State independence for tunnelling processes through black hole horizons and Hawking radiation

Valter Moretti[1,a] and **Nicola Pinamonti**[2,b]

[1] Dipartimento di Matematica, Università di Trento and Istituto Nazionale di Fisica Nucleare – Gruppo Collegato di Trento, via Sommarive 14 I-38050 Povo (TN), Italy.

[2] Dipartimento di Matematica, Università di Roma"Tor Vergata", Via della Ricerca Scientifica 1, I-00133 Roma - Italy and Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, I-16146 Genova - Italy.

E-mail: [a]moretti@science.unitn.it, [b]pinamont@dima.unige.it

Abstract. Tunnelling processes through black hole horizons have recently been investigated in the framework of WKB theory, discovering an interesting interplay with the Hawking radiation. In this paper, we instead adopt the point of view proper of QFT in curved spacetime, namely, we use a suitable scaling limit towards a Killing horizon to obtain the leading order of the correlation function relevant for the tunnelling. The computation is done for certain large class of reference quantum states for scalar fields, including Hadamard states. In the limit of sharp localization either on the external side or on opposite sides of the horizon, the quantum correlation functions appear to have thermal nature. In both cases the characteristic temperature is referred to the surface gravity associated with the Killing field and thus connected with the Hawking one. Our approach is valid for every stationary charged rotating non-extremal black hole. However, since the computation is completely local, it covers the case of a Killing horizon which just temporarily exists in some finite region, too. These results provide strong support to the idea that the Hawking radiation, which is detected at future null infinity and needs some global structures to be defined, is actually related to a local phenomenon taking place even for local geometric structures (local Killing horizons), existing just for a while.

1 Introduction

As is known, the Hawking radiation [Ha75] is detected at *future null infinity* of a spacetime containing collapsing matter giving rise to a black hole. At least in the case of spherical symmetry, the existence and the features of that radiation are quite independent from the details of the collapse. However, the type of the short-distance behaviour of the two-point function of the reference state, employed to describe the modes of the radiation, plays a relevant role [FH90]. In recent years, attention has been focused on *local* properties of models where the Hawking radiation is manifest. Here *local* means in a neighborhood of a point on the event

1

horizon [PW00, ANVZ05, APS08, APGS08]. In this second approach the radiation appears to be related with some thermal effects associated to some tunnelling process through the horizon. In particular, the tunneling probability, computed in the framework of semiclassical WKB approach, has the characteristic thermal form e^{-E/T_H} (the Boltzmann constant being re-defined as $k = 1$) where T_H is the Hawking temperature and E is the energy of the particle crossing the horizon. More precisely, that exponential thermal factor arises when taking a limit towards the horizon for an endpoint of the path of the classical particle.

This approach is interesting because it deals with local aspects only and, in this sense, it seems to be more general than the standard one. In fact, it may be applied to pictures where a certain geometric structure, interpreted as the horizon, exists "just for a while", without extending into a true global structure up to the future null infinity where, traditionally, the Hawking radiation is detected. In [DNVZZ07, HDVNZ09], even the case of a spherically symmetric black hole *in formation* was analysed, where no proper horizon structure exists, being replaced by a *dynamical horizon*. Other interesting results, also considering the backreaction, can be found in [KM07, MV05].

Within these new remarkable approaches it is however difficult to understand how the found properties are independent form the state of the quantum system. This is essentially due to the fact that they are discussed at the *quantum-mechanical* level rather than the *quantum-field-theory* level. Indeed, in [PW00, ANVZ05] it is assumed that there is some preferred notion of quantum particle whose wavefunction satisfies the Klein-Gordon equation. That equation is actually treated as a Schrödinger equation when dealing with transition probabilities within the WKB framework. However, in curved spacetime there is no natural definition of quantum particle, unless adopting the quantum-field-theory framework, fixing a preferred quasifree reference state and building up the associated Fock-Hilbert space. It does not seem that this relevant issue is tackled in the mentioned literature. Furthermore, the procedure exploited in [ANVZ05, DNVZZ07, HDVNZ09] needs a Feynman prescription to make harmless a divergence that pops up when performing the above-mentioned limit towards the horizon. As a matter of fact, that procedure turns the real-axis computation into a complex-plane computation and the very imaginary part of the WKB amplitude, arising that way, leads to the wanted factor e^{-E/T_H}. The reason for the appearance of an imaginary part in the semi-classical action has been clarified in [APS08, APGS08], through a very careful analysis of the integration paths and the adopted coordinate system, without requiring any regularization procedure and remaining in the general WKB framework, while adopting the path integral viewpoint. In any case, that clarification does not solve the problem of the absence of a precise notion of a quantum particle in curved spacetime, necessary to exploit the WKB formalism at quantum-mechanical level. It seems however plausible that the appearance of an imaginary part in the semi-classical action, formally equivalent to the Feynman regularization procedure, is nothing but a remnant of the choice of a preferred reference state at quantum-field-theory level, whose two-point function has a short-distance divergence close to that of Minkowski vacuum.

While sticking to the local aspects associated with states showing the Hawking radiation, differently from the references quoted above, in this paper we shall deal with a definite framework at the quantum-field-theory level. More precisely, we shall focus on the two-point correlation

function $\omega(\Phi(x)\Phi(y))$ of a quantum field Φ settled in a (not necessarily *quasifree*) state ω whose short-distance divergence is, essentially, of *Hadamard type*, thus generalising the short-distance behaviour of Minkowski vacuum. This is one of the hypotheses exploited in [FH90, KW91]. It actually encompasses a huge class of states, those that are supposed to have a clear physical meaning [Wa94], especially in relation with the problem of the renormalisation of the stress-energy tensor and the computation of the quantum backreaction on the metric.

The two-point function $\omega(\Phi(x)\Phi(y))$ corresponds, up to normalization, to a probability amplitude. In this sense, it measures the tunnelling probability through the horizon when x and y are kept at the opposite sides of the horizon. In particular, if the state ω is quasifree, that probability amplitude can be interpreted as a quantum-mechanical probability amplitude, as wished in the above-mentioned literature, just referring to the natural notion of particle associated with the state ω.

From the geometric viewpoint, we shall assume to work in a sufficiently small neighbourhood $\mathcal{O}$ of a local *Killing horizon* structure $\mathcal{H}$, also supposing that the *surface gravity* κ is *nonvanishing* and *constant* along the horizon. We stress that the structure could be either part of the future Killing event horizon of a stationary black hole in the full Kerr-Newman family also obtained by matter collapse, or it could be completely local and ceasing to exist in the future of $\mathcal{O}$ in view of the general dynamics of the matter and the fields in the considered spacetime. The requirement that the surface gravity is constant on the local horizon means that, at least locally, a thermal equilibrium has been reached, since a constant surface gravity corresponds to the validity of the zero-law of black-hole thermodynamics.

The existence of a timelike Killing vector K defining $\mathcal{H}$ provides the preferred notions of time and energy we intend to consider. Notice that in [DNVZZ07, HDVNZ09], dealing with spherically-symmetric black holes in formation, the notion energy was referred to the so called *Kodama-Hayward* vector field that, in those backgrounds, generalises the notion of Killing field.

Exploiting general technical achievements about Killing horizons established in [KW91] and [RW92], we shall prove that, independently from the choice of the quantum state in above-mentioned class, when the supports of the test functions centred on the two arguments x, y of $\omega(\Phi(x)\Phi(y))$ become closer and closer to the horizon, the two-point function acquires a thermal spectrum with respect to the notion of time and energy associated with the Killing field. More precisely, if both arguments stay on the same side of the horizon, the Fourier transform of the two-point function presents the very Bose-Einstein form driven by the Hawking temperature (see however the remark at the end of the paper). Conversely, whenever the two arguments are kept at the opposite sides of the horizon, the resulting spectrum is different. It is however in agreement with the transition probability between two weakly coupled reservoirs which are in thermal equilibrium at Hawking temperature. Actually, as in the case of the Boltzmann distribution, its spectrum decays exponentially for high energies. In both cases, in order to catch the leading contribution to the two-point function, we shall exploit a suitable scaling limit procedure [HNS84, Bu96, BV95] towards the horizon. Operating in this way, the local thermal nature of the correlation functions becomes manifest as a state-independent feature when the states belong to the above-mentioned wide class.

The paper is organized as follows. In the next section, recalling some technical results

established in [KW91] and [RW92], we shall present the geometric hypotheses we shall use. We assume that the reader is familiar with the standard notions of differential geometry of spacetimes [Wa84]. In the subsequent section we shall compute the two-point function $\omega(\Phi(x)\Phi(y))$ and its limit approaching the horizon. The last section will present a summary and some general remarks.

2 Spacetime Geometry

2.1. Local geometry. We start our discussion fixing the basic geometric setup that we employ in this paper. We henceforth consider a 4-dimensional (smooth) time-oriented spacetime (M, g). Furthermore, we assume the validity of the following local geometric properties, which are the same as in [RW92].

Definition 2.1. *Let $\mathcal{O}$ be an open subset of M, the* **local general geometric hypotheses** *hold in $\mathcal{O}$ if a smooth vector field K exists thereon such that:*

- **(a)** *K is a Killing field for g in $\mathcal{O}$.*
- **(b)** *$\mathcal{O}$ contains a connected 3-submanifold $\mathcal{H}$, the* **local Killing horizon**, *that is invariant under the group of local isometries generated by K and $K^a K_a = 0$ on $\mathcal{H}$.*
- **(c)** *The orbits of K in $\mathcal{O}$ are diffeomorphic to an open interval contained in $\mathbb{R}$ and $\mathcal{H}$ admits a smooth 2-dimensional cross section which intersects each orbit of K exactly once.*
- **(d)** *The* **surface gravity** *– i.e. the function $\kappa : \mathcal{H} \to \mathbb{R}$ such that, in view of (a) and (b) $\nabla^a(K_b K^b) = -2\kappa K^a$ – turns out to be strictly positive[1] and constant on $\mathcal{H}$.*

As we said above, the local Killing horizon $\mathcal{H}$ may represent a horizon which exists "just for a while", without extending into a true global structure which reaches the future null infinity. However, our hypotheses are, in particular, valid [RW92] in a neighbourhood of any point on a black hole future horizon, once that, after the collapse, the metric has settled down to its stationary, not necessarily static, form of any non-extreme black hole in the charged Kerr-Newman family. In particular, those hypotheses and our results are valid for the Kerr black hole. There K is the Killing vector defining the natural notion of time in the external region of the black hole and $\mathcal{H}$ is part of the event horizon.

With the hypotheses (a) and (b), the integral lines of K along $\mathcal{H}$ can be re-parametrized to segments of null geodesics and $\nabla^a(K_b K^b) = -2\kappa K^a$ holds on $\mathcal{H}$ where the **surface gravity**, $\kappa : \mathcal{H} \to \mathbb{R}$, is constant along each fixed geodesic [Wa84]. The requirement (d) is not as strong as it may seem at first glance. Indeed, it is possible to prove that, whenever a spacetime admitting a Killing horizon satisfies Einstein equations and the dominant energy condition is verified, the surface gravity must be constant on the horizon [Wa84]. (However, [Wa84] a result originally obtained by Carter states that – independently of any field equations – the surface gravity of a Killing horizon must be constant if (i) the horizon Killing field is static *or* (ii) there is an additional Killing field and the two Killing fields are 2-surface orthogonal.) At least in the case

---

[1]What actually matters is $\kappa \neq 0$, since $\kappa > 0$ can always be obtained in that case by re-defining $K \to -K$.

$\kappa > 0$, this is nothing but the *zero-th law of black hole thermodynamics* where $\frac{\kappa}{2\pi}$ amounts to the *Hawking temperature* of the black hole.

2.2. Killing and Bifurcate Killing horizons. We now focus on the relation of the previously introduced local geometric hypotheses and the more rigid case of a bifurcate Killing horizon. A Killing field K determines a **bifurcate Killing horizon** [Bo69] when it vanishes on a connected 2-dimensional acausal space-like submanifold $\mathcal{B} \subset M$, called the **bifurcation surface**, and K is light-like on the two K-invariant 3-dimensional null submanifolds $\mathcal{H}_+, \mathcal{H}_- \subset M$ generated by the pairs of null geodesic orthogonally emanated from $\mathcal{B}$. In particular $\mathcal{H}_+ \cap \mathcal{H}_- = \mathcal{B}$ and the null geodesics forming $\mathcal{H}_+ \cup \mathcal{H}_-$ are re-parametrised integral lines of K on $(\mathcal{H}_+ \cup \mathcal{H}_-) \setminus \mathcal{B}$. By definition, on $\mathcal{H}_+$ the field K in the future of $\mathcal{B}$ is outward pointing. The simplest example of a bifurcate Killing horizon is that realized by the Lorentz boost K in Minkowski spacetime. Other, more interesting, cases are the bifurcate Killing horizons of maximally extended black hole geometries like the Kruskal extension of the Schwarzschild even including the non-extreme charged rotating case.

For our purposes it is important to notice that, in the case of a bifurcate Killing horizon, any neighbourhood $\mathcal{O}$ of a point on $\mathcal{H}_+$, which has empty intersection with the bifurcation surface $\mathcal{B}$, satisfies the local general hypotheses stated in definition 2.1. It is very remarkable for physical applications and for our subsequent discussion in particular, that such a result can be partially reversed as established by Racz and Wald [RW92, RW96]. Indeed, if the local general geometric hypotheses are fulfilled on $\mathcal{H}$, for a sufficiently small $\mathcal{O}$, the spacetime outside $\mathcal{O}$ can be smoothly deformed preserving the geometry inside $\mathcal{O}$ and extending K and $\mathcal{H}$ to a whole bifurcate Killing horizon in the deformed spacetime. Thus, when studying local properties, the bifurcation surface $\mathcal{B}$ can be "added" also to those spacetimes without bifurcation surface, as is the case of black holes formed by stellar collapse. One can therefore take advantage of the various technical properties of the bifurcate Killing horizon as we shall do in the rest of the paper.

2.3. Killing vector, geodesical distance in $\mathcal{O}$ and coordinates V, U, x^3, x^4. Let us focus on a relevant coordinate patch [KW91] defined in a neighbourhood of $\mathcal{H}_+$ for a bifurcate Killing horizon generated by a Killing vector field K (similar adapted coordinates exist for $\mathcal{H}_-$). Let U denote an affine parameter along the null geodesics forming $\mathcal{H}_+$ fixing the origin at the bifurcation surface $\mathcal{B}$. Each point $p \in \mathcal{H}_+$ is thus determined by a corresponding pair (U, s), where $s \in \mathcal{B}$ denotes the point which is intersected by the null geodesic generator through p.

We shall now extend those coordinate system to a neighbourhood of $\mathcal{H}_+$. To this end, for each point $q \in \mathcal{B}$, let us indicate by n the unique future-pointing null vector which is orthogonal to $\mathcal{B}$ and has inner product $-1/2$ with $\frac{\partial}{\partial U}$. We extend n on all of $\mathcal{H}_+$ by parallel transport along the null generators of $\mathcal{H}_+$. Let V denote the affine parameter along the null geodesics determined by n, with V= 0 on $\mathcal{B}$. It is clear that (V, U, s) determines a point in a sufficiently small neighbourhood of $\mathcal{H}_+$. We are thus in a position to introduce the sought coordinate patch. If (x^3, x^4) denote coordinates defined on a open neighbourhood (in $\mathcal{B}$) of a point in $\mathcal{B}$, a coordinate patch (V, U, x^3, x^4), we call **adapted to** $\mathcal{H}_+$, turns out to be defined in corresponding
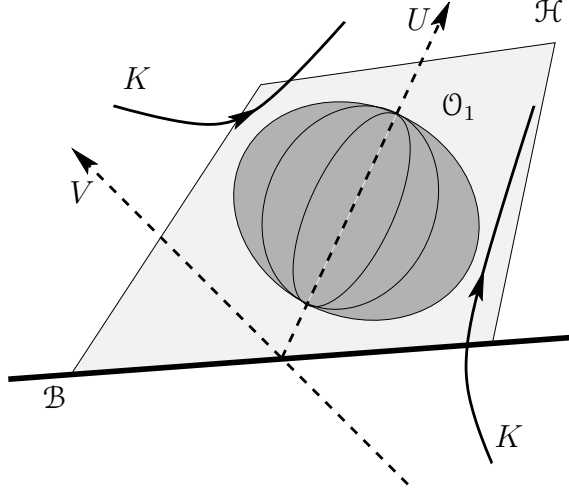
Figure 1: The thick black line is part of the bifurcation surface. The region painted in light grey corresponds to the horizon $\mathcal{H}$, while the dark-grey region represents the open set $\mathcal{O}_1$.

open neighbourhoods of points on $\mathcal{H}_+$. In these coordinates:

$$g\!\restriction_{\mathcal{H}_+} = -\frac{1}{2}dU \otimes dV - \frac{1}{2}dV \otimes dU + \sum_{i,j=3}^{4} h_{ij}(x^3, x^4)dx^i \otimes dx^j \,, \tag{1}$$

where metric h is that induced by g on $\mathcal{B}$. h is thus *positive definite* and it does not depend on V, U. We stress once more that, in view of Racz-Wald's result, such a coordinate system always exits in $\mathcal{O}$, provided the local geometric hypotheses hold in $\mathcal{O}$, regardless the actual geometry of the spacetime outside $\mathcal{O}$.

In the rest of the paper, referring to this geometric structure in $\mathcal{O}$, we shall employ the following notation. We shall indicate by $\mathcal{S}_{V,U}$ the cross section of $\mathcal{O}$ at V, U constant. Moreover, $s(p) \in \mathcal{B}$ will be the point with coordinates (x^3, x^4) when $p \in \mathcal{O}$ has coordinates (V, U, x^3, x^4) (similarly (V', U', x'^3, x'^4) will denote the analogous set of coordinates for $p' \in \mathcal{O}$). The set $G_\delta(p, V, U) \subset \mathcal{S}_{V,U}$ is defined as the set whose image $s(G_\delta(p, V, U)) \subset \mathcal{B}$ coincides with the open h-geodesical ball centred on $s(p)$ with radius δ.

We shall denote by $\sigma(p, p')$ the squared g-geodesic distance, taken with its Lorentzian sign, between any couple of points p, p' contained in some g-geodesically convex neighbourhood. Finally, we denote by $\ell(s, s')$ the squared h-geodesic distance between points s, s' in some h-geodesically convex neighbourhood contained in $\mathcal{B}$.

Making use of the preceding definitions and the introduced notations, we are going to present the following useful proposition, which is also based on achievements in [KW91].

Proposition 2.1. *Let $\mathcal{O} \subset M$ be a set on which the local general geometric hypotheses hold, and let $\mathcal{O}_1$ be another open set such that $\overline{\mathcal{O}}_1 \subset \mathcal{O}$ is compact. Define a coordinate frame adapted to $\mathcal{H}_+$ on $\mathcal{O}$ (as said at the beginning of section 2.3) so that $p \in \mathcal{O}$ has coordinates (V, U, x^3, x^4) with $p \in \mathcal{H}$ iff $V = 0$. The following holds.*

6

(a) *In $\mathcal{O}$, the decomposition $K = K^1 \frac{\partial}{\partial V} + K^2 \frac{\partial}{\partial U} + K^3 \frac{\partial}{\partial x^3} + K^4 \frac{\partial}{\partial x^4}$ is valid and, if $p \in \mathcal{O}_1$:*

$$K^1(p) = -\kappa V + V^2 R_1(p), \quad K^2(p) = \kappa U + V^2 R_2(p), \quad K^i(p) = V R_i(p), \; i = 3, 4, \quad (2)$$

where R_1, R_2, R_i are bounded smooth functions defined on $\mathcal{O}_1$.

(b) *If $\mathcal{O}_1$ is included in a g-geodesically convex neighbourhood and if $p, p' \in \mathcal{H} \cap \mathcal{O}_1$, then $\sigma(p, p') = \ell(s(p), s(p'))$.*

(c) *If $\mathcal{O}_1$ is as in (b), there exists a $\delta > 0$ such that, for every fixed $p \in \mathcal{O}_1$, the smooth map $G_\delta(p, V', U') \ni p' \mapsto \sigma(p, p')$ has vanishing gradient (with respect to the coordinates of $s(p')$) in a unique point $q(p, V', U')$ attaining its minimum there. In particular $s(q(p, V', U')) = s(p)$ if $p \in \mathcal{H}$.*

(d) *For p and $q = q(p, V', U')$ as in (c):*

$$\sigma(p, q) = \ell(s(p), s(q)) - (U - U')(V - V') + R(p, V', U'), \quad (3)$$

where $R(p, V', U') = AV^2 + BV'^2 + CVV'$, for some bounded smooth functions A, B, C of p, V', U'.

Proof. **(a)** $\nabla_a K_b + \nabla_b K_a = 0$ and $\nabla^a(K_b K^b) = -2\kappa K^a$ on $\mathcal{H}$ imply that $\nabla_K K = \kappa K$ on $\mathcal{H}$, so that $K \!\restriction_{\mathcal{H}} = \kappa U \frac{\partial}{\partial U}$ because U is an affine parameter and K vanishes on $\mathcal{B}$ where $U = 0$. If x^a is any of $x^1 = V, x^2 = U, x^3, x^4$, exploiting the parallel transport used to define the coordinates, we have

$$\Gamma^a_{21} \!\restriction_{\mathcal{H}} = \Gamma^a_{12} \!\restriction_{\mathcal{H}} = \Gamma^a_{22} \!\restriction_{\mathcal{H}} = \Gamma^a_{11} \!\restriction_{\mathcal{H}} = \Gamma^1_{2a} \!\restriction_{\mathcal{H}} = \Gamma^1_{a2} \!\restriction_{\mathcal{H}} = \Gamma^2_{a1} \!\restriction_{\mathcal{H}} = \Gamma^2_{1a} \!\restriction_{\mathcal{H}} = 0. \quad (4)$$

Above, the fifth one is equivalent to $g(\frac{\partial}{\partial U}, \nabla_{\frac{\partial}{\partial U}} \frac{\partial}{\partial x^a}) \!\restriction_{\mathcal{H}} = 0$ and it arises from $g(\frac{\partial}{\partial U}, \frac{\partial}{\partial x^a}) = -n_a$ and $\nabla_{\frac{\partial}{\partial U}} \frac{\partial}{\partial U} = 0$ on $\mathcal{H}$, the seventh one can be proved similarly. Next, taking the first-order Taylor expansion in V of both K^1 and g^{ab} about $V = 0$, we have, for some smooth function $V^2 R_1(p)$ bounded in view of the compactness of $\overline{\mathcal{O}_1}$,

$$K^1 = K^1 \!\restriction_{\mathcal{H}} + V \left(g^{1b} \!\restriction_{\mathcal{H}} \frac{\partial K_b}{\partial V} \!\restriction_{\mathcal{H}} + K_b \!\restriction_{\mathcal{H}} \frac{\partial g^{1b}}{\partial V} \!\restriction_{\mathcal{H}} \right) + V^2 R_1(p). \quad (5)$$

From $\frac{\partial g^{ab}}{\partial x^c} = -g^{bd} \Gamma^a_{cd} - g^{ad} \Gamma^b_{cd}$ and and (4), exploiting (1) and $K^1 = 0$ on $\mathcal{H}$, the identity (5) simplifies: The last derivative vanishes and $K^1 = -V \frac{\partial K_2}{\partial V} \!\restriction_{\mathcal{H}} + V^2 R^1(p)$. Furthermore, $\nabla_2 K_1 + \nabla_1 K_2 = 0$ evaluated on $\mathcal{H}$ and using (4) leads to $\frac{\partial K_2}{\partial V} \!\restriction_{\mathcal{H}} = -\frac{\partial K_1}{\partial U} \!\restriction_{\mathcal{H}}$, so that:

$$K^1 = -V \frac{\partial K_1}{\partial U} \!\restriction_{\mathcal{H}} + V^2 R_1(p) = V \left(g_{1a} \!\restriction_{\mathcal{H}} \frac{\partial K^a}{\partial U} \!\restriction_{\mathcal{H}} + K^a \!\restriction_{\mathcal{H}} \frac{\partial g_{1a}}{\partial U} \!\restriction_{\mathcal{H}} \right) + V^2 R_1(p).$$

The last derivative vanishes in view of (4), (1) and the know identity $\frac{\partial g_{ab}}{\partial x^c} = g_{bd} \Gamma^d_{ca} + g_{ad} \Gamma^d_{cb}$. Therefore the first identity in (2) holds in view of (1) and $K \!\restriction_{\mathcal{H}_R} = \kappa U \frac{\partial}{\partial U}$. The second one can be proved with the same procedure noticing that the Killing identity $\nabla_V K_1 = 0$, on $\mathcal{H}$, becomes $\frac{\partial K_1}{\partial V} \!\restriction_{\mathcal{H}} = 0$ in view of (4). The last identity in (2) is obvious.

(b) Since the geodesically convex neighbourhoods form a base of the topology and the projection $\pi : p \mapsto s(p)$ is continuous, if $\mathcal{O}_1$ is chosen to be sufficiently small, we have that $\mathcal{O}_1$ is contained in a geodesically convex neighbourhood while, at the same time, $\pi(\mathcal{O}_1)$ is contained in a h-geodesically convex neighbourhood in $\mathcal{B}$. Without loss of generality, we can further assume that the latter is included in a g-geodesically convex neighbourhood of M. Thus $\sigma(p, p')$, $\sigma(s(p)), s(p'))$ and $\ell(s(p), s(p'))$ are simultaneously well-defined if $p, p' \in \mathcal{O}_1 \cap \mathcal{H}$ for a sufficiently small $\mathcal{O}_1$. We notice that, $\sigma(p, p')$ is invariant under the action of the Killing isometry. Hence, for any $p, p' \in \mathcal{H} \cap \mathcal{O}_1$ we get the identity $\sigma(p, p') = \sigma(s(p), s(p'))$ taking the limit towards $\mathcal{B}$ of the flow generated by the Killing field K applied to p, p'. Finally $\sigma(s(p), s(p')) = \ell(s(p), s(p'))$ because $\mathcal{B}$ is totally geodesic as it can be proved by direct inspection.

(c) Let $(V, U, s) \equiv p$ and $(V', U', s') \equiv p'$. Whenever both points p and p' are contained on the horizon, namely $V = V' = 0$, the thesis holds in view of (b) and the fact that $\ell(s, s')$ is positive definite, with positive-definite Hessian matrix in the coordinates x'^3, x'^4 of s'. Furthermore, in this case $s(q(p, 0, U')) = s(p)$. By continuity, that Hessian matrix remains positive definite if p, p' stay close to $\mathcal{H}$, so that, any zero $q(p, V', U')$ of the x'^3, x'^4-gradient of $\mathcal{S}_{V',U'} \ni p' \mapsto \sigma(p, p')$ determines a minimum of $\sigma(p, p')$. Taking the Taylor expansion of $\nabla_{x'^i}\sigma(p, p')$ ($i = 3, 4$) centred on a point in $\mathcal{H} \times \mathcal{H}$ with respect to all the coordinates of p and p', the equation for $q(p, V', U')$ can easily be handled by exploiting Banach's fix point theorem, proving the existence and the uniqueness of $q(p, V', U')$ for $p \in \mathcal{O}_1$ sufficiently shrunk around $\mathcal{H}$, and p' varying in a neighbourhood $G_\delta(p, V', U')$ of $(0, U', s)$ in $\mathcal{S}_{V',U'}$. We recall that $G_\delta(p, V', U')$ is the pre-image through $\mathcal{S}_{V',U'} \ni p' \mapsto s(p')$ of a geodesic ball on $\mathcal{B}$ centred on $s(p)$. The compactness of $\overline{\mathcal{O}_1}$ and a continuity argument assures that $\delta > 0$ can be chosen uniformly in p.

(d) Keeping U, U', x^3, x^4 fixed, the expansion (3) is nothing but the first-order (V, V')-Taylor expansion of $\sigma(p, q(p, V', U'))$ at $V = V' = 0$, paying attention to the fact that the coordinates x'^3, x'^4 of $q(p, V', U')$ depend on V through p. $\square$

3   Correlations across the Killing horizon

3.1. General outlook.   We wish to compute the correlation functions of a real scalar quantum field, Φ, for field observables localized in a region intersecting a Killing horizon. Thus, we assume that a quantum system is described by a corresponding *-algebra $\mathcal{A}$ generated by the unit I and field operators $\Phi(f)$ for all $f \in C_0^\infty(M)$. For our purposes only minimal properties of such a quantum field theory are necessary, namely that the (abstract) field operator Φ is smeared with compactly supported smooth functions $f \in C_0^\infty(M)$ and furthermore that: (i) the map $f \mapsto \Phi(f)$ is linear, (ii) $\Phi(f)^* = \Phi(\overline{f})$, and (iii) $[\Phi(f), \Phi(g)] = 0$ when $supp(f)$ and $supp(g)$ are causally separated. *We stress that do not assume that Φ satisfies any specific field equation, so that, in principle our approach may encompass interacting fields.*
If ω is an (algebraic) state on $\mathcal{A}$, the *correlation functions* we are interested in are the bilinear functionals that map *real* smooth functions f, f' to $\omega(\Phi(f)\Phi(f'))$. We shall specify shortly the form of the test functions f, f'. For the moment we only say that their supports are taken in a open region $\mathcal{O}$ containing a Killing field K and which satisfies the local general geometric

8

hypotheses. We finally assume that $\mathcal{O}$ can be covered by *coordinates adapted to* $\mathcal{H}_+$ (defined at the beginning of section 2.3) with U and V increasing towards the future.

Later we shall restrict $\mathcal{O}$ to a subregion $\mathcal{O}_1$ as in (b) of proposition 2.1 because we want to use the expression (3) for the geodesic distance. The region $\mathcal{O}_1$ when considered in coordinates (V, U, s), can always be taken of the form $(-a, a) \times (b, c) \times S$, where $S \subset \mathcal{B}$ is an open relatively compact subset. Notice that, shrinking $\mathcal{O}_1$ around the region of $\mathcal{H} \cap \mathcal{O}_1$ determined by $(b, c) \times S$ means taking $a > 0$ smaller and smaller.

In the sufficiently small neighbourhood $\mathcal{O}_1$ one finds $g(K, K) = \kappa^2 UV + O(V^2)$ in view of (1), (a) in proposition 2.1, and $\partial_V g_{22}|_{\mathcal{H}} = 0$ (arising from $\Gamma_{22}^2|_{\mathcal{H}} = 0$). As a consequence, K turns out to be spacelike in $\mathcal{O}_s \equiv \{p \in \mathcal{O}_1 \mid V(p) > 0\}$ and timelike in $\mathcal{O}_t \equiv \{p \in \mathcal{O}_1 \mid V(p) < 0\}$. Referring to stationary black holes, $\mathcal{O}_1$ can be interpreted as a sufficiently small region around a point on the future horizon, the only horizon existing when the black hole is produced by collapsing matter. There, $\mathcal{O}_s$ is part of the *internal* region, containing the singularity, while $\mathcal{O}_t$ stays in the *external* region, stationary with respect to the Killing time associated to K. In this way, a notion of *energy* related to K can be defined, in $\mathcal{O}_t$ at least, and we will take advantage of it shortly.

We recall that, in the GNS representation of a state ω on the *-algebra of field observables, the expectation value of the product of two fields $\omega\left(\Phi(f)\Phi(f')\right)$ (with f, f' real) is equal to $\langle \widehat{\Phi}(f)\Psi_\omega | \widehat{\Phi}(f')\Psi_\omega \rangle$, where Ψ_ω is the cyclic vector and $\widehat{\Phi}(f)$ is the field operator represented as a proper operator on the GNS Hilbert space. Hence, up to normalization, $|\omega\left(\Phi(f)\Phi(f')\right)|^2$ can be interpreted as a transition probability between the states $\widehat{\Phi}(f)\Psi_\omega$ and $\widehat{\Phi}(f')\Psi_\omega$. In this sense, when f and f' are localized on the opposite sides of the horizon, the regions $\mathcal{O}_s$ and $\mathcal{O}_t$, the correlation function $\omega\left(\Phi(f)\Phi(f')\right)$ provides a measure of the *transition probability* through the horizon.

Inspired by the ideas proper of the *scaling-limit procedure* [HNS84, Bu96, BV95], in order to obtain the leading order to that probability, we shall consider some sequences of smearing functions f_λ and f'_λ whose support become closer and closer to the horizon $\mathcal{H}$ in the limit $\lambda \to 0^+$. Thus we are going to compute the limit:

$$\lim_{\lambda \to 0^+} \omega\left(\Phi(f_\lambda)\Phi(f'_\lambda)\right)$$

where f_λ and f'_λ are smooth functions supported in $\mathcal{O}_1$ whose supports become closer and closer to the horizon as long as $\lambda \to 0^+$.

In contrast to the discussion presented in [HNS84], where the scaling limit *towards a point* is analysed, with the proposed procedure we are instead considering scaling limits *towards a null surface*, so that the results presented in [HNS84] cannot be automatically applied to the present case.

Since only the short distance behaviour of the two-point function of the state is relevant for our computation, we select the class of allowed states looking at their ultraviolet features. We assume that the two-point function of ω is a distribution of $\mathcal{D}'(M \times M)$ defined as

$$\omega\left(\Phi(f)\Phi(f')\right) = \lim_{\epsilon \to 0^+} \int_{M \times M} \omega_\epsilon(x, x')f(x)f'(x')dx dx'$$

9

where the integral kernels ω_ϵ have the following form

$$\omega_\epsilon(x, x') = \frac{\Delta(x, x')^{1/2}}{4\pi^2 \sigma_\epsilon(x, x')} + w_\epsilon(x, x') \,, \tag{6}$$

whenever the test functions are supported in a fixed, relatively compact, geodesically convex neighbourhood. In the previous expression, $\sigma_\epsilon(x, x') = \sigma(x, x') + 2i\epsilon(T(x) - T(x')) + \epsilon^2$ and T is a fixed (arbitrarily chosen) time function [KW91]. The smooth strictly-positive function Δ is the so-called Van Vleck-Morette determinant [Wa94, KW91]. We finally assume that w_ϵ has a "less singular" behaviour with respect to that of $\sigma_\epsilon(x, x')^{-1}$ in the sense we are going to specify. An important case is:

$$w_\epsilon(x, x') = v(x, x') \ln \sigma_\epsilon(x, x') + w(x, x') \quad \text{for some fixed smooth functions } v, w. \tag{7}$$

With this form of w_ϵ, the right-hand side of (6) is a straightforward generalization of the short distance structure of the two-point function of Minkowski vacuum for a Klein-Gordon scalar field, if one also suppose that the field Φ satisfies the Klein-Gordon field equation $\Box\Phi + V\Phi = 0$ (where $V : M \to \mathbb{R}$ is any fixed smooth function). In particular, (6)-(7) are fulfilled by all the quasifree states of *Hadamard type* [Wa94] that are defined by (6)-(7) with a further requirement on the form of v related to the Klein-Gordon equation. Those states are supposed to be the most significant states for free QFT in curved spacetime [Wa94] and are very often employed in the rigorous description of thermal properties of quantum fields in the presence of black holes [KW91, FH90, DMP09]. However, since we do not need to consider free fields or any precise field equation, we further relax the requirements, adopting (6) but, in place of (7), assuming a weaker pair of requirements (in the following $p = (V, U, s)$, $p' = (V', U', s')$):

(W1) $w_\epsilon(p, p') \to w'(p, p')$ as $\epsilon \to 0^+$, almost everywhere in (p, p'), for some function w' and w_ϵ is ϵ-uniformly bounded by a locally M^2-integrable function;

(W2) $w'(V, U, s, V', U', s') \to w''(U, s, U', s')$ almost everywhere in (U, s, U', s') when $(V, V') \to (0, 0)$ for some function w'' on $\mathcal{H}^2$, and w' is (V, V')-uniformly bounded by a locally $\mathcal{H}^2$-integrable function.

Notice that (W1) and (W2) are satisfied by w_ϵ whenever it satisfies (7) and thus they are valid for Hadamard states in particular.

To conclude, few words about the precise construction of the functions f_λ, f'_λ are necessary. Let f and f' be some smooth functions with compact support contained respectively in the regions $\mathcal{O}_s$ and $\mathcal{O}_t$. The associated functions f_λ, f'_λ are defined as

$$f_\lambda(V, U, x^3, x^4) := \frac{1}{\lambda} f\left(\frac{V}{\lambda}, U, x^3, x^4\right) \,, \quad f'_\lambda(V, U, x^3, x^4) := \frac{1}{\lambda} f'\left(\frac{V}{\lambda}, U, x^3, x^4\right) \,, \quad \lambda > 0 \,. \tag{8}$$

As in other scaling-limit procedures, the pre-factor λ^{-1} is introduced in order to keep the result finite. Finally, to avoid divergences due to zero-modes in the limit $\lambda \to 0^+$, we assume that f, f'

are of the form[2]:

$$f = \frac{\partial F}{\partial V}, \quad f' = \frac{\partial F'}{\partial V}, \quad \text{for fixed } F, F' \in C_0^\infty(\mathcal{O}_1). \tag{9}$$

Alternatively, sticking to generic smooth compactly supported functions f and f', the divergent contribution of zero-modes has to be subtracted at the end of the computations. With the assumptions (8) and (9), the $\lambda \to 0^+$ limit of $\omega\left(\Phi(f_\lambda)\Phi(f'_\lambda)\right)$ is precisely the notion of *scaling limit of $\omega\left(\partial_V \Phi(x)\partial_V \Phi(y)\right)$ towards the horizon* we shall employ. The result of such limit represents the first contribution to the sought *transition probability* in an ideal asymptotic expansion for small λ.

3.2. Computation. We are in a position to present the most important result of this paper. The proof of the following theorem exploits techniques similar to those employed in the appendix B of [KW91]. However, our result differs from those presented there because we are not interested in studying the restriction of the states to the horizon in the symplectic approach, while we intend to compute the scaling limit of the state in the $4D$ smeared formalism. The relevant part for our computation arises from the $(V - V')$ contribution to σ in (3) whereas, in [KW91], only the $(U - U')$ contribution plays a relevant role. In this respect, there are some similarities to the analysis performed in [FH90], although here neither we restrict ourself to the spherically symmetric case nor we consider any equation of motion for the quantum field, nor we suppose that the Killing structure extends up to the future null infinity, as done in [FH90].

Theorem 3.1. *Assume that the general local geometric hypotheses as in Def.2.1 hold in $\mathcal{O}$ (covered by coordinates adapted to $\mathcal{H}_+$), suppose that $\mathcal{O}_1 \subset \mathcal{O}$ is a sufficiently small open neighbourhood of a point on $\mathcal{H}$ with $\overline{\mathcal{O}_1} \subset \mathcal{O}$ compact. Assume also that the state ω has two-point function given by a distribution as in (6) that verifies the requirements (W1) and (W2) above (in particular, w_ϵ may have the form (7) and may be a Hadamard state for Klein-Gordon fields). If f, f' are taken as in (8)-(9) and μ is the measure associated to the 2-metric on the bifurcation surface $\mathcal{B}$, it holds*

$$\lim_{\lambda \to 0^+} \omega\left(\Phi(f_\lambda)\Phi(f'_\lambda)\right) = \lim_{\epsilon \to 0^+} -\frac{1}{16\pi} \int_{\mathbb{R}^4 \times \mathcal{B}} \frac{F(V, U, s)F'(V', U', s)}{(V - V' - i\epsilon)^2} dU\, dV\, dU'\, dV'\, d\mu(s). \tag{10}$$

Proof. We start by focusing on the contribution of the most singular part of the two-point function (6), that is the iterated limit:

$$L := \lim_{\lambda \to 0^+} \lim_{\epsilon \to 0^+} \int_{\overline{\mathcal{O}_1} \times \overline{\mathcal{O}_1}} \frac{\Delta^{1/2}(p, p')f_\lambda(p)f'_\lambda(p')}{4\pi^2 \sigma_\epsilon(p, p')}\, dp dp', \tag{11}$$

where dp is a short-cut for the measure induced by the metric. Fixing $\delta > 0$ and p, V', U', consider a neighbourhood $G_\delta(p, V', U') \subset \mathcal{S}_{V', U'}$ as in (c) of proposition 2.1, and define a smooth map $\mathcal{B} \ni$

---

[2]Indeed, given $f \in C_0^\infty(\mathcal{O}_1)$, an $F \in C_0^\infty(\mathcal{O}_1)$ with $f = \frac{\partial F}{\partial V}$ exists if and only if $\int_{\mathbb{R}} f(V, U, x^3, x^4)dV = 0$ on $\mathcal{O}_1$, namely, if and only if $f(\cdot, U, x^3, x^4)$ has no zero modes referring to the V-Fourier transform.

11

$s \mapsto \chi_\delta(s, V', U', p) \geq 0$ with support completely included in $G_\delta(p, V', U')$ and $\chi_\delta(s', V', U', p) = 1$ for $0 \leq \sqrt{\lambda(s(p), s')} \leq \frac{\delta}{2} + \frac{1}{2}\sqrt{\lambda(s(p), s(q(p, V', U')))}$. In view of the smoothness of all considered functions it is possible to arrange these functions in order that $(s', V', U', p) \to \chi_\delta(s', V', U', p)$ is jointly smooth. Finally we can decompose the integral in (11) as:

$$\int_{\mathcal{O}_1} dp \left(\int_{\mathcal{O}_1} \frac{\Delta^{1/2}(p, p') f_\lambda(p) f'_\lambda(p')}{4\pi^2 \sigma_\epsilon(p, p')} \chi_\delta(s', V', U', p) \sqrt{|\det g(p')|} dV' dU' ds' \right)$$

$$+ \int_{\mathcal{O}_1} dp \left(\int_{\mathcal{O}_1} \frac{\Delta^{1/2}(p, p') f_\lambda(p) f'_\lambda(p')}{4\pi^2 \sigma_\epsilon(p, p')} (1 - \chi_\delta(s', V', U', p)) \sqrt{|\det g(p')|} dV' dU' ds' \right), \quad (12)$$

where $ds' = dx'^3 dx'^4$. Let us start from the second integral. As consequence of the compactness of $\overline{\mathcal{O}_1}$, the continuity of σ and (b) in proposition 2.1, we have that, for a fixed $\eta > 0$ and for p, p' in a sufficiently small $\mathcal{O}_1$, $\sqrt{\sigma(p, p')} \geq \eta/2$ if $\sqrt{\ell(s(p), s(p'))} > \eta$. We stress that the limit in $\lambda \to 0^+$ in (11) allows us to take $\mathcal{O}_1$ as small as we need. Thus, by definition of χ_δ and of $G_\delta(p, V', U')$, the denominator $\sigma_{\epsilon=0}(p, p') \geq \delta^2/16$ when $1 - \chi_\delta \neq 0$. Hence, the integrand of the second integral in (12) is jointly smooth in all variables including ϵ, even for $\epsilon = 0$. Then, in view of Lebesgue's dominated convergence theorem, the limit in ϵ can be computed simply taking $\epsilon = 0$ in the integrand. For the same reason, after changing the integration variables (V, V') to $(\lambda V, \lambda V')$, the subsequent limit as $\lambda \to 0^+$ can be computed under the sign of integration. The resulting integral vanish because, in the (V, V') variables, it is noting but the integral of the V, V' derivative of some compactly supported smooth function. Thus only the former integral in (12) may survive the limits in (11). Let us focus on that integral. Making use of (c) in proposition 2.1, in each set $G_\delta(p, V', U') \subset \mathcal{S}_{V', U'}$ we define the function $\rho(p') \geq 0$ such that:

$$\sigma(p, p') = \rho(p')^2 + \sigma(p, q(p, V', U')).$$

In view of (c) in proposition 2.1, the pair ρ, θ, where $\theta \in (-\pi, \pi)$ is the standard polar angle in geodesic polar coordinates centred on $q(p, V', U')$, determines an allowable local chart for any $p' \in G_\delta(p, V', U')$ (see also the appendix B of [KW91]), that is smooth barring the usual conical singularity for $\rho = 0$. Notice that, due to the last statement in (c) of proposition 2.1, when $p \in \mathcal{H}$ and $V' = 0$, ρ coincides with the standard geodesic radial coordinate centred on $s(p) \in \mathcal{B}$. In the following we shall employ that coordinate system in each $G_\delta(p, V', U')$. Making finally use of (d) in proposition 2.1 and choosing $T = (U + V)/2$, we can re-arrange the former integral in (12) so that:

$$L = \lim_{\lambda \to 0} \lim_{\epsilon \to 0^+} \int \frac{\Delta'^{1/2}(p, p') f_\lambda(p) f'_\lambda(p')}{\rho^2 - (V - V' - i\epsilon)(U - U' - i\epsilon) + R(p, V', U')} \frac{dp' dp}{4\pi^2}, \quad (13)$$

where, $\Delta'^{1/2}(p, p') := \Delta^{1/2}(p, p') \chi_\delta(p, p')$. From now on we shall assume that the integral in p' is performed before that in p. Using this coordinate system the integral in the right-hand side

12

of (13) can be rewritten[3] as

$$\int \Delta'^{1/2}(p,p')f_\lambda(p)f'_\lambda(p')\frac{\partial}{\partial\rho}\ln\left(\rho^2+\sigma(p,q(p,V',U'))\right)\frac{\sqrt{|\det g|}}{8\pi^2\rho}d\rho d\theta dU'dV'dp$$

where $\det g$ is the determinant of the metric in the coordinates ρ,θ,V',U', which parametrically depends on p. Notice that the domain of integration in ρ is bounded by the support of the function $\chi_\delta(p,p')$ embodied in Δ'. For $V=V'=0$, the metric takes the form (1) on $\mathcal{B}$ which does not depend on U,U',V,V' any more while R vanishes. By direct inspection one sees that $\frac{\sqrt{|\det g|}}{\rho}$ is continuous, tends to $1/2$ when $p\to p'$ in $\mathcal{H}$ and its ρ-derivative is continuous for $\rho\neq 0$ it being however bounded there. If Δ'_λ, R_λ, $\det g_\lambda$, dp_λ are respectively defined as Δ', R, $\det g$ and dp with V and V' rescaled by λ, changing coordinates $(V,V')\to(\lambda V,\lambda V')$ the integral in the right-hand side of (13) can be rearranged as

$$\int \partial_V F(p)\partial_{V'}F'(p')\Delta'^{1/2}_\lambda(p,p')\frac{\partial}{\partial\rho}\ln\left[\rho^2-(\lambda V-\lambda V'-i\epsilon)(U-U'-i\epsilon)+R_\lambda(p,V',U')\right]$$
$$\frac{\sqrt{|\det g_\lambda|}}{8\pi^2\rho}d\rho d\theta dU'dV'dp_\lambda.$$

Leaving unchanged the remaining integrations, we can first integrate by parts in the polar coordinate ρ. We obtain two boundary terms, which are integrals in the remaining variables evaluated respectively at $\rho=0$ and $\rho=\rho_0$ sufficiently large, and an integral in all the variable including ρ. When taking the limits as $\epsilon\to 0^+$ and $\lambda\to 0^+$, concerning the integral representing the boundary term at $\rho=\rho_0$, we can pass both limits under the sign of the integration by straightforward application of Lebesgue's dominated convergence theorem. The result is that, in the integrand, only $\partial_V F$ and $\partial_{V'}F'$ depend on V,V', hence, performing the integrations in V and V' both integrals vanish because F and F' are of compact support. The remaining boundary term leads to the limit:

$$L=\lim_{\lambda\to 0^+}\lim_{\epsilon\to 0^+}-2\pi\int\Delta'^{1/2}(\lambda V,U,s,\lambda V',U',s)\partial_V F(V,U,s)\partial_{V'}F'(V',U',s)$$
$$\left(\ln\left(-(V-V'-i\epsilon)(U-U'-i\lambda\epsilon)+\frac{R_\lambda(p,V',U')}{\lambda}\right)+\ln\lambda\right)\frac{\sqrt{|\det g_\lambda|}}{8\pi^2\rho}\Bigg|_{\rho=0}dU'dV'dp_\lambda.$$

Above, the factor 2π arises by the integration in θ at $\rho=0$, and we have safely replaced ϵ with $\lambda\epsilon$ in the integral in view of the order of the limits. Notice that, thanks to (d) of proposition 2.1, $|\lambda^{-1}R_\lambda(p,V',U')|<C\lambda$ for some constant C, when $p,p'\in\mathcal{O}_1$ and $\lambda\in[0,\lambda_0)$. Furthermore, from the (1) of the metric on $\mathcal{H}$ we get that $dp_\lambda=\frac{1}{2}(1+\lambda Vz)dUdVd\mu(s)$ for some smooth function $z=z(V,U,s)$, where $d\mu$ is the measure associated to the 2-metric h on $\mathcal{B}$. Thus the term proportional to $\log(\lambda)$ can be dropped as it gives no contribution to the final result because g_0, Δ_0 do not depend on V and the apparently divergent term results once more in the integral

_____

[3]We henceforth assume to cut the complex plane along the negative real axis to define the function $\ln z$.

13

of a derivative of a compactly supported smooth function. The limits of the remaining terms can then be computed, in the given order, exploiting Lebesgue's theorem. We eventually obtain:

$$L = -\int \partial_V F \partial_{V'} F' \left(i\pi \chi_{E_+} \chi_{A_+} - i\pi \chi_{E_+} \chi_{A_-} + \ln|V - V'||U - U'| \right) \frac{dU\,dV\,dU'\,dV'\,d\mu(s)}{16\pi} \; , \quad (14)$$

where $E_\pm$ is the subset of $\mathcal{O}_1$ with, respectively, $(V - V')(U - U') \gtrless 0$ and $A_\pm$ is the analogue with, respectively, $U - U' \gtrless 0$, and χ_S is the characteristic function of the set S. We have also used the fact that dp_λ becomes $\frac{1}{2}dU\,dV\,d\mu(s)$ for $\lambda = 0$ in view of the form (1) of the metric on $\mathcal{H}$ and that $\Delta' = \Delta = 1$ when $s = s'$ and $V = V' = 0$ (it follows from $\Delta(p,p) = 1$ and, since Δ is invariant under isometries, using an argument similar to that employed to prove (b) of proposition 2.1). For the same reason, in view of the meaning ρ, $\frac{\sqrt{\det|g_\lambda|}}{\rho} \to 1/2$ for $\rho \to 0$ when $\lambda = 0$, when working in coordinates ρ, θ, V, U. The integral in (14) can equivalently be re-written introducing another ϵ-prescription as:

$$L = \lim_{\epsilon \to 0^+} -\frac{1}{16\pi} \int \partial_V F \partial_{V'} F' \ln(-(V - V' - i\epsilon)(U - U'))dU\,dV\,dU'\,dV'\,d\mu(s) \; .$$

Eventually, integrating by parts, we obtain (10) concluding the proof, provided the contribution of the term w_ϵ in (6) to the left-hand side of (10) vanishes. Indeed this is the case. Taking the requirements (W1) and (W2) into account, applying Lebesgue's dominated convergence theorem, changing variables $V \to \lambda V$, $V' \to \lambda V'$ and exploiting Lebesgue's theorem again, we have:

$$\lim_{\lambda \to 0^+} \lim_{\epsilon \to 0^+} \int f_\lambda(p) f_\lambda'(p') w_\epsilon(p, p') \, dp\,dp' = \int w''(U, s, U', s') \frac{\partial F}{\partial V} \frac{\partial F'}{\partial V'} dU\,dV\,d\mu(s)dU'\,dV'\,d\mu(s') = 0$$

in view of the fact that the result of the integrations in V and V' vanishes because F and F' are smooth with compact support. □

3.3. The correlation functions and their thermal spectrum. As is known (e.g., see [Wa84]) a timelike Killing vector field on the one hand provides a natural notion of time, which is nothing but the parameter of the integral lines of the field. On the other hand it gives a natural notion of conserved energy for fields and matter propagating in the region where the Killing vector is present. We are interested in computing the energy spectrum of the correlation functions $\omega(\Phi(f_\lambda)\Phi(f_\lambda'))$ seen by an observer that moves along the curves generated by the Killing field K and computed with respect to the associated Killing time. More precisely, exploiting Theorem 3.1, we intend to compute that energy spectrum in the limit of test functions squeezed on the local Killing horizon. As the supports of the test functions are infinitesimally close to the horizon, we have to focus on what happens for $V \sim 0$. Therefore we truncate every component of the formula (2) for the Killing vector at the dominant order in powers of V and we make use of the right-hand side of (10) as definition of correlation two-point function. Now τ is the Killing time, namely the integral parameter of the curves tangent to K. In the said approximation, the first identity in the right-hand side of (2) implies:

$$V(\tau) = -e^{-\kappa\tau} \quad \text{for } V < 0 \text{ (that is in } \mathcal{O}_t) \text{ and} \quad V(\tau) = e^{-\kappa\tau} \quad \text{for } V > 0 \text{ (that is in } \mathcal{O}_s), \quad (15)$$

up to an additive constant in the definition of τ which in principle could depend on the integral curve. Our choice is coherent with the standard definitions of τ in Schwarzschild spacetime where τ is the Killing time in the external region. Indeed we recover those cases in the limit where the τ-constant 2-surfaces are close to the Killing horizon. We now examine two situations.

(a) *Both the supports of f_λ and f'_λ stay in $\mathcal{O}_t$.*
In that case, thinking of the functions F, F' as functions of τ, τ' instead of V, V' in view of (15), we can re-arrange the found expression for the correlation function as

$$\lim_{\lambda \to 0} \omega(\Phi(f_\lambda)\Phi(f'_\lambda)) = \lim_{\epsilon \to 0^+} -\frac{\kappa^2}{64\pi} \int \frac{F(\tau, U, x)F'(\tau', U', x)}{(\sinh(\frac{\kappa}{2}(\tau - \tau')) + i\epsilon)^2} d\tau dU d\tau' dU' d\mu(x) , \qquad (16)$$

where we have used the fact that the functions F and F' are compactly supported by construction even adopting the new coordinate frame. It is known that, in the sense of the Fourier transform of the distributions, $\int_{\mathbb{R}} \frac{d\tau}{\sqrt{2\pi}} \frac{e^{-iE\tau}}{(\sinh(\frac{\tau}{2}) + i0^+)^2} = -\sqrt{2\pi} \frac{Ee^{\pi E}}{e^{\pi E} - e^{-\pi E}}$ (e.g., see the appendix of [DMP09]). That identity and the convolution theorem lead to

$$\lim_{\lambda \to 0} \omega(\Phi(f_\lambda)\Phi(f'_\lambda)) = \frac{1}{32} \int_{\mathbb{R}^2 \times \mathcal{B}} \left(\int_{-\infty}^{\infty} \frac{\overline{\hat{F}(E, U, x)}\hat{F}'(E, U', x)}{1 - e^{-\beta_H E}} E dE \right) dU dU' d\mu(x) , \qquad (17)$$

where $\beta_H = 2\pi/\kappa$ is the *inverse Hawking temperature* and we have defined the smooth function:

$$\hat{F}(E, U, x) := \int_{\mathbb{R}} \frac{d\tau}{\sqrt{2\pi}} e^{-iE\tau} F(\tau, U, x) ,$$

that, for $E \to \pm\infty$, vanishes faster than E^{-n}, $n = 0, 1, 2, \ldots$ uniformly in the remaining coordinates. The thermal content of the found correlation function is manifest in view of the Bose factor $(1 - e^{-\beta_H E})^{-1}$ where the Hawking temperature $1/\beta_H$ takes place - fixed with respect to a (generally arbitrary) choice of the scale necessary to define the Killing field K (see the remark below).

(b) *The support of f_λ stays in $\mathcal{O}_s$, while that of f'_λ stays in $\mathcal{O}_t$: Tunnelling processes.*
As previously remarked, up to normalization, $|\omega(\Phi(f_\lambda)\Phi(f'_\lambda))|^2$ can be interpreted as a tunnelling probability through the horizon. Employing (15) once more, we end up with:

$$\lim_{\lambda \to 0} \omega(\Phi(f_\lambda)\Phi(f'_\lambda)) = \lim_{\epsilon \to 0^+} \frac{\kappa^2}{64\pi} \int \frac{F(\tau, U, x)F'(\tau', U', x')}{\cosh(\frac{\kappa}{2}(\tau - \tau') + i\epsilon)^2} d\tau dU d\tau' dU' d\mu(x) . \qquad (18)$$

As expected from the fact that, in this case, the support of f_λ is always disjoint from the support of f'_λ, we can directly pass the limit $\epsilon \to 0^+$ under the sign of integration simply dropping $i\epsilon$ in the denominator. Taking advantage of the convolution theorem, the final result reads:

$$\lim_{\lambda \to 0} \omega(\Phi(f_\lambda)\Phi(f'_\lambda)) = \frac{1}{16} \int_{\mathbb{R}^2 \times \mathcal{B}} \left(\int_{-\infty}^{\infty} \frac{\overline{\hat{F}(E, U, x)}\hat{F}'(E, U', x)}{\sinh(\beta_H E/2)} E dE \right) dU dU' d\mu(x) . \qquad (19)$$

15

Notice that, if the arbitrary additive constant defining τ in $\mathcal{O}_t$ were different from that in $\mathcal{O}_s$, then a further exponential $\exp{(icE)}$ would take place in the numerator for some real constant c. In any case, the energy spectrum does not agree with the Bose law. However, considering packets concentrated around to a high value of the energy E_0, (19) leads to the estimate for the tunnelling probability:

$$\lim_{\lambda \to 0} |\omega(\Phi(f_\lambda)\Phi(f'_\lambda))|^2 \sim \text{const. } E_0^2 \, e^{-\beta_H E_0} \, ,$$

in agreement with the ideas in [PW00, ANVZ05]. It is nevertheless worth remarking that the interpretation of E as an energy is questionable for the packet in the internal region $\mathcal{O}_s$ since the Killing vector K is spacelike therein.

Remark[4]. It is worth stressing that if K is a (future-directed) Killing field, for every constant $c > 0$, cK is a (future-directed) Killing field again. In general, there is no way to fix the scale c. This arbitrariness in the definition of K enters the definition $\nabla^a(K_b K^b) = -2\kappa K^a$ of the associated surface gravity κ and, in turn, it affects the definition of the Hawking temperature $T_H = \kappa/(2\pi)$. In the case of an asymptotically flat (say, Schwarzschild) black hole, one uses the asymptotic behaviour of the Killing isometries to normalize the "Schwarzschild time" parameter to correspond to "ordinary time" at infinity (and of course energy is also defined with respect to infinity), fixing the scale c. However, if one is assuming only the existence of a local horizon and not necessarily any connection with any "infinity" region, in the absence of further physical requirement, there is nothing to set the scale. So, while it may be correct to say that one has a thermal spectrum in the found expressions for $\lim_{\lambda\to 0}\omega(\Phi(f_\lambda)\Phi(f'_\lambda))$, it does not seem very meaningful to claim that this corresponds to any particular value for the Hawking temperature in the general case.

4   Conclusions

In this paper we have analysed the correlation functions and the tunnelling amplitude through a Killing horizon for quantum states in a certain, physically relevant, class of states that includes the states of Hadamard type. Although the equation of motion does not play a relevant role, the results are in particular valid for scalar Klein-Gordon particles. The computation is performed in the limit of test functions squeezed on the Killing horizon. The considered local Killing horizon with positive constant surface gravity may be a part of the complete horizon of a black hole, including non-static black holes as the non-extremal charged rotating one, or it may just temporarily exist in a finite region. The considered states are generally not required to be invariant with respect to the isometry group generated by the Killing field. We have established that, in the limit of test functions sharply localized on the opposite sides of the horizon, the correlation functions have a thermal nature, namely they have a spectrum which decays exponentially as $\exp\{-\beta_{\text{Hawking}}E\}$ for high energies. The energy E and the inverse

---

[4]The authors are grateful to one of the referees for this important remark.

16

Hawking temperature are defined with respect to the Killing field generating the horizon (see however the remark at the end of the previous section). This achievement is in agreement with the result obtained in other recent papers, although here it is obtained in the framework of the rigorous formulation of quantum field theory on curved spacetime. While, in the mentioned literature quantum-mechanical approaches are exploited, leaving unresolved the issue concerning the strongly ambiguous notion of quantum particle in curved spacetime.

Furthermore, we have also established that, when both test functions are localized in the external side of the horizon, a full Bose spectrum at the Hawking temperature arises in the expression of the correlation functions. In both cases the computation is completely local, i.e. the nature of the geometry at infinity does not matter and the results do not depend on the employed states provided they belong to the mentioned class. These results give a strong support to the idea that the Hawking radiation, that it is usually presented as a radiation detected at future (lightlike) infinity and needs the global structure of a black hole Killing horizon, can also be described as a local phenomenon for local geometric structures (local Killing horizons) existing just "for a while". A fundamental ingredient in our computation is the fact that the nonvanishing surface gravity is constant on the Killing horizon. Thus, we could exploit the result in [RW92] and, in turn, some technical constructions of [KW91]. Even if this requirement can easily be physically interpreted as the geometrical description of the thermodynamic equilibrium, it would be interesting to consider, from our viewpoint, the case of a black hole in formation, where there are no Killing horizons at all. The latter situation has already been investigated, at least in the presence of spherical symmetry, as in [DNVZZ07, HDVNZ09]. In those papers the WKB approach as well as the theory of Kodama-Hayward and the associated notion of dynamical horizon are exploited. As a preliminary comment, we notice that the computation of the scaling limit towards the horizon does not seem to require the presence of a proper Killing horizon, which could be replaced by some more generic null hypersurface.

Acknowledgements.

References

[APS08] E.T. Akhmedov, T. Pilling, D. Singleton *"Subtleties in the quasi-classical calculation of Hawking radiation,"* Int. J. Mod. Phys. D17, 2453-2458 (2008). arXiv:0805.2653v2 [gr-qc]

[APGS08] E.T. Akhmedov, T. Pilling, A. de Gill, D. Singleton *"Temporal contribution to gravitational WKB-like calculations*, Phys. Lett. **B666**, 269-271 (2008). arXiv:0804.2289 [hep-th]

[ANVZ05] M. Angheben, M. Nadalini, L. Vanzo, S. Zerbini, *"Hawking radiation as tunneling for extremal and rotating black holes,"* JHEP**0505** 014 (2005).

[Bo69] R.H. Boyer, *"Geodesic Killing orbits and bifurcate Killing horizons"*. Proc. Soc. London **A 311**, 245-252 (1969)

[BV95] D. Buchholz, R. Verch, *"Scaling algebras and renormalization group in algebraic quantum field theory,"* Rev. Math. Phys. **7** 1195 (1995).

[Bu96] D. Buchholz, *"Quarks, gluons, color: Facts or fiction?"* Nucl. Phys. **B469** 333 (1996).

[DMP09] C. Dappiaggi, V. Moretti and N. Pinamonti: *"Rigorous construction and Hadamard property of the Unruh state in Schwarzschild spacetime"*. Adv. Theor. Math. Phys., **Issue 2** (March 2011), in print. arXiv:0907.1034 [gr-qc]

[DNVZZ07] R. Di Criscienzo, M. Nadalini, L. Vanzo, S. Zerbini, G. Zoccatelli, *"On the Hawking radiation as tunneling for a class of dynamical black holes,"* Phys. Lett. **B657**, 107-111, (2007). arXiv:0707.4425 [hep-th]

[FH90] K. Fredenhagen and R. Haag, *"On the derivation of Hawking radiation associated with the formation of a black hole,"* Commun. Math. Phys. **127** 273 (1990).

[HNS84] R. Haag, H. Narnhofer and U. Stein, *"On quantum field theory in gravitational background,"* Commun. Math. Phys. **94** 219 (1984).

[HDVNZ09] S.A. Hayward, R. Di Criscienzo, L. Vanzo, M. Nadalini, S. Zerbini, *"Local Hawking temperature for dynamical black holes"*. Class. Quant. Grav. **26** 062001 (2009). arXiv:0806.0014 [gr-qc]

[Ha75] S. W. Hawking, *"Particle Creation by Black Holes,"* Commun. Math. Phys. **43** 199 (1975).

[KW91] B. S. Kay, R. M. Wald, *"Theorems On The Uniqueness And Thermal Properties Of Stationary, Nonsingular, Quasifree States On Space-Times With A Bifurcate Killing Horizon"*, Phys. Rept. **207** 49 (1991).

[KM07] R. Kerner, R. B. Mann, *"Fermions tunnelling from black holes."* Class. Quant. Grav. **25** 095014 (2008).

[MV05] A. J. M. Medved, E. C. Vagenas *"On Hawking Radiation as Tunneling with Back-Reaction"* Mod. Phys. Lett. **A20** 2449-2454 (2005).

[PW00] M. K. Parikh, F. Wilczek, *"Hawking Radiation as Tunneling",* Phys. Rev. Lett. **85** 5042 (2000).

[RW96] I. Racz, R. M. Wald, *"Global extensions of space-times describing asymptotic final states of black holes,"* Class. Quant. Grav. **13** 539 (1996).

[RW92] I. Racz, R. M. Wald, *"Extension of space-times with Killing horizon,"* Class. Quant. Grav. **9** 2643 (1992).

[Wa84] R. M. Wald: *"General Relativity"*, Chicago University Press, (1984).

[Wa94] R. M. Wald: *"Quantum field theory in curved space-time and black hole thermodynamics"*, The University of Chicago Press (1994).