# The relation between Hawking radiation via tunnelling and the laws of black hole thermodynamics

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**Abstract**

In Parikh and Wilczek's original works, the laws of black hole thermodynamics are not referred and it seems that there is no relation between Hawking radiation via tunnelling and the laws of black hole thermodynamics in their works. However, taking examples for the R-N black hole and the Kerr black hole, we find that they are correlated and even consistent if the tunnelling process is a reversible process.

## 1 Introduction

In 2000, Parikh and Wilczek reconsidered Hawking radiation. They argued that the Hawking radiation is a tunnelling process and the barrier is created just by the outgoing particle itself. In this way, They calculated the emission rate from some static black holes and obtained the corrected spectrum[1, 2, 3, 4]. Particularly, their result is consistent with the underlying unitary theory and support to the information conservation. Following this method, many other static and stationary rotating black holes are also studied[5, 6, 7, 8, 9, 10, 11, 12, 13, 14], and the results are the same as that in Parikh and Wilczek's original works. In this paper, basing on those works, and as a further study, we reinvestigate the Hawking radiation via tunnelling. As we know, the laws of black hole thermodynamics are not referred neither in Parikh and Wilczek's original works nor the extended works based on them. And it seems that there is no relation between Hawking radiation via tunnelling and the laws of black hole thermodynamics. Is it true that there is no relation between them[15]? In the following discussion, we first take an example of the Reissner-Nordstrom black

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hole. we reinvestigate the Hawking radiation of charged particles via tunnelling from it[6], and rewrite the imaginary part of the action, which is viewed from the laws of black hole thermodynamics. Then, in the same way we reinvestigate the Hawking radiation of particles with angular momentum via tunnelling from the Kerr black hole[7]. Finally, we give a brief conclusion and some discussion about our result.

## 2 Taking the R-N black hole for example

Recently, Parikh and Wilczek's original works have been extended to the R-N black hole. As a typical static spherically symmetric solution of Einstein's field equation, we first take it for example. According to Ref[6], the imaginary part of the action for the classically forbidden trajectory is

$$\begin{aligned}
\operatorname{Im} S &= \operatorname{Im}\{\int_{r_i}^{r_f}[p_r - \frac{p_{A_t}\dot{A}_t}{\dot{r}}]dr\} \\
&= -\operatorname{Im}\{\int_{r_i}^{r_f}\int_{(M,Q)}^{(M-\omega,Q-q)}[\frac{2r\sqrt{2Mr-Q^2}}{r^2-2Mr+Q^2}dM - \frac{2\sqrt{2Mr-Q^2}Q}{r^2-2Mr+Q^2}dQ]dr\} \\
&= -\pi\int_{(M,Q)}^{(M-\omega,Q-q)}[\frac{(M+\sqrt{M^2-Q^2})^2}{\sqrt{M^2-Q^2}}dM - \frac{(M+\sqrt{M^2-Q^2})Q}{\sqrt{M^2-Q^2}}dQ] \\
&= -\frac{\pi}{2}\{[(M-\omega)+\sqrt{(M-\omega)^2-(Q-q)^2}]^2 - [M+\sqrt{M^2-Q^2}]^2\} \\
&= -\frac{1}{2}\Delta S_{BH}. \qquad (1)
\end{aligned}$$

where $A_t = \frac{Q}{r}$ is the first component of the 4-Dimensional electromagnetic potential, and $p_{A_t}$ is the corresponding canonical momentum conjugate.

Usually, investigating (1), we can easily obtain that the result is consistent with the underlying unitary theory, so the information is conserved. Our work is also to reinvestigate (1), which we will discuss in the following.

As we know, for the R-N black hole, when a charged massive particle tunnels across the event horizon, the mass and the charge of black hole will be changed as a consequence. According to the first law of black hole thermodynamics, the differential Bekenstein-Smarr equation of the R-N black hole is[16]

$$dM = \frac{\kappa}{8\pi}dA + VdQ \ (J=0), \qquad (2)$$

Furthermore, if the tunnelling process is considered as a reversible process, according to the second law of black hole thermodynamics, (2) can be rewritten as

$$dM = TdS + VdQ. \qquad (3)$$

Equally, it can be rewritten as

$$dS = \frac{dM}{T} - \frac{VdQ}{T}. \tag{4}$$

The temperature and the potential are respectively[6]

$$T = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2}, V = \frac{Q}{M + \sqrt{M^2 - Q^2}}. \tag{5}$$

Substituting (5) into (4), we can obtain

$$dS = \frac{2\pi(M + \sqrt{M^2 - Q^2})^2}{\sqrt{M^2 - Q^2}} dM - \frac{2\pi(M + \sqrt{M^2 - Q^2})Q}{\sqrt{M^2 - Q^2}} dQ. \tag{6}$$

Thus, using (6), we can rewrite (1) as

$$\begin{aligned} \operatorname{Im} S &= -\pi \int_{(M,Q)}^{(M-\omega, Q-q)} [\frac{(M + \sqrt{M^2 - Q^2})^2}{\sqrt{M^2 - Q^2}} dM - \frac{(M + \sqrt{M^2 - Q^2})Q}{\sqrt{M^2 - Q^2}} dQ] \\ &= -\frac{1}{2} \int_{S_i}^{S_f} dS = -\frac{1}{2} \Delta S_{BH}. \end{aligned} \tag{7}$$

Which is also the same result as (1). The difference is that the result in (7) implicates that Hawking radiation via tunnelling is correlated with the laws of black hole thermodynamics.

# 3 Taking the Kerr black hole for example

Having taking the R-N black hole for example, instead, we study another typical solution, a stationary axially symmetrical Kerr black hole. And for the sake of simplicity, we only reinvestigate the case of a massless particle with angular momentum tunnelling across the event horizon. According to Ref[7], the imaginary part of the action is

$$
\begin{aligned}
\operatorname{Im} S &= \operatorname{Im}[\int_{r_i}^{r_f} p_r dr - \int_{\varphi_i}^{\varphi_f} p_\varphi d\varphi] \\
&= \operatorname{Im}[\int_{r_i}^{r_f} \int_{M}^{M-\omega} \frac{\sqrt{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}}{\rho^2 - \sqrt{\rho^2(\rho^2-\Delta)}} dr dM \\
&\quad - \int_{r_i}^{r_f} \int_{M}^{M-\omega} \frac{\sqrt{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}}{\rho^2 - \sqrt{\rho^2(\rho^2-\Delta)}} a\Omega dr dM] \\
&= \int_{M}^{M-\omega} \frac{-2\pi(M^2 + M\sqrt{M^2-a^2})}{\sqrt{M^2-a^2}} dM + \int_{M}^{M-\omega} \frac{\pi a^2}{\sqrt{M^2-a^2}} dM \\
&= \pi[M^2 - (M-\omega)^2 + M\sqrt{M^2-a^2} - (M-\omega)\sqrt{(M-\omega)^2-a^2} \\
&= -\frac{1}{2}\Delta S_{BH}.
\end{aligned} \tag{8}
$$

As the same discussion in section 2, Our next task is to view and rewrite (8) from the laws of black hole thermodynamics. For the Kerr black hole, the first law of black hole thermodynamics is[16]

$$
dM = \frac{\kappa}{8\pi} dA + \Omega dJ \ (Q = 0), \tag{9}
$$

And it will remain the same relationship if the tunnelling process is a reversible process, we can obtain

$$
dS = \frac{dM}{T} - \frac{\Omega dJ}{T}. \tag{10}
$$

The temperature, the angle velocity and the angular momentum of Kerr black hole are respectively[7]

$$
T = \frac{\sqrt{M^2-a^2}}{4\pi(M^2 + M\sqrt{M^2-a^2})}, \Omega = \frac{a}{r_+^2 + a^2} = \frac{a}{2(M^2 + M\sqrt{M^2-a^2})}, J = aM. \tag{11}
$$

Thus, substituting (11) into (10), we get

$$
dS = \frac{4\pi(M^2 + M\sqrt{M^2-a^2})}{\sqrt{M^2-a^2}} dM - \frac{2\pi a^2}{\sqrt{M^2-a^2}} dM. \tag{12}
$$

Comparing (12) with (8), it is easy to find that we can also rewrite the imaginary part of the action as follows

$$
\begin{aligned}
\operatorname{Im} S &= \int_{M}^{M-\omega} \frac{-2\pi(M^2 + M\sqrt{M^2-a^2})}{\sqrt{M^2-a^2}} dM + \int_{M}^{M-\omega} \frac{\pi a^2}{\sqrt{M^2-a^2}} dM \\
&= -\frac{1}{2}\int_{M}^{M-\omega} [\frac{4\pi(M^2 + M\sqrt{M^2-a^2})}{\sqrt{M^2-a^2}} dM - \frac{2\pi a^2}{\sqrt{M^2-a^2}} dM] \\
&= -\frac{1}{2}\int_{S_i}^{S_f} dS = -\frac{1}{2}\Delta S_{BH}. \qquad (13)
\end{aligned}
$$

Which shows that the Hawking radiation via tunnelling is also relative with the laws of black hole thermodynamics for the stationary axial symmetry case.

# 4 Conclusion and Discussion

In section 2 and 3, we respectively take R-N black hole and Kerr black hole for examples. And we obtain the result that the Hawking radiations via tunnelling, both in the static spherically symmetric case and the stationary axially symmetric case, are related to the laws of black hole thermodynamics. In other words, Parikh and Wilczek's original works have already made the assumption of the laws of black hole thermodynamics, in detail, the first and the reversible second law of black hole thermodynamics. More general speaking, Parikh and Wilczek's original works are only suitable for the reversible process. However, in fact, because of the negative heat capacity, an evaporating black hole is a highly unstable system, and there is no stable thermal equilibrium between black hole and the outside. Thus, the tunnelling process is usually an irreversible process in principle. That is, it is a little early for Parikh and Wilczek to say that the tunnelling process is consistent with the underlying unitary theory.

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