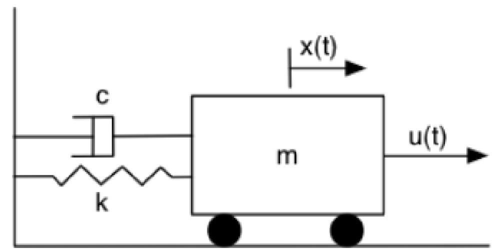
Ordinary Differential Equations: Mass-Spring-Damper

Consider the mass-spring-damper system described by the following differential equation and initial conditions:



$$m\ddot{x} + c\dot{x} + kx = u(t)$$

$$x = x_0, \dot{x} = \dot{x}_0 \text{ when } t = 0$$

a) Sketch the cause/effect diagram for this system.
b) Write a function to numerically solve this ODE. Use `ode45()` and a time vector with a step size of 0.01 s. Change the relative and absolute tolerances to `1e-6` each.
c) Consider the *unforced case* where $u(t) = 0\ N$. Use your function to solve the ODE given the following parameters: $x_0 = 1\ m$, $\dot{x}_0 = 0\frac{m}{s}$, $m = 20\ kg$, $k = 20\frac{N}{m}$, and c as described in the table below. Put all three curves for the different c values on a single figure, with x and $\dot{x}$ on separate subplots. See Figure 9.7 in the Vick text for an example.

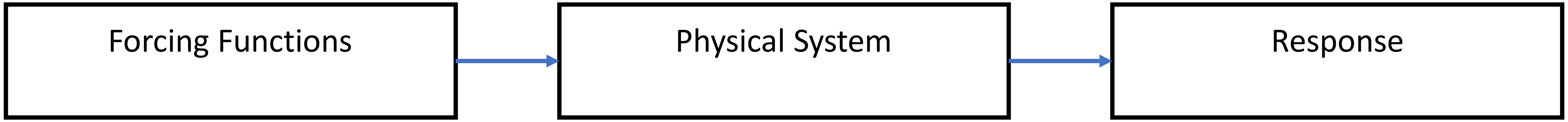
Type of Damping	$\boldsymbol{c}\ \left(\frac{N\cdot s}{m}\right)$
Case 1 (Undamped)	0
Case 2 (Underdamped)	8
Case 3 (Critically Damped)	$40\ (= 2\sqrt{mk})$

d) Now consider the *forced case* where $u(t) \neq 0\ N$. Use your function to solve the ODE given the following parameters: $x_0 = 1\ m$, $\dot{x}_0 = 0\frac{m}{s}$, $m = 20\ kg$, $k = 20\frac{N}{m}$, $c = 4\ \frac{N\cdot s}{m}$, and $u(t)$ as described in the table below. Put each result on a separate figure (3 figures total for this part), with $u(t)$, x, and $\dot{x}$ on separate subplots.

Type of Forcing	**Forcing Function $\boldsymbol{u(t)}$ $(\boldsymbol{N})$**
Constant Forcing	$u(t) = u_0$ $u_0 = 10\ N$
Harmonic Forcing	$u(t) = u_0 \sin(\omega t)$ $u_0 = 10\ N; \omega = 1\frac{rad}{s}$
Pulsed Forcing*	$u(t) = u_0\big(H(t) - H(t - t_{on})\big)$ $u_0 = 10\ N, t_{on} = 20\ s$

*Recall $H(t)$ is the Heaviside Function/Unit Step Function.

Mass-Spring Damper Equations



Mass-Spring Damper Equations

$$m\ddot{x} + c\dot{x} + kx = u$$

Introduce two new variables:

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cdot \end{bmatrix}$$

Differentiate z, substitute into original ODE, and express in terms of z:

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \dot{} \\ \ddot{} \end{bmatrix}$$

$$\rightarrow \dot{z} = \begin{bmatrix} \quad \end{bmatrix} = \boxed{\begin{bmatrix} \qquad\qquad \\ \\ \end{bmatrix}}$$

Mass-Spring Damper Equations

Forcing Functions
$u(t), x_0, \dot{x}_0$

$\rightarrow$

Physical System
m, c, k

$\rightarrow$

Response
$x(t), \dot{x}(t)$

Mass-Spring Damper Equations

$$m\ddot{x} + c\dot{x} + kx = u$$

Introduce two new variables:

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Differentiate z, substitute into original ODE, and express in terms of z:

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$$

$$\rightarrow \dot{z} = \begin{bmatrix} z_2 \\ \dfrac{u - c\dot{x} - kx}{m} \end{bmatrix} = \boxed{\begin{bmatrix} z_2 \\ \dfrac{u - cz_2 - kz_1}{m} \end{bmatrix}}$$

Mass-Spring Damper Equations

$$\dot{z} = \begin{bmatrix} z_2 \\ \dfrac{u - c\dot{x} - kx}{m} \end{bmatrix} = \boxed{\begin{bmatrix} z_2 \\ \dfrac{u - cz_2 - kz_1}{m} \end{bmatrix}}$$

Alternatively:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\dfrac{1}{m}\right) \end{bmatrix} u$$

$$\dot{z} = Az + U$$