Numerical Errors

Part 2: Truncation Error and Total Numerical Error

Outline (Part 1, Part 2)

- 1.1: Definitions
- 1.2: Roundoff Errors

- 2.1: Truncation Errors
- 2.2: Total Numerical Error

This video

2.1: Truncation Errors

Truncation Errors

- Truncation Error: results from using an approximation in place of an exact mathematical procedure
- *Example*: Taylor Series

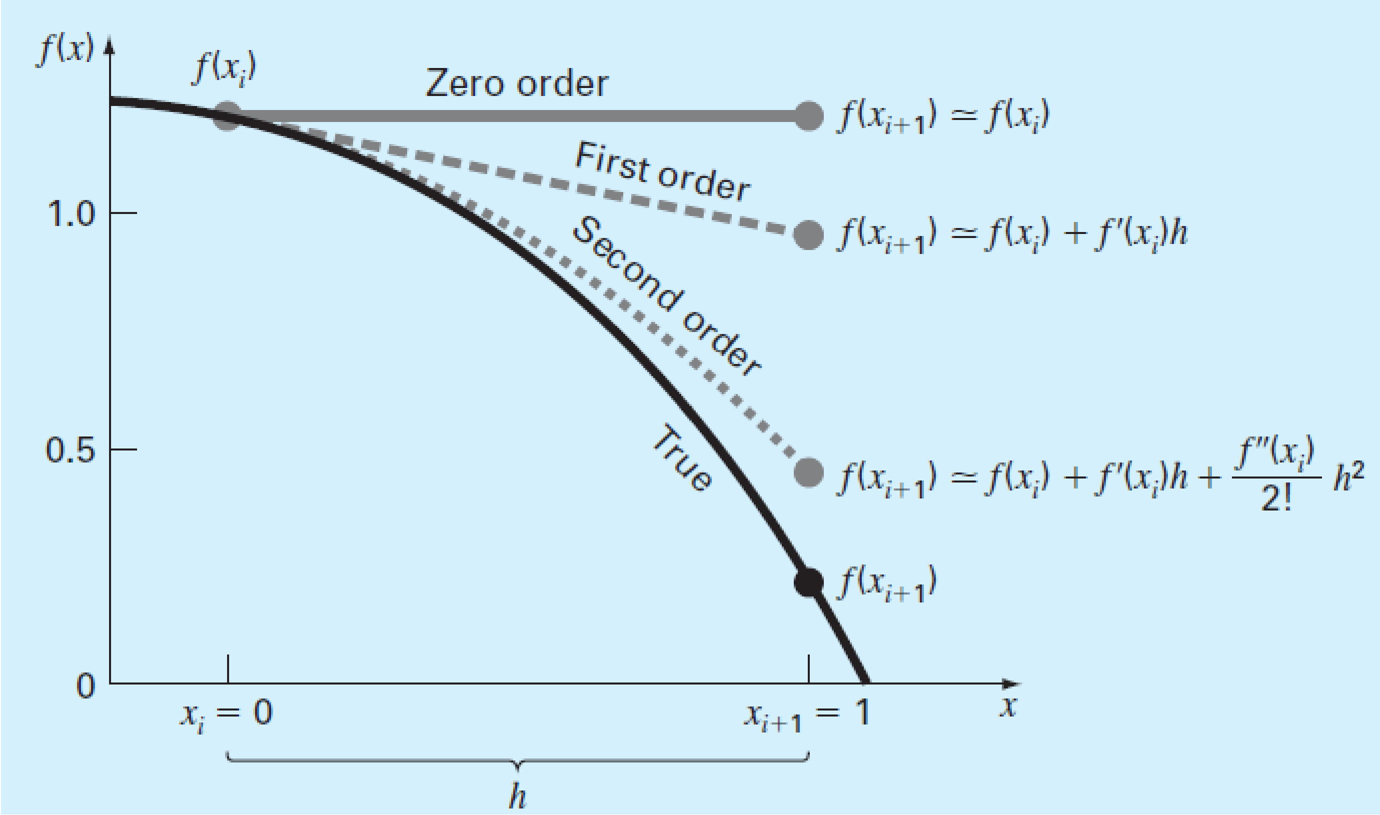
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{3.6}$$

- *Example*: Finite-difference derivative approximation:

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \tag{3.7}$$

Taylor Series

- Taylor Series allows us to approximate any smooth function as a polynomial



Taylor Series

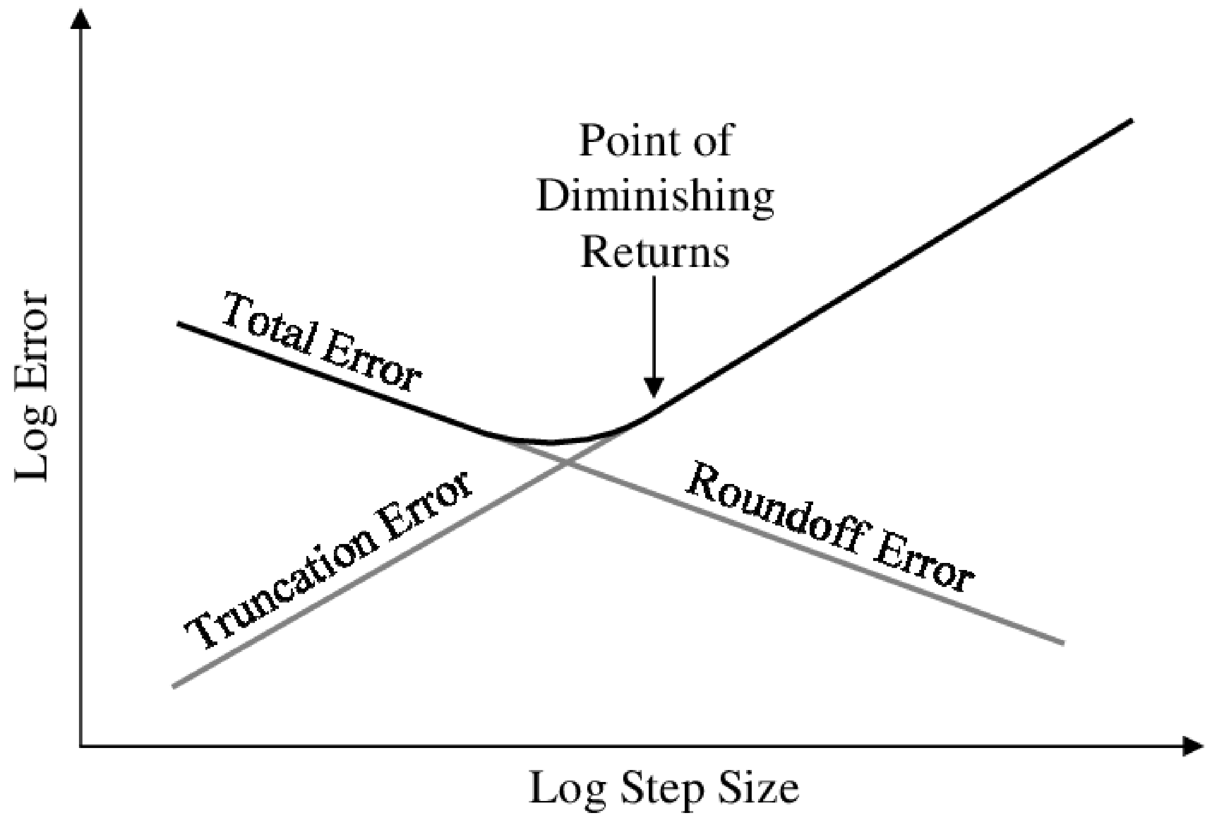
- Big O notation: allows us to judge the comparative error of numerical methods based on Taylor Series expansions

First Derivative		
Method	**Formula**	**Truncation Error**
Two-point forward difference	$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$	$O(h)$
Three-point forward difference	$f'(x_i) = \frac{-3f(x_i) + 4f(x_{i+1}) - f(x_{i+2})}{2h}$	$O(h^2)$
Two-point backward difference	$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$	$O(h)$
Three-point backward difference	$f'(x_i) = \frac{f(x_{i-2}) - 4f(x_{i-1}) + 3f(x_i)}{2h}$	$O(h^2)$
Two-point central difference	$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$	$O(h^2)$
Four-point central difference	$f'(x_i) = \frac{f(x_{i-2}) - 8f(x_{i-1}) + 8f(x_{i+1}) - f(x_{i+2})}{12h}$	$O(h^4)$

2.2: Total Numerical Error

Total Numerical Error

- Total Numerical Error = roundoff error + truncation error



Total Numerical Error

- Potential ways to reduce error:
  - Fully understand your mathematical model and the underlying assumptions
  - Leverage the Taylor Series to estimate the theoretical error

  - Avoid subtracting 2 nearly equal numbers (subtractive cancellation)
  - Operate on smaller numbers first
  - Have intermediate checks in your code to prevent large-scale error cascades

  - Perform sensitivity analyses
  - Obtain independent audits

Summary

- Numerical Methods = approximating something → errors
- If you don’t understand how errors affect your approximation, your work is meaningless!

- Roundoff errors: stems from computer’s inability to represent numbers exactly
- Truncation errors: stems from using an approximation instead of the analytical procedure
- Tradeoff between roundoff/truncation errors