Vector and Matrix Norms

ME 2004

Outline

1.1: Vector Norms

Vector Norms

- Norm: measures the size or "length" of vectors/matrices

$$P = [1 \quad 0 \quad -1]$$
$$Q = [1 \quad 4 \quad 0]$$

Are P and Q similar?

- Generalized form of a vector p-norm:

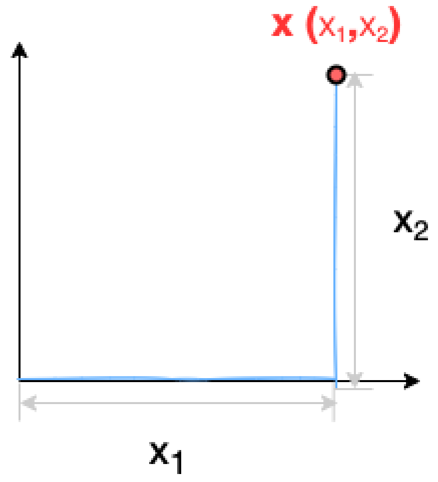
$$||X||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

Vector Norms

- 1-norm/L^1 norm/Manhattan norm:

$$||x||_1 = |x_1| + |x_2|$$

$$\|X\|_1 = \sum_{i=1}^{n} |x_i|$$



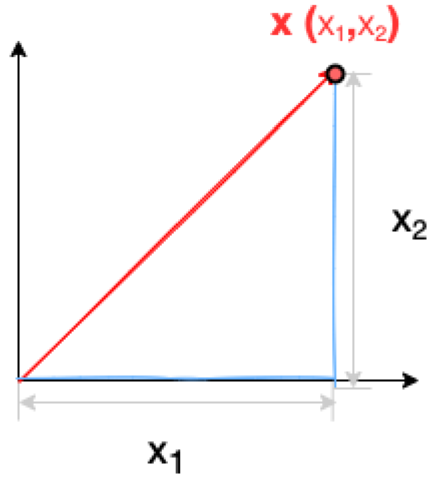
- "Total length" of the vector

Vector Norms

- 2-norm/L^2 norm/Euclidean norm:

$$||x||_2 = \sqrt{x_1^2 + x_2^2}$$

$$||X||_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$



- Shortest distance from the origin

Vector Norms

- ∞-norm/L^∞ norm/max norm:

$$\|X\|_\infty = \max_{1 \le i \le n} |x_i|$$

- Absolute value of the largest element in the vector

Vector Norms

- Given $X = [-1,4,2]$, calculate the 1-norm, 2-norm, and ∞-norm.
- 1-norm:

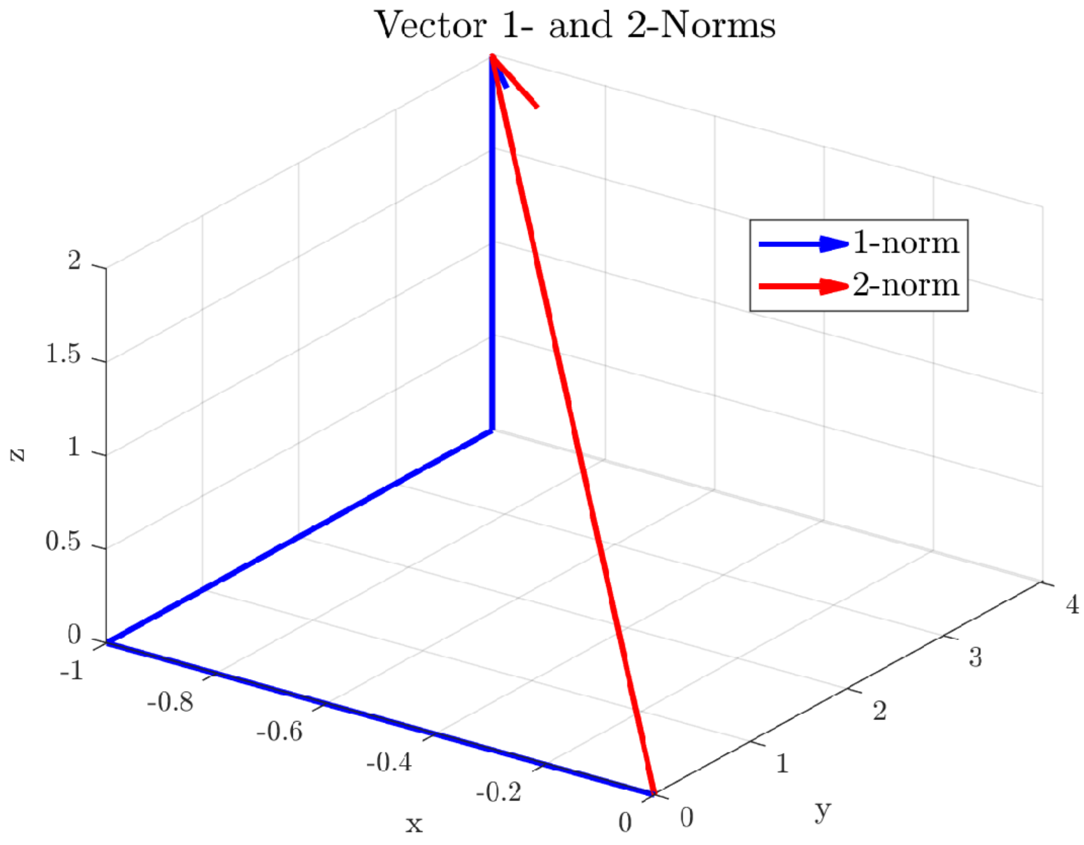
$$\|X\|_1 = \sum_{i=1}^{n} |x_i| = 1 + 4 + 2 = \boxed{7}$$

- 2-norm:

$$\|X\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{(-1)^2 + 4^2 + 2^2} = \boxed{4.58}$$

- ∞-norm:

$$\|X\|_\infty = \max_{1 \le i \le n} |x_i| = \max(-1,2,4) = \boxed{4}$$



1.2: Matrix Norms

Matrix Norms

- 1-norm/column-sum norm:

$$||A||_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

- Sum the absolute values of each column, then take the largest summation

Matrix Norms

- ~~2-norm/Frobenius norm~~
- ∞-norm/row-sum norm:

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

- Sum the absolute values of each row, then take the largest summation

Matrix Norms

- Given $A = \begin{bmatrix} 7 & 1 & 8 \\ 4 & 5 & 8 \\ 10 & 4 & 2 \end{bmatrix}$, calculate the 1-norm and ∞-norm.
- 1-norm:

$$||A||_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

$$= \max([(7+4+10) \quad (1+5+4) \quad (8+8+2)])$$

$$\rightarrow \boxed{||A||_1 = 21}$$

Matrix Norms

- Given $A = \begin{bmatrix} 7 & 1 & 8 \\ 4 & 5 & 8 \\ 10 & 4 & 2 \end{bmatrix}$, calculate the 1-norm and ∞-norm.
- ∞-norm:

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

$$= \max\left(\begin{bmatrix} (7+1+8) \\ (4+5+8) \\ (10+4+2) \end{bmatrix}\right) = \max\left(\begin{bmatrix} 16 \\ 17 \\ 16 \end{bmatrix}\right)$$

$$\rightarrow \boxed{\|A\|_\infty = 17}$$

Matrix Norms

- In MATLAB:

```
norm(Z,p)
```

- `Z` = vector or matrix
- `p` = `1,2` (default),`inf` = which norm to compute
  - Can also supply `-inf` and other positive scalars (vectors only)
  - Can also supply `'fro'` to compute matrix Frobenius norm (not used in this class)

Matrix Norms

p	Matrix	Vector
1	`max(sum(abs(X)))`	`sum(abs(v))`
2	`max(svd(X))`	`sum(abs(v).^2)^(1/2)`
Positive, real-valued numeric scalar	—	`sum(abs(v).^p)^(1/p)`
`Inf`	`max(sum(abs(X')))`	`max(abs(v))`
`-Inf`	—	`min(abs(v))`

Matrix Norms

Command Window

```
>> x = [-1 4 2];
>> x1 = norm(x,1)

x1 =

     7

>> x2 = norm(x,2)

x2 =

    4.5826

>> x_inf = norm(x,inf)

x_inf =

     4

>> x_ninf = norm(x,-inf)

x_ninf =

     1
```

Command Window

```
>> A = [7 1 8; 4 5 8; 10 4 2]

A =

     7     1     8
     4     5     8
    10     4     2

>> A1 = norm(A,1)

A1 =

    21

>> A_inf = norm(A,inf)

A_inf =

    17
```

1.3: Purpose of Norms

Applications

- Many linear algebra applications (machine learning, image classification, etc.) rely on computing the (dis)similarity between items
- Norms are useful because they're a single number related to the size of the vector/matrix, so they can be used as a comparison metric

Norm	Application
L^1	LASSO (Least Absolute Shrinkage and Selection Operator) (machine learning)
$\|\|A\|\|_\infty$	Optimizing performance objectives (control systems)

Applications

- In MATLAB Grader (Robot Navigation):

```
out = norm(p_or_soln-p_or,inf) <= 1e-3;
if out
    msg = 'Test %d passed! \n';
    fprintf(msg,testcase)
else
    msg = 'Test %d failed. Check your entries in p_or again. \n';
end
```

- Absolute value of the largest element in the error vector

Applications

- Condition number: measures how much the x changes if b changes slightly

$$Cond(A) = \|A\| * \|A^{-1}\| \geq 1$$

- Large condition number: small error in b → large change in x

Summary

- Norms measure the "size" of a vector/matrix
  - Useful to directly compare vectors/matrices (such as the ones in your Workshops ☺)
- Different types of norms: 1-norm, 2-norm, ∞-norm
  - Can be computed by hand or in MATLAB

p	Matrix	Vector
1	`max(sum(abs(X)))`	`sum(abs(v))`
2	`max(svd(X))`	`sum(abs(v).^2)^(1/2)`
Positive, real-valued numeric scalar	—	`sum(abs(v).^p)^(1/p)`
`Inf`	`max(sum(abs(X')))`	`max(abs(v))`
`-Inf`	—	`min(abs(v))`