Black hole motion in Euclidean space as a diffusion process

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Abstract

A diffusion equation for a black hole is derived from the Bunster-Carlip equations. Its solution has the standard form of a Gaussian distribution. The second moment of the distribution determines the quantum of black hole area. The entropy of diffusion process is the same, apart from the logarithmic corrections, as the Bekenstein-Hawking entropy.

Bunster (Teitelboim) and Carlip showed [1] that the wave function of a black hole with the Arnowitt-Deser- Misner (ADM) mass M and area A evolves according to the Schrödinger-type equations

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t} + M\psi = 0, \tag{1}$$

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial\Theta} - \frac{A}{8\pi G}\psi = 0, \tag{2}$$

where t is the lapse of asymptotic proper time at spatial infinity and Θ is the lapse of the hyperbolic angle at the horizon. Under Euclidean continuation Θ transforms to an angle variable. As a result, as pointed out in [1], $A/8\pi G$ and Θ become conjugate exactly like M and t. This means that the area is the operator-valued quantity. It was shown in [2] that $A/8\pi G$ can be interpreted as the z component of an internal angular momentum of a black hole L_z. Indeed, in the semiclassical approximation

$$\psi = a \exp\left(\frac{i}{\hbar} I\right), \tag{3}$$

where I is the action of a black hole. Substituting this in (2) we obtain

$$\psi \frac{\partial I}{\partial \Theta} = \frac{A}{8\pi G}\psi; \tag{4}$$

the slowly varying amplitude a need not be differentiated. Under Euclidean continuation $\Theta_{\mathrm{E}} = i\Theta$ and $I_{\mathrm{E}} = iI$,

$$\psi \frac{\partial I_{\mathrm{E}}}{\partial \Theta_{\mathrm{E}}} = \frac{A}{8\pi G}\psi. \tag{5}$$

The derivative $\partial I_{\mathrm{E}}/\partial \Theta_{\mathrm{E}}$ is just a generalized momentum corresponding to the angle of rotation about one of the axes (say, the z^{th}) for a mechanical system. Therefore the operator $A/8\pi G$ is what corresponds in quantum mechanics to the z component of angular momentum L_z. Quantization of L_z gives the equidistant area spectrum of a black hole

$$A_m = \Delta A \cdot m, \quad m = 0, 1, 2, ... \tag{6}$$

with the area quantum

$$\Delta A = 8\pi l_{\mathrm{P}}^2. \tag{7}$$

Medved [3] found this value immediately from the Bunster-Carlip action [1]. Note that Bekenstein [4] was the first to determine the quantum of area. Later on, the value (7) was obtained using different approaches and techniques [5].

In this note I derive a diffusion equation for a black hole from the Bunster-Carlip equations and show that the black hole motion in Euclidean space exhibits a diffusion process. Moreover I find that the entropy of the process is the same, apart from the logarithmic corrections, as the Bekenstein-Hawking entropy.

I begin with the Bunster-Carlip equations (1) and (2). Analytically continuing Θ and t to the real values of $\Theta_{\mathrm{E}} = i\Theta$ and $t_{\mathrm{E}} = it$ we obtain

$$\hbar \frac{\partial \psi}{\partial t_{\mathrm{E}}} + M\psi = 0, \tag{8}$$

$$\hbar \frac{\partial \psi}{\partial \Theta_{\rm E}} - \frac{A}{8\pi G}\psi = 0. \tag{9}$$

Taking the complex conjugate of equation (6) we get

$$\hbar \frac{\partial \psi^*}{\partial \Theta_{\rm E}} - \frac{A}{8\pi G}\psi^* = 0. \tag{10}$$

Multiplying (9) by ψ^* and (10) by ψ and then adding, we find

$$\hbar \left(\psi^* \frac{\partial \psi}{\partial \Theta_{\rm E}} + \psi \frac{\partial \psi^*}{\partial \Theta_{\rm E}} \right) = \left(\psi^* \frac{A}{8\pi G}\psi + \psi \frac{A}{8\pi G}\psi^* \right). \tag{11}$$

In the spherical polar coordinates, i.e. in terms of $\Theta_{\rm E}$, the z component of the internal angular momentum $A/8\pi G$ is not a product of operators. Therefore

$$\hbar \frac{\partial (\psi\psi^*)}{\partial \Theta_{\rm E}} = \frac{A}{8\pi G}(\psi\psi^*) \tag{12}$$

or

$$\hbar \frac{\partial \rho}{\partial \Theta_{\rm E}} = \frac{A}{8\pi G}\rho, \tag{13}$$

where $\rho = |\psi(t_{\rm E},\, \Theta_{\rm E})|^2$ is the probability density of finding the black hole at point $(t_{\rm E},\, \Theta_{\rm E})$ in the Euclidean manifold. On the other hand, for a Schwarzschild black hole

$$\frac{A}{8\pi G} = 2GM^2. \tag{14}$$

Since, according to (8),

$$M^2 \equiv \hbar^2 \frac{\partial^2}{\partial t_{\rm E}^2}, \tag{15}$$

the equation (13) reads

$$\frac{\partial \rho}{\partial \Theta_{\rm E}} = D \frac{\partial^2 \rho}{\partial t_{\rm E}^2}. \tag{16}$$

This is an one-dimensional diffusion equation in the temporal $\Theta_{\rm E}$ and spatial $t_{\rm E}$ coordinates with the diffusion coefficient $D = 2G\hbar$. The solution to the equation is

$$\rho = \frac{1}{\sqrt{4\pi D \Theta_{\rm E}}} e^{-t_{\rm E}^2/(4D\Theta_{\rm E})}, \tag{17}$$

which is normalized such that

$$\int_{-\infty}^{+\infty} \rho(t_{\rm E}, \Theta_{\rm E}) dt_{\rm E} = 1. \tag{18}$$

It follows that the black hole spreads out with increasing "time" $\Theta_{\rm E}$. The mean square value of $t_{\rm E}$ is given by

$$\langle t_{\rm E}^2 \rangle = \int_{-\infty}^{+\infty} t_{\rm E}^2 \rho(t_{\rm E}, \Theta_{\rm E}) dt_{\rm E} = 2D\Theta_{\rm E}. \tag{19}$$

This result shows that the width of distribution increases as $\Theta_{\rm E}^{1/2}$, which is a general characteristic of diffusion and random walk problems in one dimension. Since regularity of the Euclidean manifold at the horizon imposes the fixed Euclidean angle $\Theta_{\rm E} = 2\pi$, we get

$$\langle t_{\rm E}^2 \rangle = 8\pi l_{\rm P}^2. \tag{20}$$

This value coincides with (7). But (7) is a result of a true quantum-mechanical quantization of area; it arises due to the periodicity of $\Theta_{\rm E}$. In contrast, (20) can be viewed as a result of discreteness of $\Theta_{\rm E}$ in a random walk model. In the model, $\langle t_{\rm E}^2 \rangle = N l_0^2$, where l_0 is the length of each step of a random walk and N is the number of steps. The quantum of area $8\pi l_{\rm P}^2$ arises if we let 2π be the duration of a step; then $\langle t_{\rm E}^2 \rangle = (\Theta_{\rm E}/2\pi) l_0^2$ which being combined with (19) gives $l_0^2 = 8\pi l_{\rm P}^2$. By abuse of language, we call $8\pi l_{\rm P}^2$ in (20) the quantum of area. Therefore the motion of a black hole in Euclidean space exhibits a diffusion process. Thus the elementary act which changes the size of a black hole is the gain or loss by it, of one quantum of area $8\pi l_{\rm P}^2$ during one period $\Theta_{\rm E} = 2\pi$.

Analytical continuation to the cyclic imaginary time $\Theta_{\rm E}$ means that we deal with a quantum system at a finite temperature. In this case, the system is described not by the probability density $\rho(x)$ but by the density matrix $\rho(x, x'; \beta)$, where x is a spatial coordinate and $\beta \equiv T^{-1}$ is the inverse temperature. The partition function $Z(\beta)$ is the trace of the density matrix

$$Z(\beta) = \int \rho(x, x; \beta) dx. \tag{21}$$

In (16), $t_{\rm E}$ plays the role of the spatial coordinate. The temporal coordinate $\Theta_{\rm E} = k t_{\rm E}$, where $k = 1/4GM$ is the surface gravity. Since $\Theta_{\rm E}$ has the interpretation as an angular coordinate with periodicity 2π, t_{E} itself has then periodicity $8\pi GM$ which, when set equal to $\hbar/T_{\mathrm{H}}$, gives the Hawking temperature T_{H}. Since $t_{\mathrm{E}} \sim t_{\mathrm{E}} + 8\pi GM$, we have

$$\rho(t_{\mathrm{E}}, t_{\mathrm{E}}; \beta) = \frac{1}{\sqrt{4\pi D\Theta_{\mathrm{E}}}} e^{-\beta^2/(4D\Theta_{\mathrm{E}})} \tag{22}$$

Therefore

$$Z(\beta) = \frac{\beta}{\sqrt{4\pi D\Theta_{\mathrm{E}}}} e^{-\beta^2/(4D\Theta_{\mathrm{E}})}. \tag{23}$$

The internal energy is given by

$$E = -\frac{\partial \ln Z(\beta)}{\partial \beta} = M - \frac{1}{8\pi GM}. \tag{24}$$

It is the same, apart from a term of order of the Hawking temperature, as the ADM mass of a black hole. Finally, the entropy is given by

$$S = \ln Z + \beta E = \frac{A}{4l_{\mathrm{P}}^2} + \frac{1}{2}\ln\left(\frac{A}{4l_{\mathrm{P}}^2}\right) + \ln\left(\frac{1}{e\sqrt{\pi}}\right). \tag{25}$$

It is the same, apart from the terms $\mathcal{O}(\ln(A/4l_{\mathrm{P}}^2))$, as the Bekenstein-Hawking entropy $S_{\mathrm{BH}} = A/4l_{\mathrm{P}}^2$.

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