

Black hole lasers

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High frequency dispersion does not alter the low frequency spectrum of Hawking radiation from a single black hole horizon, whether the dispersion entails subluminal or superluminal group velocities. We show here that in the presence of an inner horizon as well as an outer horizon the superluminal case differs dramatically however. The negative energy partners of Hawking quanta return to the outer horizon and stimulate more Hawking radiation if the field is bosonic or suppress it if the field is fermionic. This process leads to exponential growth or damping of the radiated flux and correlations among the quanta emitted at different times, unlike in the usual Hawking effect. These phenomena may be observable in condensed matter black hole analogs that exhibit “superluminal” dispersion.

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I. INTRODUCTION

Recent work has shown that Hawking radiation is highly insensitive to modifications of the short distance physics of the quantum field. In these models linear fields are considered, and the field equation is modified at high wavevectors in some preferred frame, yielding a nonlinear dispersion relation $\omega(k)$ relating frequency to wavevector. Models with both subluminal [1–3] and superluminal [4,5] group velocities at high wavevectors have been studied, including lattice black hole spacetimes [6] (which have subluminal dispersion). The picture that emerges from these studies is that the thermal Hawking spectrum is very robust for black holes with temperature much less than the energy scale of the new physics. Although the short distance physics does modify this spectrum, the modifications are so slight at the frequencies of interest that they seem well nigh impossible to observe.

We have found a dramatic exception to this rule however. If there is both an outer and an inner horizon, and if the dispersion is superluminal, then the Hawking process for a bosonic field is self-amplifying and the radiated flux grows exponentially in time, while for a fermionic field the process is self-attenuating. What happens is that

the negative energy partner of a Hawking particle, after falling to the inner horizon, “bounces” and returns to the outer horizon on a superluminal trajectory, where it either stimulates or suppresses more Hawking radiation in the bosonic or fermionic case respectively. This secondary radiation is not only different than the usual Hawking flux, but it is correlated to the prior radiation. In the bosonic case the process continues to amplify at least until the back reaction becomes important.

Charged black holes have inner horizons, but astrophysical ones would lose their charge very rapidly, so it is difficult to imagine how this runaway Hawking effect could ever be observed for real black holes. Even so, it provides an interesting theoretical laboratory in which to explore the effects of short distance physics. Moreover, it is conceivably relevant to string theory, and it might be observable in a condensed matter analogue of a black hole. Let us briefly indicate these ideas in turn.

In spite of many points of close agreement between the physics of near extremal D-branes and black holes, a glaring discrepancy persists. If a radiating near extremal D-brane state is maintained at fixed energy by a constant influx of energy in a pure state, then the entropy in the radiation will be constant and there will be correlations in the radiation that emerges at different times. For a black hole, on the other hand, the usual Hawking process leads to uncorrelated thermal radiation for all time. The effects of superluminal dispersion invalidate the usual Hawking picture because the negative energy partners return to the event horizon. If there is something analogous to the superluminal dispersion of our model in string theory, then perhaps that could eliminate the discrepancy between the string and black hole pictures. This may not be so far-fetched. String theory is, after all, non-local in some sense, and there is some evidence [7] suggesting that it supports superluminal effects.

A condensed matter analog—Unruh’s sonic black hole [8,1,9]—was the original stimulus for the development of the dispersive models. In this model, a sonic horizon occurs where the flow velocity of an inhomogeneous fluid exceeds the speed of sound. Although it seems unlikely that this situation can be experimentally realized for a low temperature superfluid, there are variations of the idea that might be realizable, involving quasiparticles other than phonons in different systems. For example, this may occur for fermion quasiparticles in rotating superfluid vortex cores with gap nodes such as $^3\text{He-A}$ or d -wave superconductors [10], or in moving $^3\text{He-A}$ tex-

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tures [11]. In both these examples there are both inner and outer horizons. Moreover, the quasiparticle dispersion relation is “relativistic” sufficiently near a gap node, and the group velocity increases (i.e. becomes “superluminal”) as the difference between the momentum and the gap node increases, so the effective field theory has “superluminal” dispersion. Thus it is not inconceivable that the phenomena discussed here may someday be observable.

This paper is organized as follows. In section II the superluminal dispersion model for both bosons and fermions is discussed. The propagation of wavepackets in the black hole spacetime with inner and outer horizons is analyzed qualitatively in section III and the implications for the amplification or suppression and the correlations in the Hawking radiation are drawn in section IV. Section V renders the previous discussion quantitative by using explicit wavepacket solutions (derived in the appendix) to find expressions for the number and correlations between the radiated quanta. Open issues concerning the boundary conditions on the quantum state and the gravitational back reaction are discussed briefly in section VI.

We use units with $\hbar = c = 1$ and metric signature $(+---)$.

II. SUPERLUMINAL DISPERSION MODEL

A 2-dimensional model suffices to illustrate the essential physics. We assume the spacetime metric is static, and therefore [12] coordinates can be chosen (at least locally) so that the line element takes the form

$$ds^2 = dt^2 - (dx - v(x)dt)^2. \quad (2.1)$$

A special case is the line element of the $t - r$ subspace of the Reissner-Nordström black hole spacetime in Painlevé-Gullstrand coordinates, where $v(r) = -\sqrt{2GM/r - Q^2/r^2}$. (These coordinates cover the black hole interior down to where $v(r) = 0$, at $r = Q^2/2GM$.) More generally, we consider any $v(x)$ which is negative, vanishes as $x \rightarrow +\infty$, and is greater than -1 except between inner and outer horizons, located at x_i and x_o , where $v(x_{i,o}) = -1$.

A. Boson field

We adopt a linear field theory with higher spatial derivatives included in the action in order to provide a superluminal dispersion relation. In this section we restrict to the case of a real bosonic field. The case of a Majorana fermion field will be discussed in section II B. The action for the field is given by

$$S_\phi = \frac{1}{2} \int d^2x \left[\left((\partial_t + v\partial_x)\phi \right)^2 + \phi \hat{F}(\partial_x)\phi \right]. \quad (2.2)$$

In the ordinary relativistic action one has $\hat{F}(\partial_x) = \partial_x^2$. In this paper we take

$$\hat{F}(\partial_x) = \partial_x^2 - \frac{1}{k_0^2} \partial_x^4. \quad (2.3)$$

To motivate this action we note that the black hole defines a preferred frame, the frame of freely falling observers. In the Painlevé-Gullstrand coordinate system, $(\partial_t + v\partial_x)$ is the unit tangent to free fall world lines that start from rest at infinity, and ∂_x is its unit, outward pointing normal. Our action comes from modifying the derivative operator, only along the unit normal ∂_x , by the addition of higher derivative terms which become important only when the wavelength is of order $1/k_0$ or shorter. We will assume that this length scale of “new physics” is much shorter than the length scale of the metric (2.1), i.e. $k_0 \gg |v'/v|$. (In particular, we assume $k_0 \gg \kappa$, where $\kappa = |v'(x_{i,o})|$ is the surface gravity of the horizon.) The idea is that the microstructure of spacetime, or of a condensed matter analog, might give rise to such higher derivative terms in the effective action. The choice (2.3) is just the generic form for the lowest order such term that is reflection invariant and produces superluminal group velocities.

The action (2.2,2.3) produces the equation of motion

$$(\partial_t + \partial_x v)(\partial_t + v\partial_x)\phi = \partial_x^2\phi - \frac{1}{k_0^2}\partial_x^4\phi. \quad (2.4)$$

To derive the dispersion relation for this equation we look for solutions of the form

$$\phi(t, x) = \exp\left(-i\omega t + i \int^x k(x') dx'\right) \quad (2.5)$$

where $k(x)$ is a position dependent wavevector. Substituting this ansatz into the equation of motion (2.4) and neglecting derivatives of $v(x)$ and $k(x)$ results in the dispersion relation

$$(\omega - vk)^2 = F^2(k) \quad (2.6)$$

where

$$F^2(k) = k^2 + k^4/k_0^2. \quad (2.7)$$

The group velocity in the free-fall frame is dF/dk , so wavepackets with $k \ll k_0$ propagate near the speed of light, whereas wavepackets with $k \gtrsim k_0$ propagate superluminally.

The dispersion relation (2.6) is a fourth order polynomial equation in the wavevector k so it has four solutions for k at given values of ω and v . The nature of these roots is revealed by a graphical method. In figure 1 we plot the straight line $(\omega + |v|k)$ for one value of ω (satisfying $0 < \omega \ll k_0$) and two values of v , and the curve $\pm F(k)$, as functions of k . (We define $F(k)$ as the *positive* square root of (2.7).) The intersection

points are the allowed real wavevector roots to the dispersion relation. When $|v| < 1$ there are only two *real* roots (corresponding to the two roots for the ordinary dispersion relation with $F_{ord}(k) = \pm k$), the other two being complex. The positive wavevector is denoted k_{+s} . When $|v| \gtrsim 1 + \frac{3}{2}(\omega/k_0)^{2/3} \approx 1$, on the other hand, all four roots are real, with one positive and three negative. The positive wavevector is denoted k_+ in this case and, in decreasing magnitude, the negative wavevectors are denoted k_- and k_{-s} respectively (the other negative wavevector corresponds to an ingoing wave that plays no role in this paper so we do not give it a name). These roots are labeled in figure 1.

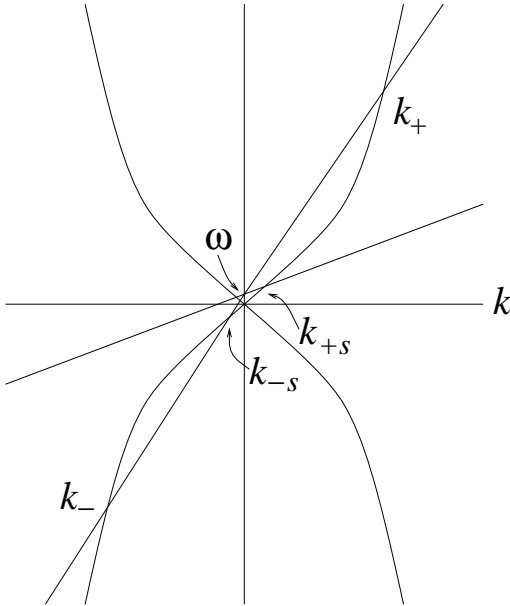


FIG. 1. Plot of $(\omega + |v|k)$ (for one value of ω and two values of v) and $F(k)$ as functions of k . The intersection points of the curves are the allowed wavevector roots of the dispersion relation (2.6).

The dispersion relation plot in figure 1 is also quite convenient for tracing the motion of wavepackets in the background spacetime. The coordinate group velocity $v_g = dx/dt$ of a wavepacket centered on a given wavevector is given by

$$v_g = \frac{d\omega}{dk} = -|v| \pm \frac{dF}{dk}, \quad (2.8)$$

where $\pm dF/dk$ is the group velocity in the free-fall frame. Thus at any wavevector v_g is just the slope of the $\pm F(k)$ curve minus the slope of the straight line $(\omega + |v|k)$. For all four types of wavevectors $k_{\pm s, \pm}$ of interest to us, $\pm dF/dk$ is positive, hence the sign of v_g is determined by which of the two slopes is larger, something that is easily read from the figure. For $\omega > 0$, the group velocity for k_{+s} and k_{\pm} is positive, whereas for k_{-s} it is negative.

When generalized to a complex scalar field, the action (2.2) is invariant under constant phase transformations of the field. This implies the existence of a conserved

current j^μ . The integral of the time component j^0 over a spatial slice serves as a conserved inner product when evaluated on complex solutions to the equation of motion (2.4). For the metric (2.1), this inner product takes the form

$$(f, g) = i \int dx \left(f^* (\partial_t + v \partial_x) g - g (\partial_t + v \partial_x) f^* \right), \quad (2.9)$$

where $f(t, x)$ and $g(t, x)$ are solutions to (2.4).

Two classes of complex solutions to the field equation (2.4) are of special interest for quantization. The first are the positive free fall frequency wavepackets. They can be written as sums of solutions satisfying

$$(\partial_t + v \partial_x) f(t, x) = -i\omega' f(t, x) \quad (2.10)$$

where $\omega' > 0$. The second are the positive Killing frequency wavepackets. These are sums of solutions of the form $e^{-i\omega t} \varphi(x)$ where $\omega > 0$. A positive free fall frequency wavepacket confined to a constant $v(x)$ interval at one time necessarily has a positive norm under (2.9), as does a positive Killing frequency wavepacket confined to a region where $v(x) = 0$ (where Killing frequency coincides with free-fall frequency). Since the norm is conserved, it is positive at all times if it is at one time, even when the wavepacket does not remain in an interval of constant or vanishing $v(x)$. Note that if the wavelength is small compared to the scale of variations of $v(x)$, then a positive free-fall frequency wavepacket will have positive norm even if $v(x)$ is not constant.

To quantize the field we assume that $\hat{\phi}(t, x)$ is a self-adjoint operator solution to the field equation that satisfies the canonical commutation relations. We define the annihilation operator $a(f)$ associated to a normalized complex solution to the wave equation $f(t, x)$ by

$$a(f) \equiv (f, \hat{\phi}). \quad (2.11)$$

The commutation relations for the field operator are equivalent to the relations

$$[a(f), a^\dagger(g)] = (f, g) \quad (2.12)$$

for all f and g . If $f(t, x)$ is a positive norm solution, then $a(f)$ behaves as an annihilation operator. If $f(t, x)$ is a negative norm solution, $f^*(t, x)$ has positive norm, so $a(f) = -a^\dagger(f^*)$ behaves as a creation operator.

B. Fermion field

For simplicity we consider two-dimensional massless Majorana fermions. Following the conventions of [13], the action in a general curved spacetime is given by¹

¹In higher dimensions there would be a spin connection term as well. In two-dimensions it is easy to show that this term vanishes identically.

$$S_\psi = \frac{i}{2} \int d^2x \sqrt{-g} \bar{\psi} \Gamma^\mu \partial_\mu \psi \quad (2.13)$$

where $\Gamma^\mu = \Gamma^a e_a^\mu$ and e_a^μ is the zweibein. We take the flat space gamma matrices as

$$\Gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (2.14)$$

Decomposing the spinor ψ as

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \quad (2.15)$$

and expanding the action in the metric (2.1) using the zweibein $(e_0, e_1) = (\partial_t + v\partial_x, \partial_x)$ we find

$$S_\psi = \frac{i}{2} \int d^2x \left(\psi_+ (\partial_t + (1+v)\partial_x) \psi_+ + \psi_- (\partial_t - (1-v)\partial_x) \psi_- \right). \quad (2.16)$$

In this form it is clear that ψ_+ and ψ_- do not mix. Furthermore at infinity, where $v(x) = 0$, ψ_+ is right-moving while ψ_- is left-moving. We therefore drop ψ_- in the remainder as it plays no role in the Hawking radiation calculation.

Following the same motivation described in section II, we now modify the action for ψ_+ by subtracting the higher derivative term $k_0^{-2} \psi_+ \partial_x^3 \psi_+$, obtaining the action

$$S_\psi = \frac{i}{2} \int d^2x \left(\psi_+ (\partial_t + (1+v)\partial_x - k_0^{-2} \partial_x^3) \psi_+ \right). \quad (2.17)$$

Varying with respect to ψ_+ results in the equation of motion

$$(\partial_t + v\partial_x + \partial_x v/2 + \partial_x - k_0^{-2} \partial_x^3) \psi_+ = 0. \quad (2.18)$$

Substituting $\psi_+(t, x) = \exp(-i\omega t + i \int^x k(x') dx')$ into the equation of motion and dropping derivatives of $k(x)$ and $v(x)$ results in the dispersion relation

$$\omega - vk = k + k^3/k_0^2. \quad (2.19)$$

This is the same (up to the coefficient of the k^3 term and higher order terms) as the branch of the scalar field dispersion relation corresponding to positive group velocity in the free-fall frame given in (2.6) and displayed in figure 1. The classification of scalar wavepacket types in section (II A) therefore applies to fermion wavepackets as well. In particular, the higher derivative term leads to superluminal propagation at large wavevectors.

To quantize the field we assume that $\hat{\psi}_+(t, x)$ is a self-adjoint operator solution to the field equation that satisfies the canonical anti-commutation relations $\{\hat{\psi}_+(t, x), \hat{\psi}_+(t, x')\} = \delta(x, x')$. The conserved inner product is the integral of the time component of the conserved current associated with phase invariance of the action (2.17) (generalized to complex fermions), and takes the form

$$\langle \psi_1, \psi_2 \rangle = \int dx \psi_1^* \psi_2. \quad (2.20)$$

We define the annihilation operator $b(f)$ associated to a normalized complex solution to the wave equation $f(t, x)$ by

$$b(f) \equiv \langle f, \hat{\psi}_+ \rangle. \quad (2.21)$$

The anti-commutation relations for the field operator are then equivalent to the relations

$$\{b(f), b^\dagger(g)\} = \langle f, g \rangle \quad (2.22)$$

for all f and g . We represent the operators $b(f)$ on the fermionic Fock space generated by positive free-fall frequency solutions to the equation of motion (2.18). If $f(t, x)$ is a positive free-fall frequency solution then $b(f)$ behaves as an annihilation operator on this space. If $f(t, x)$ is a negative free-fall frequency solution, then $f^*(t, x)$ has positive free-fall frequency, so $b(f) = b^\dagger(f^*)$ behaves as a creation operator.

III. WAVEPACKET PROPAGATION

In this section we give a qualitative analysis of the role of the inner horizon in modifying the Hawking radiation. This analysis will exploit a WKB description of wavepacket propagation, allowing for non-WKB “mode conversion” in the vicinity of the horizons. The analysis applies equally well for the bosonic and fermionic quantum fields. Scattering of waves on account of the background curvature of the metric (2.1) is negligible as long as the radius of curvature is much greater than $1/k_0$. For small wavevectors, $k \ll k_0$, this is because the wave equation is approximately conformally invariant and the metric (like any two-dimensional metric) is conformally flat. For large wavevectors, $k \gtrsim k_0$, it is because the wavelength is much smaller than the radius of curvature.

We begin outside the outer horizon with a low frequency outgoing wavepacket peaked around a wavevector of type k_{+s} (see figure 1), and we follow this wavepacket backwards in time. A sketch of what we find is given in figure 2. The final wavepacket (i.e. the one we begin with) is labeled $+s$ in figure 2. This packet has positive group velocity and therefore is right-moving, as can be seen from the graph of the dispersion relation (Fig. 1). Following this packet backward in time it moves toward the black hole and blueshifts. The Killing frequency ω is conserved, so the increase in the wavevector can be seen from Fig. 1 by increasing the slope of the straight line while keeping the intercept fixed. As the wavevector grows, the group velocity increases in the free-fall frame, and so the packet becomes superluminal and crosses the horizon (backward in time), becoming a packet with wavevectors of type k_+ (see figure 2).

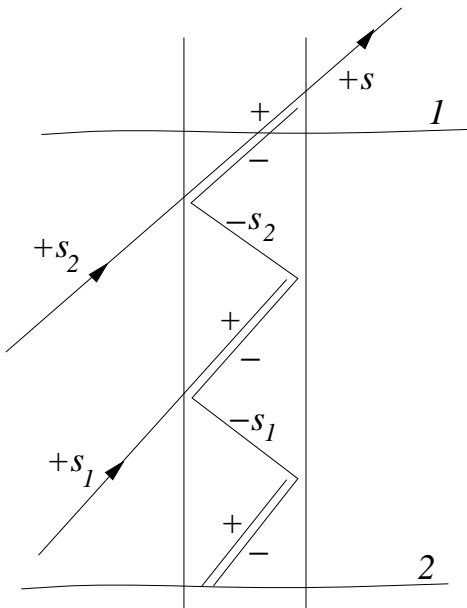


FIG. 2. Spacetime sketch of the evolution of an outgoing k_{+s} wavepacket backward in time. A line end indicates a wavepacket arising from mode conversion, while an unbroken line indicates that the wavevector evolves continuously on the dispersion curve.

The wavepacket inside the horizon also has a k_- component, which is not obvious if we simply follow continuously along the dispersion curve. In fact, the WKB approximation breaks down near the horizon, and “mode conversion” from the positive wavevector to the negative wavevector, negative free-fall frequency, branch of the dispersion relation occurs. This is easily shown analytically, and is made plausible by the fact that, around the horizon, the straight line of figure 1 nearly coincides with a large portion of the curved line of the dispersion curve, thus allowing other wavevectors to become mixed in. The dispersion relation allows wavevectors of types k_+ , k_- , and k_{-s} in between the horizons, however only the first two are right moving, whereas the last type is left moving. Since our final wavepacket is by assumption purely outgoing outside the horizon, there can be no k_{-s} component generated here. The k_+ and k_- wavepackets are labeled + and - in figure 2. In this figure a line end indicates a wavepacket arising from mode conversion, while an unbroken line indicates that the wavevector evolves continuously on the dispersion curve.

The k_+ and k_- packets propagate backward in time toward the inner horizon where they both undergo partial mode conversion. The group velocity of the k_+ packet remains positive around the inner horizon and therefore it can cross, becoming a k_{+s} packet, labeled $+s_2$ in figure 2. As before though, there is also some mode conversion from the positive to the negative wavevector branch of the dispersion relation, and a left-moving k_{-s} packet ($-s_2$ in figure 2) is generated which propagates backward in time back toward the outer horizon. The k_- packet on the other hand cannot cross the inner horizon on the

negative wavevector branch because its group velocity drops to zero at the horizon. Indeed the group velocity goes through zero and becomes negative, so the k_- packet turns around and propagates back toward the outer horizon as a k_{-s} packet still on the negative wavevector branch. In addition, some mode conversion from the negative to the positive wavevector branch of the dispersion relation occurs at the inner horizon. Therefore part of the k_- packet does cross the horizon as a k_{+s} packet and is superposed with the k_{+s} packet that evolved from the k_+ packet.

The k_{+s} packet inside the inner horizon continues propagating to the left backward in time. The k_{-s} packet however returns to the outer horizon, near which its group velocity drops to zero. Again, partial mode conversion to the positive wavevector branch occurs, so the k_{-s} packet evolves backward in time to a pair of k_+ and k_- packets which are heading back to the inner horizon. This is now almost the same situation we started with, since the original k_{+s} packet also evolved into a pair of k_+ and k_- packets between the horizons (although with a different relative weight). The analysis given above thus tells us qualitatively what happens when they reach the inner horizon, namely, the same thing as happened before. The general pattern that emerges is shown in figure 2.

We have so far discussed the history of an outgoing k_{+s} wavepacket followed backward in time. It is also instructive to look at the *future* evolution of a k_{-s} wavepacket in between the horizons, since the negative energy partner of a Hawking particle is such a wavepacket. This evolution can be inferred by the same sort of analysis just given, or simply by time and space reversal of that analysis, and is shown in figure 3.

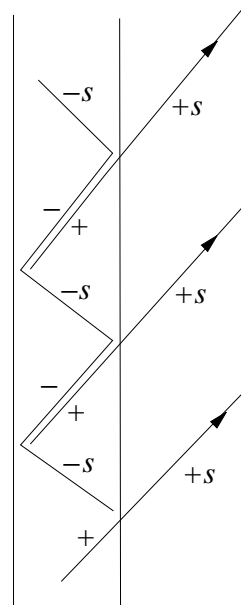


FIG. 3. Spacetime sketch of the trajectory of a Hawking particle and its partner forward in time. A line end indicates a wavepacket arising from mode conversion, while an unbroken line indicates that the wavevector evolves continuously on the dispersion curve.

IV. PARTICLE CREATION: ORIGIN OF THE AMPLIFICATION OR SUPPRESSION OF HAWKING RADIATION

The amount of particle creation in an outgoing positive frequency wavepacket ψ is indicated by the expectation value of the number operator $N(\psi) = a^\dagger(\psi)a(\psi)$. To determine this expectation value an initial quantum state must be specified. Let us define an in-Hilbert space on some spacelike surface as the Fock space generated by positive free-fall frequency wavepackets on that surface. The corresponding ground state is then annihilated by annihilation operators of these wavepackets. We shall suppose the initial state is such a free-fall ground state associated with a given surface Σ . Decomposing $\psi = \psi^+ + \psi^-$ into its positive and negative free-fall frequency parts on Σ , the ground state condition implies in the bosonic case that $\langle N(\psi) \rangle = -(\psi^-, \psi^-)$, and in the fermionic case $\langle N(\psi) \rangle = \langle \psi^-, \psi^- \rangle$.

Suppose we choose Σ as surface 1 in figure 2, i.e., a surface that cuts through the k_+ and k_- packets first produced by propagating the k_{+s} packet back in time. Then the number expectation value for the k_{+s} packet is just (minus) the norm of the k_- packet. In [4,5] this was shown (for bosons) to be thermal at the Hawking temperature, for wavepackets with Killing frequencies ω satisfying $\kappa \lesssim \omega \ll k_0$. That is, the standard Hawking effect occurs even in the presence of superluminal dispersion, if there is only a single horizon.

When there is also an inner horizon, the particle creation depends very much on which surface is used to define the initial ground state. If we impose the ground state condition on the earlier surface 2 in figure 2, instead of surface 1, the occupation number for the final k_{+s} packet is no longer thermal. The norm of the negative frequency part of the wavepacket on surface 2 is determined not just by the final passage across the outer horizon, but also by the mode conversion processes at the inner and outer horizons.

As the time between the initial ground state and the final outgoing wavepacket grows, there is an exponential amplification or suppression in the occupation number of the final wavepacket in the boson and fermion cases respectively. To see why, note that the k_{-s} packet denoted $-s_2$ in figure 2 evolves into the orthogonal k_{+s} and k_{-s} packets denoted $+s_1$ and $-s_1$ respectively, hence the norms are related by

$$\| -s_2 \|^2 = \| +s_1 \|^2 + \| -s_1 \|^2 \quad (4.1)$$

where $\|f\|^2$ stands for (f, f) in the bosonic case and $\langle f, f \rangle$ in the fermionic case.

Consider first the bosonic case. A $k_{+s}(k_{-s})$ packet has positive (negative) free fall frequency and therefore positive (negative) norm under (2.9), so it follows from (4.1) that $\| -s_1 \|^2$ is larger in magnitude than $\| -s_2 \|^2$. Continuing into the past this process repeats, and for each “bounce” between the horizons the norm of the wavepacket between the horizons grows by some fixed multiple, resulting in exponential growth of both $\| -s_n \|^2$ and $\| +s_n \|^2$.² Since the negative frequency part of this wavepacket determines the number of created particles in the final outgoing wavepacket, that number will grow exponentially in the time between the initial surface Σ and the emergence of the outgoing wavepacket ψ . Viewed forward in time, the Hawking effect is a self-amplifying process since the negative energy partners of the Hawking particles return to the event horizon (in the form of a pair of k_+ and k_- packets) and stimulate the emission of more radiation and more partners. The wavepacket trajectories associated with this forward in time picture are shown in figure 3.

For a fermionic field, the above discussion is modified only by the fact that all wavepackets have positive norm, so equation (4.1) implies that $\| -s_1 \|^2$ is *smaller* in magnitude than $\| -s_2 \|^2$. This means that the number of created particles will be exponentially *damped* in time. In effect, the allowed states between the horizons for the negative energy partners of the Hawking particles become filled, cutting off further pair creation.

One further important point can be extracted from this analysis. Since a single particle/partner wavepacket pair evolves to a sequence of outgoing wavepackets as shown in figure 3, the states of all these outgoing wavepackets will be *correlated*. This can also be seen from the backwards in time picture. It is clear from figure 2 that successive outgoing wavepackets will have past histories that partly overlap, in particular on the initial ground state surface, so there will be correlations between the quanta emitted from the horizon at different times. These correlations are in sharp contrast to the usual Hawking effect which produces uncorrelated thermal radiation. The information loss that is normally associated with the correlations between Hawking quanta and their partners is largely eliminated, since an unending sequence of Hawking quanta is coherently correlated to the same partner degrees of freedom.

²It is perhaps surprising to have exponential growth in time when there are no imaginary frequency solutions to the dispersion relation (2.6). There is no contradiction however, since these exponentially growing wavepacket solutions cannot be Fourier transformed in time, so need not be expressible as superpositions of time-independent mode solutions.

V. QUANTITATIVE ANALYSIS

The qualitative analysis of the previous section will now be sharpened by explicitly constructing the wavepacket solutions discussed there. This will allow us to quantify the amount of amplification, suppression, and correlation of the black hole radiation. In the first two subsections we treat only the bosonic case, and in the last subsection we discuss the fermionic case.

A. Wavepacket solutions

The basic idea applied here is to patch together local wavepacket solutions with the aid of “evolution formulae”. The derivation of these evolution formulae is discussed in the Appendix of this paper. They are derived using connection formulae for time-independent mode functions which are obtained by matching WKB solutions to near-horizon approximations. Forming wavepackets with the mode functions we then obtain the evolution formulae for the wavepackets.

Evolution formulae are needed for two different boundary conditions at both the inner and outer horizons, corresponding to the spacetime diagrams in figures 4a-d. Using the notation “ $f \rightarrow g$ ” to denote “ f evolves to g ” (forward in time), the evolution formulae about the outer horizon are:

$$\psi_{n,+} + \psi_{n,-} \rightarrow \psi_{n,+s} \quad (5.1a)$$

$$\chi_{n,+} + \chi_{n,-} \rightarrow \psi_{n+1,-s}, \quad (5.1b)$$

while about the inner horizon they are

$$\psi_{i,n,+s} + \psi_{n,-s} \rightarrow \psi_{n,+} + \psi_{n,-} \quad (5.1c)$$

$$\chi_{i,n,+s} + \psi_{n,-s} \rightarrow \chi_{n,+} + \chi_{n,-}, \quad (5.1d)$$

where all packets have been left unnormalized in order to keep the formulae as simple as possible. The evolution formulae given here are preferred for evolving packets backwards in time. Following the same techniques described in the appendix evolution formulae more conducive to evolving wavepackets forward in time can be derived. The $+$, $-$, $+s$ and $-s$ subscripts denote which type of wavevector the packet is peaked about. $\psi_{n,+s}$ lies outside the outer horizon, while $\psi_{i,n,+s}$ and $\chi_{i,n,+s}$ both lie inside the inner horizon. The subscript n is a sort of time variable. Translation of n by one unit has the effect of translating the wavepacket in time by a certain amount and also distorting the wavepacket. Note that in the evolution formula at the outer horizon (5.1b) n increases by one unit on the $\psi_{n+1,-s}$ wavepacket. The evolution formulae (5.1a-d) are basically scattering solutions about a black hole event horizon with (5.1a-b) corresponding to scattering the $\psi_{n,+s}$ and $\psi_{n,-s}$ packets off the outer horizon backward in time (see figures 4a and b respectively) and (5.1c-d) corresponding to scattering

the resulting combinations of $+$ and $-$ packets off the inner horizon backward in time as well (see figures 4c and d respectively).

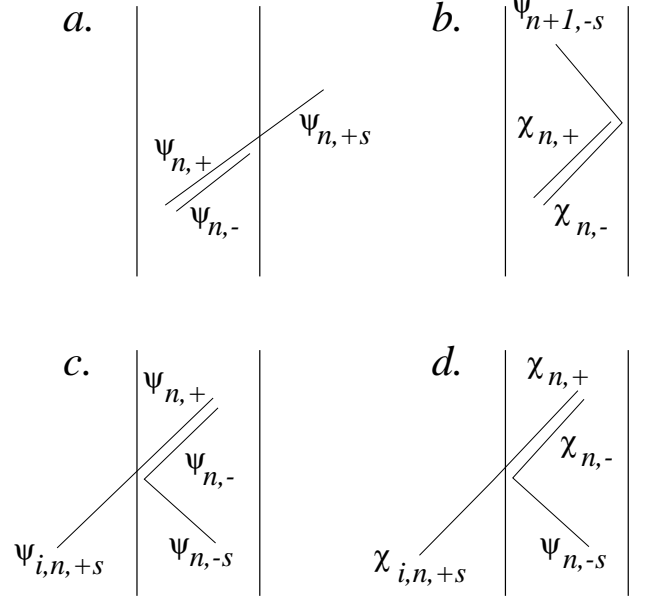


FIG. 4. Spacetime sketches of the local wavepacket evolutions given by (5.1a-d) respectively.

To construct the wavepacket solution with final data consisting of a k_{+s} packet outside the outer horizon we start with the local solution (5.1a) (figure 4a). This clearly is not a global solution since $\psi_{n,\pm}$ do not solve the equation of motion (2.4) about the inner horizon. The combination may be replaced by (5.1c) which does however. This results in the evolution formula

$$(\psi_{i,n,+s} + \psi_{n,-s}) \rightarrow \psi_{n,+s}. \quad (5.2)$$

The evolution (5.2) is also not a global solution to the equation of motion (2.4) however since $\psi_{n,-s}$ is not a solution about the outer horizon. Using (5.1b) to evolve $\psi_{n,-s}$ about the outer horizon, followed by (5.1d) to evolve the resulting wavepackets $\psi_{n-1,+}$ and $\psi_{n-1,-}$ about the inner horizon, we obtain an evolution formula that can be iterated indefinitely,

$$(\chi_{i,n-1,+s} + \psi_{n-1,-s}) \rightarrow \psi_{n,-s}. \quad (5.3)$$

Beginning with (5.2) and iterating (5.3) $(n-m)$ times yields

$$\left(\psi_{i,n,+s} + \sum_{j=1}^{n-m} \chi_{i,n-j,+s} + \psi_{m,-s} \right) \rightarrow \psi_{n,+s}. \quad (5.4)$$

In this manner we can evolve the final wavepacket packet back to the spacelike surface where the initial ground state boundary condition is defined. Solutions of the form (5.4) correspond to those used in the qualitative discussion of section III, which are depicted in Fig. 2

(wherein the first k_{-s} packet has been traded for a k_{\pm} pair using (5.1b).)

If the horizons are not sufficiently widely separated then intermediate wavepackets that arise between the initial and final packets of (5.4) will overlap with the initial and final wavepackets, thus complicating the analysis of particle creation. We can avoid such complications by constructing a different solution. Setting n equal to m in (5.2) and subtracting from (5.4), we obtain

$$\left(\psi_{i,n,+s} + \sum_{j=1}^{n-m} \chi_{i,n-j,+s} \psi_{i,m,+s} \right) \rightarrow (\psi_{n,+s} - \psi_{m,+s}). \quad (5.5)$$

This solution corresponds to sending $(n-m)$ $\chi_{i,k,+s}$ packets and a pair of $\psi_{i,k,+s}$ packets into the inner horizon and getting a pair of ψ_{+s} packets out of the outer horizon.

B. Particle creation

We can now compute the average number of particles in the wavepacket³ $\hat{\psi}_{n,+s}$ and the correlations between emitted particles for different values of the time index n . To begin with let us evaluate the occupation number $\langle 0|N(\hat{\psi}_{0,+s})|0\rangle$ of the first outgoing packet after the initial state condition is assumed. Then the only evolution formula we need is (5.1a), with $n = 0$. The annihilation operator for a normalized wavepacket f is given by $a(f) = (f, \hat{\phi})$ (2.11). Taking the inner product of (5.1a) with the quantum field $\hat{\phi}$, and using the ground state conditions

$$a(\hat{\psi}_{0,+})|0\rangle = 0 = a(\hat{\psi}_{0,-}^*)|0\rangle, \quad (5.6)$$

we obtain

$$\begin{aligned} \langle 0|N(\hat{\psi}_{0,+s})|0\rangle &= -\frac{(\psi_{0,-}, \psi_{0,-})}{(\psi_{n,+s}, \psi_{n,+s})} \\ &= \frac{1}{\omega_u - \omega_l} \int_{\omega_l}^{\omega_u} d\omega \frac{1}{e^{2\pi\omega/\kappa} - 1}. \end{aligned} \quad (5.7)$$

(The norms (A30) were used in the last equality.) This is just the Planck distribution at the Hawking temperature $T_H = \kappa/2\pi$, as was shown previously [4,5] for a superluminal dispersive field theory in the case that there is just one horizon. It holds for $\kappa \lesssim \omega \ll k_0$.

We would have obtained a different result for $\langle 0|N(\hat{\psi}_{0,+s})|0\rangle$ had we replaced the initial conditions (5.6) with, for example,

$$a(\hat{\psi}_{i,0,+s})|0\rangle = 0 = a(\hat{\psi}_{0,-s}^*)|0\rangle. \quad (5.8)$$

Indeed, from (5.2) with $n = 0$, we find that if (5.8) holds the occupation number of $\hat{\psi}_{0,+s}$ is given by

$$\begin{aligned} \langle 0|N(\hat{\psi}_{0,+s})|0\rangle &= -\frac{(\psi_{0,-s}, \psi_{0,-s})}{(\psi_{0,+s}, \psi_{0,+s})} \\ &= \frac{1}{\omega_u - \omega_l} \int_{\omega_l}^{\omega_u} d\omega \\ &\quad \times \frac{2(1 - \cos(\theta_+(\omega) - \theta_-(\omega)))}{e^{2\pi\omega/\kappa} + e^{-2\pi\omega/\kappa} - 2}. \end{aligned} \quad (5.9)$$

The phase angles $\theta_{\pm}(\omega)$ are defined implicitly in (A18), and the norms (A30) were used in the last equality of (5.9). This differs from the thermal result (5.7).

It is not yet clear to us what is the “correct” initial condition on the quantum state of the field. To determine this would require following the evolution of the field state as the black hole (or condensed matter black hole analog) forms. It does seem however that the conditions $a(\hat{\psi}_{i,k,+s})|0\rangle = 0$ are likely to hold, while the remaining specification of the state remains to be determined. Fortunately these conditions alone suffice to determine the rate of growth of the number of particles emitted and the correlations between them.

In order to find the number and correlations in the radiation for $n > 0$ we use the solution (5.5). We take $n > m \gg 1$ so that the intermediate wavepackets that entered the construction of (5.5) will not have any support on the initial surface, and we assume the ground state conditions

$$a(\hat{\psi}_{i,k,+s})|0\rangle = 0 = a(\hat{\chi}_{i,k,+s})|0\rangle \quad (5.10)$$

for $k \geq m$. Taking the inner product of (5.5) with the field operator $\hat{\phi}$ and using conditions (5.10) we obtain

$$a(\hat{\psi}_{n,+s})|0\rangle = \sqrt{\frac{(\psi_{m,+s}, \psi_{m,+s})}{(\psi_{n,+s}, \psi_{n,+s})}} a(\hat{\psi}_{m,+s})|0\rangle, \quad (5.11)$$

from which it follows that

$$\begin{aligned} \langle 0|a^\dagger(\hat{\psi}_{k,+s})a(\hat{\psi}_{n,+s})|0\rangle &= \\ &= \frac{(\psi_{m,+s}, \psi_{m,+s})}{\sqrt{(\psi_{k,+s}, \psi_{k,+s})(\psi_{n,+s}, \psi_{n,+s})}} \langle 0|N(\hat{\psi}_{m,+s})|0\rangle. \end{aligned} \quad (5.12)$$

In particular taking $k = n$ we obtain

$$\langle 0|N(\hat{\psi}_{n,+s})|0\rangle = \frac{(\psi_{m,+s}, \psi_{m,+s})}{(\psi_{n,+s}, \psi_{n,+s})} \langle 0|N(\hat{\psi}_{m,+s})|0\rangle. \quad (5.13)$$

The norm of $\psi_{n,+s}$ (A30) is given by

$$(\psi_{n,+s}, \psi_{n,+s}) = 4\pi \int_{\omega_l}^{\omega_u} d\omega \left(1 + \frac{1 - \cos(\theta_+ - \theta_-)}{2 \sinh^2(\pi\omega/\kappa)} \right)^{-n} \quad (5.14)$$

where we have used (A21). This decreases monotonically with n except for at most a discrete set of frequencies for

³We use a $\hat{\cdot}$ to denote normalized wavepackets.

which $\theta_+(\omega) = \theta_-(\omega) + 2\pi k$ for some integer k . (For these frequencies T_1 (A21) vanishes, so according to (A20) the corresponding mode is a bound state trapped between the horizons.) Therefore the particle creation in $\hat{\psi}_{n,+s}$ increases monotonically with n , diverging as $n \rightarrow \infty$. In particular, if the wavepackets are narrowly peaked about a frequency ω , (5.13) yields

$$\frac{\langle 0|N(\hat{\psi}_{n,+s})|0\rangle}{\langle 0|N(\hat{\psi}_{m,+s})|0\rangle} = \left(1 + \frac{1 - \cos(\theta_+ - \theta_-)}{2 \sinh^2(\pi\omega/\kappa)}\right)^{n-m}, \quad (5.15)$$

which grows exponentially with $n - m$.

A measure of the correlation between emitted particles is given by

$$C(m, n) := \frac{\langle 0|a^\dagger(\hat{\psi}_{m,+s})a(\hat{\psi}_{n,+s})|0\rangle}{\left(\langle 0|N(\hat{\psi}_{m,+s})|0\rangle\langle 0|N(\hat{\psi}_{n,+s})|0\rangle\right)^{1/2}} = 1, \quad (5.16)$$

independent of the difference $n - m$. This should be contrasted with the correlation obtained when $\hat{\phi}$ satisfies the ordinary wave equation,

$$C(m, n) = (\hat{\psi}_{m,+s}, \hat{\psi}_{n,+s}), \quad (5.17)$$

which is nonvanishing only to the extent that these wavepackets are not orthogonal. As $n - m$ grows the overlap of these wavepackets and hence the correlation (5.17) goes to zero, whereas the correlation (5.16) remains.

Finally let us estimate the time between the successive particle emissions (see figure 3), i.e. the difference in times when successive $\psi_{n,+s}$ packets (A22) cross a fixed x coordinate. The trajectory of the packets is given approximately by the condition of stationary phase,

$$t = \frac{d}{d\omega} \text{Arg}\left(A_n(\omega)\phi_{+s}(x, \omega)\right) \quad (5.18)$$

and therefore the time Δt between the n^{th} and $(n+1)^{\text{th}}$ packets crossing the coordinate x is given approximately by

$$\Delta t \approx \frac{d}{d\omega} \text{Arg}(T_3) = \frac{d}{d\omega} \left(\gamma + \text{Arg}\left(e^{-\pi\omega/\kappa}e^{-i\theta_+} - e^{\pi\omega/\kappa}e^{-i\theta_-}\right) \right) \quad (5.19)$$

where we have substituted for A_n using (A23), and T_3 is given by (A21).

Using the results given in the appendix the ω -dependence of the phase factors $\gamma(\omega)$ and $\theta_\pm(\omega)$ can be computed. Rather than carrying out this calculation—which we can only do explicitly for any particular $v(x)$ in some approximation anyway—let us make a rough estimate. The interval Δt is determined by the time it takes a wavepacket to “bounce” back and forth between the horizons. If $v(x) + 1$ is of order unity between the horizons, then using the group velocity of the k_{-s} and k_\pm waves one finds that this bounce time is of order a , the coordinate distance between the horizons.

C. Fermionic case

In this section we briefly describe the differences in the quantitative analysis of the fermion case. The derivation of the wavepacket solutions for fermions parallels that given for the scalar field in section V A and yields an evolution formula very similar to (5.5). The final expression for the average value of the number operator is identical in form to (5.13), however the norm $N_{n,+s}$ now increases monotonically with n so that the number expectation value decreases exponentially in n . This is to be expected since unlike the scalar case, the conserved norm $\langle f, f \rangle$ (2.20) is positive definite. The effect of this crucial difference is that instead of exponential growth of Hawking radiation we now get exponential decay.

VI. DISCUSSION

We left the question of the “correct” initial condition unanswered. For a condensed matter black hole it should be straightforward to deduce this by following the state of the field as the “black hole” forms. It seems fairly clear that the $\psi_{i,+s}$ wavepackets inside the inner horizon will be in their ground states. What is less clear is the state of the wavepackets ψ_+ , ψ_-^* and ψ_-^* between the horizons. For a real black hole—if one wants to entertain the possibility of superluminal dispersion—the same may be true. The $\psi_{i,+s}$ wavepackets inside the inner horizon arise from ingoing waves that scatter around or through the central singularity of a Reissner-Nordström black hole in the manner discussed in [14]. Since the region inside the inner horizon is static, it would seem plausible that these are in their ground state as well.

Another issue we have not touched is that of the gravitational back-reaction to the radiation studied here. In the bosonic case the exponential growth of the number of negative energy Hawking partners between the horizons would surely rapidly entail a strong gravitational reaction. In the fermionic case, the exponential suppression of radiation leads quickly to a state with no radiation at all. This is hard to reconcile with the usual picture in which Hawking radiation is a robust consequence of a general “well-behaved” state near the horizon. One would expect that although the negative energy states of the Hawking partners in the ergoregion between the horizons become filled, there is not all that much energy in these states (since the partners at late times are the same as the partners at earlier times due to the “bounce” between the horizons) so the back-reaction should be limited. If so, then why doesn’t the Hawking radiation continue? The answer, it would seem, is that although the state is reasonably well-behaved in terms of energy density, it has peculiar features in just those modes relevant to the Hawking effect.

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APPENDIX: WAVEPACKET SOLUTIONS

In this appendix we explain how the evolution formulae (5.1a-d) for wavepacket solutions are derived with the help of the results of [5]. We treat only the bosonic case, although the fermionic case is essentially identical.

The wavepacket evolution formulae are inferred from connection formulae for mode solutions to the field equation (2.4) of the form

$$u(t, x) = e^{-i\omega t} \phi(x, \omega), \quad (\text{A1})$$

where $\phi(x, \omega)$ satisfies the ordinary differential equation (ODE)

$$\begin{aligned} -\phi^{(iv)}(x) + (1 - v^2(x))\phi''(x) + 2v(x)(i\omega - v'(x))\phi'(x) \\ - i\omega(i\omega - v'(x))\phi(x) = 0. \end{aligned} \quad (\text{A2})$$

In [5] such solutions were constructed for a black hole spacetime with a single horizon. The basic technique used was to find approximate solutions to (A2) using the WKB approximation away from the horizon, and to match these solutions across the horizon by comparing to the near horizon solution obtained by the method of Laplace transforms.

1. Outer horizon connection formulae

Assuming that the horizon is located at $x = 0$, and that the metric in the vicinity of the horizon is given by

$$v(x) \approx -1 + \kappa x, \quad (\text{A3})$$

the analysis of [5] leads to the following two connection formulae:

$$K(e^{\pi\omega/2\kappa}\phi_+ + e^{-\pi\omega/2\kappa}\phi_-) \leftrightarrow \phi_{+s} \quad (\text{A4a})$$

$$-\phi_{-s} + K(e^{-\pi\omega/2\kappa}\phi_+ + e^{\pi\omega/2\kappa}\phi_-) \leftrightarrow 0, \quad (\text{A4b})$$

where

$$K = (\omega/2 \sinh(\pi\omega/\kappa))^{1/2}. \quad (\text{A5})$$

The notation “ $\phi_1(x) \leftrightarrow \phi_2(x)$ ” denotes that the approximate WKB solution $\phi_1(x)$ behind the horizon connects

to the approximate WKB solution $\phi_2(x)$ outside the horizon. The modes $\phi_{\pm}, \phi_{\pm s}$ are approximate WKB solutions to (A2) and are given by⁴

$$\begin{aligned} \phi_{\pm}(x) \approx C_{\pm}(v(x)^2 - 1)^{-3/4} \\ \times \exp\left(i \int_{-\epsilon}^x ds k_{\pm}(v(s), \omega)\right) \end{aligned} \quad (\text{A6})$$

$$\phi_{\pm s}(x) \approx C_{\pm s} \exp\left(i \int_{-\epsilon}^x ds k_{\pm s}(v(s), \omega)\right), \quad (\text{A7})$$

where the approximate WKB wavevectors are given by

$$k_{\pm} \approx \pm k_0 \sqrt{v^2 - 1} + \omega v / (1 - v^2) \quad (\text{A8})$$

$$k_{\pm s} \approx \omega / (1 + v), \quad (\text{A9})$$

provided we assume $\omega \ll k_0$ and choose $|x|, \epsilon \gg (\omega/k_0)^{2/3}/\kappa$. The WKB solutions are only valid for $|x| \gg \kappa^{-1/3}k_0^{-2/3}$. The coefficients $C_{\pm, \pm s}$ are necessary to match these WKB solutions to the near-horizon Laplace transform solutions. They can be determined by comparing the Laplace transform solutions given in [5] to the matching formulae (A4a-b) with the WKB modes (A6, A7) evaluated in the small x limit.⁵ We find that the coefficients are given by

$$\begin{aligned} C_{\pm} = i^{(1 \mp 1)/2} \exp\left(\mp i \frac{2}{3} \sqrt{2\kappa/k_0} (k_0 \epsilon)^{3/2}\right) \\ \times \exp\left(-i \frac{\omega}{2\kappa} \ln(2\kappa \epsilon)\right) \end{aligned} \quad (\text{A10})$$

$$C_{\pm s} = \exp\left(i \frac{\omega}{\kappa} \ln(k_0 \epsilon) + i \frac{\pi}{4} - i \arg\left(\Gamma(1 + i\omega/\kappa)\right)\right). \quad (\text{A11})$$

2. Inner horizon connection formulae

In the case of a black hole with both inner and outer horizons, the connection formulae (A4a-b) remain valid

⁴We have changed notation slightly from that in [5]. We have added the lower limit of integration $\pm\epsilon$ to the integrals appearing in the exponents and consequently the coefficients $C_{\pm, \pm s}$ acquire some ϵ dependence to compensate. Furthermore a factor of i appearing in the matching formulae of [5] has been absorbed in ϕ_- and the phase of N as defined in [5] has been absorbed into $\phi_{\pm s}$. We have also renamed the ϕ_{-m} solution in [5] as ϕ_{-s} here.

⁵ x cannot be arbitrarily small however because the WKB approximation breaks down as $x \rightarrow 0$. In [5] it was shown that the WKB and Laplace transform approximate solutions are both valid when $\kappa^{-1/3}k_0^{-2/3} \ll |x| \ll \kappa^{-1}$, and therefore the matching can be done in this range. We also choose ϵ to satisfy the same inequality.

locally about each horizon (after some slight modifications to be discussed presently), but the solutions are no longer global. Assume that the outer horizon is located at $x_o = 0$, with $v(x)$ taking the same form as given by (A3), and that the inner horizon is at $x_i = -a$ with $v(x)$ near the inner horizon taking the form

$$v(x) \approx -1 - \kappa(x + a). \quad (\text{A12})$$

(We assume that the surface gravity of the inner horizon has the same magnitude as that of the outer horizon to simplify the results. There is no difficulty however in allowing the surface gravities to be different.) Then the connection formulae (A4a-b) are valid for equation (A2) locally about the outer horizon, where the notation “ $\phi_1(x) \leftrightarrow \phi_2(x)$ ” now denotes that $\phi_1(x)$ is valid between the horizons and $\phi_2(x)$ is valid outside the outer horizon.

To find the “local” mode solutions about the inner horizon we reexpress the mode equation (A2) in terms of the new coordinate $y := -(x + a)$. The resulting y -equation is the complex conjugate of the x -equation (A2), with $v(x)$ replaced by $\tilde{v}(y) := v(-y - a)$. We denote the WKB mode solutions to this y -equation by $\tilde{\phi}_\pm$ and $\tilde{\phi}_{\pm s}$. Since we have chosen the surface gravities to have the same magnitude, $\tilde{v}(y)$ takes the same form near the inner horizon as $v(x)$ does near the outer horizon (A3). Therefore the mode solutions near the inner horizon are the complex conjugates of those about the outer horizon, with x replaced by y , so they satisfy the same connection formulae:

$$K(e^{\pi\omega/2\kappa}\tilde{\phi}_+ + e^{-\pi\omega/2\kappa}\tilde{\phi}_-) \leftrightarrow \tilde{\phi}_{i,+s} \quad (\text{A13a})$$

$$-\tilde{\phi}_{-s} + K(e^{-\pi\omega/2\kappa}\tilde{\phi}_+ + e^{\pi\omega/2\kappa}\tilde{\phi}_-) \leftrightarrow 0. \quad (\text{A13b})$$

The notation “ $\tilde{\phi}_1(y) \leftrightarrow \tilde{\phi}_2(y)$ ” now denotes that $\tilde{\phi}_1(y)$ is valid between the horizons and $\tilde{\phi}_2(y)$ is valid inside the inner horizon. We have included a subscript “ i ” in $\tilde{\phi}_{i,+s}(y)$ to make it clear that this solution is valid only inside the inner horizon.

The WKB solutions $\tilde{\phi}_{\pm,-s}(y(x))$ are in fact the same functions of x , up to ω -dependent phases, as the WKB solutions $\phi_{\pm,-s}(x)$ respectively. To see this, note that the WKB modes $\phi_{\pm,-s}(x)$ given in (A6,A7) all take the form⁶

$$\phi(x) = C(\omega)f(v(x)) \exp\left(i \int_{-\epsilon}^x ds k(v(s), \omega)\right) \quad (\text{A14})$$

where C is x -independent and f and k are real functions of $v(x)$. Since, as discussed above, the x and y equations are related by substituting $v(x) \rightarrow \tilde{v}(y) = v(-y - a)$ and complex conjugating, the WKB modes $\tilde{\phi}_{\pm,-s}(y)$ are given by

$$\tilde{\phi}(y) = C^*(\omega)f(\tilde{v}(y)) \exp\left(-i \int_{-\epsilon}^y ds k(\tilde{v}(s), \omega)\right) \quad (\text{A15})$$

Using $y = -x - a$ and $\tilde{v}(y) = v(x)$, and changing the integration variable in (A15) to $s' = -s - a$, we obtain

$$\tilde{\phi}(-x - a) = C^*(\omega)f(v(x)) \exp\left(i \int_{-a+\epsilon}^x ds' k(v(s'), \omega)\right), \quad (\text{A16})$$

which differs from (A14) only by an ω -dependent phase factor, i.e.,

$$\tilde{\phi}(-x - a) = \left(\frac{C^*(\omega)}{C(\omega)} \exp\left(i \int_{-a+\epsilon}^{-\epsilon} ds k(v(s), \omega)\right)\right) \phi(x). \quad (\text{A17})$$

We shall not attempt to compute these phase factors, but rather shall assume the generic form

$$\begin{aligned} \tilde{\phi}_\pm(-x - a) &= e^{i\theta_\pm(\omega)} \phi_\pm(x), \\ \tilde{\phi}_{-s}(-x - a) &= e^{i\gamma(\omega)} \phi_{-s}(x), \end{aligned} \quad (\text{A18})$$

where θ_\pm, γ also depend on $v(x)$, k_0 and a (the coordinate distance between the horizons) but do not depend on ϵ as long as it is chosen large enough so that the WKB approximation holds.

Using the phase relations (A18) the inner horizon connection formulae (A13a,A13b) can be reexpressed in terms of the same linear combinations of ϕ_\pm appearing in the outer horizon formulae (A4a,A4b):

$$K(e^{\pi\omega/2\kappa}\phi_+ + e^{-\pi\omega/2\kappa}\phi_-) + e^{i\gamma}T_1\phi_{-s} \leftrightarrow T_2\tilde{\phi}_{i,+s} \quad (\text{A19})$$

$$K(e^{-\pi\omega/2\kappa}\phi_+ + e^{\pi\omega/2\kappa}\phi_-) + T_3\phi_{-s} \leftrightarrow T_1\tilde{\phi}_{i,+s} \quad (\text{A20})$$

with

$$\begin{aligned} T_1 &= \omega^{-1}K^2(e^{-i\theta_+} - e^{-i\theta_-}) \\ T_2 &= \omega^{-1}K^2(e^{\pi\omega/\kappa}e^{-i\theta_+} - e^{-\pi\omega/\kappa}e^{-i\theta_-}) \\ T_3 &= \omega^{-1}K^2e^{i\gamma}(e^{-\pi\omega/\kappa}e^{-i\theta_+} - e^{\pi\omega/\kappa}e^{-i\theta_-}). \end{aligned} \quad (\text{A21})$$

3. Wavepacket evolution formulae

We can now form wavepackets from the modes and use the mode connection formulae to obtain wavepacket evolution formulae. To keep the latter simple, the ω -dependence of the coefficients in the connection formulae is built into the definition of the wavepackets as follows. Define

⁶We have dropped the $\pm, -s$ subscripts on $\phi(x)$, C , f , and k .

$$\begin{aligned}
\psi_{n,+s} &= \int_{\omega_l}^{\omega_u} \frac{d\omega}{\sqrt{\omega}} A_n e^{-i\omega t} \phi_{+s} \\
\psi_{n,-s} &= \int_{\omega_l}^{\omega_u} \frac{d\omega}{\sqrt{\omega}} B_n e^{-i\omega t} \phi_{-s} \\
\psi_{n,\pm} &= \int_{\omega_l}^{\omega_u} \frac{d\omega}{\sqrt{\omega}} C_{n,\pm} e^{-i\omega t} \phi_{\pm} \\
\chi_{n-1,\pm} &= \int_{\omega_l}^{\omega_u} \frac{d\omega}{\sqrt{\omega}} D_{n-1,\pm} e^{-i\omega t} \phi_{\pm} \\
\psi_{i,n,+s} &= \int_{\omega_l}^{\omega_u} \frac{d\omega}{\sqrt{\omega}} E_n e^{-i\omega t} \tilde{\phi}_{i,+s} \\
\chi_{i,n-1,+s} &= \int_{\omega_l}^{\omega_u} \frac{d\omega}{\sqrt{\omega}} F_{n-1} e^{-i\omega t} \tilde{\phi}_{i,+s}, \quad (A22)
\end{aligned}$$

where the coefficients $A \dots F$ depend on ω , and the mode functions ϕ depend on both ω and x .

With these definitions the evolution formulae about the outer horizon (5.1a,5.1b) follow immediately from the outer horizon connection formulae (A4a,A4b) provided we choose $C_{n\pm} = K \exp(\pm\pi\omega/2\kappa) A_n$ and $D_{n,\pm} = -K \exp(\mp\pi\omega/2\kappa) B_{n+1}$. Similarly, the inner horizon evolution formula (5.1c) follows provided $B_n = e^{i\gamma} T_1 A_n$ and $E_n = T_2 A_n$, while (5.1d) requires $F_n = -T_1 B_{n+1}$ and $B_n = -T_3 B_{n+1}$. The solution (up to an undetermined overall constant) is given by

$$\begin{aligned}
A_n &= (-T_3)^{-n} \\
B_n &= e^{i\gamma} T_1 (-T_3)^{-n} \\
C_{n,\pm} &= K e^{\pm\pi\omega/2\kappa} (-T_3)^{-n} \\
D_{n,\pm} &= -K e^{\mp\pi\omega/2\kappa} e^{i\gamma} T_1 (-T_3)^{-n-1} \\
E_n &= T_2 (-T_3)^{-n} \\
F_n &= -e^{i\gamma} (T_1)^2 (-T_3)^{-n-1} \quad (A23)
\end{aligned}$$

4. Norm of the wavepackets

The wavepackets defined in (A22) are not normalized. Their norms can be determined as follows. A generic one of these wavepackets has the form

$$\psi = \int_{\omega_l}^{\omega_u} \frac{d\omega}{\sqrt{\omega}} G e^{-i\omega t} \phi, \quad (A24)$$

which has the norm (cf. (2.9))

$$\begin{aligned}
(\psi, \psi) &= i \int dx \int \frac{d\omega}{\sqrt{\omega}} \int \frac{d\omega'}{\sqrt{\omega'}} \left\{ G_{\omega}^* G_{\omega'} e^{i(\omega - \omega')t} \right. \\
&\quad \left. \left[\phi_{\omega}^* (\partial_t + v \partial_x) \phi'_{\omega} - \phi'_{\omega} (\partial_t + v \partial_x) \phi_{\omega}^* \right] \right\}. \quad (A25)
\end{aligned}$$

The norm is conserved under time evolution so it suffices to evaluate it at any one time.

The key assumption we need in order to evaluate the norm is that at some time the wavepacket is confined to

a constant v region. This is certainly the case for the $+s$ wavepackets, since they are outgoing and eventually reach the asymptotic region. If the region between the horizons is large and has a large constant velocity region, then it may similarly hold for the $-s$ and \pm wavepackets as well. Alternatively, these wavepackets spend some time squeezed near the horizon, with wavelengths much smaller than the length scale over which $v(x)$ changes (but, in the case of k_s , still much longer than k_0^{-1} , so we can nevertheless use the small k approximation). If the wavepacket is contained in a constant v region then, for the purposes of evaluating the norm, we can imagine this region to extend to infinity in both directions. The fixed ω mode equation (A2) in a constant v region has solutions $\phi_{\omega} = C_{\omega} \exp(ikx)$, where $\omega - vk = \pm F(k)$ with $F(k)$ given by (2.7). Matching to the WKB modes (A6,A7) we see that $|C_{\omega}| = \{1, (v^2 - 1)^{-3/4}\}$ for the $\pm s$ modes and \pm modes respectively. Thus we have

$$\int dx \phi_{\omega'}^* \phi_{\omega} = 2\pi \{1, (v^2 - 1)^{-3/4}\} \left| \frac{d\omega}{dk} \right| \delta(\omega' - \omega). \quad (A26)$$

Using (A26) in (A25) yields

$$(\psi, \psi) = 4\pi \{1, (v^2 - 1)^{-3/2}\} \int d\omega |G_{\omega}|^2 \frac{\pm F}{\omega} \left| \frac{d\omega}{dk} \right|. \quad (A27)$$

Using the small and large k approximations for $k_{\pm s}$ and k_{\pm} respectively, we find that

$$\{1, (v^2 - 1)^{-3/2}\} \frac{\pm F}{\omega} \left| \frac{d\omega}{dk} \right| = \pm \{1, \omega^{-1}\} \quad (A28)$$

respectively. Thus, finally,

$$(\psi, \psi) = \pm 4\pi \int d\omega |G_{\omega}|^2 \{1, \omega^{-1}\}. \quad (A29)$$

With (A29) and the coefficients (A23) for the wavepackets (A22) we obtain the norms needed in section VB:

$$\begin{aligned}
(\psi_{0,-}, \psi_{0,-}) &= -4\pi \int_{\omega_l}^{\omega_u} d\omega \frac{1}{e^{2\pi\omega/\kappa} - 1}, \\
(\psi_{n,+s}, \psi_{n,+s}) &= 4\pi \int_{\omega_l}^{\omega_u} d\omega |T_3|^{-2n}, \\
(\psi_{0,-s}, \psi_{0,-s}) &= -4\pi \int_{\omega_l}^{\omega_u} d\omega |T_1|^2 \\
&= -4\pi \int_{\omega_l}^{\omega_u} d\omega \frac{2(1 - \cos(\theta_+(\omega) - \theta_-(\omega)))}{e^{2\pi\omega/\kappa} + e^{-2\pi\omega/\kappa} - 2}. \quad (A30)
\end{aligned}$$

- [1] W.G. Unruh, Phys. Rev. D **51**, 2827 (1995).
- [2] R. Brout, S. Massar, R. Parentani and Ph. Spindel, Phys. Rev. D **52**, 4559 (1995).
- [3] S. Corley and T. Jacobson, Phys. Rev. D **54**, 1568 (1996).
- [4] W. G. Unruh, personal communication.
- [5] S. Corley, Phys. Rev. D **57**, 6280 (1998).
- [6] S. Corley and T. Jacobson, Phys. Rev. D **57**, 6269 (1998).
- [7] D.A. Lowe *et al.*, Phys. Rev. D **52**, 6997 (1995).
- [8] W.G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981).
- [9] M. Visser, Class. Quantum Grav. **15**, 1767 (1998).
- [10] N.B. Kopnin and G.E. Volovik, Phys. Rev. B **57**, 8526 (1998); Phys. Rev. Lett. **79**, 1377 (1997).
- [11] T.A. Jacobson and G.E. Volovik, Phys. Rev. D **58**, 064021 (1998); Pis'ma ZhETF **68**, 833-838 (1998).
- [12] See e.g. Appendix A of ref. [6].
- [13] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory, v.1*, (Cambridge University Press, 1987).
- [14] T. Jacobson, Phys. Rev. D **57**, 4890 (1998).