Linear Algebra

ME 2004

Outline

1.1: Interpreting Matrices

Understand the system → **Do solutions exist?** → **If solutions exist, are they unique?**

Introduction

- General form of m simultaneous linear equations with n unknowns:

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m
\end{gathered}
$$

- m rows, n columns

Introduction

- These linear equations are condensed into:

$$Ax = b$$

$$A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Introduction

- We are usually given A and b from the governing equations, so we need to find x:

$$Ax = b \rightarrow x = A^{-1}b$$

- Many ways to compute x:
  - Row reduction/Gauss-Jordan Elimination
  - Computing A^{-1}
  - LU/QR Decomposition
  - Gauss-Seidel
  - etc.

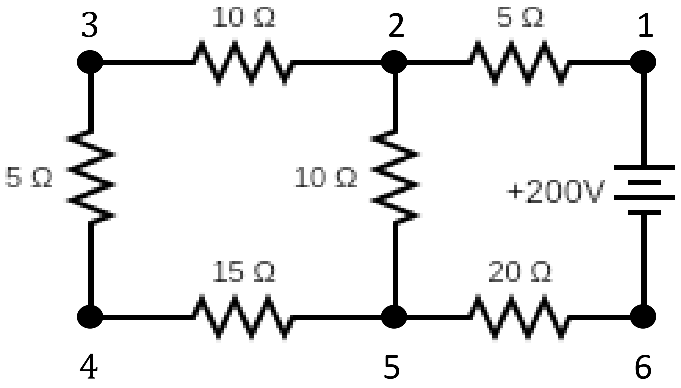
Cause-Effect Relationship

- Linear systems represent a cause-effect relationship:
  - A: contains the system parameters. How do parts of the system interact?
  - b: forcing functions. What is acting on the system?
  - x: response. How to the independent variables react to the forcing functions?

Forcing Functions $\mathbf{b}$ $\Rightarrow$ Physical System $\mathbf{A}$ $\Rightarrow$ Response $\mathbf{x}$

Cause-Effect Relationship

- Finding the currents in a circuit:



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 10 & -10 & 0 & -15 & -5 \\ 5 & -10 & 0 & -20 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_{12} \\ i_{52} \\ i_{32} \\ i_{65} \\ i_{54} \\ i_{43} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 200 \end{bmatrix}$$

$$\boxed{\begin{array}{c}\text{Forcing Functions} \\ V\end{array}} \rightarrow \boxed{\begin{array}{c}\text{Physical System} \\ R_{12}, R_{23}, \ldots R_{56}\end{array}} \rightarrow \boxed{\begin{array}{c}\text{Response} \\ i_{12}, i_{32}, \ldots i_{65}\end{array}}$$

Geometric Interpretations

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- **Row interpretation:** interpreting each row of the system as lines

$$\begin{aligned} A_{11}x_1 + A_{12}x_1 &= b_1 \\ A_{21}x_1 + A_{22}x_2 &= b_2 \end{aligned}$$

  - Solution: intersection of the lines

- **Column interpretation:** interpreting each column of the system as a vector

$$\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} x_1 + \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

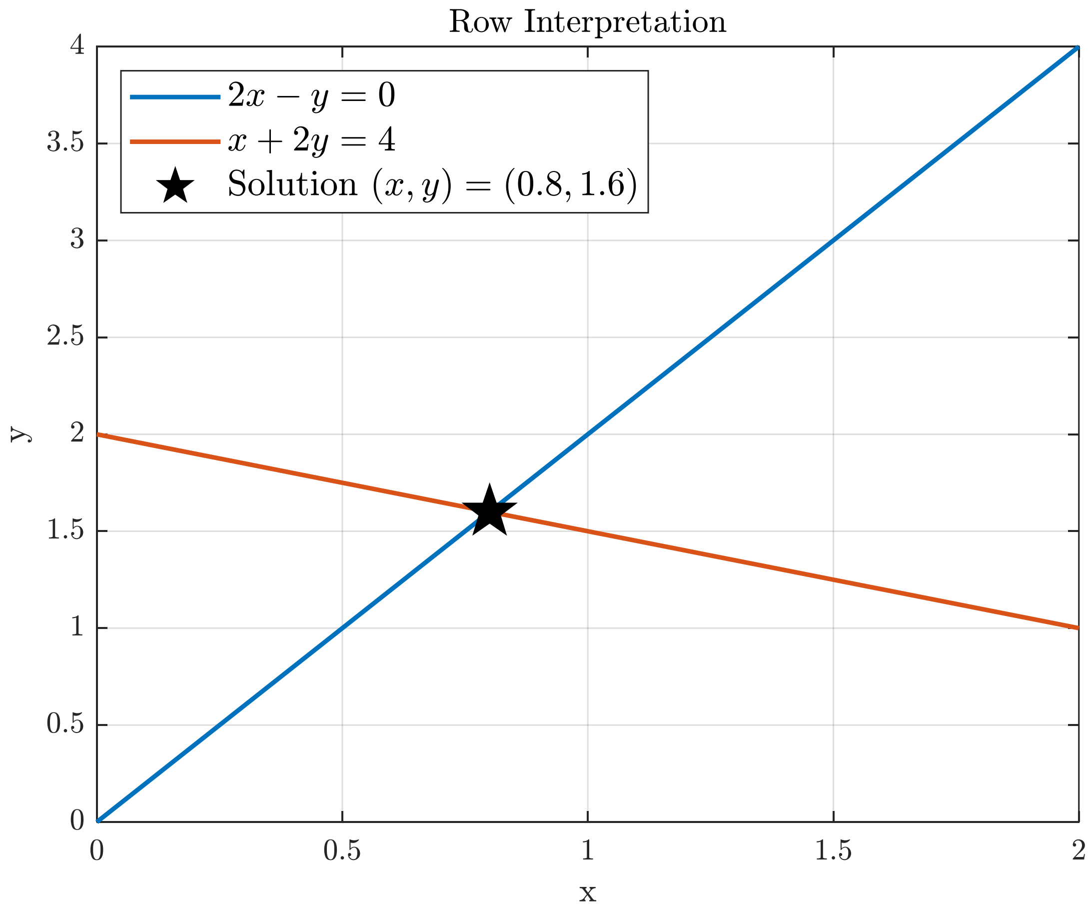
  - Solution: values which form b as a linear combination of the column vectors

Geometric Interpretations

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$2x - y = 0$$
$$x + 2y = 4$$

$$\rightarrow (x, y) = (0.8, 1.6)$$

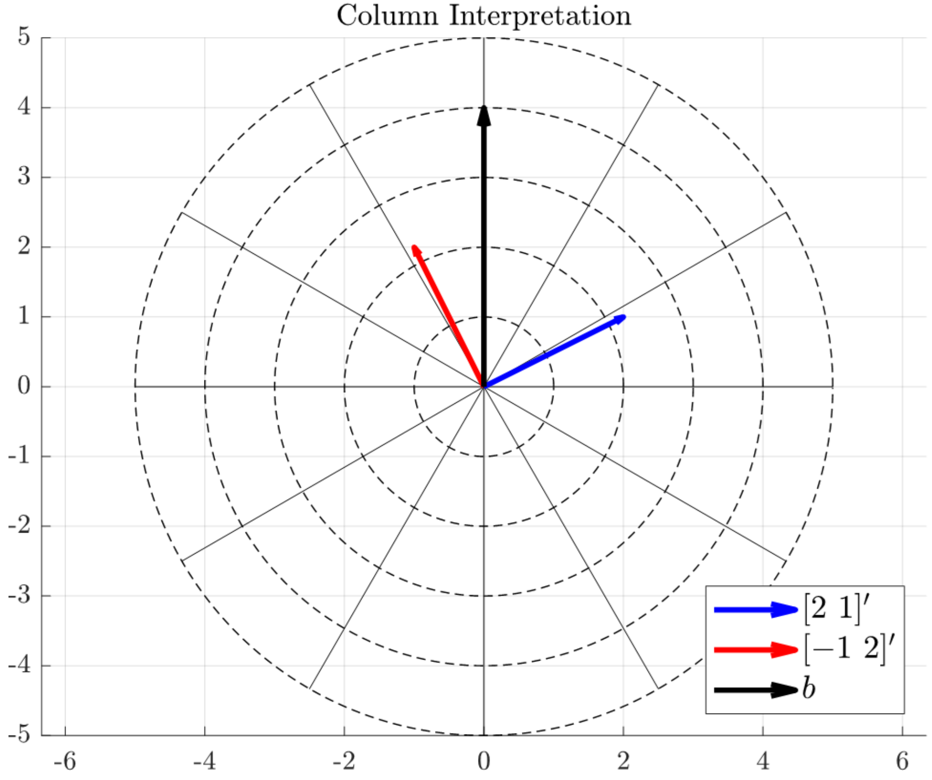


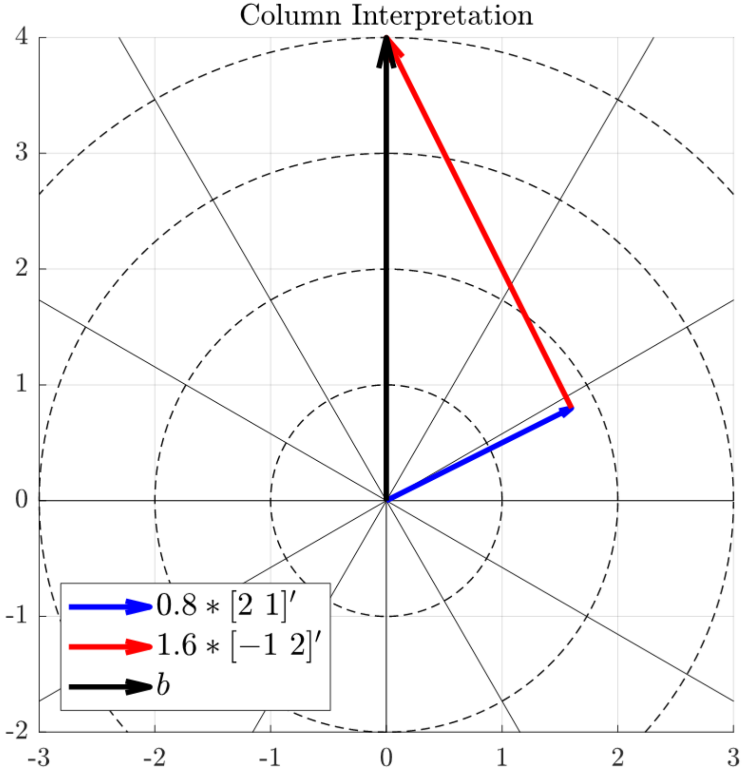
Geometric Interpretations

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\rightarrow (x, y) = (0.8, 1.6)$$



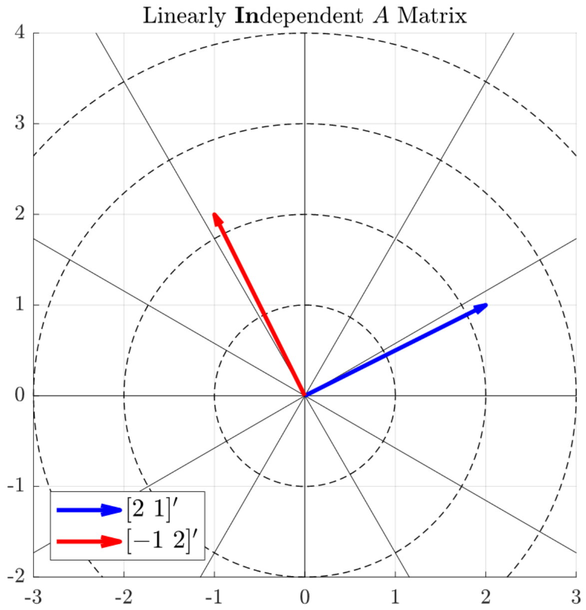


1.2: Existence of Solutions

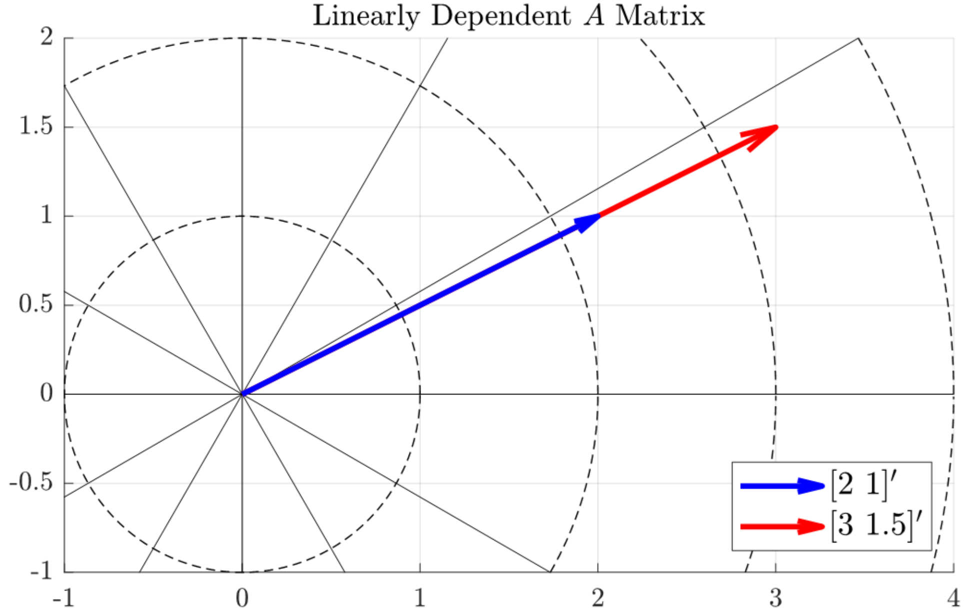
Understand the system → **Do solutions exist?** → **If solutions exist, are they unique?**

Linear (In)Dependence

- A is linearly INdependent if no columns are a linear combination of the other columns (A has a pivot in each row; $\det(A) \neq 0$, etc.)



$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \rightarrow rref(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1.5 \end{bmatrix} \rightarrow rref(A) = \begin{bmatrix} 1 & 1.5 \\ 0 & 0 \end{bmatrix}$$

Rank and Consistency

- Rank: number of linearly INdependent columns of A
  - Full rank: $rank(A)$ = # columns of A
  - Rank deficient: $rank(A) <$ # columns of A

- Consistency: characterizes whether a system of equations has solutions
  - Augmented matrix: $\tilde{A} = [A\ b]$
  - System is consistent (at least one solution exists) if $rank(A) = rank(\tilde{A})$
  - System is inconsistent (solutions do not exist) if $rank(A) \neq rank(\tilde{A})$

Rank and Consistency

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

- Compute $rank(A)$
- Compute $rank(\tilde{A})$
- Is the system consistent?

Command Window

```
>> A = [2 -1; 1 2]; b = [0 4]';
>> rref(A)

ans =

     1     0
     0     1

>> rank(A)

ans =

     2

>> rank([A b])

ans =

     2
```

A is linearly independent

A is full rank

$rank(\tilde{A}) = rank(A) = 2$
$\rightarrow$ system is consistent

Rank and Consistency

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1.5 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

- Compute $rank(A)$
- Compute $rank(\tilde{A})$
- Is the system consistent?

Command Window

```
>> A = [2 3; 1 1.5]; b = [-1 6]';
>> rref(A)

ans =

    1.0000    1.5000
         0         0

>> rank(A)

ans =

     1

>> rank([A b])

ans =

     2
```

A is linearly dependent

A is rank deficient

$rank(\tilde{A}) \neq rank(A)$
$\rightarrow$ system is inconsistent

1.3: Uniqueness of Solutions

Uniqueness of Solutions

- A unique solution exists if:
  - $rank(A) = n$
  - $rank(\tilde{A}) = rank(A)$

- Infinite solutions exist if:
  - $rank(A) < n$
  - $rank(\tilde{A}) = rank(A)$

- No solutions exist if:
  - $rank(A) < n$
  - $rank(\tilde{A}) = rank(A) + 1$

Uniqueness of Solutions

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

- Compute $rank(A)$
- Compute $rank(\tilde{A})$
- Is the system consistent?
  - If the system is consistent, what is/are the solution(s)?

```
>> rref([A b])

ans =

    1.0000         0    0.8000
         0    1.0000    1.6000
```

Command Window

```
>> A = [2 -1; 1 2]; b = [0 4]';
>> rref(A)

ans =

     1     0
     0     1

>> rank(A)

ans =

     2

>> rank([A b])

ans =

     2
```

A is linearly independent

A is full rank

$rank(\tilde{A}) = rank(A) = 2$
→ system is consistent
and there is a unique solution

Uniqueness of Solutions

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1.5 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

- Compute $rank(A)$
- Compute $rank(\tilde{A})$
- Is the system consistent?
  - If the system is consistent, what is/are the solution(s)?

```
>> rref([A b])

ans =

    1.0000    1.5000         0
         0         0    1.0000
```

Command Window

```
>> A = [2 3; 1 1.5]; b = [-1 6]';
>> rref(A)

ans =

    1.0000    1.5000
         0         0

>> rank(A)

ans =

     1

>> rank([A b])

ans =

     2
```

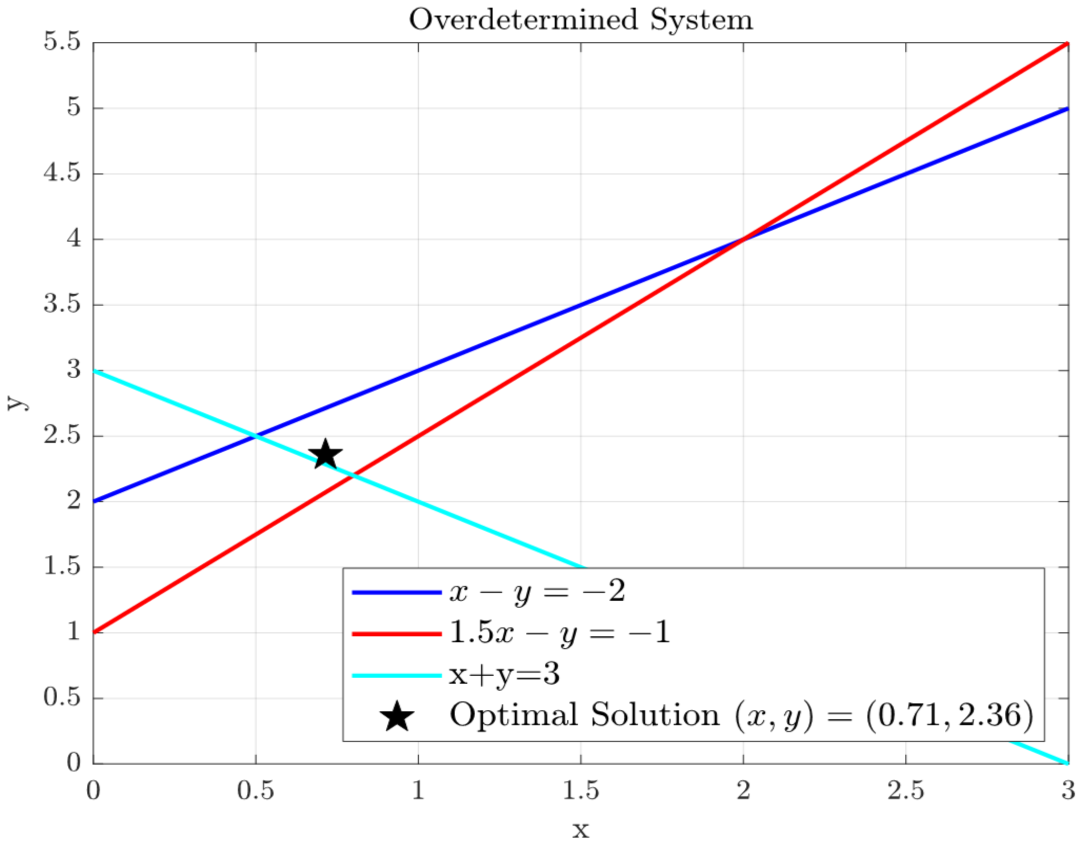
A is linearly dependent

A is rank deficient

$rank(\tilde{A}) \neq rank(A)$
$\rightarrow$ system is inconsistent

Over/Underdetermined Systems

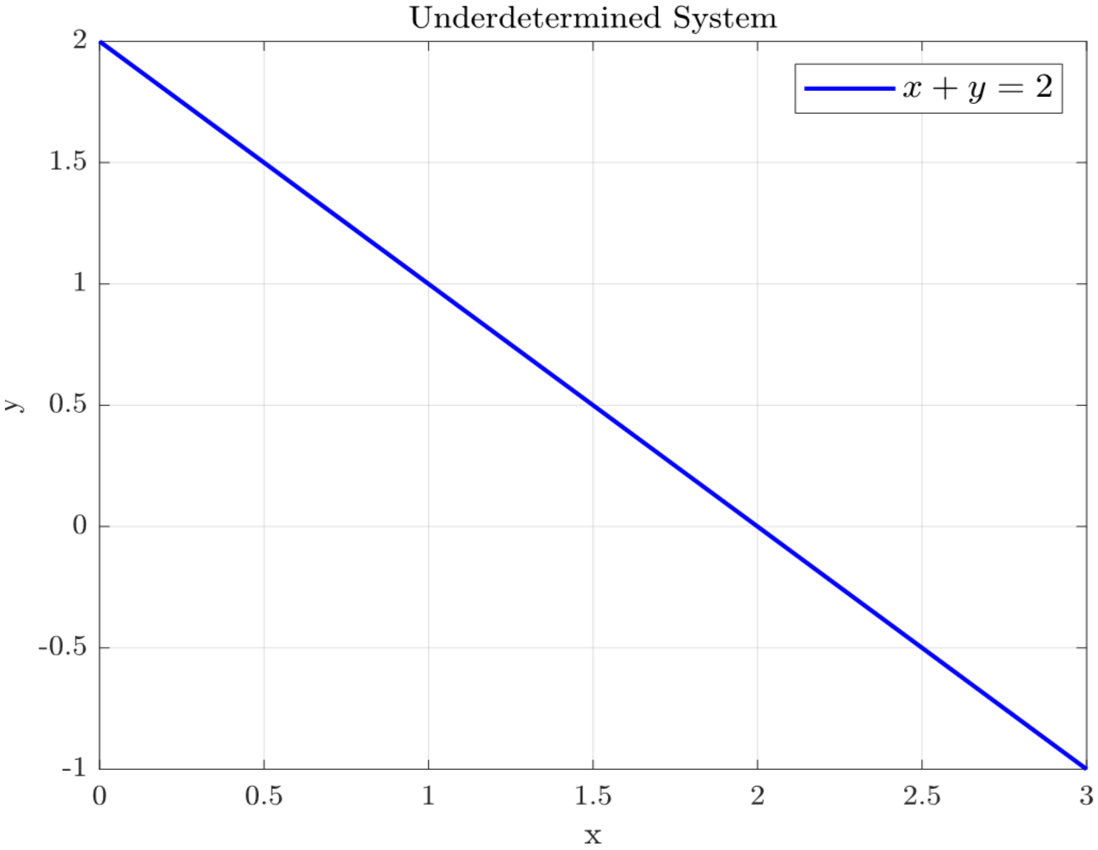
- Overdetermined system: more equations than unknowns $(m > n)$
- In general, *no solution* satisfies the system
- Application: curve fitting (least-squares regression)



$$\begin{bmatrix} 1 & -1 \\ 1.5 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

Over/Underdetermined Systems

- Underdetermined system: more unknowns than equations $(n > m)$
- In general, *infinite solutions* satisfy the system
- Application: integer programming



$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

Summary

- Express m simultaneous linear equations with n unknowns in matrix form:

$$A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Forcing Functions **b**	⇒	Physical System **A**	⇒	Response **x**

Summary

Understand the system

- What are A, b, x?
- What are m, n?

Do solutions exist?

- Consistency
  - Rank
    - Linear independence

If solutions exist, are they unique?

- Compare:
  - $rank(A)$ vs n
  - $rank(A)$ vs $rank(\tilde{A})$
- Over/underdetermined

Summary

Review Linear Algebra terminology if you haven’t already!!!

In particular:

[Invertible Matrix Theorem]