Almost Certain Escape from Black Holes

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Abstract: Recent models of the black-hole final state suggest that quantum information can escape from a black hole by a process akin to teleportation. These models require a specific final state and restrictions on the interaction between the collapsing matter and the incoming Hawking radiation for quantum information to escape. This paper investigates escape from black holes for arbitrary final states and for generic interactions between matter and Hawking radiation. Classical information, including the result of any computation performed by the matter inside the hole, escapes from the hole with certainty. Quantum information escapes with fidelity $\approx (8/3\pi)^2$: only half a bit of quantum information is lost on average, independent of the number of bits that escape from the hole.

It has been proposed that black holes could function as quantum computers [1-2]; the computational capacity of black holes can be calculated in terms of their mass and lifetime [1-3]. In order to function as a useful computer, however, a black hole must permit information to escape as the black hole evaporates. Recently, Horowitz and Maldacena proposed a model of black hole evaporation that imposes a final state boundary condition at the black-hole singularity [4]. The result is a nonlinear time evolution for the quantum states in and outside of the black hole, which permits quantum information to escape from the black hole by a process akin to teleportation. Because it allows information to escape, such a model naturally allows the black hole to function as a computer whose output is written in the outgoing Hawking radiation produced during evaporation, as envisioned in [1]. The Horowitz-Maldacena model requires a specific final state which is perfectly entangled between the matter that formed the black hole and the incoming Hawking radiation. Whether or not quantum gravity supports such a final state remains

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to be seen. In addition, even with the proper final state, interactions between the incoming Hawking radiation and the collapsing matter can spoil the unitary nature of the black-hole evaporation [5], destroying some or all of the quantum information inside the hole [6-7].

The purpose of this paper is to examine the robustness of the escape of quantum information during black hole evaporation in final state projection models. In particular, I show that for projection onto *any* final state at the singularity (independent of the exact details of quantum gravity) and for *almost all* interactions between the matter and incoming Hawking radiation, properly encoded classical information escapes from the hole with certainty. Of the quantum information that escapes from the hole, only one half a qubit is lost on average, regardless of the number of bits of quantum information in the hole to begin with. More precisely, the state of the matter that formed the hole is preserved under black hole evaporation with a fidelity of $f \approx (8/3\pi)^2 \approx .85$. This is the fidelity of escape of the entire state of the collapsing matter: individual quantum bits escape with a fidelity that approaches one as the number of bits in the hole becomes large. Since the fidelity is above the threshold for the use of quantum-error correcting codes, properly encoded quantum information can escape from the hole with fidelity arbitrarily close to 1.

The Horowitz-Maldacena model [4] is described concisely in [5]. Black holes evaporate by absorbing negative-energy 'incoming' Hawking radiation and by emitting positive-energy 'outgoing' Hawking radiation. Let the dimension of the Hilbert space for the collapsing matter inside the black hole be N. In the ordinary semi-classical treatment of black-hole evaporation, the incoming and outgoing Hawking radiation is in the maximally entangled state

$$|\phi\rangle_{in\otimes out} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |j\rangle_{in} \otimes |j\rangle_{out}, \tag{1}$$

where $\{|j\rangle_{in}\}$ is an orthonormal basis for the Hilbert space H_{in} of the incoming Hawking radiation and $\{|j\rangle_{out}\}$ is an orthonormal basis for the Hilbert space H_{out} of the outgoing Hawking radiation.

Let $|\phi\rangle_{matter\otimes in} \in H_{matter} \otimes H_{in}$ be the final state onto which the collapsing matter together with the incoming Hawking radiation is projected at the singularity. Horowitz and Maldacena postulated a form for this state of

$$|\phi\rangle_{matter\otimes in} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (S|k\rangle_{matter}) \otimes |k\rangle_{in}, \tag{2}$$

where S is a unitary transformation acting on the matter states alone. The usual analysis

of quantum teleportation shows that for states of this form, the transformation from the state of the collapsing matter to the state of the outgoing Hawking radiation is

$$T = {}_{matter \otimes in}\langle\psi|\phi\rangle_{in \otimes out} = S/N. \tag{3}$$

The factor $1/N$ reflects the fact that if this were conventional teleportation, then this particular final state would occur only with probability $1/N^2$. In final-state projection however, only one final state can occur: accordingly, the final transformation from collapsing matter to outgoing Hawking radiation is renormalized, and the net result is the unitary transformation S. In the H-M model, final-state projection leads to a unitary transformation between collapsing matter and the outgoing Hawking radiation.

Comparing final-state projection to conventional teleportation, we see that the main difference is that final-state projection mandates a single outcome, while teleportation allows N^2 outcomes. In teleportation, $\log_2 N^2$ bits must be sent from the input of the teleporter to its output in order to reconstruct the input state. In final-state projection $\log_2 1 = 0$ bits must be sent from inside the black hole to outside the black hole to reconstruct the input state.

Final state projection is an intrinsically nonlinear process and shares the virtues and vices of other proposals for nonlinear quantum mechanical processes. Escape of quantum information from black holes via final state projection is similar to the use of nonlinear quantum mechanics to provide superluminal communication as described (and rejected) in [8-10], to violate the second law of thermodynamics [11], or to solve NP-complete problems [12]. Such nonlinear quantum effects have been investigated experimentally under non-Planckian conditions and ruled out to a high degree of accuracy [13-16] ([6-7] proposes similar tests of such nonlinear quantum effects in a 'normal' environment). In fact, the nonlinearity that arises from projection onto a state is a particularly powerful type, capable of allowing superluminal communication and time travel if it occurs under normal conditions. Indeed, this nonlinearity must allow information to propagate over spacelike intervals if the information is to escape from a black hole. Because it occurs at a singularity beyond an event horizon, however, final state projection does not obviously allow causal paradoxes such as time travel.

Despite its somewhat dubious provenance, nonlinear quantum mechanics including final state projection might hold sway in extreme Planckian regimes such as the black-hole singularity. At any rate, in the absence of a full theory of quantum gravity, we are certainly free to postulate such an effect and to investigate its consequences.

Even in the presence of final-state projection, without further assurances that go beyond the H-M model the escape of quantum information from a black hole is by no means certain. Gottesman and Preskill [5] point out that if the incoming Hawking radiation interacts with the collapsing matter within the black hole (as is likely), then the H-M model no longer preserves quantum information. In particular, let the interaction between incoming Hawking radiation and matter be given by a unitary transformation U. The transformation between the state of the collapsing matter and the state of the outgoing Hawking radiation is then

$$T = {}_{matter \otimes in} \langle \phi | U | \phi \rangle_{in \otimes out}. \tag{4}$$

Gottesman and Preskill note that if all U's are allowed, T can be *any* matrix satisfying $\sum_{m,n} |\langle m | T | n \rangle|^2 = 1$, including transformations that completely destroy the quantum information in the matter, leading to purely thermal Hawking radiation. In general, if the state $U | \phi \rangle_{matter \otimes in}$ is not perfectly entangled, then some quantum information in the matter is lost.

For the purposes of using a black hole as a quantum computer, the key question is how much quantum information is lost on average due to such interactions. I'll now show that for *any* final state, not just the special H-M states, and for *almost any U*, classical information escapes from the hole with certainty, and quantum information escapes from the hole with fidelity $\approx (8/3\pi)^2 \approx .85$. Essentially, all but half a qubit of the quantum information escapes. This fidelity holds in the limit $N >> 1$ and is independent of the exact number of bits escaping from the hole: it is the fidelity of escape for the entire state of the collapsing matter. Individual quantum bits inside the hole escape with higher fidelity. In the limit $N >> 1$, individual quantum bits escape from the hole with fidelity arbitrarily close to 1.

Let $|\phi\rangle_{matter \otimes in}$ be any final state, including a product state, and let U be a random unitary transformation on the matter and incoming Hawking radiation, selected according to the Haar measure. (The Haar measure is the unique measure over $U(n)$ that is invariant with respect to unitary transformation.) In particular, the final state could be the as yet unknown correct final state specified by the as yet unknown correct theory of quantum gravity. Because U is selected according to the Haar measure, the state

$$|\psi\rangle_{matter \otimes in} = U | \phi \rangle_{matter \otimes in} \tag{5}$$

is a random pure state of the matter and incoming Hawking radiation, i.e., a pure state selected according to the uniform measure on the sphere in N^2 dimensions. That is, it is

a random state selected according to the Hilbert-Schmidt measure. The random nature of U implies that the escape of quantum information from a black hole does not depend on details of the final state.

Because $|\psi\rangle_{matter \otimes in}$ is random, it is not perfectly entangled. As a result, black hole evaporation will not preserve all the quantum information in the collapsing matter. But by the same token, because $|\psi\rangle_{matter \otimes in}$ is random, it is almost perfectly entangled for large N. In particular, a typical random state is within one half a qubit of maximum entanglement.

More precisely, a random state in $H_{matter} \otimes H_{in}$ can be written in Schmidt form as

$$|\psi\rangle_{matter \otimes in} = \sum_{\ell} \lambda_{\ell} |\ell\rangle'_{matter} \otimes |\ell\rangle'_{in}. \tag{6}$$

The distribution of the Schmidt coefficients λ_{ℓ} for random states is known [17-19]. A random state is almost perfectly entangled [20-21]: the average entropy of entanglement, $-\sum_{\ell} \lambda_{\ell}^2 \log_2 \lambda_{\ell}^2$, is within one half bit of its maximum possible value, $\log_2 N$. It is the high entanglement of random states that leads to the escape of information from the hole.

We now can calculate the average fidelity with which a state for the collapsing matter fields

$$|\mu\rangle_{matter} = \sum_{\ell} \mu_{\ell} |\ell\rangle'_{matter} \tag{7}$$

is transferred to the outgoing Hawking radiation.

First, look at what happens to the information inside the hole under final state projection. Action of U on $|\mu\rangle$ together with the incoming Hawking radiation, followed by projection onto the final state $|\phi\rangle_{matter \otimes in}$, yields a transformation from the matter to the outgoing Hawking radiation

$$T = {}_{matter \otimes in}\langle \psi | \phi \rangle_{in \otimes out} \tag{8}$$

The (unnormalized) state of the outgoing Hawking radiation is

$$|\phi\rangle_{out} = \frac{1}{\sqrt{N}} \sum_{\ell} \lambda_{\ell} \mu_{\ell} |\ell\rangle'_{out}, \tag{9}$$

where $\{|\ell\rangle'_{out}\}$ is a basis for the Hilbert space of outgoing Hawking radiation, related to the basis $\{|\ell\rangle'_{matter}\}$ for the Hilbert space for the collapsing matter via a unitary transformation T'. Because the normalization of this state depends in a nonlinear fashion on the μ_{ℓ}, this is a nonlinear transformation of the input state of the matter.

Now that we know what happens to quantum states under final state projection, we can determine what happens to the information inside the hole. Start by looking at the escape of classical information from the hole. If this information is encoded in the states $|\ell\rangle'_{matter}$, then the perfect correlation embodied in the Schmidt decomposition, combined with the nonlinear effect of the final state projection implies that *all* classical information escapes from the hole, down to the last bit. Interestingly, the complete escape of classical information from the hole does not depend on the precise distribution of the Schmidt coefficients λ_ℓ: it only requires that they all be non-zero, which occurs with probability equal to 1. Properly encoded, classical information escapes from the hole with certainty.

Now look at how quantum information escapes from the hole. Comparing the (normalized) outgoing state of the Hawking radiation with T' times the state of the collapsing matter, we obtain

$$|_{out}\langle\phi|T'|\mu\rangle_{matter}|^2 = (\sqrt{N}\sum_\ell \lambda_\ell|\mu_\ell|^2)^2. \tag{10}$$

Since a typical state has $|\mu_\ell|^2 \approx 1/N$, the state of the collapsing matter is transferred to the state of the outgoing Hawking radiation with a fidelity

$$f \approx (\frac{1}{\sqrt{N}}\sum_\ell \lambda_\ell)^2. \tag{11}$$

This approximate result can be confirmed using standard treatments of teleportation with imperfectly entangled states [22]. The maximum mean teleportation fidelity attainable using imperfectly entangled states with Schmidt coefficients λ_ℓ is

$$\bar{f} = \frac{1}{N+1}\left[1 + \left(\sum_\ell \lambda_\ell\right)^2\right]. \tag{12}$$

This fidelity is attained for the standard teleportation protocols. Because escape from a black hole via final state projection is equivalent to teleportation with a fixed measurement outcome, this is also the mean fidelity for escape from a black hole.

The techniques of [19] now allow us to estimate the value of $\bar{f}$. for $N >> 1$, we have

$$\langle\sum_\ell \lambda_\ell\rangle = \sqrt{N}\frac{\Gamma(2)}{\Gamma(3/2)\Gamma(5/2)}(1 + O(\frac{1}{N})) \approx \frac{8}{3\pi}\sqrt{N}. \tag{13}$$

As a result, for $N >> 1$, we have

$$\bar{f} \approx \left(\frac{8}{3\pi}\right)^2 \approx .85 \quad . \tag{14}$$

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Quantum information escapes from the hole with fidelity $\approx .85$. (Note that in this estimate we are approximating $\langle (\sum_\ell \lambda_\ell)^2 \rangle$ by $\langle (\sum_\ell \lambda_\ell) \rangle^2$ in the limit that $N >> 1$.)

This fidelity is the fidelity for escape of the entire state of the collapsing matter. The fidelity of escape of individual quantum bits is higher and approaches 1 asymptotically as N becomes large. Because the escape fidelity lies above the threshold required for quantum error correction [23], suitably encoded quantum information escapes from the black hole with fidelity arbitrarily close to 1. Given final-state projection, escape from a black hole is almost certain.

Note that in the above demonstration of almost certain escape from a black hole via final state projection relies on a random interaction between the collapsing matter and the incoming Hawking radiation. As only a finite proper time exists for interaction between the matter and the incoming Hawking radiation, this interaction is not truly random. What is important for the escape of the quantum information is not true randomness, however, but entanglement. We have recently demonstrated both theoretically and experimentally that *pseudorandom* states and transformations, implemented by quantum logic circuits with $O(2n)$ gates for $n = \log_2 N$ qubits, exhibit the same Schmidt coefficient statistics as true random states and transformations [24]. Accordingly, we may reasonably hope that the final state projection, whatever it is, is sufficiently entangled to give high fidelity transfer of the state of the matter within the hole to the state of the outgoing Hawking radiation.

The results of this paper suggest that if black holes evaporate via final state projection, they might make good quantum computers. The fidelity of transfer of quantum information is better than what is required for robust quantum computation. Indeed, if all one wants is a Yes/No answer from the computation, i.e., a classical bit, then the black hole can deliver the answer with certainty.

Note, however, that for information to escape from the hole under the final state projection model, the time evolution apart from the final projection must remain unitary as the densities of matter and energy approach the Planck scale near the singularity. That is, the strategy for escaping from a black hole presented here assumes that the only source of nonlinearity is the final state projection. Even if the time evolution apart from the projection is unitary, a person outside the hole must know the exact interaction that occurred between the collapsing matter and the incoming Hawking radiation in order to reconstruct the information escaping from the hole. Final state projection will have to await experimental and theoretical confirmation before black holes can be used as quantum

computers. It would be premature to jump into a black hole just now.

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