The Levy-area process A of a d-dimensional Brownian motion W is given by $A_t^{(i,j)} := \frac{1}{2}(\int_0^t W_u^{(i)} dW_u^{(j)} - \int_0^t W_u^{(j)} dW_u^{(i)})$. It is the unique process (in law) which obeys the following version of Chen's relation:

$$A_{0,t}^{(i,j)} = A_{0,s}^{(i,j)} + A_{s,t}^{(i,j)} + \frac{1}{2}\left(W_{0,s}^{(i)} W_{s,t}^{(j)} - W_{0,s}^{(j)} W_{s,t}^{(i)}\right)$$

for $0 \leq s \leq t$.

1 Brownian bridge flipping

bsz

$$A_{s,t} = H_{s,t} \otimes W_{s,t} - W_{s,t} \otimes H_{s,t} + b_{s,t}$$

$$A_{s,t}^{(i,j)} := \frac{1}{2}\left(\int_s^t W_{s,u}^{(i)} dW_u^{(j)} - \int_s^t W_{s,u}^{(j)} dW_u^{(i)}\right)$$

$$\widetilde{A}_{s,t}^{(i,j)} = H_{s,t}^{(i)} W_{s,t}^{(j)} - W_{s,t}^{(i)} H_{s,t}^{(j)} + \widetilde{b}_{s,t}^{(i,j)} \quad \text{where } \widetilde{b}_{s,t}^{(i,j)} \sim \mathcal{N}\left(0, \frac{1}{12}(t-s)^2\right)$$

$$b_{s,t}^{(i,j)} := \int_s^t B_{s,u}^{(i)} \circ dB_u^{(j)}$$

$$\widetilde{A}_{s,t}^{(i,j)} = c_i H_{s,t}^{(i)} W_{s,t}^{(j)} - c_j W_{s,t}^{(i)} H_{s,t}^{(j)} + c_i c_j \widetilde{b}_{s,t}^{(i,j)} \quad \text{for } \mathbf{c} \in \{-1, 1\}^d$$

$$\mathbb{E}[\widetilde{A}] = \mathbb{E}[A] = 0$$

$$\widetilde{H} \sim \mathcal{N}^d\left(0, \tfrac{1}{12}\right)$$

$$s = 2^d, \quad a = \frac{d(d-1)}{2}, \quad h = d, \quad W \leftarrow (b \times d), \quad H \leftarrow (b \times h)$$

$$T = \left(t_{i,j,k}\right)_{\substack{1\leq i\leq d\\ 1\leq j\leq h\\ 1\leq k\leq a}} \quad \text{where } t_{i,j,k} = \begin{cases} 1 & \text{if } j < i \text{ and } k = \mathrm{a}_{\text{index}}(i,j) \\ -1 & \text{if } i < j \text{ and } k = \mathrm{a}_{\text{index}}(j,i) \\ 0 & \text{otherwise} \end{cases}$$

$$M = \left(m_{l,k}\right)_{\substack{1\leq l\leq s\\ 1\leq k\leq a}} \quad \text{where } m_{l,k} = S_{l,i}S_{l,j} \text{ if } k = \mathrm{a}_{\text{index}}(i,j)$$

$$S = (x \in \{-1,1\}^{h,\text{ row}}) \in \mathcal{M}^{s\times h}$$

```
W.shape = (bsz,w); T.shape = (w,h,a)
WT= tensordot(W,T,dims=1), WT.shape = (bsz,h,a)
S.shape = (1,s,h); H.shape = (bsz,1,h)
SH = mul(S,H), SH.shape = (bsz,s,h)
WTH = matmul(SH, WT), WTH.shape = (bsz,s,a)
M.shape = (s,1,a); B.shape = (1,bsz,a)
MB = mul(M,B), MB.shape = (s,bsz,a)
WTHMB = flatten(WTH) + flatten(MB.permute(1,0,2))
WTHMB.shape = (s*bsz,a)
```

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

$$\widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + \mu(\widehat{X}_{t_k}, t_k)\ \Delta t_k + \sigma(\widehat{X}_{t_k}, t_k)\Delta W_k + 1$$

$$\mathbb{E}X = \mathbb{E}Y = 0,\ \ \mathbb{E}X^2 = \mathbb{E}Y^2 \text{ and } \mathbb{E}\left[XY\right] \implies \left\|X+Y\right\|_{L^2} = \sqrt{2}\left\|X\right\|_{L^2}$$

$$\theta^{(i,j)} \coloneqq \int_s^t W_{s,u}^{(i)} dW_u^{(j)} = \int_s^t \int_s^u dW_v^{(i)} dW_u^{(j)}$$

$$A_{s,t}^{(i,j)} = \frac{1}{2}\left(\int_s^t \int_s^u dW_v^{(i)} dW_u^{(j)} - \int_s^t \int_s^u dW_v^{(j)} dW_u^{(i)}\right) = \frac{1}{2}\left(\theta^{(i,j)} - \theta^{(j,i)}\right)$$

$$\left\|A_{s,t}^{(i,j)} - F(L_1(\mathbf{W}, \ldots, L_n(\mathbf{W}))\right\|_{L^2(\mathbb{P})} \geq \frac{1}{\pi\sqrt{n}} \qquad \left\|A_{s,t}^{(i,j)} - \mathbb{E}[A_{s,t}^{(i,j)} \mid \mathcal{G}_n]\right\|_{L^2(\mathbb{P})} \geq \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{n}}$$

$$\left\|A_{s,t}^{(i,j)} - \mathbb{E}[A_{s,t}^{(i,j)} \mid \mathcal{G}_n]\right\|_{L^2(\mathbb{P})} \geq \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{n}}$$

$$\mathcal{I}(h) = \frac{1}{2}(W_h W_h^\top - hI_d) + A(h) = (\theta_h^{(i,j)})_{1\leq i,j\leq d}$$

$$d \quad a = d(d-1)/2$$

$$A_{0,t} \stackrel{d}{=} tA_{0,1} \quad W_{0,t} \stackrel{d}{=} \sqrt{t}W_{0,1} \quad \times\frac{1}{\sqrt{2}} \quad \times\frac{1}{2} \quad W_{0,\frac{1}{2}} \quad A_{0,\frac{1}{2}} \quad W_{\frac{1}{2},1} \quad A_{\frac{1}{2},1}$$

$$W_{0,1} = W_{0,\frac{1}{2}} + W_{\frac{1}{2},1}$$

$$d \to 2d \quad \implies \quad \epsilon \to \frac{\epsilon}{\sqrt{2}}$$

$$C_{\text{GAN,time}} = \epsilon_{\text{GAN}}\sqrt{t_{\text{GAN}}} \quad C_{\text{julia,time}} = \epsilon_{\text{julia}}\sqrt{t_{\text{julia}}} \quad C_{\text{GAN,noise}} = \epsilon_{\text{GAN}}\sqrt{n} \ \text{ where } n \coloneqq [\text{noise size}]$$

$$C_{\text{GAN,time}} = \epsilon_{\text{GAN}}\sqrt{t_{\text{GAN}}} \tag{1}$$
$$C_{\text{julia,time}} = \epsilon_{\text{julia}}\sqrt{t_{\text{julia}}} \tag{2}$$
$$C_{\text{GAN,noise}} = \epsilon_{\text{GAN}}\sqrt{n} \qquad \text{where } n \coloneqq [\text{noise size}] \tag{3}$$