

IEOR 4739

Further optimization tasks

A version of portfolio rebalancing. Suppose that on day t we own quantity q_j of asset j for $1 \leq j \leq n$. As before the current price (on day t) of asset j equals $p_j^{(t)}$. Thus the current fraction (percentage) of the portfolio that we own, in asset j , equals

$$\tilde{x}_j \doteq \frac{p_j^{(t)} q_j}{\sum_{i=1}^n p_i^{(t)} q_i}.$$

We want to buy/sell assets so as to convert these fractions into other fractions that are as close as possible to a previously computed portfolio given by fractions x_j^* . How do we do so?

Denote by d_j the number of shares that we buy/sell of asset j . So what we want is: .

$$p_j^{(t)}(q_j + d_j) = x_j^* \sum_{i=1}^n p_i^{(t)}(q_i + d_i) \quad 1 \leq j \leq n, \quad (1a)$$

$$q_j + d_j \geq 0, \quad 1 \leq j \leq n \quad (1b)$$

This basic set of equations describes the needed changes. But there may be many ways to attain (1) – hence we can talk about optimization.

a. Minimize value change needed to attain rebalancing.

Here we would solve the problem

$$\min \sum_{i=1}^n p_i^{(t)}(d_i^+ + d_i^-) \quad (2a)$$

Subject to:

$$d_i = d_i^+ - d_i^-, \quad d_i^+, d_i^- \geq 0, \quad 1 \leq i \leq n, \quad (2b)$$

$$\text{Constraints (1).} \quad (2c)$$

b. Minimize (weighted)f number of shares needed to attain rebalancing.

Here instead we would solve the problem

$$\min \sum_{i=1}^n W_i(d_i^+ + d_i^-) \quad (3a)$$

Subject to:

$$d_i = d_i^+ - d_i^-, \quad d_i^+, d_i^- \geq 0, \quad 1 \leq i \leq n, \quad (3b)$$

$$\text{Constraints (1).} \quad (3c)$$

In this case the W_i are given weights. For example if $W_i = 1$ for all i we are simply minimizing the number of traded shares. But in general we would have different weights – used to model difficulty in trading different assets.

The first two problems, in any case, may give rise to a large amount of trading. Rather than aiming for perfect rebalancing, we might instead ask for near-perfect rebalancing. As an example, suppose we are given (small) values $\varepsilon_i > 0$ for all i . Then we simply ask that

$$\left| \frac{p_j^{(t)}(q_j + d_j)}{\sum_{i=1}^n p_i^{(t)}(q_i + d_i)} - x_j^* \right| \leq \varepsilon_j$$

for all j . This allows some errors (relative to the x_j^*) in the final positions. Instead of (1) we will now have the inequalities

$$p_j^{(t)}(q_j + d_j) \leq (x_j^* + \varepsilon_j) \sum_{i=1}^n p_i^{(t)}(q_i + d_i) \quad 1 \leq j \leq n \quad (4a)$$

$$-p_j^{(t)}(q_j + d_j) \leq (-x_j^* + \varepsilon_j) \sum_{i=1}^n p_i^{(t)}(q_i + d_i) \quad 1 \leq j \leq n \quad (4b)$$

$$q_j + d_j \geq 0, \quad 1 \leq j \leq n \quad (4c)$$

Using this criterion we obtain problems similar to those considered in (a) and (b) above.

First version of robust portfolio optimization. In previous lectures we considered the classical mean-variance portfolio optimization problem, which in the long-only variant takes the form

$$\begin{aligned} \min \quad & \lambda x^T Q x - \sum_{j=1}^N \mu_j x_j \\ \text{Subject to:} \quad & \sum_j x_j = 1 \end{aligned} \quad (5a)$$

$$Ax \geq b \quad (5b)$$

$$x \geq 0. \quad (5c)$$

Here $\lambda \geq 0$ is the risk aversion parameter, Q is the covariance of returns, μ is the vector of average asset returns, and $Ax \geq b$ are structural constraints. Note that $Ax \geq b$ can be used to model many requirements, including bounds on the x_j .

What we are interested in here is the behavior of the portfolio when the μ_j are noisy estimates of the “true” expected asset returns, that is to say, they include errors. Thus, we run the risk of constructing a portfolio that will behave *very* differently when the real returns are realized – even if the real returns are slightly different from the values μ_j .

We can study this problem by imagining that future average returns will be set by an imaginary adversary. This is best considered as a two-step process:

- (1) First, a decision-maker (you) picks the asset weights x_j .
- (2) Second, the adversary picks an expected returns vector $\hat{\mu}$. This vector is chosen from some set $\mathcal{M} \subseteq \mathbb{R}^n$. The goal of the adversary is to *minimize* the return earned by x , that is to say, the adversary solves the optimization problem

$$r(x) \doteq \min \sum_{i=1}^n \hat{\mu}_i x_i : \hat{\mu} \in \mathcal{M}. \quad (6)$$

Here, $r(x)$ is the *worst-case* expected return earned by the weights x , given the set \mathcal{M} . Using these ideas, we can now formulate a “robust portfolio optimization” problem, which is the variant of the mean-variance problem where you protect yourself against asset average return mis-estimations. This updated problem is

as follows

$$\min \lambda x^T Q x - \rho \quad (7a)$$

Subject to:

$$\sum_j x_j = 1 \quad (7b)$$

$$Ax \geq b \quad (7c)$$

$$x \geq 0 \quad (7d)$$

$$\rho \leq \sum_j \hat{\mu}_j x_j \quad \forall \hat{\mu} \in \mathcal{M} \quad (7e)$$

In this formulation, ρ is an added variable that represents the value $r(x)$ as per equation (6). Constraint (7e) makes sure that indeed, at optimality we will have $\rho = r(x)$ (can you see why?)

There remains the task of solving the optimization problem (7). In general this is a challenging task because this is a nonstandard optimization problem – it is an optimization problem with another optimization problem as a constraint. This is called a bilevel problem.

However, there are cases where problem (7) can be reduced to a standard problem. An interesting case arises when the set \mathcal{M} is as follows. First we have nonnegative constants $\gamma_1, \gamma_2, \dots, \gamma_n$, and an additional constant Γ . Then $\hat{\mu} \in \mathcal{M}$ whenever

$$(i) \text{ First, } |\hat{\mu}_j - \mu_j| \leq \gamma_j \text{ for } 1 \leq j \leq n.$$

$$(ii) \text{ Second, } \sum_{j=1}^n |\hat{\mu}_j - \mu_j| \leq \Gamma.$$

Rule (i) states that none of the return estimates is “very” wrong – the error in asset j is at most γ_j . Rule (ii) states that the sum of all errors is also not very large.

Example. Suppose $n = 3000$, $\gamma_j = 0.001$ for all j and that $\Gamma = 0.1$. Then, for example, you could have 100 assets each exhibiting an error $|\hat{\mu}_j - \mu_j| = 0.001$ – but all other assets would have to have no errors at all, i.e. they would have to have $|\hat{\mu}_j - \mu_j| = 0$. Or, you could have 1000 assets each attaining $|\hat{\mu}_j - \mu_j| = 0.0001$. Or, you could have every asset attaining an error $|\hat{\mu}_j - \mu_j| = 0.1/3000$. Or many other combinations that satisfy (i) and (ii).

Next we show to solve problem (7) as a standard optimization problem. To this end, let us write the problem solved by the adversary, *given a vector of asset weights* x , as follows

$$r(x) = \min \sum_{j=1}^n x_j \hat{\mu}_j \quad (8a)$$

Subject to:

$$\delta_j - \hat{\mu}_j \geq -\mu_j \quad 1 \leq j \leq n \quad (8b)$$

$$\delta_j + \hat{\mu}_j \geq \mu_j \quad 1 \leq j \leq n \quad (8c)$$

$$0 \leq \delta_j \leq \gamma_j \quad 1 \leq j \leq n \quad (8d)$$

$$\sum_{j=1}^n \delta_j \leq \Gamma. \quad (8e)$$

In this formulations the only variables are the $\hat{\mu}_j$ and the δ_j – everything else is data. Note that (8b) and (8c) imply $\delta_j \geq |\hat{\mu}_j - \mu_j|$. In fact, because of (8e), *without loss of generality* at optimality we will have $|\hat{\mu}_j - \mu_j| = \delta_j$ for each j .

What we have attained so far is to represent the worst case adversarial return, $\rho(x)$, as the value of a linear program. But how do we use this fact to convert (7) into a standard problem? Answer: we use LP-duality. Denote the dual variable associated with (8b), (8c), (8d) and (8e) by p_j , q_j , $-s_j$ and $-t$, respectively.

Then, using LP duality, we can write LP (8) in the equivalent form

$$r(x) = \max \sum_{j=1}^n \{-\mu_j p_j + \mu_j q_j - \gamma_j s_j + \Gamma t\} \quad (9a)$$

Subject to:

$$p_j + q_j - s_j - t \leq 0, \quad 1 \leq j \leq n \quad (9b)$$

$$-p_j + q_j \leq x_j, \quad 1 \leq j \leq n \quad (9c)$$

$$p, q, s \geq 0, \quad t \geq 0. \quad (9d)$$

So now we can rewrite the robust problem (7) in the following form:

$$\min \lambda x^T Q x - \rho \quad (10a)$$

Subject to:

$$\sum_j x_j = 1 \quad (10b)$$

$$Ax \geq b \quad (10c)$$

$$x \geq 0 \quad (10d)$$

$$\rho \leq \sum_{j=1}^n \{-\mu_j p_j + \mu_j q_j - \gamma_j s_j + \Gamma t\} \quad (10e)$$

$$p_j + q_j - s_j - t \leq 0, \quad 1 \leq j \leq n \quad (10f)$$

$$-p_j + q_j \leq x_j, \quad 1 \leq j \leq n \quad (10g)$$

$$p, q, s \geq 0, \quad t \geq 0. \quad (10h)$$

In this problem the variables are the x_j , ρ , as well as the p_j, q_j, s_j and t . Do you see why this formulation works?