# Black Hole Evaporation along Macroscopic Strings

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We develop the quantization of a macroscopic string which extends radially from a Schwarzschild black hole. The Hawking process excites a thermal bath of string modes that causes the black hole to lose mass. The resulting typical string configuration is a random walk in the angular coordinates. We show that the energy flux in string excitations is approximately that of spacetime field modes.

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## 1. Introduction

The collision of quantum mechanics and black hole physics has captured the interest of the theoretical physics community. The discovery of two-dimensional model systems [1] sharing many of the features of the Hawking problem has spurred the hope that one might learn something about information storage and retrieval in black holes, and perhaps about some essential features of quantum gravity. Before this hope evaporates (with or without a remnant) we would like to point out another context in which such two-dimensional systems crop up: the quantization of the collective modes of a macroscopic string trapped by a black hole.

Often one ties two-dimensional model systems to four-dimensional physics when, at the core of a spherically symmetric object, all but S-wave modes of the four-dimensional system are effectively massive [2] [3] and may be integrated out. For a macroscopic string, a somewhat different mechanism occurs. The internal excitations of the string core are irrelevant to the physics of the Nambu-Goldstone transverse oscillation modes (for a macroscopic fundamental string, there are no such core excitations). The dominant dynamical effects come from the projection of the spacetime geometry onto the world sheet of the propagating string. When the string is trapped by the event horizon of a black hole, the event horizon is projected onto the world sheet. Hence one expects Hawking radiation of transverse string oscillations to occur upon quantization. Since the essence of the Hawking paradox is whether information follows energy, it would seem that one ought to understand it in this somewhat simpler two-dimensional context. Any remnant scenario would entail a huge expansion of the string density of states, since the remnant would be threaded by the string (although such remnants might be hard to produce, as may be the case for recently considered species of remnant [4]). If black holes violate quantum mechanical unitarity, we will have to face the problem already in the quantization of a single string.

Hawking radiation along the string leads to a number of interesting questions, of which we shall address two, which involve effects of the radiation on the string and its back-reaction on spacetime. We will show that the thermal bath of emitted string modes causes the mean square transverse extent of the string to grow linearly with its length (or some other infrared cutoff) $L$

$$\langle : (\xi_\perp)^2 : \rangle \sim \frac{\ell_s^2 L}{r_h} \ ,$$

and that the power radiated in the string Hawking process is

$$\left( \frac{\delta M_{\text{bh}}}{\delta t} \right)_{\text{string modes}} \sim \frac{\hbar c^2}{r_h^2} \ , \tag{1.1}$$

where $r_h$ is the horizon radius. This is to be compared with, *e.g.*, the total radiated power a spacetime massless scalar field in the presence of a black hole, which is the same within numerical factors.

We begin in section 2 with the analysis of the small fluctuations of a single straight, infinitely long, bosonic string in a Schwarzschild black hole background in the critical dimension of string theory by expanding the $\sigma$-model action to quadratic order in Riemannian normal coordinates. In section 3 we discuss the linearized classical equations of motion and show that the physical solutions are stable to this order. In section 4 we quantize the transverse fluctuations in a physical gauge, demonstrating both through this gauge fixing and through path integral techniques in conformal gauge that there are $(d_{cr}-2)$ independent physical coordinates in the critical dimension. We choose the Unruh vacuum as the string state and construct the Bogolubov transformations used to describe Hawking radiation. Those who are familiar with this material may wish to skip directly to section 5, where we compute some observable manifestations of the Hawking process: the mean square deviation from the equilibrium position and the radiation of spacetime energy onto the string, both characteristic of a string thermally excited at the Hawking temperature.

## 2. The string action in a Schwarzschild background

*2.1. The background and the $\sigma$-model action*

The metric of the Schwarzschild black hole in D dimensions is derivable in the same fashion as the 4-dimensional case, and differs only in the power of the radial coordinate dependence of the metric for the $r-t$ plane. (see for example [5]):

$$ds^2 = -\left(1 - \frac{C}{r^{D-3}}\right)dt^2 + \left(1 - \frac{C}{r^{D-3}}\right)^{-1} dr^2 + r^2 d\Omega^2 \ . \tag{2.1}$$

Here $d\Omega^2$ is the metric for the D-2 sphere; $C$ is a constant of integration arising from solving Einstein's equations and is related to the mass via [5]

$$M = \frac{C(D-2)A_{D-2}}{16\pi G} \ , \tag{2.2}$$

where $A_{D-2}$ is the area of a unit D-2 sphere and G is the D-dimensional generalization of Newton's constant, defined via the Einstein action with dimensions [length]$^{D-2}$ ($\hbar = c = 1$). The radius of the horizon is of course $r_h = C^{1/(D-3)}$.
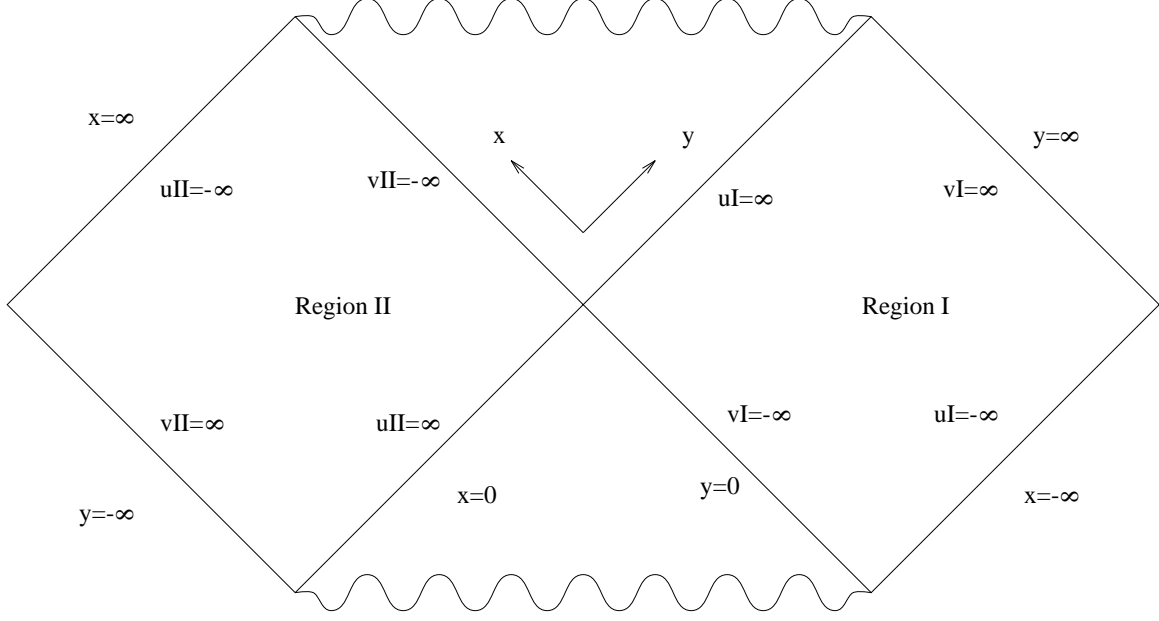
**Figure 1** Penrose diagram for the x-y plane of the extended Kruskal manifold.

In this paper we will be switching between Schwarzschild null coordinates and Kruskal null coordinates. The former are found by solving for $ds^2 = 0$ in the r-t plane, and are

$$u = t - r_*$$
$$v = t + r_* \ ,$$

(2.3)

where $r_*$ is the D-dimensional generalization of the Regge-Wheeler tortoise coordinate [6]

$$r_* = \int^r \frac{dr}{1 - \left(\frac{r_h}{r}\right)^{D-3}} = r + r_h \int^{r/r_h} \frac{dz}{z^{D-3} - 1} \ .$$

(2.4)

The indefinite integral may be computed exactly by expanding the denominator in partial fractions, with the result:

$$r_* = r + \frac{r_h}{(D-3)} \sum_{k=0}^{D-4} \ln\left(\frac{r}{r_h} - e^{i2\pi k/(D-3)}\right) \ .$$

(2.5)

The metric may then be written as:

$$ds^2 = -\left[1 - \left(\frac{r_h}{r}\right)^{D-3}\right] dudv + r^2 d\Omega^2 \ .$$

(2.6)

Kruskal null coordinates are used here in dimensionless form,

$$x = -e^{-\left(\frac{D-3}{2r_h}\right)u}$$
$$y = \ e^{\left(\frac{D-3}{2r_h}\right)v} \ ;$$

(2.7)

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the $r - t$ metric is, in these coordinates,

$$ds^2 = -2f(x,y) \ dxdy$$

$$= -\frac{4r_h^2}{(D-3)^2 xy} \left[ 1 - \left( \frac{r_h}{r} \right)^{D-3} \right] \ dxdy \ . \tag{2.8}$$

Figure 1 shows the standard Penrose diagram for the "r-t" or "x-y" plane of the black hole, labelled with our notational conventions.

We wish to study an infinitely long string fluctuating in this background; we will assume that the string and its fluctuations have a negligible effect on the spacetime. We will also assume that the black hole has a large mass or equivalently a large horizon radius $r_h$, which will allow us to expand the action out in inverse powers of $r_h$; in addition, since the Hawking temperature goes as the inverse of the horizon radius, we may treat adiabatically the change in the background metric due to the shrinking of the black hole. The action may be written as the $\sigma$-model action[1]

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \left\{ h^{\alpha\beta} g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \right\} + S_{\text{ghosts}} \ . \tag{2.9}$$

The ghost action will arise from fixing to conformal gauge with some fiducial metric $\hat{h}_{\alpha\beta}$ multiplied by an arbitrary conformal factor $\rho$. The ghost action will be the same as that of a string in Minkowski spacetime; it arises purely from the worldsheet geometry, and will depend explicitly on the fiducial metic and implicitly (through ultraviolet regulation) on $\rho$. Note that since the space-time metric is Ricci-flat, this theory is free of the conformal anomaly in 26 dimensions [7]. The generalization to the supersymmetric $\sigma$-model in this background is straightforward.

## 2.2. Normal coordinate expansion

We would like to examine small fluctuations around an equilibrium position of the string, *i.e.* about some solution to the classical equations of motion; thus, we want some

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[1] We adopt the following conventions. For vector and tensor indices: latin indices in the first half of the alphabet $(a, b...)$ are worldsheet tangent space indices, greek indices in the first half of the alphabet $(\alpha, \beta,...)$ are worldsheet coordinate indices, latin indices in the last half of the alphabet $(m,n,...)$ are spacetime tangent space indices, and greek indices in the last half of the alphabet $(\mu, \nu...)$ are spacetime coordinate indices. The flat space metric signature is $\eta_{\alpha\beta} = (-,+)$ on the worldsheet and $\eta_{\mu\nu} = (-,+,...,+)$ in space time.

sort of Taylor expansion of $X^\mu$ around the equlibrium configuration $X_{\text{cl}}^\mu$. We find that, in Kruskal coordinates,

$$
\begin{aligned}
X_{\text{cl}}^x &= \sigma^- \\
X_{\text{cl}}^y &= \sigma^+ \\
X_{\text{cl}}^{\theta_k} &= \frac{\pi}{2}
\end{aligned}
\tag{2.10}
$$

is the coordinate embedding for an infinitely long string in conformal gauge. This embedding defines a one-to-one mapping between the $x - y$ plane of spacetime and the classical worldsheet of the string. We may also pull back the metric for the spacetime $x - y$ plane coordinates onto worldsheet, fixing the fiducial metric as:

$$
\hat{h}_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = -2f d\sigma^- d\sigma^+ \ .
\tag{2.11}
$$

Riemannian normal coordinates are a standard and natural way to carry out a Taylor expansion of the $\sigma$-model action in a manifestly coordinate invariant fashion (see [8] for an explanation of this expansion in the bosonic and supersymmetric $\sigma$-models). In particular, it is an expansion in derivatives of the metric; one finds that a term in the action of order $n$ in small fluctuations is accompanied by geometric coefficients (derivatives of and combinations of derivatives of the Riemann tensor) containing $n$ derivatives of the metric. Since the only dimensionful quantity in the metric is $r_h$, we may regard this expansion as an expansion in the fluctuation size divided by $r_h$; then the expansion is good if the fluctuations are small on the scale of the black hole or if one is in the asymptotic region. Classically, this means simply that we wish to perturb the string (shake it) with small amplitude. Quantum mechanically, the strength of worldsheet fluctuations is given by $1/\ell_s^2$ which appears in front of the nonlinear $\sigma$-model action; thus, perturbations to expectation values coming from nonlinear terms will be of higher order in $\ell_s/r_h$.

Following [8], let us expand the coordinate fields and the spacetime metric in normal coordinates, which are contravariant vectors $\xi^\mu$ in spacetime. We may rewrite these as spacetime tangent frame quantities,

$$
\xi^m = e_\mu^m(X_{\text{cl}})\xi^\mu \ .
\tag{2.12}
$$

where $e_\mu^m(X_{\text{cl}})$ is the vielbein field of the spacetime manifold at the spacetime coordinate $X_{\text{cl}}$. By working with these vielbein fields we will have a more standard kinetic term

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and thus a more standard propagator. The final form of the action for small coordinate fluctuations to quadratic order is:

$$S = -\frac{1}{2}\int d^2\sigma\sqrt{h}\Big\{\eta_{mn}h^{\alpha\beta}(D_\alpha\xi)^m(D_\beta\xi)^n$$
$$R_{\mu mn\nu}\xi^m\xi^n h^{\alpha\beta}\partial_\alpha X_{cl}^\mu\partial_\beta X_{cl}^\nu\Big\} + S_{\text{ghosts}} \ . \tag{2.13}$$

Here $\eta$ is a flat Minkowski metric for the tangent frames, and the derivatives $D$ are defined by

$$(D_\alpha\xi)^m = \partial_\alpha\xi^m + \omega_\mu^{mn}\partial_\alpha X_{cl}^\mu\xi_n \ , \tag{2.14}$$

where $\omega_\mu^{mn}$ is the spacetime spin connection.

The vielbein components are:

$$e_x^0 = \sqrt{\frac{f}{2}} = e_y^0 = e_y^1$$
$$e_x^1 = -\sqrt{\frac{f}{2}} \tag{2.15}$$
$$e_{\theta_k}^m = r(x,y)\sin\theta_{D-2}\cdots\sin\theta_{k+1} \qquad \text{for } 2 \leq k \leq D-1$$

($r$ is defined implicitly via (2.3) - (2.7)). We may use these to calculate $\omega_\mu^{mn}$ and $R_{\mu\nu\rho\sigma}$. The explicit form of (2.13) in Kruskal coordinates is found to be:

$$S = -\tfrac{1}{2}\int d^2\sigma\Big\{2\eta_{ab}\partial_-\xi^a\partial_+\xi^b$$
$$+\frac{1}{f}\left[\xi^1(\partial_- f\partial_+ - \partial_+ f\partial_-)\xi^0 - \xi^0(\partial_- f\partial_+ - \partial_+ f\partial_-)\xi^1\right]$$
$$+\left(\frac{\partial_- f\partial_+ f - \tfrac{1}{2}f\partial_-\partial_+ f}{f^2}\right)(-\xi^0\xi^0 + \xi^1\xi^1) \tag{2.16}$$
$$+2\frac{\partial_-\partial_+ r}{r}\sum_{k=2}^{D-1}\xi^k\xi^k\Big\}$$
$$+S_{\text{ghosts}} \ .$$

(In fact, if we let $\sigma^-.\sigma^+ = X_{cl}^u, X_{cl}^v$, $f = -\tfrac{1}{2}(1 - (r_h/r)^{D-3})$, then (2.16) is the action in Schwartzchild null coordinates as well.) Note that this action has the standard kinetic term for the transverse coordinates; in general, the curvature terms will be some finite r-dependent piece (vanishing as a power of $r_h/r$ asymptotically) times a factor of $1/r_h^2$. Hence the kinetic term dominates the action far from the horizon. On the other hand,

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near the horizon the modes relevant to Hawking radiation are highly blue-shifted and also dominate the curvature terms.

The world sheet stress tensor comes from varying (2.9) with respect to $h_{\alpha\beta}$:

$$T_{\pm\pm} = g_{\mu\nu}\partial_\pm X^\mu \partial_\pm X^\nu$$
$$T_{+-} = 0 \qquad \text{(classically)} .$$
(2.17)

Using [8] to expand this to quadratic order, we find that:

$$T_{--} = 2g_{xy}\partial_- X^x_{\text{cl}} D_-\xi^y + 2g_{xy}D_-\xi^x D_-\xi^y + D_-\xi^k \cdot D_-\xi^k + R_{x\kappa\lambda x}\xi^\kappa\xi^\lambda$$
$$T_{++} = 2g_{xy}\partial_+ X^y_{\text{cl}} D_+\xi^x + 2g_{xy}D_+\xi^x D_+\xi^y + D_+\xi^k \cdot D_+\xi^k + R_{y\kappa\lambda y}\xi^\kappa\xi^\lambda ,$$
(2.18)

where $k$ is summed over the angular variables. Here we have written the longitudinal modes as coordinate vector fields rather than as tangent frame vectors, as we will find this form to be easier to use in explicitly solving the constraints. For the Schwarzschild metric, the curvature term couples to the longitudinal modes only, and the explicit form of $T_{\pm\pm}$ is

$$T_{--} = -2f\partial_-\xi^y - 2f\left(\partial_-\xi^x + \frac{\partial_- f}{f}\xi^x\right)\partial_-\xi^y$$
$$+ \frac{2(D-2)r_h^2}{(D-3)^3(\sigma^+\sigma^-)^2}\frac{D-2}{D-3}\left(\frac{r_h}{r}\right)^{D-1}\left[1-\left(\frac{r_h}{r}\right)^{D-3}\right]^2(\xi^y)^2$$
$$+ \partial_-\xi^k \cdot \partial_-\xi^k$$
$$T_{++} = T_{--}(\partial_- \leftrightarrow \partial_-, x \leftrightarrow y) .$$
(2.19)

## 3. Classical physics

### 3.1. Mode solutions

The equations of motion for the angular coordinates are

$$\partial_+\partial_-\xi^k - \frac{\partial_+\partial_- r}{r}\xi^k = 0, \qquad 2 \geq k \geq D-1 .$$
(3.1)

One may write the second (curvature) term more explicitly:

$$\frac{\partial_+\partial_- r}{r} = \left(\frac{r_h}{r}\right)^{D-1}\frac{1}{(D-3)\sigma^+\sigma^-}\left[1-\left(\frac{r_h}{r}\right)^{D-3}\right] ,$$
(3.2)

again with $r = r(x = \sigma^-, y = \sigma^+)$. It is easy to see that (3.2) vanishes for large $r$. For $r$ close to $r_h$, we may substitute in (2.7) and (2.3), and use (2.5), to find that the zero

in $\sigma^+\sigma^-$ cancels the zero in $[1 - (r_h/r)^{D-3}]$, and that the result is a finite number. As we shall see, all the modes will be blueshifted infinitely at the horizon, so that the kinetic term will then dominate close to the horizon. Thus we will, after ensuring stability, use the zeroth-order approximation that the transverse modes satisfy the 2-d massless Klein-Gordon equation for the purpose of computing Bogolubov transformations between mode bases. The inaccuracy of this procedure lies in the fact that there is mixing between left- and right-moving modes due to scattering off the background curvature. Hence the Bogolubov coefficients will mix left- and right-moving creation and annihilation operators.

We were unable to solve (3.1) exactly, but one can find solutions that work asymptotically as $r_* \to -\infty$ $(r \to r_h)$ and as $r, r_* \to \infty$. To do this, transform the variables as follows:

$$q = \sqrt{-\sigma^-\sigma^+}$$
$$s = \sqrt{-\frac{\sigma^+}{\sigma^-}} \; . \tag{3.3}$$

From (2.7) one can see that

$$q = e^{\frac{D-3}{2r_h} r_* (x=\sigma^-, y=\sigma^+)}$$
$$s = e^{\frac{D-3}{2r_h} t (x=\sigma^-, y=\sigma^+)} \; . \tag{3.4}$$

Note that these coordinates are only defined in one or the other regions of Figure I which are outside the horizon; we will define an appropriate analytic continuation below.

In these coordinates the equations of motion separate:

$$s^2\left(\frac{\partial^2}{\partial s^2} - \frac{1}{s}\frac{\partial}{\partial s}\right)\xi^k = q^2\left(\frac{\partial^2}{\partial q^2} - \frac{1}{q}\frac{\partial}{\partial q}\right)\xi^k - 4\frac{1}{D-3}\left(\frac{r_h}{r}\right)^{D-1}\left[1 - \left(\frac{r_h}{r}\right)^{D-3}\right]\xi^k \; . \tag{3.5}$$

One finds that at null infinity and in towards the horizon, the last term vanishes. Thus, in this asymptotic sense, the wave equation for the transverse coordinates is

$$s^2\left(\frac{\partial^2}{\partial s^2} - \frac{1}{s}\frac{\partial}{\partial s}\right)\xi^k = q^2\left(\frac{\partial^2}{\partial q^2} - \frac{1}{q}\frac{\partial}{\partial q}\right)\xi^k = -\omega^2\xi^k \; , \tag{3.6}$$

where $\omega$ should be real for stable solutions. Solutions to this approximate equation are

$$\xi^k = s^{i\omega}q^{\pm i\omega} \tag{3.7}$$

for any $\omega$. Translated back into Kruskal coordinates, the solutions become

$$\xi^k = (\sigma^-)^{\pm i\omega} \qquad \text{or} \qquad (\sigma^+)^{\pm i\omega} \; . \tag{3.8}$$

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These mode solutions are standard in Hawking radiation calculations [9] [10], and following Unruh and Birrell and Davies, we put the logarithmic branch cut for (3.8) in the upper half plane by making the choice

$$-1 = e^{-i\pi} . \tag{3.9}$$

With this choice, the solutions in (3.8) are analytic in the lower half plane of $x$, and thus their Fourier transforms vanish for $\omega < 0$, so that the modes shown are positive frequency in $\sigma^{\pm}$ for all $\omega$. Note also that for $r \to \infty$, these look like plane wave solutions in Schwarzschild coordinates.

The equations of motion for the longitudinal modes are most easily solved in their coordinate vector form. The equations are:

$$\partial_+ \partial_- \xi^x + \frac{\partial_- f}{f} \partial_+ \xi^x = 0$$
$$\partial_+ \partial_- \xi^y + \frac{\partial_+ f}{f} \partial_- \xi^y = 0 , \tag{3.10}$$

with the solutions

$$\xi^x = \phi^x(\sigma^-) + \int_{-\infty}^{\sigma^+} d\sigma^{+\prime} \frac{\psi^x(\sigma^{+\prime})}{f(\sigma^-, \sigma^{+\prime})}$$
$$\xi^y = \phi^y(\sigma^+) + \int_{-\infty}^{\sigma^-} d\sigma^{-\prime} \frac{\psi^y(\sigma^{-\prime})}{f(\sigma^{-\prime}, \sigma^+)} , \tag{3.11}$$

where $\phi^{x,y}$ and $\psi^{x,y}$ are arbitrary functions.

The actual classical solutions to this equation should include the satisfaction of the Virasoro constraints. If the gradients of the coordinate fluctuation fields are moderate (as they are asymptotically), the constraints will be self-consistently solved by equating the linear term in longitudinal modes to the quadratic term in transverse modes; the terms quadratic in longitudinal modes will be of fourth order in the fluctuation size and should be dealt with in the next order of the calculation. Thus we find that in the asymptotic region where $\xi^k \sim (\sigma^+)^{i\omega}$,

$$\psi^x \sim (\sigma^+)^{2i\omega - 2}$$
$$\xi^x \sim \sigma^-(\sigma^+)^{2i\omega} \quad , \qquad r \to \infty \quad .$$

As $r$ approaches $r_h$, however, $f\partial\xi^x$ will become large and the term in (2.18) which is second order in derivatives of longitudinal fluctuations will become even larger and will become

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the dominant contribution to the longitudinal stress tensor. Thus, near the horizon we should equate this quadratic term with $\partial_\pm \xi^k \partial_\pm \xi^k$. Specifically, the piece of the stress tensor quadratic in derivatives of longitudinal fluctuations is

$$f\partial_+\xi^x(\partial_+\xi^y + \frac{\partial_+ f}{f}\xi^y) = \psi^x\left(\partial_+\phi^y - \int^{\sigma^-} d\sigma^{-\prime} \frac{\psi^y}{f}\frac{\partial_+ f}{f} + \frac{\partial_+ f}{f}\xi^y\right) . \tag{3.12}$$

Now

$$\frac{\partial_\pm f}{f} = -\frac{1}{\sigma^\pm}\left[1 - \left(\frac{r_h}{r}\right)^{D-2}\right] . \tag{3.13}$$

As $r \to r_h$, we can find by writing out $\sigma^\pm$ in terms of $r$ and $t$ that $\partial_\pm f/f$ vanishes as $\sqrt{r - r_h}$ as we approach the horizon; thus in this limit,

$$f\partial_+\xi^x\left(\partial_+ + \frac{\partial_+ f}{f}\right)\xi^y \sim \psi^x\partial_+\phi^y , \tag{3.14}$$

and similarly for $x \leftrightarrow y$, $\sigma^+ \leftrightarrow \sigma^-$. Matching these forms to $\partial_\pm\xi^k\partial_\pm\xi^k \sim (\sigma^\pm)^{2i\omega-2}$, we can solve the constraints to lowest order by setting

$$\begin{aligned}\psi^{x,y} &\sim (\sigma^{+,-})^{i\omega-1} \\ \phi^{x,y} &\sim (\sigma^{-,+})^{i\omega} \qquad r \to r_h .\end{aligned} \tag{3.15}$$

Note that with this solution the term quadratic in the derivatives of longitudinal fluctuations indeed diverges faster than both the term linear in the derivatives and the non-derivative quadratic term.

## 3.2. Classical linear stability

Before we quantize the quadratic theory (2.16), we need to know whether it is classically consistent – that is, if small fluctuations stay small over time. We should note that simply looking at the size of $\xi^\mu$ over space and time is incorrect since the metric is invariably singular in some region of spacetime for whatever coordinate system we might choose – in $H^\pm$ for Schwarzschild coordinates and in $\mathcal{I}^\pm$ for Kruskal coordinates. The natural quantity to look at is the scalar product of the fluctuations,

$$\begin{aligned}\|\xi\|^2 &= g_{\mu\nu}\xi^\mu\xi^\nu \\ &= 2g_{xy}\xi^x\xi^y + \delta_{kl}\xi^k\xi^l \qquad (\text{k,l} \geq 2) .\end{aligned} \tag{3.16}$$

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For the transverse fluctuations, in the absence of exact solutions to (3.5), we shall try to cast the equations of motion into a Sturm-Liouville form where stability is obvious. If we make an additional change of variables

$$R = \ln q = \frac{D-3}{2r_h} r_*$$
$$T = \ln s = \frac{D-3}{2r_h} t \ ,$$

(3.17)

then the equations of motion become

$$\frac{\partial^2}{\partial T^2} \xi^k - \frac{\partial^2}{\partial R^2} \xi^k + g(R)\xi^k = 0 \ ,$$

(3.18)

where

$$g(R) = \frac{4}{D-3} \left(\frac{r_h}{r}\right)^{D-1} \left[1 - \left(\frac{r_h}{r}\right)^{D-3}\right] \ ,$$

(3.19)

with $r$ used as an implicit function of $R$. Roughly, (3.18) is the Klein-Gordon equation with a spatially varying mass term. Since the "mass squared" $g(R)$ is always positive, the normalizable transverse solutions are stable.

Knowing this, we may use $(\sigma^\pm)^{i\omega}$ for real $\omega$ as a complete set of asymptotic solutions for the transverse equations of motion, and we may then use the results of the previous subsection to examine the form of the longitudinal solutions in the asymptotic regions. For the $r \to \infty$ region,

$$g_{xy} = f \sim \frac{1}{\sigma^+ \sigma^-} \ , \qquad \xi^x \sim \sigma^-(\sigma^+)^{2i\omega} \ , \qquad \xi^y \sim \sigma^+(\sigma^-)^{2i\omega}$$

(3.20)

so that the scalar product of the longitudinal fluctuations is

$$g_{xy}\xi^x\xi^y \sim (\sigma^+\sigma^-)^{2i\omega} \ .$$

For the $r \to r_h$ region, f is a regular, finite quantity, the zero in $1/(\sigma^+\sigma^-)$ cancelling the zero in $[1 - (r/r_h)^{D-3}]$. The form of $\xi^{x,y}$ near the horizon will be

$$\xi^x \sim (\sigma^+)^{i\omega} + \int^{\sigma^-} d\sigma^{-\prime} \frac{(\sigma^{-\prime})^{i\omega-1}}{f} \ ,$$

(3.21)

and similarly for $x \leftrightarrow y$, $\sigma^+ \leftrightarrow \sigma^-$. Naive power counting tells us that the second term should be regular near the horizon, so that $g_{xy}\xi^x\xi^y$ is also regular near the horizon.

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## 4. Quantization

In this section we would like to address two separate issues: the imposition of the Virasoro constraints on the theory defined by (2.16), and the problems of selecting physically motivated "in" and "out" states. We will first discuss in general the conformal invariance of the system, and argue that the angular fluctuations are a good representation of the physical Hilbert space of the string by showing that the partition function depends only on the determinants of the transverse fluctuation operators. We will then discuss the possible definitions of the positive frequency modes of the transverse fluctuations and construct the desired vacuum states and their Bogolubov transformations. Finally, we shall argue that the Hawking radiation does not include unphysical modes if the BRST cohomology theorem holds for this system.

### 4.1. Path integral quantization

Now the full nonlinear $\sigma$-model action (2.9) satisfies the $\beta$-function equations of [7] in the standard critical dimensions to first order in $\alpha'$, since the metric coupling satisfies Einstein's equations. Thus we should be able to use the conformal invariance to decouple the negative-norm states in the Hilbert space of coordinates plus ghosts, and we expect that in principle one should be able to embark upon a standard covariant quantization program by imposing the Virasoro conditions or the corresponding BRST condition. We have not been able to successfully use this formalism to prove a no-ghost theorem due to the subtleties of constructing the vacuum, which involves correlating the fluctuations of the string near the horizon and in the asymptotic region.

If the spectrum is free of negative norm states and conformal invariance may be imposed on the quantum level, then the only physical degrees of freedom are the transverse oscillations of the string, as in flat space. We will support this supposition by showing that the partition function depends only on the determinants of the transverse fluctuations; upon integrating out the conformal ghosts and (at quadratic order) the longitudinal bosonic coordinates, we will find that determinants from these integrals will formally cancel each other leaving only the partition function for the transverse coordinates.

Consider the longitudinal $(\xi^0, \xi^1)$ part of (2.16) and integrate it by parts to put the lagrangian in the form $\mathcal{L} = \xi^m D_{mn} \xi^n$. The result is

$$
S_{\text{longitudinal}} = \frac{i}{2} \int d^2\sigma \Bigg\{ - \eta_{mn} \xi^m \left( 2\partial_- \partial_+ - \frac{\partial_- \partial_+ f}{f^2} + \frac{\partial_- f \partial_+ f}{2f^2} \right) \xi^n \\
- \epsilon_{mn} \xi^m \left( \frac{1}{f} \epsilon_{\alpha\beta} \partial_\alpha f \partial_\beta \right) \xi^n \Bigg\} ,
$$

(4.1)

where $m$ and $n$ are restricted to 0 and 1 (which the vielbeins map back to the $x - y$ plane). Integrating over $\xi^0$ and $\xi^1$ leads to the determinant of the above quadratic operator. On the other hand, Polyakov[11] showed that in writing an arbitrary metric as

$$h_{\alpha\beta} = \hat{h}_{\alpha\beta}\delta\phi + \nabla_\alpha\varepsilon_\beta + \nabla_\beta\varepsilon_\alpha \ , \tag{4.2}$$

where $\nabla$ is the worldsheet covariant derivative, the functional integration measure is

$$\mathcal{D}g_{\alpha\beta} = \mathcal{D}\phi\mathcal{D}\varepsilon_\alpha \det[L_{\alpha\beta}] \ , \tag{4.3}$$

with

$$L_{\alpha\beta} = \nabla_\beta\nabla_\alpha + \hat{h}_{\alpha\beta}\nabla^\lambda\nabla_\lambda - \nabla_\alpha\nabla_\beta \tag{4.4}$$

If we change variables from the coordinate vectors $\varepsilon_\alpha$ to the tangent space vectors $\varepsilon^a = e^a_\alpha\varepsilon^\alpha$, where $e^a_\alpha$ is the zweibein for the worldsheet with metric $\hat{h}_{\alpha\beta}$, then we get (4.3) with the indices on the right hand side changed to tangent space indices and the tangent space operator $L_{ab}$ identical to the operator in (4.1), so that the Fadeev-Popov determinant formally cancels the determinant arising from integrating out the longitudinal modes.

In performing the normal coordinate expansion, the fields became contravariant vectors in spacetime, and so the kinetic terms were endowed with a spin connection induced by the spacetime geometry. This allowed us to use the determinants which arose from integrating out the longitudinal fluctuations to cancel the Fadeev-Popov determinant written with worldsheet covariant derivatives acting on worldsheet vectors and spinors; the worldsheet metric is pulled back from the spacetime metric, so it is reasonable to state that the worldsheet tangent space structure, and thus the spin connection, is induced by the spacetime tangent space structure. Thus this cancellation is a consequence of our choice of equilibrium classical solution and of the particluar choice of fiducial metric shown in equation (2.11).

It is important to keep straight in the above argument the distinction between the intrinsic metric $h_{\alpha\beta}$ and the induced metric $\gamma_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X)$. The determinants that arise at quadratic order in the functional integral have the structure

$$\det_{FP}[L_{ab}(h)] \det^{-1}_{X\pm}[L_{mn}(\gamma)] \det^{-1}_{X\perp}[\Delta_0(G)] \tag{4.5}$$

where $\Delta_0$ is the quadratic fluctuation operator of the transverse modes. The arguments of the differential operators indicate their explicit dependences; of course each determinant

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implicitly depends on $h$ through the regulator. Since we are in the critical dimension, the *derivative* of this product of determinants with respect to $h$ vanishes. On the other hand, if we are interested in the *value* of the determinants at the point $h = \gamma$, then the above treatment shows that the first two determinants cancel and the partition function will be the determinant of the transverse fluctuation operator regulated with the induced metric.

*4.2. Operator quantization, vacuum states, and Bogolubov transformations*

In constructing the Hilbert space of the transverse fluctuations we are faced with the well-known problem of finding a reasonable physical vacuum state (see for example [10] for a general discussion) given curved worldsheet and spacetime manifolds. In the absence of an exact quantization we are restricted to constructing this vacuum perturbatively in the normal coordinate expansion. Based on the analysis of the previous section, this expansion is consistent with the solution of the Virasoro constraints in conformal gauge. Moreover we expect to be able to dress any state in the Hilbert space of the transverse modes into a physical state whose dependence on the longitudinal coordinates is a small deviation from the classical identification $\sigma^- = X^x, \sigma^+ = X^y$ which may be corrected for systematically (order by order). In particular, we may use the classical worldsheet $\leftrightarrow$ spacetime correspondence to ask what worldsheet fluctuations might be positive frequency with respect to a spacetime observer. Here we will assume that the aforementioned solution to the Virasoro constraints may be arrived at quantum mechanically in fixing to a physical, light-cone-like gauge along the lines of section 3.1. The problem then reduces, in lowest order, to that of considering the transverse fluctuations in (2.16) as a field theory in a 2d spacetime corresponding to the $r$ - $t$ or $x$ - $y$ plane of a Schwarzschild black hole; scalar field theories in this 2-D spacetime have been well studied (see for example [9][10]and references therein).

We wish to assume that the "in" vacuum consists of no excitations in the far past in Schwarzschild time. On the Kruskal manifold, this corresponds to asking that there are no field excitations leaving $H^-$ and $\mathcal{I}^-$, in region I of figure 1, and no excitations leaving $H^+$ and $\mathcal{I}^+$ in region II of figure 1. An observer at $\mathcal{I}_I^-$, living in an asymptotically flat region, will be a Schwarzschild observer and thus will define positive frequency modes with respect to the Schwarzschild time $t$. An observer leaving or falling into the horizon, or at least one for whom the horizon is a finite time away, will be a Kruskal observer and thus will define positive frequency modes with respect to a characteristic Kruskal time; in particular, on $H^\pm$ and $\mathcal{I}^\pm$, the timelike Killing vector becomes lightlike so that the

characteristic Kruskal time will be $\sigma^+$ or $\sigma^-$ depending on the surface. Thus, the "in" vacuum is in region I annihilated by left-moving modes which are positive-frequency with respect to $t$ and right-moving modes which are positive frequency with respect to Kruskal time. We are only asking questions about observers in region I of our Kruskal diagram, so we have chosen the modes in region II for mathematical convenience – phenomena in region I will not depend on how we treat region II. This vacuum is the "Unruh vacuum," used to mock up gravitational collapse [9][10][12]. We might use other vacuums, for example by asking that there is no radiation entering or leaving the horizon $H_I^- \cup H_I^+$ – the "Hartle-Hawking vacuum" [10] – but the Unruh vacuum is physically compelling and at any rate the extension of our calculations to the Kruskal vacuum is not difficult.

The "out" vacuum we choose corresponds to an observer in the far future $(\mathcal{I}_I^+)$ who, living in an asymptotically flat region, will define positive frequency modes in terms of Schwarzschild time; thus the "out" vacuum is that annihilated by all modes which are positive frequency with respect to $t$. This is known as the "Schwarzschild vacuum."

For the right-moving modes used to define the Unruh vacuum we use the solutions $(\sigma^-)^{i\omega}$ which are, as we have stated below (3.9), are positive frequency for all (real) $\omega$. Negative frequency modes are defined by taking the complex conjugate, which is most easily computed by writing $x^{i\omega}$ properly in terms of Schwarzschild modes in region I and II,

$$x^{ia\omega} = e^{\pi\omega a}\, e^{-i\omega u_I}\, \theta(-x) + e^{-i\omega u_{II}}\, \theta(x)$$

$$y^{1\omega a} = e^{i\omega v_I}\theta(y) + e^{\pi\omega a}e^{i\omega v_{II}}\theta(-y) \tag{4.6}$$

$$a = \frac{2r_h}{D-3}\ ,$$

where $u_I$ and $u_{II}$ are the Schwarzschild null coordinates in region I and II, respectively, of figure I, and $\theta$ is the Heaviside function. Note that due to the real exponential factor, the complex conjugate of $x^{i\omega}$ is different from $x^{-i\omega}$. More formally, complex conjugation will move the logarithmic branch cut in $x^{i\omega}$ from the upper half plane as defined in (3.9) to the lower half plane, so that the complex conjugate is upper half plane analytic and thus is composed of negative frequency modes.

The transverse fluctuation operators may now be written in terms of the two different mode decompositions. The frequency-dependent coefficients in the sums over modes arise

from requiring that the single-particle wavefunction in front of each operator is normalized to a delta function under the Klein-Gordon norm [10], [9]. These decompositions are:

$$
\begin{aligned}
\xi^k = & \int_0^\infty \frac{d\omega}{\sqrt{4\pi|\omega|}} \left[ e^{-i\omega v_I} a_I^k(\omega) + e^{i\omega v_I} a_I^{k\dagger}(\omega) \right] \theta(y) \\
& + \int_{-\infty}^\infty \frac{d\omega}{\sqrt{4\pi|\omega|\sinh(2\pi|\omega|)}} e^{-\frac{1}{2}\pi\omega a} \left[ x^{i\omega a} b^k(\omega) + (x^{i\omega a})^* b^{k\dagger}(\omega) \right] \\
& + \int_0^\infty \frac{d\omega}{\sqrt{4\pi|\omega|}} \left[ e^{i\omega v_2} a_{II}^k(\omega) + e^{-i\omega v_{II}} a_{II}^{k\dagger}(\omega) \right] \theta(-y)
\end{aligned} \tag{4.7}
$$

in the modes by which we define the Unruh vacuum, and

$$
\begin{aligned}
\xi^k = \int_0^\infty \frac{d\omega}{\sqrt{4\pi|\omega|}} \Bigg[ & \left( e^{-i\omega v_I} a_I^k(\omega) + e^{i\omega v_I} a_I^{k\dagger}(\omega) \right) \theta(y) \\
& + \left( e^{-i\omega u_I} c_I^k(\omega) + e^{i\omega u_I} c_I^{k\dagger}(\omega) \right) \theta(-x) \\
& + \left( e^{i\omega u_{II}} c_{II}^k(\omega) + e^{-i\omega u_{II}} c_{II}^{k\dagger}(\omega) \right) \theta(x) \\
& + \left( e^{i\omega v_{II}} a_{II}^k(\omega) + e^{-i\omega v_{II}} a_{II}^{k\dagger}(\omega) \right) \theta(-y) \Bigg]
\end{aligned} \tag{4.8}
$$

in the modes by which we define the Schwarzschild vacuum. The operators $a_{I,II}, b, c_{I,II}$ each satisfy canonical commutation relations

$$
[a_I^k(\omega), a_I^{k'\dagger}(\omega')] = \delta^{kk'} \delta(\omega - \omega') . \tag{4.9}
$$

The modes describing left-moving excitations, created by $a_I^{k\dagger}$ and $a_{II}^{k\dagger}$ are the same for both decompositions so that the only difference is in our definition of left-moving excitations. The Bogolubov transformations between the $b$-operators defining the Unruh vacuum and the $c_{I,II}$-operators defining the Schwarzschild vacuum (we shall call these "Kruskal" and "Schwarzschild" operators, respectively) are easily found by expanding $x^{i\omega a}$ as in (4.6), and are [10]:

$$
\begin{aligned}
c_I^k(\omega) &= \frac{e^{\frac{1}{2}\pi\omega a}\, b_I^k(\omega) + e^{-\frac{1}{2}\pi\omega a} b^{k\dagger}(-\omega)}{\sqrt{2\sinh(\pi|\omega|a)}} \\
c_{II}^k(\omega) &= \frac{e^{\frac{1}{2}\pi\omega a}\, b^k(-\omega) + e^{-\frac{1}{2}\pi\omega a}\, b^{k\dagger}(\omega)}{\sqrt{2\sinh(\pi|\omega|a)}}
\end{aligned} \tag{4.10}
$$

$$
\omega \geq 0 .
$$

We shall use these transformations in the next section to compute the Hawking radiation of the black hole into spatial fluctuations of the string.

*4.3. Physical states and Hawking radiation*

One of the original motivations of this work was to figure out whether the Virasoro constraints could be consistently imposed in the covariant formalism in the presence of Hawking radiation: one might think that the Hawking radiation would populate all modes – ghosts, longitudinal modes, *etc.* – thermally and so there is no way to guarantee that an asymptotic observer would not see unphysical modes radiated by the black hole. However, in order to ask physical questions, one must either work in a physical gauge or work with physical operators in a covariant gauge, and define the vacuum in a physical fashion.

The first question that should be asked is whether one may legitimately fix to the physical gauge we have chosen, or in covariant language, whether one may consistently impose the BRST condition

$$Q_{\text{BRST}}|\text{physical}\rangle = 0$$

and find an appropriate representation for the BRST cohomology classes. We have not been able to rigorously prove either, but given that classically one may fix a physical gauge; that the theory is anomaly free to one loop (since the one-loop $\sigma$-model beta functional vanishes); and that the longitudinal vacuum fluctuations explicitly cancel off the ghost vacuum fluctuations in the partition function (again, to one loop), it seems that we have sufficient circumstantial evidence that the constraints may be solved explicitly and consistently.

Given a consistent physical gauge, we may simply work with the transverse oscillators and solve for the longitudinal oscillators order by order. This is fine for computing quantities such the size of transverse fluctuations $\langle \xi^2 \rangle$ or the expectation value of the transverse stress tensor, at least to lowest order in the normal coordinate expansion, as we shall below. Alternatively, if BRST quantization is consistent, we expect that we can represent the BRST cohomology classes by building states with DDF-like operators[13][14][15][16]. This also amounts to dressing the transverse oscillators with longitudinal modes to make a physical state. In building such states we are faced with questions of what a physical (spacetime) observer sees as positive frequency; the worldsheet coordinates have no meaning in covariant gauge, so in particular the question of what is a positive frequency mode on the worldsheet has no inherent spacetime meaning. However, since the frequency of a DDF operator is a spacetime quantity, we use it to define the vacuum directly with these manifestly physical operators. (This is exactly what Schoutens *et.al.* [17] have done for their model of 2-D black hole evaporation.) We see why our original fear was naive; a

proper formulation of the theory avoids any question of unphysical modes before Hawking radiation ever becomes an issue.

This argument can be recast in terms of more physical or heuristic descriptions of Hawking radiation. If one considers Hawking radiation as due to pair production near the horizon [10] then the argument means essentially that pair production involves only physical modes, so that only physical states are radiated to $\mathcal{I}^+$. Similarly, if one describes Hawking radiation via a density matrix due to a summation over states at or behind the horizon, hidden from our asymptotic observer at $\mathcal{I}^+$, then one will sum only over states in the physical Hilbert space – the full Hilbert space including negative norm states being in some sense a fiction arising from the gauge freedom remaining after passing to a covariant gauge.

## 5. Physical manifestations of the Hawking process

Having discussed quantization and the difference between the "in" and "out" vacuums, one may begin to ask some physical questions about the string. The Hawking radiation in the 2d worldsheet field theory will manifest itself in some thermal population of the transverse modes in the "out" basis, which means that an asymptotic (Schwarzschild) observer will see the string fluctuating as if it was thermally excited with the worldsheet temperature equal to the Hawking temperature

$$k_B T_H = \frac{\hbar c (D-3)}{2\pi r_h} \ . \tag{5.1}$$

The three most apparent (to us) detectable manifestations of this radiation are the thermal wandering of the string, the energy that is radiated in string fluctuations out of the black hole towards $\mathcal{I}^+$, and the production of physical spacetime modes (microscopic strings) due to transitions of the excited string. We discuss the first two of these for the bosonic string in the following subsections.

### 5.1. Wandering of the string in the Schwarzschild background

As an estimation of the wandering of the string, we may calculate the mean square deviation from the classical background solution,

$$\langle \xi^2 \rangle = \lim_{x \to x', y \to y'} \sum_{k=2}^{D-1} \langle \Psi | : \xi^k(x,y) \xi^k(x',y') : | \Psi \rangle \ . \tag{5.2}$$

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We need to fix both the state $|\Psi\rangle$ and the normal ordering prescription. $|\Psi\rangle$ we take to be the Unruh vacuum, but we wish to normal order this expression with respect to the Schwarzschild modes, on the assumption that normal ordering is a function of the observer. This means that we begin by expanding the operators $\xi^k$ according to (4.8), pushing the Schwarzschild creation operators to the left of the destruction operators, and then using (4.10) to expand out $c^k$ in terms of $b^k$, so as to be able to calculate the expectation value in the "in" state which is annihilated by $a^k$ and $b^k$. The normal ordered correlator is

$$\langle : \xi(x,y)\xi(x',y') : \rangle = \frac{(D-2)\ell_s^2}{4\pi} \int_0^\infty \frac{d\omega \, 2\cos[\omega(u-u')]}{\omega \left(e^{2\pi\omega a} - 1\right)} \ , \tag{5.3}$$

where we have put in a factor of $\ell_s^2 = \hbar^2 c^2 2\pi\alpha' = \hbar c/T$, where $T$ is the string tension [18], to give $\langle\xi^2\rangle$ the correct dimensions. We note immediately that the normal ordering has removed the ultraviolet divergences but a nasty infrared divergence remains. Eq. (5.3) has the form $\int d\omega D(\omega)\langle n(\omega, T_H)\rangle$ (with $D$ being the density of states) of a 2d Bose gas of $D-2$ massless particles, where $k_B T_H = 1/\beta = 1/(2\pi a)$. The IR divergences arising from the density of states and from the Planck distribution function are characteristic of this general statistical mechanical system in the infinite volume limit, and are entirely physical. Also note that the transverse stress tensor $\partial\xi\partial\xi$ is infrared finite at this order. To make sense of (5.3), we need a physically motivated finite-size or finite-time cutoff – an example of the latter is the finite lifetime of the black hole.

Thus if one regulates (5.3) with an infrared momentum cutoff at momentum $\epsilon$, one finds that $\langle\xi^2\rangle$ diverges linearly with the cutoff,

$$\langle\xi^2\rangle \sim \frac{(D-2)\ell_s^2}{2\pi\beta\epsilon} \ . \tag{5.4}$$

The deviation then goes as the square root of the finite-size or finite-time cutoff $1/\epsilon$, so that the average string configuration is that of a random walk. This is a result we might have expected from studies of average string configurations at finite temperature; in particular, Mitchell and Turok [19] found that for closed bosonic strings at finite temperature, the mean square radius of the string scaled as the length $L$ of the string,

$$\langle\Delta r^2\rangle \sim L \ .$$

Nevertheless the angular variation $\delta\theta \sim \delta\xi/r$ is well-behaved, so although the string fluctuates far from its center of mass, it does not cover a large solid angle of the black hole.

The presence of this infrared catastrophe indicates that although the equations of motion derived from (2.13) are stable, we expect that quantum fluctuations will require us to include some nonlinear effects in the worldsheet effective action. One might worry that, since the thermal propagator for the transverse string fluctuations is quadratically infrared divergent, that higher order corrections to *e.g.* the stress tensor will be much larger than the effects we have calculated (see below). We believe that what is breaking down is not the order of magnitude of the radiated Hawking flux, but rather the Taylor expansion of certain quantities in the action. For example, part of the small fluctuation expansion involves the Taylor expansion of terms like $G(X_{\rm cl} + \xi)\partial\xi\partial\xi$, which has a radius of convergence of order $r_{\rm cl}$. So instead of treating perturbatively the quantum fluctuations of all the modes, we should introduce an infrared cutoff and treat the low frequency oscillations adiabatically and classically (since they have large occupation numbers) while quantizing all the higher frequency modes. This scheme should work quite well if we stay far from the horizon.

### 5.2. Energy radiated onto the string

We would like some idea of the energy radiated onto the string in the Hawking process. The world-sheet stress tensor measures the world-sheet energy radiated out along the string, so a definite relation between world-sheet and spacetime time coordinates will relate world-sheet and spacetime energy densities and fluxes. Such a relation is provided by the classical solution (2.10)– hence we expect that the spacetime and world-sheet energy fluxes should coincide.

We shall work entirely in the asymptotically flat regime of spacetime, in Schwartzchild null coordinates. In this region the Lagrangian (2.9) is manifestly (approximately) Lorentz invariant and so there is an asymptotically well defined Noether current

$$P_{\alpha\mu} = Tg_{\mu\nu}\partial_\alpha X^\nu \ , \tag{5.5}$$

where $T$ is the string tension. The total momentum radiated to future infinity will then be

$$P_\mu^{\rm rad} = \int_{\mathcal{I}^+} d\sigma^- P_{-\mu} \ . \tag{5.6}$$

Expanding in normal coordinates, we find that

$$P_{\alpha\mu} = P_{\rm (cl)\alpha\mu} + \delta P_{\alpha\mu} = T(\partial_\alpha X_0^\mu + D_\alpha\xi^\mu + \dots) \ . \tag{5.7}$$

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Now $-P^\mu P_\mu$ should be the total mass-squared radiated away to $\mathcal{I}^+$, so that

$$
\begin{aligned}
-2fP^u P^v + r^2 \left(P^\perp\right)^2 &= -2fP_c^u P_c^v - 2fP_c^u \delta P^v - 2fP_c^v \delta P^u - \\
&\quad - f\delta P^u \delta P^v + (P_c^\perp)^2 + 2(P_c^\perp \cdot \delta P^\perp) + (\delta P^\perp)^2 \qquad (5.8) \\
&= -(M_c + \delta M)^2 \ .
\end{aligned}
$$

where the subscript $c$ denotes "classical." We use the classical mass shell condition and our explicit values for $X_c^\mu$ to write

$$
-2fP_c^u \delta P^v + (\delta P^\perp)^2 = -2M_c \delta M - \delta M^2 \ . \qquad (5.9)
$$

The integrals in (5.6) used to define the total momentum $P^\mu$ will be infinite and will furthermore select out the zero modes of $\partial_- X^\mu$. We thus find an expression for mass per unit interval along $\mathcal{I}^+$:

$$
-2f\partial_- X_c^u (\partial_- \xi^v)_0 + (\partial_- \xi_0^\perp)^2 + \ldots = -\frac{2M\delta M}{(\text{length})^2} - \left(\frac{\delta M}{\text{length}}\right)^2 \ , \qquad (5.10)
$$

where the subscript 0 indicates the space-independent part. We will drop the $\delta M^2$ term as being higher order in our expansion. Substituting in the lowest order light-cone condition in Schwarzchild null coordinates, we find that

$$
2f\partial_- X_c^u \partial_- \xi^v = T_{--}^\perp
$$

(where here we have set $\sigma^- = u$, $\sigma^+ = v$), and thus

$$
T_{--,0}^\perp - (\partial_- \xi_0^\perp)^2 = \frac{1}{T^2}\frac{2M\delta M}{(\text{length})^2}. \qquad (5.11)
$$

The second term on the left hand side should vanish since we are looking at oscillating transverse solutions with zero total transverse momentum. Now $(M/\text{length})$ is just the string tension, so that, converting energy density to energy flux,

$$
\frac{\delta M}{\text{time}} = \frac{c}{4\pi\hbar c\alpha'}T_{--,0}^\perp \qquad (5.12)
$$

In order to compute the Hawking radiation, we wish to take the vacuum expectation value of $: T^\perp :$ in the Unruh vacuum, with normal ordering performed as before with respect to

the Schwarzschild vacuum. With this vacuum and normal ordering prescription, only $T^\perp_{--}$ will be non-zero, and we find that to lowest order,

$$\langle : T^\perp_{--} : \rangle = (D - 2) \frac{\pi \ell_s^2}{12 \beta^2} \ . \tag{5.13}$$

The mass per unit time radiated out onto the string is thus found to be:

$$\text{energy flux} = \frac{\delta M}{\text{time}} \sim \frac{\hbar c^2}{r_h^2} \ , \tag{5.14}$$

where we have put all of the physical constants back in.

We may ask when the energy radiated onto the string modes is larger than the radiation into spacetime modes. The energy flux into, for example, a scalar field such as the string dilaton is essentially the same as calculated above since the radiation is mostly into S-wave states:

$$\mathcal{P} \sim \frac{\hbar c^2}{r_h^2} \ . \tag{5.15}$$

Thus the energy flux of radiation onto string modes is proportional to the energy flux of spacetime scalars.

In fact both the world-sheet and spacetime field energy fluxes differ from the estimates (5.14) and (5.15) in several ways. Both the spacetime field modes and the angular string fluctuations will suffer backscattering from the gravitational field near the horizon. This backscattering will be frequency-dependent, and for energies typical of the Hawking radiation leads to a non-negligible fraction of the radiation to be reabsorbed by the black hole. Curiously, although the shape of this barrier is in general different for spacetime S-waves and string angular fluctuations, the two barriers are identical in $D = 4$. The spacetime field modes of nonzero angular momentum also contribute some flux (decreasing exponentially in the angular momentum) which does not significantly alter the total flux. Finally, it should be emphasized that the string is generally propagating in a spacetime $\mathcal{M}^D_{\text{schw}} \times \mathcal{K}^{d_{cr} - D}$, the product of the $D$-dimensional Schwarzschild geometry we have been discussing and an unspecified internal manifold $\mathcal{K}$ that together with $\mathcal{M}_{\text{schw}}$ makes up the total effective central charge $d_{cr}$ of the sigma model in which the string propagates. The string coordinates of this internal space will also Hawking radiate, and in this case there is no barrier from the Schwarzschild curvature. The corrections to the flux in internal coordinates come from the difference between the density of states of the sigma model on $\mathcal{K}$ and that of free fields. If the sigma model on $\mathcal{K}$ is weakly coupled, in the normal

coordinate expansion the corrections to the stress tensor will be down by powers of $\ell_s/r_\mathcal{K}$ where $r_\mathcal{K}$ is the typical radius of curvature of $\mathcal{K}$. Thus the total flux in spacetime modes will be roughly proportional to the number of massless spacetime fields (remembering to count polarization states and correct for the angular momentum barrier in case of intrinsic spin), while the flux in string modes will be roughly proportional to $d_{cr} - 2$.

Finally, one should remember that when the black hole entraps the macroscopic string, two semi-infinite pieces protrude from the horizon. This doubles the flux into string modes – or does it? For chirally asymmetric strings, one of the two pieces propagates 'left-moving' string modes in the radially outward direction; the other piece propagates 'right-moving' string modes radially outward. Thus the energy-momentum flux in general differs for the two halves. This effect is most pronounced for the heterotic string in the critical dimension, where the left-moving (bosonic string) flux is that of 24 bosons, while that of the right-moving (fermionic string) flux is effectively that of 12 bosons (*e.g.* by bosonizing the eight transverse fermions). Curiously, the back-reaction of the Hawking radiation will not only evaporate the black hole, but due to the asymmetric radiation pressure will accelerate the black hole down the string!

## 6. Discussion

One issue we have not really settled is how to restrict the string functional integral to the region of field space outside the horizon. In principle the string fluctuates away from the classical solution in such a way that part of the string dips into the black hole. If we take seriously the ultraviolet fluctuations of the world-sheet, these 'dips' are ubiquitous (and indeed extend to the singularity), although incoherent. A bit of averaging over short distances replaces such fluctuations by a renormalized wandering of order the renormalization scale. We leave a detailed investigation of the structure of physical states near the horizon to future work.

Another issue is the relation between the Bekenstein-Hawking entropy as measured by the world sheet, in comparison to that measured by spacetime physics. The former will be either $\log M$ or constant as in the two-dimensional model system of [1], whereas the latter will be proportional to the $(D-2)$-dimensional area of the event horizon; what states of the black hole does the string have access to? The naive calculation would suggest only the 'states near the horizon' where the string is attached to the black hole. It has been claimed recently that the divergent quantum corrections to the entropy are

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cut off in string theory [20] because of string theory's soft spacetime high energy behavior. However string theory still has divergences on the world sheet, and those presumably still contribute a logarithmic divergence to the macroscopic string's entropy, unless there is a relation between the world sheet cutoff and the spacetime cutoff provided by the soft high-energy behavior of strings. But how does such a relation come about? Again it is not clear.

A logical extension of the present work (currently being contemplated) involves the construction of the physical vacuum. According to our discussion of section (4.3), this vacuum is not an incoherent superposition of all modes, but rather of a thermal distribution of transverse modes which are *coherently* dressed by ghosts and longitudinal modes. We intend to build such a state out of DDF-style operators for the dressed transverse oscillation modes, with the ultimate goal of calculating the amplitudes for emission and scattering of microscopic strings.

One may also contemplate a number of other model calculations which go beyond the scope of the present work. One would like to study not just the evaporation problem but also the formation of black holes in the macroscopic string model, as in the two-dimensional system of Callan *et.al.*[1]. This might be achieved by an appropriate calculation of the gravitational field back-reaction of a macroscopic string. Naively the left- and right-moving modes of a fundamental string in flat space are free fields. However, we expect two counter-propagating pulses ('shock waves') to interact gravitationally, and if sufficiently energetic, to form a black hole upon collision. Generally one must also take into account the other long-range fields carried by the string (*e.g.* the axion and dilaton charge per unit length carried by macroscopic fundamental strings [21]). The appropriate starting point for such a calculation might be the family of exact string solutions of Garfinkle [22][23].

The proper relation between the string and spacetime fields were only partially addressed here. It would be interesting to exhibit a self-consistent world-sheet calculation that would incorporate the back-reaction of the macroscopic string dynamics on the spacetime fields, for instance to show in a world-sheet beta function calculation how the energy radiated along the string contributes to a decrease in the black hole mass. We imagine such effects should appear in a term of the Polyakov type $(\partial \log[f(X)])^2$ in the world-sheet effective action, which then couples to the spacetime effective action in the manner described in [23]. This mechanism is somewhat confusing in that the macroscopic string is a single quantum from the point of view of the string field theory, and we do not ordinarily incorporate the classical field of a single quantum in the classical background (*e.g.* we do

not treat single electrons as small black holes or background Coulomb charges). Presumably one needs to resum the emission of soft gravitons from the world sheet into an effective classical field, and such nearly on-shell gravitons will give logarithmic almost-divergences in the world-sheet theory which resum to a modified classical field potential. From the worlsheet point of view these microscopic strings that dress the spacetime geometry are a resummation of 'baby universe' processes. It would be nice to understand the details.

## 7. Acknowledgements

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