IEOR 4739
Trade impact models

Suppose we have N shares to sell over T time periods. Here we describe a set of models that address *price impact*, i.e. the adverse impact on price that the announcemnt of the sale will have. Corresponding to any time t we will denote the price of the asset at the start of time t by p_t, with p_1 known (the initial price).

The models described here have several attributes:

1. For every integer $0 \le k \le N$ we have a value $0 \le d(k) \le 1$ with the following property: if at the start of time t we **announce** that we want to sell k shares, then the asset price will be reduced by a factor of $d(k)$. That is to say, we will have

$$p_{t+1} \;=\; d(k)p_t. \tag{1}$$

   As an example, suppose that $\alpha > 0$ is a sufficiently small value. Then we could have $d(k) = 1 - \alpha \log k$. A more drastic example is $d(k) = (k+1)^{-1/2}$.

2. The second ingredient is that, having announced that we want to sell k shares, the shares that we do sell at time t will be sold at price p_{t+1}, i.e. at price $d(k)p_t$ rather than at p_t.

3. The final ingredient concerns the *number* of shares we manage to sell at time t. We want to model a lack of liquidity and thus a *stochastic* number of shares that actually get sold. For each announced number of shares that we wish to sell, k, and for each $0 \le k' \le k$, we assume that we know (or have a good estimate for)

$$\text{Prob}(k'|k)$$

   which is the probability that k' shares will get sold, given that we announced that k was the intended number to sell.

   Note that the total number of probabilities we need to know specify $(N+1)N/2$. In general we may wish to assume a specific functional form. For example, we may wish to assume that

$$\text{Prob}(k'|k) \;=\; \frac{\beta}{(k-k'+1)^2},$$

   with β set so that $\sum_{k'} \text{Prob}(k'|k) = 1$. Under this specific model we are more likely to actually sell k shares than any other fixed number of shares, but the probability that we do not sell k shares is large.

4. The combination of the above two items yields that if at time t we announce that we have k shares to sell, then the revenue we obtain is

$$p_t d(k) k', \quad \text{with probability} \quad P(k'|k).$$

5. Our goal is to choose what action to take in every time period, and in any state we find ourselves in (i.e. given any quantity of outstanding shares) so as to **maximize** the expected revenue we get.

Let us introduce some notation. Suppose that at time $t \le T$ we have $n \le N$ shares left to sell. We will call this situation a *state*, and represent it by the pair (t,n). Let us denote by $F(t,n)$ the maximum expected revenue we can get, between time periods t and T, if we start in this state at time t, *assuming* that $p_t = 1$. This assumption is justified because equation (1) adjusts prices multiplicatively (so if p_t were any price other than unity then to get the correct revenue we would just multiply $F(t,n)$ by the actual p_t). Then our main problem is to compute $F(1,N)$. This can be done using a simple dynamic programming algorithm.

Suppose we find ourselves in state (T,n). If we choose to announce that we k want to sell k shares, then our expected revenue equals

$$\sum_{k'=0}^{k} d(k)\,\mathrm{Prob}(k'|k)\,k'.$$

Therefore,

$$F(T,n) \quad = \quad \max_{0\le k\le n}\left\{\sum_{k'=0}^{k} d(k)\,\mathrm{Prob}(k'|k)\,k'\right\}. \tag{2}$$

Next, suppose we are in state (t,n) with $t<T$. Again suppose announce that we k want to sell k shares. Given this choice our maximum expected revenue equals

$$\sum_{k'=0}^{k} d(k)\,\mathrm{Prob}(k'|k)\left[k' + F(t+1,n-k')\right].$$

The justification for this formula is that if we announce a sale of k units, then first of all the asset price will be reduced to $d(k)$ (from unity). Further, with probability $\mathrm{Prob}(k'|k)$ we only sell k' shares, giving us a revenue of $d(k)k'$ now; we then move to state $(t+1,n-k')$ and the best we can do from that state onwards is to get revenue $d(k)F(t+1,n-k')$. Hence,

$$F(t,n) \quad = \quad \max_{0\le k\le n}\left\{\sum_{k'=0}^{k} d(k)\,\mathrm{Prob}(k'|k)\left[k' + F(t+1,n-k')\right]\right\}. \tag{3}$$

Applying formula (2) for all $0\le n\le N$ and then applying (3) for $t=T-1,T-2,\ldots,1$ and for each t, for all $0\le n\le N$ we will complete the desired task.

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