**Linear Equation** is equation where $n$ variables $x_1, x_2, ..., x_n$ can be expressed in the form $a_1x_1 + a_2x_2 + ... + a_nx_n = b$
where $a_1, a_2, ..., a_nx_n$ and $b$ are constants, and $a's$ are not all zero.

**System of linear equations** or **linear system** is a finite test of linear equations.The variables are called **unknowns**

**Solution** of a linear system with $n$ unknonws $x_1, x_2, ..., x_n$ is a sequance of $n$ numbers $s_1, s_2, ...s_n$ for which the substitution $x_1 = s_1, x_2 = s_2, ..., x_n = s_n$ makes each equation a true statement.

*Every system of linear equations has zero, one or infinitely many solutions. There are no other possibilities.*

**Consistent** linear system is the one that has at least one solution. They are two equations in this system, one solution of infinitely many.

**Inconsistent** linear system is the one that has no solutions.

**Augmented Matrix** is a matrix obtained by appending the columns of two given matrices.
Given the matrices A and B, where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 5 & 2 & 2 \end{bmatrix} \quad , B = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

the augmented matrix $(A|B)$ is written as

$$(A|B) = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & 3 \\ 5 & 2 & 2 & 1 \end{bmatrix}$$

Augmented Matrix can be used to represent sytem equations. For example :

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

**Elementary Row Operations** The basic method for solving a linear system is to perform appropiate algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simple system, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are as follows:

1. Multiply an equation through by a nonzero constant
2. Interchange two equations
3. Add a constant times of one equation to another

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operation on the rows of the augmented matrix

1. Multiply a row through by a nonzero constant
2. Interchange two rows
3. Add a constant times of one row to another

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first equation to all equation

Add -2 times the first row to all row

$$\begin{aligned} -x - y - 2z &= -9 \\ 2y - 7z &= -17 \\ x + 4y - 9z &= -18 \end{aligned} \qquad \begin{bmatrix} -1 & -1 & -2 & -9 \\ 0 & 2 & -7 & -17 \\ 1 & 4 & -9 & -18 \end{bmatrix}$$

Add the first equation the third equation

Add the first row to the third row

$$\begin{aligned} -x - y - 2z &= -9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned} \qquad \begin{bmatrix} -1 & -1 & -2 & -9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -1 of the third equation the second equation

Add -1 of the third row to the second row

$$\begin{aligned} -x - y - 2z &= -9 \\ -y + 4z &= 10 \\ 3y - 11z &= -27 \end{aligned} \qquad \begin{bmatrix} -1 & -1 & -2 & -9 \\ 0 & -1 & +4 & 10 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -1 of the second equation the first equation

Add -1 of the second row to the first row

$$\begin{aligned} -x + 0y - 6z &= -19 \\ -y + 4z &= 10 \\ 3y - 11z &= -27 \end{aligned} \qquad \begin{bmatrix} -1 & 0 & -6 & -19 \\ 0 & -1 & 4 & 10 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add 3 times of the second equation the third equation

Add 3 times of the second row to the third row

$$\begin{aligned} -x + 0y - 6z &= -19 \\ -y + 4z &= 10 \\ z &= 3 \end{aligned} \qquad \begin{bmatrix} -1 & 0 & -6 & -19 \\ 0 & -1 & 4 & 10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add 6 times of the third equation the first equation

Add 6 times of the third row to the first row

$$\begin{aligned} -x &= -1 \\ -y + 4z &= 10 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 4 & 10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -4 times of the third equation the second equation

Add -4 times of the third row to the second row

$$\begin{aligned} -x &= -1 \\ -y &= -2 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Multiply -1 times for each equation 1 and 2

Multiply -1 times for each row 1 and 2

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

**Echelon Forms and Reduced Echelon Forms**

1. If row does not consists of entirely zeroes, then the first nonzero number in the row is a 1. This is called a *leading 1.*
2. It there are any rows that consist entirely of zeroes, the they are grouped together a the bottom of the matrix.
3. In any two succesive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right the the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeroes everywhere else in the column.

1 to 3 is property of *row echelon form*, meanwhile 1 to 4 is property of *reduced row echelon form*.

Example of Reduced Echelon Form

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example of Echelon Form but not Reduced

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**General Solutions**. If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is called **general solution** to the system.

**Elimination Procedure** is steps that can be used to reduce matrix to reduced echelon form

**Gauss-Jordan Elimination**.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

1. Locate the leftmost column that does not consist entirely zeros

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

2. Interchange the top row with another row, if necessary, to bring a nonzeor entry to the top of the column found in 1

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

3. If the entry that is now at the top of the column found in 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

4. Add a suitable multiples of the top row to the rows below so that all entries below the leading 1 becomes zeros

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

5. Now cover the top row in the matrix and begina again with step 1 applied to the submatrix that remains. Continue in this way untuk the entire matrix in row echelon form

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

**Example**

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 && (1) \\ 2x_1 + 6x_2 - 5x_3 + 2x_4 + 4x_5 - 3x_6 &= -1 && (2) \\ 5x_3 + 10x_4 + 15x_6 &= 5 && (3) \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6 && (4) \\ & && (5) \end{aligned}$$

**Inner Product** The standard *inner product* (also called *dot product*) of two $n$-vectors is defined as the scalar

$$a^T b = a_1 b_1 + a_2 b_2 + ... + a_n b_n$$

**Linear combinations.** if $a_1 ....a_m$ are n-vectors, and $\beta_1 ...\beta_m$ are scalars, the $n - vector$

$$\beta_1 a_1 + ... + \beta_m a_m$$

is called a *linear combination* of vectors $a_1 ... a_n$. The scalars $\beta_1 ... \beta_m$ are called the *coefficients* of the lienar combination.

**The inner product function.** Suppose $a$ is an $n - vector$. We can define scalar-valued function $f$ of $n - vectors$, given by

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + ... + a_n x_n \tag{6}$$

for any $n - vectorx$. This function gives the inner product of its $n - vector$ argument $x$ with some (fixed) $n - vector$.

**Superposition and linearity**. The inner product function $f$ defined above satisfies property

$$\begin{aligned} f(\alpha x + \beta y) &= a^T(\alpha x + \beta y) \\ &= a^T(\alpha x) + a^T(\beta y) \\ &= \alpha(a^T x) + \beta(a^T y) \\ &= \alpha f(x) + \beta f(x) \end{aligned}$$

for all $n - vectorsx, y$, and all scalars $\alpha, \beta$. This property is called *superposition.* A function that satisfies the superposition property is called *linear.*

**Superposition equality** is thus

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \tag{7}$$

A function $f : \mathbf{R}^n \to \mathbf{R}$ is linear if satisfies

- *Homogenity.* For any $n$-vector $x$ and any scala $\alpha$, $f(\alpha x) = \alpha f(x)$
- *Additivity.* For any $n$-vector $x$ and $y$, $f(x + y) = f(x) + f(y)$

**Inner product representation of linear function** A function is linear if it is defined as inner product of it's argument with some fixed vector.
$f(x) = a^T x$ for all $x$. $a^T x$ is inner product representation of $f$

**Affine functions**. A linear function plus a constant is called an *affine* function. A function $f : \mathbf{R}^n \to \mathbf{R}$ is affine if an only if it can be expressed as $f(x) = a^T x + b$