be expressed in exactly one way in th form $u = w_1 + w_2$, where w_1 is a scalar multiple of a and w_2 is orthogonal to a.



Proof

$$k = \|u\| \cos(\theta)$$
$$u \cdot a = \|u\| \|a\| \cos(\theta)$$
$$k \|a\| = \frac{u \cdot a}{\|a\|}$$
$$k = \frac{u \cdot a}{\|a\|^2}$$

The inner product function. Suppose a is an $n - vector$. We can define scalar-valued function f of $n - vectors$, given by

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + ... + a_n x_n \tag{2}$$

for any $n - vectorx$. This function gives the inner product of its $n - vector$ argument x with some (fixed) $n - vector$.

Superposition and linearity. The inner product function f defined above satisfies property

$$\begin{aligned} f(\alpha x + \beta y) &= a^T(\alpha x + \beta y) \\ &= a^T(\alpha x) + a^T(\beta y) \\ &= \alpha(a^T x) + \beta(a^T y) \\ &= \alpha f(x) + \beta f(x) \end{aligned}$$

for all $n - vectorsx, y$, and all scalars α, β. This property is called *superposition.* A function that satisfies the superposition property is called *linear.*

Superposition equality is thus

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \tag{3}$$

A function $f : \mathbf{R}^n \to \mathbf{R}$ is linear if satisfies

- *Homogenity.* For any n-vector x and any scala α, $f(\alpha x) = \alpha f(x)$
- *Additivity.* For any n-vector x and y, $f(x + y) = f(x) + f(y)$

Inner product representation of linear function A function is linear if it is defined as inner product of it's argument with some fixed vector.
$f(x) = a^T x$ for all x. $a^T x$ is inner product representation of f

Affine functions. A linear function plus a constant is called an *affine* function. A function $f : \mathbf{R}^n \to \mathbf{R}$ is affine if an only if it can be expressed as $f(x) = a^T x + b$