Schwinger Mechanism and Hawking Radiation as Quantum Tunneling

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(Dated:)

The common interpretation of the Hawking radiation as quantum tunneling has some ambiguity such as coordinate-dependence of tunneling rate and non-invariance of the action under canonical transformations. It is shown that the tunneling process of black holes can be successfully described by Rindler coordinates in analogy with the Schwinger mechanism for pair production. We study the tunneling process of a charged black hole and a BTZ black hole.

Keywords : Hawking radiation, quantum tunneling, Rindler coordinates, Schwinger mechanism.

arXiv:0709.4313v1 [hep-th] 27 Sep 2007

I. INTRODUCTION

More than a quarter century ago Hawking discovered that black holes could radiate thermal radiations with the Hawking temperature determined by the surface gravity at the event horizon [1]. The surface gravity is the acceleration of a static particle just outside the event horizon as measured at the spatially infinity. Also Unruh observed that a uniformly accelerated particle would experience a thermal state from the Minkowski vacuum with the temperature determined by the acceleration [2]. Since then there have been many attempts to derive or reinterpret the Hawking radiation.

Recently the study of the Hawking radiation has been revived partly because the Hawking radiation could be derived from the motion of spherically emitted radiation including the back reaction of radiation [3] and partly because quantum tunneling of fluctuations or waves across the event horizon could lead to the Hawking radiation [4]. In both cases the imaginary part of action was responsible for tunneling of Hawking radiation.

Hawking radiation was originally derived by calculating the amplitude ratio of an outgoing wave to an incoming wave scattered by the event horizon. The recent quantum tunneling interpretation differs from the early interpretation in that quantum fluctuations just inside the horizon can cross the horizon and the tunneling amplitude determines the Boltzmann distribution for Hawking radiation [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] (for review, see Ref. [19]). However, there is some ambiguity in the tunneling interpretation of Hawking radiation [16]. The Boltzmann distribution depends on the coordinates used to express the field equation. For instance, the Schwarzschild coordinate leads to a temperature twice of the Hawking temperature.

In this paper we show that the Rindler coordinates correctly provide the tunneling rate for Hawking radiation and resolve the ambiguity. We then apply the method to a charged black hole and a BTZ black hole. The idea is based on the observation that the field equation in the Rindler coordinates of a non-extremal black hole takes a similar form as the field equation in an external electric field for the Schwinger mechanism [20]. In the space-dependent (Coulomb) gauge the field equation conceptually leads to the tunneling problem, in which virtual pairs of charged particles in the Dirac sea experience a potential barrier lowered by the electric field [21, 22, 23, 24, 25, 26]. Further the tunneling rate is given by an instanton action in the complex plane [24, 25]. The tunneling rate expressed by a contour integral in the complex plane is invariant under canonical transformations.

The organization of this paper is as follows. In Sec. II, we briefly review the Schwinger mechanism and introduce the tunneling rate as a contour in the complex plane. In Sec. III, the tunneling rate for Hawking radiation is formulated as a contour integral in Rindler coordinates of black holes. We then apply the formula to a charged black hole and a BTZ black hole.

II. SCHWINGER MECHANISM AND INSTANTON ACTIONS

Schwinger used the proper-time method to calculate the effective action of a charged particle in an external electromagnetic field and found an imaginary part for a uniform electric field [20]. This leads to the vacuum decay through pair production. The Schwinger mechanism for pair production by an external field can be physically understood as follows. Virtual pairs of charged particles from vacuum fluctuations can be separated to become real pairs when the potential energy over the Compton wavelength is greater than or equal to the rest mass energy. In the tunneling interpretation the virtual pairs pass through the potential barrier lowered by the electric field [21, 22, 23, 24, 25, 26]. Roughly speaking, the barrier width is inversely proportional to the strength of the electric field.

The similarity between the Schwinger mechanism and the Hawking radiation has been studied for many years [5, 22, 23]. A naive interpretation would be separation of virtual pairs by an electric field in the former and by the
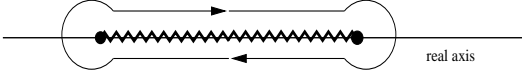
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FIG. 1: The contour $\mathcal{C}$ in the complex x-plane excluding a branch cut connecting two roots for tunneling.

event horizon in the latter. Another interpretation would be that virtual pairs accelerate by the electric field and these pairs would feel thermal state from the Minkowski vacuum. Likewise the Hawking radiation may be interpreted as the Unruh effect of a static particle just outside the horizon with the acceleration of the surface gravity measured at the infinity [2]. In fact, the Rindler spacetime of the accelerated particle also has the horizons.

Now we further elaborate the analogy between the Schwinger mechanism and the Hawking radiation. A difference from other studies is that the Hawking radiation is interpreted as quantum tunneling in the same way as the Schwinger mechanism. We observe that, for instance, the Klein-Gordon equation for charged boson pairs in the space-dependent (Coulomb) gauge takes the same form as the field equation in the Rindler coordinates of black holes. Each Fourier mode becomes a tunneling problem, the sub-barrier penetration. The essence of the instanton action method is that the tunneling states of this equation lead to the vacuum decay and pair production of charged particles [24, 25]. A similar form of tunneling rate is provided by the worldline instanton [26].

A constant electric field along the x-direction has the space-dependent gauge potential $A_\mu = (-Ex, 0)$. The Fourier mode of the Klein-Gordon equation for charge e ($e > 0$) and mass μ takes the form [in units with $\hbar = c = 1$ and with metric signature $(-, +)$]

$$\left[-\frac{\partial^2}{\partial x^2} - q(x)\right]\varphi_\omega(x) = 0, \quad q(x) = (\omega + eEx)^2 - \mu^2. \quad (1)$$

In quantum mechanics, Eq. (1) is a tunneling problem with the inverted potential barrier $-q(x)$ and $q(x)$ corresponds to p_x^2 in the WKB approximation. Using the phase-integral formula [27], the wave function can be written as

$$\varphi_\omega(x) = Af_\omega + Bf_\omega^*, \quad (2)$$

where $f_\omega(x)$ is an asymptotic solution with unit incoming flux and

$$A = (e^{2S_\omega} + 1)^{1/2}, \quad B = e^{S_\omega}. \quad (3)$$

Here the action can also be defined in the complex x-plane by the contour integral [25]

$$\mathcal{S}_\omega = -i \oint_{\mathcal{C}} \sqrt{q(x)}dx = \frac{\pi m^2}{eE}, \quad (4)$$

where the integral is along the contour in Fig. 1.

III. QUANTUM FIELD THEORY FOR HAWKING RADIATION

Now we turn to quantum tunneling of Hawking radiation. Let us first consider a charged black hole with mass m and charge q

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \quad (5)$$

where

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} = \frac{(r - r_+)(r - r_-)}{r^2}. \quad (6)$$

Here $r_\pm = m \pm \sqrt{m^2 - q^2}$ is the outer (inner) horizon. The surface gravity is

$$\kappa = \frac{f'(r_+)}{2} = \frac{r_+ - r_-}{2r_+^2}, \quad (7)$$

from which follows the Hawking temperature

$$T_H = \frac{\kappa}{2\pi}. \quad (8)$$

The s-wave of a scalar field Φ with mass μ satisfies

$$\left[-\frac{1}{f}\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(fr^2\frac{\partial}{\partial r}\right) - \mu^2\right]\Phi = 0. \quad (9)$$

The solution of the form $\Phi = e^{-i\omega t + iS(r)}$ may describe quantum tunneling of Hawking radiation when the action S has an imaginary part and thus the amplitude squared $|\Phi|^2 = e^{-2S_I}$ shows tunneling across the horizon. The action in the WKB approximation

$$S(r) = \pm \int_{r_0}^{r} \left(\omega^2 - m^2 f\right)^{1/2}\frac{dr}{f} \quad (10)$$

with r_0 inside the horizon has a simple pole at the horizon r_+. The positive (negative) sign is for an outgoing (incoming) wave.

The imaginary part obtained by taking a semi-circle over the horizon r_+ is

$$S_I = \pi\omega\left(\frac{r_+^2}{r_+ - r_-}\right) = \frac{\pi\omega}{2\kappa}. \quad (11)$$

Then the tunneling rate is given by

$$P(\omega) = e^{-\frac{\pi\omega}{\kappa}} = e^{-\omega/T}. \quad (12)$$

Here the temperature $T = \kappa/\pi$ is the twice of the Hawking temperature (8). This ambiguity is closely related with coordinates used to calculate the tunneling rate. Several ways have been proposed so far to remedy this discrepancy such as the ratio of emission to absorption [5] and a proper distance along the radial direction [10]. Another ambiguity is that the action (10) and the imaginary part is not invariant under canonical transformations [16].

We calculate the tunneling rate of s-waves (two-dimensional sector) in Rindler coordinates of a non-extremal charged black hole and a BTZ black hole. Introducing $r^* = \int dr/f(r)$, we write the metric in a conformal spacetime

$$ds^2 = -f(r^*)(dt^2 - dr^{*2}), \tag{13}$$

where the Regge-Wheeler coordinate for a non-extremal black hole $(m^2 > q^2)$ is

$$r^* = r + \frac{r_+^2}{r_+ - r_-}\ln(r - r_+) - \frac{r_-^2}{r_+ - r_-}\ln(r - r_-). \tag{14}$$

Then the spatial part of the scalar, $\Phi = e^{-i\omega t}\varphi_\omega(r^*)$, takes the form

$$\left[-\frac{\partial^2}{\partial r^{*2}} - q_\omega(r^*)\right]\varphi_\omega = 0, \quad q_\omega(r^*) = \omega^2 - \mu^2 f(r^*). \tag{15}$$

The waves (particles) can be observed at spatial infinity only for $\omega \geq m$. From the analogy with the Schwinger mechanism, the tunneling rate (imaginary part of the action) is determined by the contour integral in the complex r-plane

$$\mathcal{S}_\omega = i\oint_{\mathcal{C}} \sqrt{q_\omega(r^*)}dr^*. \tag{16}$$

Here the contour integral is taken the inside of $\mathcal{C}$ in contrast with the outside of $\mathcal{C}$ in Sec. II.

However, as the simple pole occurs at $r^* = -\infty$, the contour integral (16) may not be appropriate for practical calculations. Instead, in the Rindler coordinate

$$f(r) = \frac{r_+^2}{r^2}(\kappa x)^2, \tag{17}$$

the black hole spacetime becomes

$$ds^2 = -\frac{(\kappa r_+)^2}{(m + \sqrt{\bar{m}^2 + (\kappa r_+ x)^2})^2}(xdt)^2 + \frac{(\kappa r_+)^2(m + \sqrt{\bar{m}^2 + (\kappa r_+ x)^2})^2}{\bar{m}^2 + (\kappa r_+ x)^2}dx^2, \tag{18}$$

where $\bar{m} = (r_+ - r_-)/2 = \sqrt{m^2 - q^2}$. Near the event horizon the spacetime (18) approximately takes the form of $ds^2 = -(\kappa x)^2 dt^2 + dx^2$, a Rindler spacetime with acceleration κ. We extend the right wedge of the Minkowski spacetime (18) exterior to the horizon to the left wedge, which is an analytical continuation of the interior to the horizon. The simple pole is located at $x = 0$. Now the action for tunneling

$$\mathcal{S}_\omega = i\oint_{\mathcal{C}} \sqrt{q_\omega(x)}\frac{dx}{(\kappa x)} = \frac{2\pi\omega}{\kappa}, \tag{19}$$

leads to the tunneling rate

$$P(\omega) = e^{-\mathcal{S}_\omega} = e^{-\omega/T_H}. \tag{20}$$

We compare with other coordinates. In Ref. [10] the proper distance is used as a coordinate to calculate the tunneling rate. The proper distance, $\sigma = \int dr/\sqrt{f}$, measured from the event horizon is

$$\sigma = \sqrt{(r - r_+)(r - r_-)} + m\ln\left(\frac{r - m + \sqrt{(r - r_+)(r - r_-)}}{m}\right). \tag{21}$$

Then the two-dimensional metric near the event horizon

$$ds^2 = -(\kappa\sigma)^2 dt^2 + d\sigma^2, \tag{22}$$

is the Rindler spacetime. This is the reason why the proper distance method recovers the correct tunneling rate. On the other hand, the isotropic coordinate defined as

$$\int \frac{d\rho}{\rho} = \int \frac{dr}{r\sqrt{f(r)}}, \tag{23}$$

is

$$r = \frac{(\bar{\rho} + \bar{m})^2 - q^2}{2(\bar{\rho} - r_-)}, \tag{24}$$

where $\bar{\rho} = \rho/c$ and c is a constant. Later we choose $c = r_+/m$ to match the isotropic spacetime near the horizon with the Rindler spacetime. The metric is

$$ds^2 = -\frac{[(\bar{\rho} + \bar{m})^2 - q^2 - 2r_+(\bar{\rho} - r_-)]}{[(\bar{\rho} + \bar{m})^2 - q^2]} \times \frac{[(\bar{\rho} + \bar{m})^2 - q^2 - 2r_-(\bar{\rho} - r_-)]}{[(\bar{\rho} + \bar{m})^2 - q^2]}dt^2 + \frac{m^2[(\bar{\rho} + \bar{m})^2 - q^2]^2}{4r_+^2\bar{\rho}^2(\bar{\rho} - r_-)^2}d\rho^2. \tag{25}$$

Near the horizon it becomes again the Rindler spacetime witt the acceleration κ. Thus the isotropic coordinates also recover the correct result.

Finally, we consider the BTZ black hole with the metric

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2\left(d\theta - \frac{j}{2r^2}dt\right)^2, \tag{26}$$

where

$$g(r) = -m + \frac{j^2}{4r^2} + \lambda r^2. \tag{27}$$

The BTZ black hole has the outer (inner) horizon at $r_\pm^2 = (m \pm \sqrt{m^2 - \lambda j^2})/(2\lambda)$. The angular part $d\chi = d\theta - jdt/(2r^2)$ describes a rotation. For the s-wave the angular part does not play any role. The remaining two-dimensional sector can be treated in a similar way. Introducing the coordinate

$$g = \frac{\lambda(r^2 - r_+^2)(r^2 - r_-^2)}{2r^2} = (\kappa x)^2, \tag{28}$$

we find the metric

$$ds^2 = -(\kappa x)^2 dt^2 + \frac{8\kappa^2}{\lambda}$$

$$\times \frac{\left(1 + \frac{m + 4(\kappa x)^2}{\sqrt{(m+4(\kappa x)^2)^2 - \lambda j^2}}\right)^2}{m + 4(\kappa x)^2 + \sqrt{(m+4(\kappa x)^2)^2 - \lambda j^2}} dx^2 \quad (29)$$

When κ is the surface gravity

$$\kappa = \frac{\lambda}{4} \frac{r_+^2 - r_-^2}{r_+} = \frac{\sqrt{m^2 - \lambda j^2}}{4r_+}, \quad (30)$$

the spacetime (29) near the horizon becomes a Rindler spacetime. Therefore the tunneling rate of BTZ black hole is also given by (20).

IV. CONCLUSION

We critically reviewed the current study of the Hawking radiation as quantum tunneling. The common method to calculate the tunneling rate of a field across the event horizon has been confronted with two ambiguities: the coordinate-dependence of the action and the temperature different from the Hawking temperature [16]. Though some prescriptions have been put forth such as the proper distance and the ratio of emission to absorption, the proper choice of coordinate has been an issue of debate.

In this paper we used the Rindler coordinates as an appropriate spacetime to describe the tunneling problem of waves across the event horizon in the black hole spacetime. The reasons are that tunneling occurs across the event horizon and the tunneling rate is determined by the geometric property at the horizon while the Rindler spacetime approximately describes the spacetime near the horizon. Further the Hawking radiation can be interpreted as the Unruh effect of an observer accelerating with the surface gravity. Another interesting observation is that the field equation in a Rindler spacetime is reminiscent of the field equation for the Schwinger mechanism. The rate of pair production is obtained by a contour integral of the action in the complex plane and similarly the tunneling rate for black hole is given by the contour integral in the complex plane of the Rindler coordinate.

We applied the tunneling rate in the form of contour integral to a charged black hole and a BTZ black hole. In both cases the black hole has the Rindler coordinates in an explicit form except for the extremal black hole. Our tunneling formula leads to the correct Boltzmann distribution and the Hawking temperature. Further this formula avoids the ambiguity of tunneling interpretation. A caveat is that the formula cannot be applied to the extremal black hole because it does not have a Rindler spacetime near the horizon. The Hawking temperature vanishes there and thus no Hawking radiation. The development of field theoretical interpretation of tunneling in Rindler coordinates and applications to other black holes will be addressed in a future publication [28].

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