Root Finding: Newton-Raphson Method

ME 2004

Outline

- 1.1: Newton-Raphson Method

Newton-Raphson Method

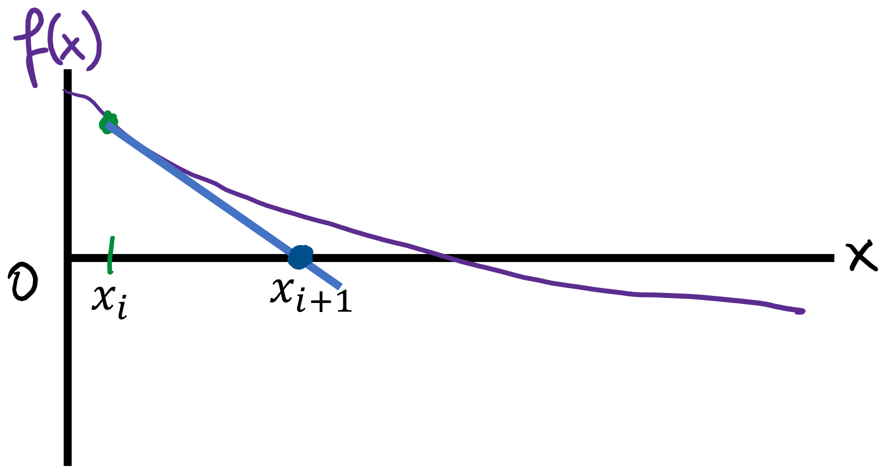
- Newton-Raphson/Newton's Method: another root finding technique
- Premise: combine an initial starting point with the derivative of the function to iteratively trace the root
  - WARNING: May diverge!!! (unable to locate root)
- Called an *open method* because the user must supply the algorithm with a singular starting point
  - Contrast: Bisection is a *bracketing method* because it needs two initial guesses which bracket the root
- Converges quadratically: (current error) $\propto$ (previous error) ^{2}
  - Contrast: Bisection Method exhibits linear convergence (slower)

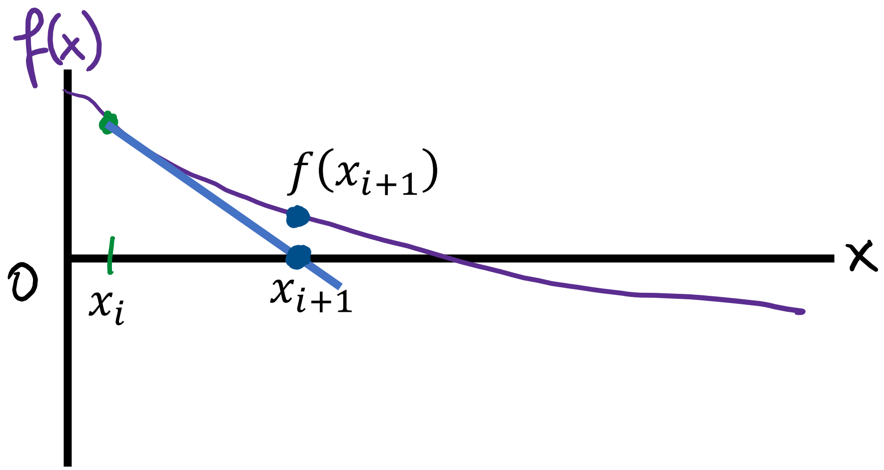
Newton-Raphson Method

- Brief algorithm overview:
  1) Input initial guess x_i and compute $f'(x_i)$
  2) From the starting point x_i, extend the tangent $f'(x_i)$ until it hits the x-axis
  3) This new location is the new estimate of the root, x_{i+1}
  4) Compute error* and iterate until the stopping criterion e_s is met

*error can be a tolerance in x: $err = 100\% * \left|\frac{x_{i+1}-x_i}{x_{i+1}}\right| \leq e_s$

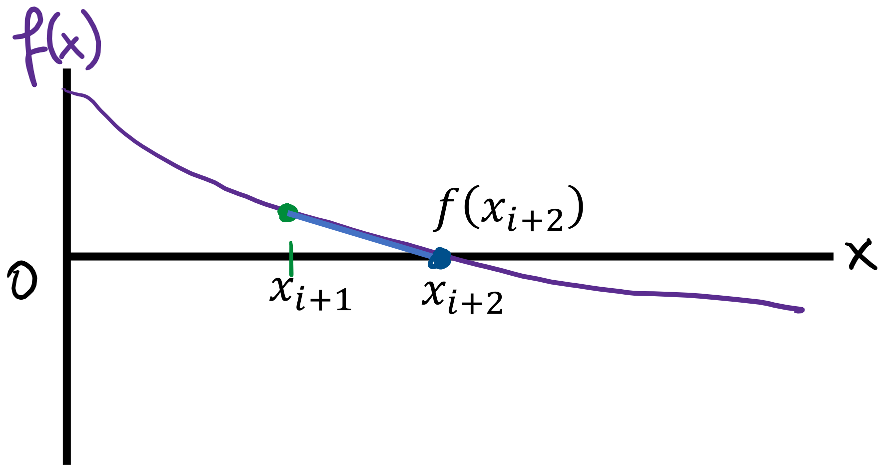
or a tolerance in y: $err = |f(x_{i+1})| \leq e_s$





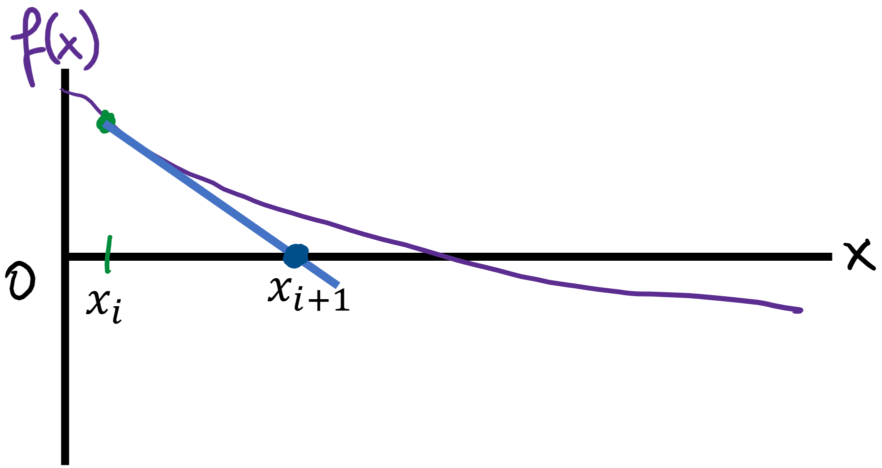
- Start with an initial guess x_i
- Draw the tangent $(f'(x))$ to the curve at that point and “follow” it until it hits the x-axis
- This point is x_{i+1} (new root estimate)

- Evaluate $|f(x_{i+1})|$. Assume it doesn’t meet our stopping criterion (e_s) → must iterate.



- We draw the tangent at x_{i+1} and extend it until it crosses the x-axis
- This point is x_{i+2} (new root estimate)
- Assume $|f(x_{i+2})| \leq e_s$ $\rightarrow$ we've located the root!

Newton-Raphson Method



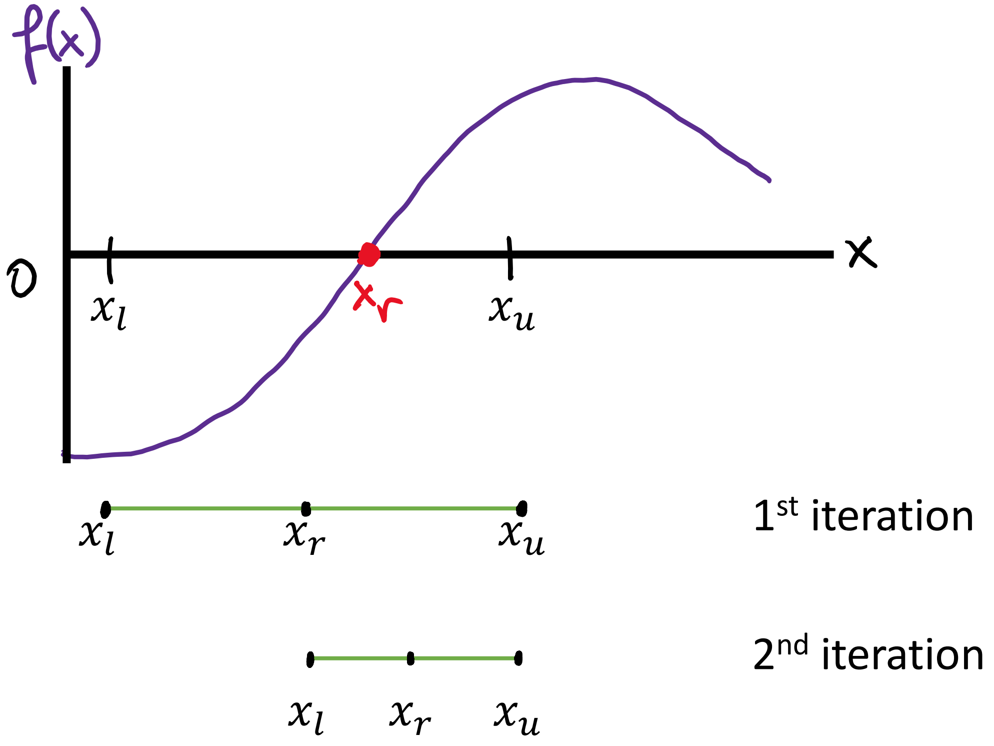
- Definition of slope:

$$f'(x) = \frac{f(x_i) - \cancel{f(x_{i+1})}^{0}}{x_i - x_{i+1}}$$

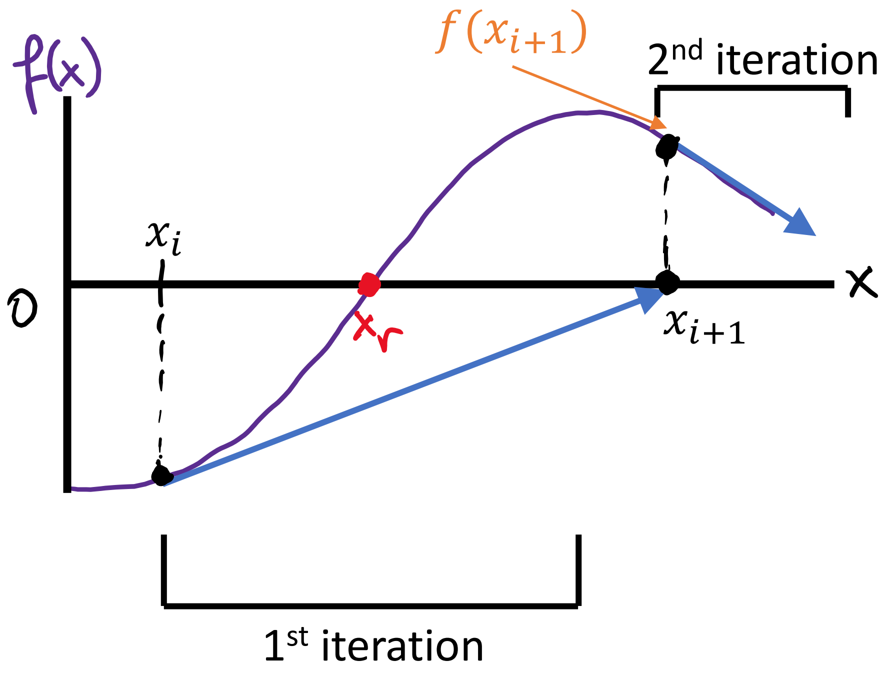
- Rearranged:

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Newton-Raphson Method



Bisection always converges (given a valid initial bracket)!

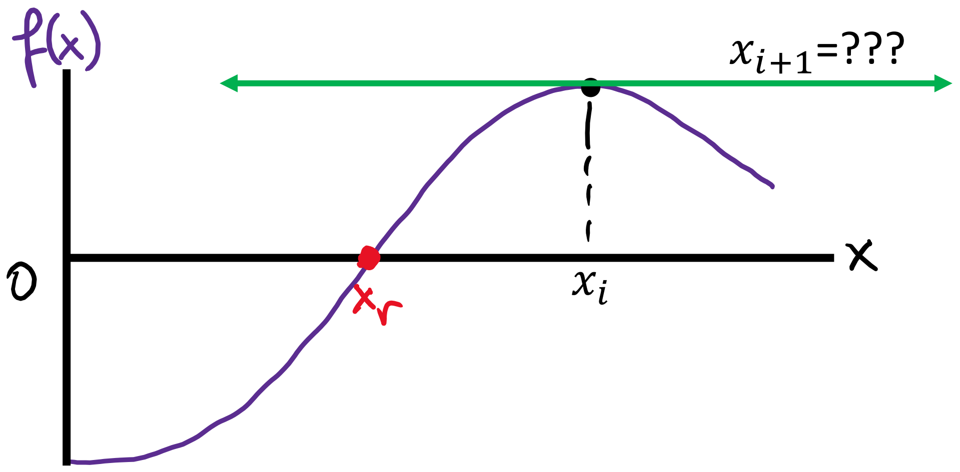


NR may diverge!

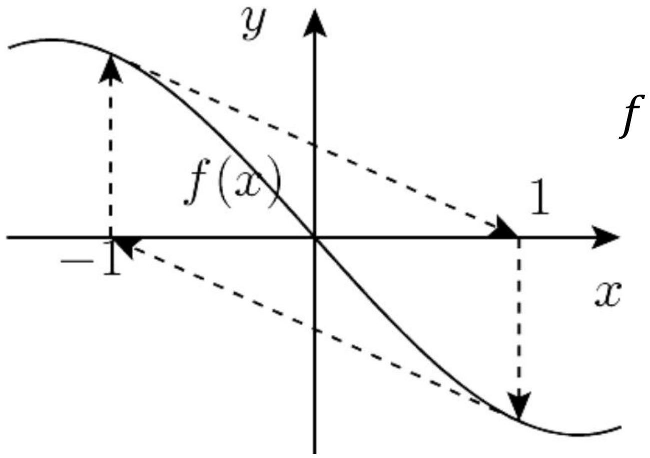
Newton-Raphson Method

- A pitfall of Newton-Raphson is the possibility of division-by-zero

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



- Or, NR could cycle around a point (occurs when an inflection point is in the vicinity of a root)



$f(x) = x^3 - 5x,$
$x_0 = 1$

Newton-Raphson Method

Consider Picking Bisection If…

- You MUST converge on x_r
- You don't care about convergence speed
- You don't know the behavior of $f'(x)$
- $f'(x)$ is hard to compute

Consider Picking NR If…

- You accept the possibility of divergence
- You want to quickly find x_r
- You know $f'(x)$ won't return a division-by-0 error
- $f'(x)$ is relatively easy to compute

Summary

- Newton-Raphson only requires 1 initial guess, but it requires you to compute $f'(x)$
- Newton-Raphson is not guaranteed to converge, but *generally* converges faster than Bisection (quadratic vs. linear convergence)
- Newton-Raphson is prone to division-by-zero errors and may not be suitable if $f'(x)$ is difficult to compute