EECE.5200 - Homework 5

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Accessing Source Code

Source code is available at: https://github.com/tjkessler/eece5200/tree/main/hw5

To run the code, first run ***make*** in the *hw5* directory. This will create one output file, **q1.o**. This executable will output all information relevant to the assignment, and export data to two files for plotting with **plot_results.py**.

Question 1

Given the function e^z, determine the $N+1$ term Chebyshev expansion valid for $0 \leq z \leq 1$:

$$e^z = \sum_{n=0}^{N} a_n T_n(x)$$

Assume a map between z and x is given algebraically as $x = 2z - 1 \implies z = (x+1)/2 \implies$

$$e^{(x+1)/2} = \sum_{n=0}^{N} a_n T_n(x)$$

where $-1 \leq x \leq 1$.

a.

Given that $N = 8$, the coefficients $\underline{a} = [a_0, ..., a_N]$ are found by minimizing the residual at the extreme points of T_N, defined as $x_i = cos(\theta_i)$ where $\theta_i = i\pi/N$ for $i = (0, N)$. The system of variables can be re-written as:

$$\underline{f} = [T]\underline{a}$$

where $\underline{f}$ is a vector of solutions for $e^{(x_i+1)/2}$, $\underline{a}$ are unknown coefficients, and $[T]$ is a matrix representation of first-kind Chebyshev polynomials. Upon calling *gess*, the coefficients $\underline{a}$ were found to be:

```
 Coefficients a:
a          0  =    1.75338769
a          1  =   0.850391567
a          2  =   0.105208702
a          3  =    8.72212369E-03
a          4  =    5.43408096E-04
a          5  =    2.71295994E-05
a          6  =    1.12497901E-06
a          7  =    3.78512972E-08
a          8  =   -1.62590919E-08
```

it is observed that the magnitudes of the first three coefficients are considerably larger than the remaining coefficients. This likely indicates that a lower-order Chebyshev polynomial system, perhaps $N = 3$, can be used to adequately represent the function e^z.

b.

The error between approximate and exact results calculated using M uniformly distributed values of z where $0 \leq z \leq 1$ and $i = (1, M)$ is determined using the mean absolute error, defined as:

$$MAE = \frac{\sum_{i=1}^{M} |y_i - \hat{y}_i|}{M}$$

where y_i is an exact solution for e^{z_i} and $\hat{y}_i$ is an approximate solution given known coefficients $\underline{a}$. The approximate solution is computed using:

$$\hat{y}_i = \sum_{j=0}^{N} \underline{a}_j T_j$$

where T is a vector length N with initial conditions $T_0 = 1.0$ and $T_1 = 2z_i - 1$. The remaining $j = (2, N)$ values are calculated with:

$$T_j = 2T_{j-1}(2z_i - 1) - T_{j-2}$$

The following output was achieved when determining the error of the approximate solution with $M = 16$, and the parity between values is displayed in Figure 1:

```
Interval=16; actual e^z compared to Cheb. derived:
Actual  Chebyshev-Derived
1.00000000        1.00000012
1.06449449        1.06449461
1.13314843        1.13314867
1.20623028        1.20623052
1.28402543        1.28402543
1.36683798        1.36683798
1.45499146        1.45499146
1.54883027        1.54883039
1.64872122        1.64872122
1.75505471        1.75505459
1.86824596        1.86824596
1.98873746        1.98873734
2.11700010        2.11699986
2.25353479        2.25353503
2.39887524        2.39887547
2.55358934        2.55358958
2.71828175        2.71828175
Given 16 samples of z, mean absolute error =     2.02655792E-06
```

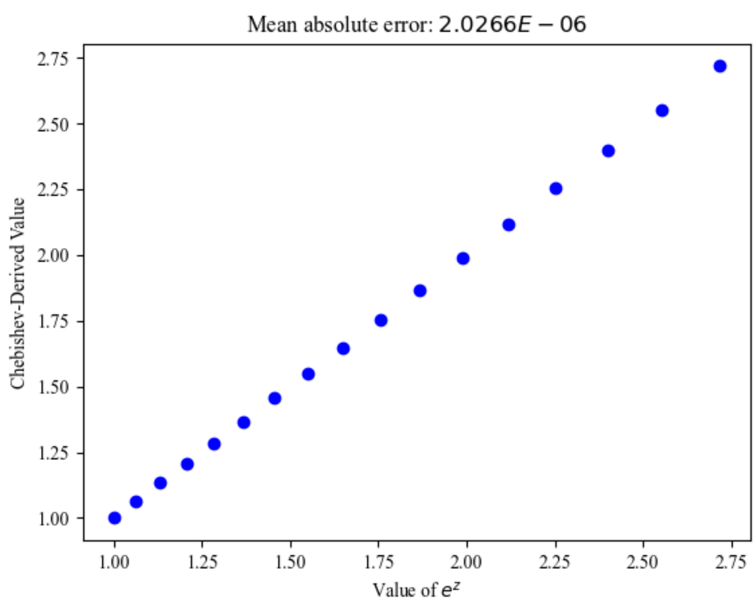

Figure 1: Parity between approximate results and exact results

c.

In a discrete system, we define the derivative as the forward difference operator, Δ, applied to two consecutive points in a uniformly distributed series of length N:

$$\Delta f = n \to f_{i+1} - f_i$$

In this application, the difference operator is applied to both exact and approximate data from subsection (b). Upon executing the previously mentioned error calculation procedure, the following output was achieved when determining the error of the approximate solution's derivative, and the parity between exact and approximate derivatives is displayed in Figure 2:

```
dz=16; deriv. of actual compared to Cheb. derived:
Actual Chebyshev-Derived
6.44944906E-02    6.44944906E-02
6.86539412E-02    6.86540604E-02
7.30818510E-02    7.30818510E-02
7.77951479E-02    7.77949095E-02
8.28125477E-02    8.28125477E-02
8.81534815E-02    8.81534815E-02
9.38388109E-02    9.38389301E-02
9.98909473E-02    9.98908281E-02
0.106333494       0.106333375
0.113191247       0.113191366
0.120491505       0.120491385
0.128262639       0.128262520
0.136534691       0.136535168
0.145340443       0.145340443
0.154714108       0.154714108
0.164692402       0.164692163
Deriv. mean absolute error:    1.11758709E-07
```

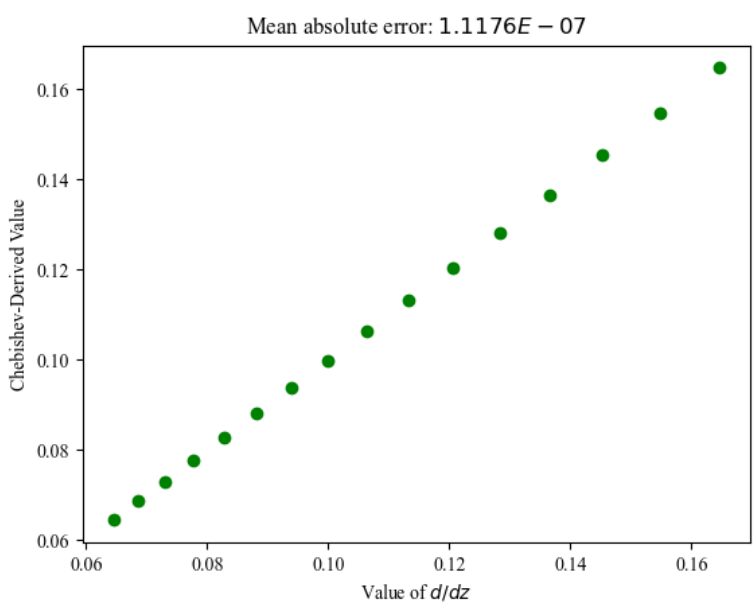

Figure 2: Parity between approximate derivative and exact derivative

Bibliography

[1] Thompson, C. *University of Massachusetts Lowell Department of Electrical and Computer Engineering 16.520 Computer Aided Engineering Analysis Problem Set 6*. Retrieved March 30, 2021, from http://morse.uml.edu/Activities.d/16.520/S2021.d/HW5.pdf