Matrix Inverse

ME 2004

Outline

1.1: Matrix Inverse

Matrix Inverse

- Express m simultaneous linear equations with n unknowns in matrix form:

$$A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Forcing Functions **b**	$\Rightarrow$	Physical System **A**	$\Rightarrow$	Response **x**

Matrix Inverse

- The matrix Z which satisfies $AZ = I$ is the inverse of matrix A
- A is not always invertible! See [Invertible Matrix Theorem]
- Quick ways to check if A^{-1} exists:
  - $\det(A) \neq 0$
  - $rank(A) = n$
  - $rref(A) = I$

For the remainder of this PPT, assume A is invertible

Matrix Inverse

- Superposition: if a system is subject to several different forcing functions (b), the responses (x) can be computed individually and the results can be summed to obtain a total response.

$$f(b_1) + f(b_2) = f(b_1 + b_2)$$

Application to linear systems: $x_{total} = \sum x$

- Proportionality: scaling the forcing function causes the response to be scaled by the same amount.

$$f(\alpha b) = \alpha f(b)$$

Application to linear systems: $Ax = b, b \to \alpha b, x \to \alpha x$

Matrix Inverse

- Superposition example:

$$f(b) = 2b$$
$$f(3) + f(5) = 6 + 10 = 16$$
$$f(3+5) = f(8) = 16$$
$$f(b) = \sin(b) \rightarrow \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right) \neq \sin\left(\frac{3\pi}{4}\right)$$

- Proportionality example:

$$f(b) = ???, f(5) = 10, \alpha = 2$$
$$f(5\alpha) = \alpha f(5) = 2(10) = 20$$
$$f(b) = \sin(b) \rightarrow \sin\left(\alpha\frac{\pi}{4}\right) \neq \alpha\sin\left(\frac{\pi}{4}\right)$$

Matrix Inverse

- To solve $Ax = b$, compute $x = A\backslash b = A^{-1}b$

$$A^{-1} = \begin{bmatrix} a_{11}^{-1} & a_{21}^{-1} & \cdots & a_{1n}^{-1} \\ a_{21}^{-1} & a_{22}^{-1} & \cdots & a_{2n}^{-1} \\ & & \ddots & \\ a_{m1}^{-1} & a_{m2}^{-1} & \cdots & a_{mn}^{-1} \end{bmatrix}$$

$$\begin{gathered} x_1 = a_{11}^{-1}b_1 + a_{12}^{-1}b_2 + a_{13}^{-1}b_3 + \cdots + a_{1n}^{-1}b_m \\ x_2 = a_{21}^{-1}b_1 + a_{22}^{-1}b_2 + a_{23}^{-1}b_3 + \cdots + a_{2n}^{-1}b_m \\ \vdots \\ x_n = a_{m1}^{-1}b_1 + a_{m2}^{-1}b_2 + a_{m3}^{-1}b_3 + \cdots + a_{mn}^{-1}b_m \end{gathered}$$

Matrix Inverse

$$x_n = \underbrace{a_{m1}^{-1}}_{\text{Effect of a unit change in } b_1} b_1 + a_{m2}^{-1} b_2 + \underbrace{a_{m3}^{-1} b_3} + \cdots + \underbrace{a_{mn}^{-1} b_m}$$

Add the contributions from the other various b's *(superposition)*

- Each element of A^{-1} is a proportionality constant which represents the response of a single part of the system to a unit change in the stimulus of any other part of the system!

$$a_{ij}^{-1} = \text{change in } x_i \text{ due to a unit change in } b_j$$

- a_{11}^{-1} = change in x_1 due to a unit change in b_1
- a_{13}^{-1} = change in x_1 due to a unit change in b_3

Matrix Inverse

- For non-unity changes to b, superposition and proportionality can be used to quickly analyze x without re-solving the system

$$x_n = a_{m1}^{-1} b_1 + a_{m2}^{-1} b_2 + a_{m3}^{-1} b_3 + \cdots + a_{mn}^{-1} b_m$$

becomes

$$\Delta x_{n,tot} = a_{m1}^{-1}(\Delta b_1) + a_{m2}^{-1}(\Delta b_2) + \cdots + a_{mn}^{-1}(\Delta b_m)$$

$$\Delta x_{3,tot} = \underbrace{a_{31}^{-1}}\,\underbrace{(\alpha b_1)} + \underbrace{a_{32}^{-1}(\beta b_2)} + \underbrace{a_{33}^{-1}(\gamma b_3)}$$

Effect of a unit change in b_1

The change is scaled by α, not unity *(proportionality)*

Add the contributions from the other various changes in the b's *(superposition)*

Matrix Inverse

- 2x2 matrix: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \left(\frac{1}{\det(A)}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\rightarrow \boxed{A^{-1} = \left(\frac{1}{ad - bc}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

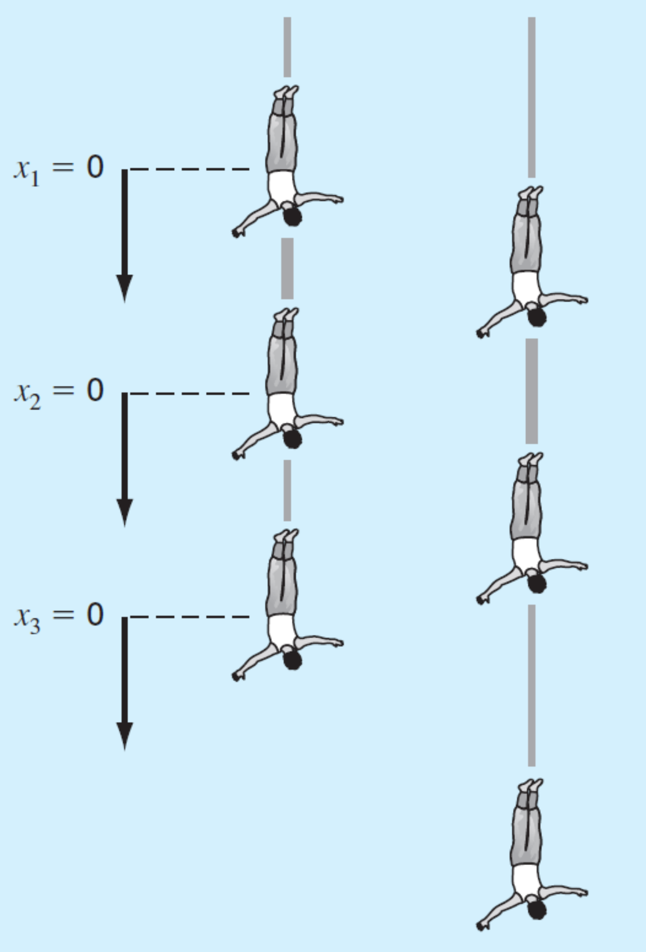
- If A is 3x3 or larger, compute A^{-1} in MATLAB via `inv()`

1.2: Example

Example

- 3 bungee jumpers ($m_1 = 60, m_2 = 70, m_3 = 80\ kg$) are connected by 20 m long bungee cords of various elasticities $\left(k_1 = 50, k_2 = 100, k_3 = 50 \frac{N}{m}\right)$
- After jumping, they come to rest at some distance (x_1, x_2, x_3) past their equilibrium $(x_1, x_2, x_3 = 0)$

$$\begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} m_1 g \\ m_2 g \\ m_3 g \end{bmatrix}$$



(a) Unstretched (b) Stretched

Example

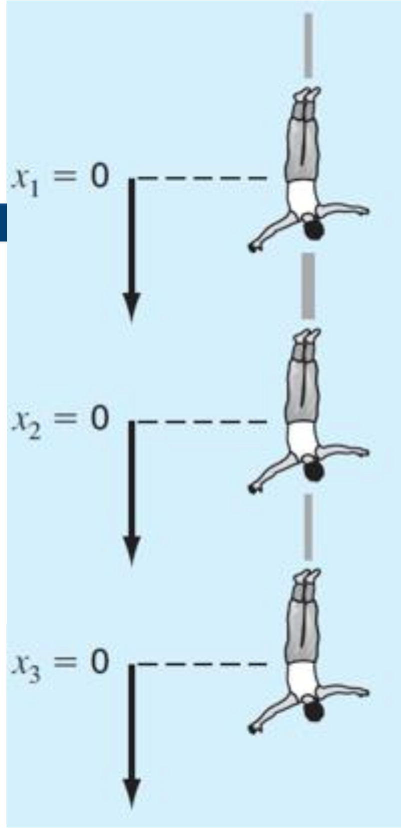
- a_{ij}^{-1} = change in x_i due to unit change in force applied to jth jumper
- a_{12}^{-1} = change in x_1 due to unit change in force applied to 2nd jumper
- a_{31}^{-1} = change in x_3 due to unit change in force applied to 1st jumper

Command Window

```
>> A = [150 -100 0;-100 150 -50;0 -50 50];
>> inv(A)

ans =

    0.0200    0.0200    0.0200
    0.0200    0.0300    0.0300
    0.0200    0.0300    0.0500
```



Example

- It is known that $g_{moon} = \frac{g_{Earth}}{6}$. How would the jumpers' deflections change if they jumped on the moon?

$$b_{Earth} = g_{Earth}\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

$$\rightarrow b_{moon} = \frac{b_{Earth}}{6}$$

$\rightarrow$By proportionality: $x_{moon} = \frac{x_{Earth}}{6}$

Command Window

```
>> A = [150 -100 0;-100 150 -50;0 -50 50]; b = 9.81*[60:10:80]';
>> x_Earth = A\b

x_Earth =

   41.2020
   55.9170
   71.6130

>> x_moon = x_Earth/6

x_moon =

    6.8670
    9.3195
   11.9355

>> b_moon = (9.81/6)*[60:10:80]';
>> x_moon = A\b_moon

x_moon =

    6.8670
    9.3195
   11.9355
```

Example

- How would the jumpers' deflections change if additional forces of $\Delta F_1 = 10\ N, \Delta F_2 = 50\ N$, and $\Delta F_3 = 20\ N$ were applied?

$$\Delta x_{n,tot} = a_{m1}^{-1}(\Delta b_1) + a_{m2}^{-1}(\Delta b_2) + \cdots + a_{mn}^{-1}(\Delta b_m)$$

$$\rightarrow \Delta x_{3,tot} = (0.02)(10) + (0.03)(50) + (0.05)(20)$$

$$\rightarrow \boxed{\Delta x_{3,tot} = 2.7\ m}$$

Command Window

```
>> delta_F = [10 50 20]';
>> delta_x = inv(A)*delta_F

delta_x =

    1.6000
    2.3000
    2.7000

>> x_tot = (A\b) + delta_x

x_tot =

   42.8020
   58.2170
   74.3130
```

Summary

- $a_{ij}^{-1} =$ change in x_i due to a unit change in b_j
- Superposition and proportionality are useful when analyzing linear systems
- Inverse of 2x2 matrix can be computed by hand; larger inverse can be computed in MATLAB