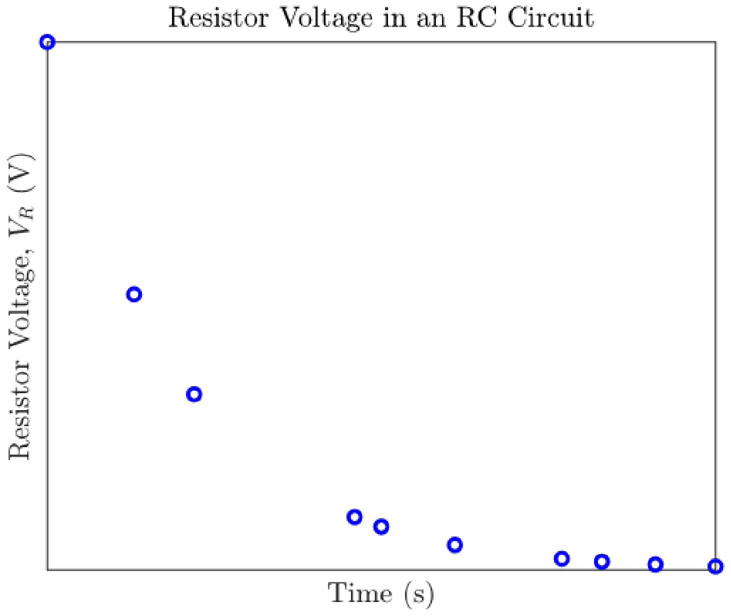
Linearizing Nonlinear Equations for Least-Squares Regression

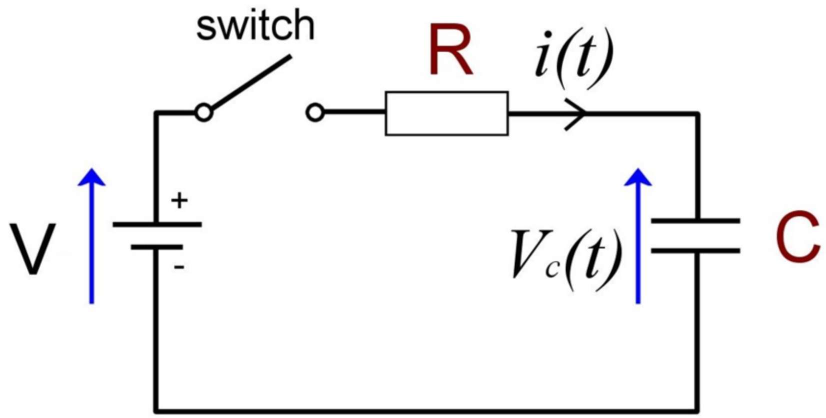
ME 2004

Outline

Linearizing Nonlinear Equations

- There are many nonlinear functions in engineering, so we cannot directly apply a linear curve fit





- How could we find C given experimental V_R vs. t data, V, and R?

Linearizing Nonlinear Equations

- Alternative approach: linearize the equation, then apply least-squares regression
- Goal: obtain a function in the form $Y = a_1 X + a_0$

$$V_R = Ve^{-\left(\frac{t}{RC}\right)} \rightarrow \ln(V_R) = \ln(V) + \left(-\frac{t}{RC}\right)$$

$$\rightarrow \boxed{\underbrace{\ln(V_R)}_{Y} = \underbrace{\left(-\frac{1}{RC}\right)}_{a_1}\underbrace{t}_{X} + \underbrace{\ln(V)}_{a_0}}$$

Linearizing Nonlinear Equations

- Once linearized, apply least-squares regression to the transformed parameters

$$\underbrace{\ln(V_R)}_{Y} = \underbrace{\left(-\frac{1}{RC}\right)}_{a_1} \underbrace{t}_{X} + \underbrace{\ln(V)}_{a_0}$$

$$y_i \rightarrow \ln\left(V_{R_i}\right); x_i \rightarrow t_i$$

$$\begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix} \rightarrow \begin{bmatrix} n & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \Sigma \ln\left(V_{R_i}\right) \\ \Sigma\left(t_i \ln\left(V_{R_i}\right)\right) \end{bmatrix}$$

Linearizing Nonlinear Equations

- Once the numerical values of a_1 and a_0 are obtained:

$$\underbrace{\ln(V_R)}_{Y} = \underbrace{\left(-\frac{1}{RC}\right)}_{a_1}\underbrace{t}_{X} + \underbrace{\ln(V)}_{a_0}$$

$$\rightarrow a_1 = -\frac{1}{RC} \rightarrow \boxed{C = -\frac{1}{Ra_1}}$$

Linearizing Nonlinear Equations

Equation	Linearized	Coefficients $(Y = a_1 X + a_0)$	Values for Least-Squares	Suggested Plot (on linear x, y axes)
$y = px^q$	$\ln(y) = q\ln(x) + \ln(p)$	$Y = \ln(y)\,; X = \ln(x)$ $a_1 = q, a_0 = \ln(p)$	$\ln(x_i)\,, \ln(y_i)$	$\ln(y)$ vs $\ln(x)$
$y = pe^{qx}$	$\ln(y) = qx + \ln(p)$	$Y = \ln(y)\,; X = x$ $a_1 = q, a_0 = \ln(p)$	$x_i, \ln(y_i)$	$\ln(y)$ vs x
$y = \dfrac{1}{px + q}$	$\dfrac{1}{y} = px + q$	$Y = \dfrac{1}{y}, X = x$ $a_1 = p, a_0 = q$	$x_i, \dfrac{1}{y_i}$	$\frac{1}{y}$ vs x
$y = \dfrac{px}{q + x}$	$\dfrac{1}{y} = \left(\dfrac{q}{p}\right)\dfrac{1}{x} + \dfrac{1}{p}$	$Y = \dfrac{1}{y}, X = \dfrac{1}{x}$ $a_1 = \dfrac{q}{p}, a_0 = \dfrac{1}{p}$	$\dfrac{1}{x_i}, \dfrac{1}{y_i}$	$\frac{1}{y}$ vs $\frac{1}{x}$

Linearizing Nonlinear Equations

- Which nonlinear function should you use? Refer to the underlying theory!
- If theory is unknown:
  - Power equation ($y = px^q$)
  - Plot!! Try various combinations of transformations

Suggested Plot (on linear x, y axes)
$\ln(y)$ vs $\ln(x)$
$\ln(y)$ vs x
$\frac{1}{y}$ vs x
$\frac{1}{y}$ vs $\frac{1}{x}$

Linearizing Nonlinear Equations

- Theoretical considerations:
  - Power function: $y = 0$ when $x = 0$
  - Exponential functions do not pass through the origin
  - Exponential functions can only be used if all y's are positive or all y's are negative
  - Reciprocal function cannot have $y = 0$

Equation
$y = px^q$
$y = pe^{qx}$
$y = \frac{1}{px + q}$
$y = \frac{px}{q + x}$

Summary

- To apply least-squares regression to a nonlinear function, it must first be linearized
- After linearization, proceed with least-squares regression using the transformed variables
- Always plot!