The Levy-area process A of a d-dimensional Brownian motion W is given by $A_t^{(i,j)} := \frac{1}{2}(\int_0^t W_u^{(i)} dW_u^{(j)} - \int_0^t W_u^{(j)} dW_u^{(i)})$. It is the unique process (in law) which obeys the following version of Chen's relation:

$$A_{0,t}^{(i,j)} = A_{0,s}^{(i,j)} + A_{s,t}^{(i,j)} + \frac{1}{2}\left(W_{0,s}^{(i)} W_{s,t}^{(j)} - W_{0,s}^{(j)} W_{s,t}^{(i)}\right)$$

for $0 \leq s \leq t$.

1 Brownian bridge flipping

bsz

$$A_{s,t} = H_{s,t} \otimes W_{s,t} - W_{s,t} \otimes H_{s,t} + b_{s,t}$$

$$H_{s,t} \sim \mathcal{N}^d\left(0, \frac{1}{12}(t-s)\right), \quad H_{s,t} \perp W_{s,t}$$

$$A_{s,t}^{(i,j)} := \frac{1}{2}\left(\int_s^t W_{s,u}^{(i)} dW_u^{(j)} - \int_s^t W_{s,u}^{(j)} dW_u^{(i)}\right)$$

$$\widetilde{A}_{s,t}^{(i,j)} = H_{s,t}^{(i)} W_{s,t}^{(j)} - W_{s,t}^{(i)} H_{s,t}^{(j)} + \widetilde{b}_{s,t}^{(i,j)} \quad \text{where } \widetilde{b}_{s,t}^{(i,j)} \sim \mathcal{N}\left(0, \frac{1}{12}(t-s)^2\right)$$

$$b_{s,t}^{(i,j)} := \int_s^t B_{s,u}^{(i)} \circ dB_u^{(j)}$$

$$\widetilde{A}_{s,t}^{(i,j)} = c_i H_{s,t}^{(i)} W_{s,t}^{(j)} - c_j W_{s,t}^{(i)} H_{s,t}^{(j)} + c_i c_j \widetilde{b}_{s,t}^{(i,j)} \quad \text{for } c_i \sim \mathrm{Rad}(1/2) \text{ i.i.d.}$$

$$\mathbb{E}\left[\widetilde{A}_{s,t} \mid W_{s,t}\right] = \mathbb{E}\left[A_{s,t} \mid W_{s,t}\right] = 0$$

$$\xi_k = \widetilde{Y}_k - Y_k, \quad A_{t_k, t_{k+1}} \perp \mathcal{F}_{t_k}$$

$$\widetilde{H} \sim \mathcal{N}^d\left(0, \tfrac{1}{12}\right)$$

$$s = 2^d, \quad a = \frac{d(d-1)}{2}, \quad h = d, \quad W \leftarrow (b \times d), \quad H \leftarrow (b \times h)$$

$$T = \left(t_{i,j,k}\right)_{\substack{1 \le i \le d \\ 1 \le j \le h \\ 1 \le k \le a}} \quad \text{where } t_{i,j,k} = \begin{cases} 1 & \text{if } j < i \text{ and } k = \mathrm{a}_{\text{index}}(i,j) \\ -1 & \text{if } i < j \text{ and } k = \mathrm{a}_{\text{index}}(j,i) \\ 0 & \text{otherwise} \end{cases}$$

$$M = \left(m_{l,k}\right)_{\substack{1 \le l \le s \\ 1 \le k \le a}} \quad \text{where } m_{l,k} = S_{l,i} S_{l,j} \text{ if } k = \mathrm{a}_{\text{index}}(i,j)$$

$$S = (x \in \{-1, 1\}^{h, \text{ row}}) \in \mathcal{M}^{s \times h}$$

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W.shape = (bsz,w); T.shape = (w,h,a)

WT= tensordot(W,T,dims=1), WT.shape = (bsz,h,a)

S.shape = (1,s,h); H.shape = (bsz,1,h)

SH = mul(S,H), SH.shape = (bsz,s,h)

WTH = matmul(SH, WT), WTH.shape = (bsz,s,a)

M.shape = (s,1,a); B.shape = (1,bsz,a)

MB = mul(M,B), MB.shape = (s,bsz,a)

WTHMB = flatten(WTH) + flatten(MB.permute(1,0,2))

WTHMB.shape = (s*bsz,a)
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$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

$$\widehat{X}_{t_{k+1}} = \widehat{X}_{t_k} + \mu(\widehat{X}_{t_k}, t_k)\ \Delta t_k + \sigma(\widehat{X}_{t_k}, t_k)\Delta W_k + 1$$

$$\mathbb{E}X_i = 0,\ \ \mathbb{E}X_i^2 = \ \text{ and } \mathbb{E}\left[XY\right] = 0 \implies \ \|X+Y\|_{L^2} = \sqrt{2}\,\|X\|_{L^2}$$

$$\theta^{(i,j)} := \int_s^t W_{s,u}^{(i)} dW_u^{(j)} = \int_s^t \int_s^u dW_v^{(i)} dW_u^{(j)}$$

$$A_{s,t}^{(i,j)} = \frac{1}{2}\left(\int_s^t \int_s^u dW_v^{(i)} dW_u^{(j)} - \int_s^t \int_s^u dW_v^{(j)} dW_u^{(i)}\right) = \frac{1}{2}\left(\theta^{(i,j)} - \theta^{(j,i)}\right)$$

$$\left\|\theta - F(L_1(\mathbf{W}), \ldots, L_n(\mathbf{W}))\right\|_{L^2(\mathbb{P})} \geq \frac{C}{\sqrt{n}} \approx \frac{\sqrt{2}}{\pi\sqrt{n}} \qquad \left\|A_{s,t}^{(i,j)} - \mathbb{E}[A_{s,t}^{(i,j)} \mid \mathcal{G}_n]\right\|_{L^2(\mathbb{P})} \geq \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{n}}$$

$$\left\|A_{s,t}^{(i,j)} - \mathbb{E}[A_{s,t}^{(i,j)} \mid \mathcal{G}_n]\right\|_{L^2(\mathbb{P})} \geq \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{n}}$$

$$\mathcal{I}(h) = \frac{1}{2}(W_h W_h^\top - hI_d) + A(h) = (\theta_h^{(i,j)})_{1\leq i,j\leq d}$$

$$d \quad a = d(d-1)/2 \qquad n = 2dp,\ h = 1 \qquad \ldots = \frac{\sqrt{d}}{\pi\sqrt{n}}$$

$$A_{0,t} \overset{d}{=} tA_{0,1} \quad W_{0,t} \overset{d}{=} \sqrt{t}W_{0,1} \quad \times\frac{1}{\sqrt{2}} \quad \times\frac{1}{2} \quad W_{0,\frac{1}{2}} \quad A_{0,\frac{1}{2}} \quad W_{\frac{1}{2},1} \quad A_{\frac{1}{2},1}$$

$$W_{0,1} = W_{0,\frac{1}{2}} + W_{\frac{1}{2},1}$$

$$d \to 2d \quad \implies \quad \epsilon \to \frac{\epsilon}{\sqrt{2}}$$

$$C_{\text{GAN,time}} = \epsilon_{\text{GAN}}\sqrt{t_{\text{GAN}}} \quad C_{\text{julia,time}} = \epsilon_{\text{julia}}\sqrt{t_{\text{julia}}} \quad C_{\text{GAN,noise}} = \epsilon_{\text{GAN}}\sqrt{n} \ \text{ where } n := [\text{noise size}]$$

$$C_{\text{time}} \coloneqq \epsilon\sqrt{t}, \quad C_{\text{true}} \coloneqq \epsilon\sqrt{n} \quad \epsilon_{\text{joint}} \quad \epsilon_{\text{coord-wise}}$$

$$C_{\text{julia,true}} = \epsilon_{\text{julia}}\sqrt{n} \tag{1}$$
$$C_{\text{GAN,noise}} = \epsilon_{\text{GAN}}\sqrt{n} \tag{2}$$
$$C_{\text{julia,time}} = \epsilon_{\text{julia}}\sqrt{t_{\text{julia}}} \tag{3}$$
$$C_{\text{GAN,time}} = \epsilon_{\text{GAN}}\sqrt{t_{\text{GAN}}} \tag{4}$$
$$C_{\text{4mom,time}} = \epsilon_{\text{4mom}}\sqrt{t_{\text{4mom}}} \tag{5}$$
$$C_{\text{F\&L,time}} = \epsilon_{\text{F\&L}}\sqrt{t_{\text{F\&L}}} \tag{6}$$
$$\tag{7}$$

$$C_{\text{julia,true}} \approx \frac{\sqrt{2}}{\pi} \approx 0.45 \tag{8}$$
$$C_{\text{GAN,noise}} = 0.36 \tag{9}$$
$$C_{\text{julia,time}} = 1.38 \times 10^{-4} \tag{10}$$
$$C_{\text{GAN,time}} = 1.15 \times 10^{-4} \tag{11}$$
$$C_{\text{4mom,time}} = 6.6 \times 10^{-5} \tag{12}$$
$$C_{\text{F\&L,time}} = 5.5 \times 10^{-5} \tag{13}$$

$$p_1 = 1.5 \tag{14}$$
$$p_2 = 1 \tag{15}$$
$$p = 0.5 \tag{16}$$

$$\text{where } n \coloneqq [\text{noise size}]$$

$$d_{\text{1-Wass}}(\boldsymbol{x}, \widetilde{\boldsymbol{x}}) \approx$$