

Two-Dimensional Dilatonic Black Holes and Hawking Radiation

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Abstract

Hawking radiation emanating from two-dimensional charged and uncharged dilatonic black holes - dimensionally reduced from $(2 + 1)$ spinning and spinless, respectively, BTZ black holes - is viewed as a tunnelling process. Two dimensional dilatonic black holes (AdS(2) included) are treated as dynamical backgrounds in contrast to the standard methodology where the background geometry is fixed when evaluating Hawking radiation. This modification to the geometry gives rise to a nonthermal part in the radiation spectrum. Nonzero temperature of the extremal two-dimensional charged black hole is found. The Bekenstein-Hawking area formula is easily derived for these dynamical geometries.

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Introduction

In 1992 Bañados, Teitelboim and Zanelli (BTZ) [1, 2] showed that $(2 + 1)$ -dimensional gravity has a black hole solution. This black hole is described by two parameters, its mass M and its angular momentum (spin) J . It is locally anti-de-Sitter space and thus it differs from Schwarzschild and Kerr solutions in that it is asymptotically anti-de-Sitter instead of flat. Additionally it has no curvature singularity at the origin. AdS black holes, which are members of the two-parametric family of BTZ black holes, play a central role in AdS/CFT conjecture [3, 4] and also in brane-world scenarios [5, 6]. Specifically AdS(2) black hole is most interesting in the context of string theory and black hole physics [7, 8]. In this paper motivated by this recent interest in two-dimensional black hole backgrounds we treat the two-dimensional charged and uncharged dilatonic black holes (including AdS(2)) as radiating sources.

Concerning the quantum process called Hawking effect [9] much work has been done using a fixed background during the emission process. The idea of Keski-Vakkuri, Kraus and Wilczek (KKW) [10, 11, 12, 13] is to view the black hole background as dynamical by treating the Hawking radiation as a tunnelling process. The energy conservation is the key to this description. The total (ADM) mass is kept fixed while the mass of the two-dimensional dilatonic black hole decreases due to the emitted radiation. The effect of this modification gives rise to additional terms in the formulae concerning the known results for the two-dimensional charged and uncharged black holes; these additions are analogous to those found in [14, 15, 16, 17] for the respective geometries; a nonthermal partner to the thermal spectrum of the Hawking radiation shows up. We explore the consequences to vacuum states of two-dimensional black holes (extremal black holes) since black holes are in general regarded as highly excited states. The extremality of the two-dimensional charged black holes is now shifted since the charge J of the charged black hole is approached by the mass M earlier. This alteration produces a non -“frozen” extremal two-dimensional charged black hole characterized by a constant Hawking temperature $T_H^{ext} \neq 0$. KKW method provides an easy way to derive the entropy of the two-dimensional charged and uncharged black holes.

In section 1 we make a short review of the two-dimensional charged and uncharged dilatonic black holes and their properties. We present for the two-dimensional charged black hole expressions for its temperature, area and entropy. The extremal two-dimensional charged black hole is derived and its zero temperature is verified. In section 2 we implement the KKW method. Using the imaginary part of the action of an outgoing positive-energy particle the temperature of the two-dimensional charged black hole is evaluated and its dependence on the energy of the emitted massless particle is explicitly shown. This modified temperature due to the self-gravitation effect leads to a nonthermal spectrum. The extremal (vacuum state) two-dimensional charged black hole is no more “frozen”. Its temperature is nonzero. Corresponding results for the two-dimensional uncharged black holes and for the AdS(2) spacetime are deduced. In section 3 we calculate the entropy of two-dimensional charged and uncharged black holes. Finally in section 4 we summarize our results and give our conclusions.

1 (1 + 1) Dilatonic Black Holes

The black hole solutions of Bañados, Teitelboim and Zanelli in (2 + 1) spacetime dimensions are derived from a three dimensional theory of gravity:

$$S = \int dx^3 \sqrt{-g} ({}^{(3)}R + 2\Lambda) \quad (1)$$

with a negative cosmological constant ($\Lambda > 0$). The corresponding line element is:

$$ds^2 = - \left(-M + \Lambda r^2 + \frac{J^2}{4r^2} \right) dt^2 + \frac{dr^2}{\left(-M + \Lambda r^2 + \frac{J^2}{4r^2} \right)} + r^2 \left(d\theta - \frac{J}{2r^2} dt \right)^2. \quad (2)$$

There are many ways to reduce the three dimensional BTZ black hole solutions to the two dimensional charged and uncharged dilatonic black holes [18, 19]. The Kaluza-Klein reduction of the (2 + 1)-dimensional metric (2) yields a two-dimensional line element:

$$ds^2 = -g(r)dt^2 + g(r)^{-1}dr^2 \quad (3)$$

where

$$g(r) = \left(-M + \Lambda r^2 + \frac{J^2}{4r^2} \right) \quad (4)$$

with M the ADM mass, J the charge of the two-dimensional charged black hole, a U(1) gauge field:

$$A_t = -\frac{J}{2r^2} \quad (5)$$

and a dilaton field:

$$\Phi = r. \quad (6)$$

For the positive mass black hole spectrum with charge ($J \neq 0$), the line element (3) has two horizons:

$$r_{\pm}^2 = \frac{M \pm \sqrt{M^2 - \Lambda J^2}}{2\Lambda} \quad (7)$$

with r_+ , r_- the outer and inner horizon respectively. The area \mathcal{A}_H and Hawking temperature T_H of the event (outer) horizon are [20, 21]:

$$\begin{aligned} \mathcal{A}_H &= 2\pi \left(\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2} \\ &= 2\pi r_+ \end{aligned} \quad (8)$$

$$\begin{aligned} T_H &= \frac{\sqrt{2\Lambda}}{2\pi} \frac{\sqrt{M^2 - \Lambda J^2}}{(M + \sqrt{M^2 - \Lambda J^2})^{1/2}} \\ &= \frac{\Lambda}{2\pi} \left(\frac{r_+^2 - r_-^2}{r_+} \right). \end{aligned} \quad (9)$$

The entropy of the two-dimensional charged black hole is:

$$S_{bh} = 4\pi r_+ \quad (10)$$

and if we reinstate the Planck units $8\hbar G = 1$ we get:

$$S_{bh} = \frac{1}{4\hbar G} \mathcal{A}_H = S_{BH} \quad (11)$$

which is the well-known Bekenstein-Hawking area formula for the entropy (S_{BH}) [23, 24, 25] (or [26] by counting excited states).

Concerning the extremality of the two-dimensional charged black hole:

$$M = \sqrt{\Lambda} J \quad (12)$$

the inner and outer horizon coincide ($r_+ = r_-$). This two-dimensional charged black hole can be viewed as the vacuum state of the positive mass spectrum of two-dimensional charged black holes which saturates the bound:

$$M^2 - \Lambda J^2 \geq 0 \Leftrightarrow M \geq \sqrt{\Lambda} |J| \quad (13)$$

imposed in (7). Obviously the extremal two-dimensional charged black hole has zero temperature ($T_H^{ext} = 0$).

The two-dimensional uncharged black hole may be obtained by a similar Kaluza-Klein reduction for a covariantly constant electric field and the resulting metric is:

$$ds^2 = -(-M + \Lambda r^2) dt^2 + (-M + \Lambda r^2)^{-1} dr^2 \quad (14)$$

which has an horizon at:

$$r_H = \sqrt{\frac{M}{\Lambda}} \quad (15)$$

and is similar to Schwarzschild black hole with the important difference that it is not asymptotically flat but it has constant negative curvature. The temperature of the two-dimensional uncharged black hole is [22]:

$$T_H = \frac{\sqrt{2\Lambda}}{2\pi} M^{1/2}. \quad (16)$$

Two-dimensional charged black holes with $M < \sqrt{\Lambda} |J|$ are discarded since they contain a naked singularity and for the same reason two-dimensional uncharged black holes with $M < 0$ have not been treated above. The only exception is the two-dimensional uncharged black hole with $M = -1$ and $J = 0$ which is the ordinary anti-de Sitter spacetime (AdS(2) black hole).

2 KKW Methodology

In order to apply the idea of Keski-Vakkuri, Kraus and Wilczek (KKW) [10, 11, 12, 13] to the two-dimensional charged black hole (3-4) we have to make a coordinate transformation. We choose the Painlevé coordinates [27] (utilized for black hole backgrounds recently in [28]) which are non-singular on the outer horizon (r_+). Thus we will be able to deal with phenomena whose main contributions come from the outer horizon. We introduce the time coordinate τ_P by imposing the ansatz:

$$\sqrt{g(r)} dt = \sqrt{g(r)} d\tau_P - \sqrt{1 + M - \Lambda r^2 - \frac{J^2}{4r^2}} \frac{dr^2}{\sqrt{g(r)}}. \quad (17)$$

The line element (3-4) is now written as :

$$ds^2 = - \left(-M + \Lambda r^2 + \frac{J^2}{4r^2} \right) d\tau_P^2 + 2 \sqrt{1 + M - \Lambda r^2 - \frac{J^2}{4r^2}} d\tau_P dr + dr^2. \quad (18)$$

It is obvious from the above expression that there is no singularity at the points r_+ and r_- . The null ($ds^2 = 0$) geodesics followed by a massless particle are:

$$\dot{r} \equiv \frac{dr}{d\tau_P} = \pm 1 - \sqrt{1 + M - \Lambda r^2 - \frac{J^2}{4r^2}} \quad (19)$$

where the signs $+$ and $-$ correspond to the outgoing and ingoing geodesics, respectively, under the assumption that τ_P increases towards future.

We fix the total ADM mass and we let the mass M of the two-dimensional charged black hole vary. If a shell of energy (mass) ω is radiated outwards the outer horizon then the two-dimensional charged black hole mass will be reduced to $M - \omega$ and the shell of energy will travel on the modified geodesics:

$$\dot{r} = 1 - \sqrt{1 + (M - \omega) - \Lambda r^2 - \frac{J^2}{4r^2}} \quad (20)$$

produced by the modified line element:

$$ds^2 = - \left(-(M - \omega) + \Lambda r^2 + \frac{J^2}{4r^2} \right) d\tau_P^2 + 2 \sqrt{1 + (M - \omega) - \Lambda r^2 - \frac{J^2}{4r^2}} d\tau_P dr + dr^2. \quad (21)$$

It is known that the emission rate from a radiating source can be expressed in terms of the imaginary part of the action for an outgoing positive-energy particle as:

$$\Gamma = e^{-2ImS} \quad (22)$$

but also in terms of the temperature and the entropy of the radiating source which in our case will be a two-dimensional charged black hole:

$$\Gamma = e^{-\beta\omega} = e^{+\Delta S_{bh}} \quad (23)$$

where β is the inverse temperature of the two-dimensional charged black hole and ΔS_{bh} is the change the entropy of the two-dimensional charged black hole before and after the emission of the shell of energy ω (outgoing massless particle). It is clear that if we evaluate the action then we will know the temperature and/or the change in the entropy of the two-dimensional charged black hole. We therefore evaluate the imaginary part of the action for an outgoing positive-energy particle which crosses the event horizon outwards from:

$$r_{in} = r_+(M, J) = \left(\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2} \quad (24)$$

to

$$r_{out} = r_+(M - \omega, J) = \left(\frac{(M - \omega) + \sqrt{(M - \omega)^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2}. \quad (25)$$

The imaginary part of the action is:

$$ImS = Im \int_{r_{in}}^{r_{out}} p_r dr = Im \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp'_r dr. \quad (26)$$

We make the transition from the momentum variable to the energy variable using Hamilton's equation $\dot{r} = \frac{dH}{dp_r}$ and equation (20). The result is:

$$ImS = Im \int_{r_{in}}^{r_{out}} \int_0^\omega \frac{(-d\omega') dr}{1 - \sqrt{1 + (M - \omega') - \Lambda r^2 - \frac{J^2}{4r^2}}} \quad (27)$$

where the minus sign is due to the Hamiltonian being equal to the modified mass $H = M - \omega$. This is not disturbing since $r_{in} > r_{out}$. After some calculations (involving contour integration into the lower half of ω' plane) we get:

$$ImS = 2\pi \left[\left(\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2} - \left(\frac{(M - \omega) + \sqrt{(M - \omega)^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2} \right]. \quad (28)$$

Apparently the emission rate depends not only on the mass M and charge J of the two-dimensional charged black hole but also on the energy ω of the emitted massless particle:

$$\begin{aligned} \Gamma(\omega, M, J) &= e^{-2ImS} \\ &= exp \left[4\pi \left(\sqrt{\frac{(M - \omega) + \sqrt{(M - \omega)^2 - \Lambda J^2}}{2\Lambda}} - \sqrt{\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda}} \right) \right]. \end{aligned} \quad (29)$$

Comparing (23) and (29) we deduce that the modified temperature (due to the self-gravitation) of the two-dimensional charged black hole is:

$$T(\omega) = \frac{\omega}{4\pi} \left[\left(\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2} - \left(\frac{(M - \omega) + \sqrt{(M - \omega)^2 - \Lambda J^2}}{2\Lambda} \right)^{1/2} \right]^{-1}. \quad (30)$$

We see that there are modifications to the result previously mentioned (9) for a fixed two-dimensional charged black hole background. The temperature of the two-dimensional charged black hole is no longer the Hawking temperature T_H .

2.1 (1 + 1) Charged Black Hole

Concerning the extremal (vacuum solution) two-dimensional charged black hole the condition for extremality now satisfied will be:

$$M - \omega = \sqrt{\Lambda} J . \quad (31)$$

This modification indicates that the mass M of the two-dimensional charged black hole cannot be less than the charge J (since $\omega = M - \sqrt{\Lambda} J > 0$) and the temperature of the extremal two-dimensional charged black hole will not be zero but:

$$T_H^{ext} = \frac{\sqrt{2\Lambda}}{4\pi} \frac{(M - \sqrt{\Lambda} J)}{(M + \sqrt{M^2 - \Lambda J})^{1/2} - (\sqrt{\Lambda} J)^{1/2}} . \quad (32)$$

2.2 (1 + 1) Uncharged Black Hole

The modified temperature (30) for the two-dimensional uncharged black hole becomes:

$$T(\omega) = \frac{\omega}{4\pi} \left[\sqrt{\frac{M}{\Lambda}} - \sqrt{\frac{M - \omega}{\Lambda}} \right] \quad (33)$$

and which in first order in ω is:

$$T_H = \frac{\sqrt{\Lambda}}{2\pi} M^{1/2} \quad (34)$$

in agreement to what was shown in [15].

For the case $M = -1$ which may be recognized as the ordinary anti-de-Sitter space (AdS(2) spacetime) the modified temperature is:

$$T(\omega) = \frac{\omega\sqrt{\Lambda}}{4\pi} (1 - \sqrt{1 - \omega})^{-1} \quad (35)$$

which, to first order in ω , gives:

$$T_H = \frac{\sqrt{\Lambda}}{2\pi} . \quad (36)$$

3 Entropy Calculation via KKW Method

It is obvious that we can have a short and direct derivation of the entropy of two-dimensional charged black hole up to a constant using equations (22) and (23) where:

$$\Delta S_{bh} = S_{bh}(M - \omega, J) - S_{bh}(M, J). \quad (37)$$

Indeed, if we combine (22), (23) and (30) we get:

$$S_{bh}(\omega, M, J) = 4\pi \left[\frac{(M - \omega) + \sqrt{(M - \omega)^2 - \Lambda J^2}}{2\Lambda} \right]^{1/2} + S_0 \quad (38)$$

where S_0 is the arbitrary constant and which to first order in ω will give the known expression (10) up to a constant for the entropy of the two-dimensional charged black hole:

$$S_{bh}(M, J) = 4\pi \left[\frac{M + \sqrt{M^2 - \Lambda J^2}}{2\Lambda} \right]^{1/2} + S_0. \quad (39)$$

For the two-dimensional uncharged black hole the entropy is:

$$S_{bh}(M) = 4\pi \sqrt{\frac{M - \omega}{\Lambda}} + S'_0 \quad (40)$$

where S'_0 is also an arbitrary constant and which will give to first order in ω :

$$S_{bh}(M) = 4\pi \sqrt{\frac{M}{\Lambda}} + S'_0. \quad (41)$$

If we adopt the conjecture that the entropy of an excited state tends to the entropy of its vacuum state [29] then the additive constants are set to zero :

$$S_0 = 0 \quad \text{and} \quad S'_0 = 0. \quad (42)$$

Discussion

In this work, we have viewed the Hawking radiation as a quantum tunnelling process. The self-gravitation of the radiation was included and this treatment introduced a nonthermal part for the radiation spectrum of the two-dimensional charged and uncharged dilatonic black holes. The temperature of the two-dimensional charged black hole is no more the Hawking temperature and the “greybody factor” showing up declares explicitly the dependence of the temperature on the emitted particle’s energy ω . The leading term in ω gives the thermal Boltzmann factor while the higher order terms represent corrections emanating from the response of the background geometry to the emission of a quantum. The extremal two-dimensional charged black hole is no more “frozen” but it carries a background Hawking temperature $T_H^{ext} \neq 0$ ensuring the validity of the third law of black

hole thermodynamics [30]. Therefore it is obvious that we again have a strong evidence to believe that black holes constitute excited states while the extremal black holes correspond to ground (vacuum) states.

The above-mentioned treatment for incorporating the effects of the emission of a shell of energy for the case of two-dimensional uncharged dilatonic black holes (including AdS(2) black hole) yields the corresponding modified temperature with the leading term in ω again being the thermal Boltzmann factor.

Additionally the imaginary part of the action of the outgoing positive-energy particle is linear in the change of the entropy. We derive in a short and direct way the modified entropy for the two-dimensional charged and uncharged dilatonic black holes due to the specific modelling of the self-gravitation effect. A welcomed but not unexpected result is that the expressions for the entropy of the two-dimensional dilatonic black holes obtained to first order in ω are in agreement with those for the corresponding fixed two-dimensional black hole backgrounds.

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References

- [1] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69** (1992) 1849.
- [2] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D**48** (1993) 1506.
- [3] D. Birmingham, I. Sachs and S. Sen, Phys. Lett. B**413** (1997) 281.
- [4] J. Maldacena, Adv. Theo. Math. Phys. **2** (1998) 231.
- [5] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370.
- [6] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690.
- [7] J. Maldacena, J. Michelson and A. Strominger, JHEP **9902** (1999) 011.
- [8] M. Spradlin and A. Strominger, JHEP **9911** (1999) 021.
- [9] S.W. Hawking, Commun. Math. Phys. **43** (1975) 199.
- [10] E. Keski-Vakkuri and P. Kraus, Phys. Rev. D**54** (1996) 7407.
- [11] P. Kraus and F. Wilczek, Nucl. Phys. B**433** (1995) 403.

- [12] P. Kraus and F. Wilczek, Nucl. Phys. B**437** (1995) 231.
- [13] E. Keski-Vakkuri and P. Kraus, Nucl. Phys. B**491** (1997) 249.
- [14] M.K. Parikh and F. Wilczek, Phys. Rev. Lett. **85** (2000) 5042.
- [15] Y. Kwon, Il Nuovo Cimento B**115** (2000) 469.
- [16] S. Hemming and E. Keski-Vakkuri, Phys. Rev. D**64** (2001) 044006.
- [17] E.C. Vagenas, Phys. Lett. B**503** (2001) 399.
- [18] A. Achúcarro and M.E. Ortiz, Phys. Rev. D**48** (1993) 3600.
- [19] D.A. Lowe and A. Strominger, Phys. Rev. Lett. **73** (1994) 1468.
- [20] A. Kumar and K. Ray, Phys. Lett. B**351** (1995) 431.
- [21] A. Kumar and K. Ray, Phys. Rev. D**51** (1995) 5954.
- [22] W.T. Kim and J.J. Oh, Phys. Lett. B**461** (1999) 189.
- [23] J.D. Bekenstein, Phys. Rev. D**7** (1973) 2333.
- [24] J.D. Bekenstein, Phys. Rev. D**9** (1974) 3292.
- [25] S.W. Hawking, Phys. Rev. D**13** (1976) 191.
- [26] A. Strominger, JHEP **9802** (1998) 009.
- [27] P. Painlevé, C.R. Acad. Sci.(Paris) **173** (1921) 677.
- [28] P. Kraus and F. Wilczek, Mod. Phys. Lett. A**9** (1994) 3713.
- [29] G. Gour and A.E. Mayo, Phys. Rev. D**63** (2001) 064005.
- [30] S. Liberati, T. Rothman and S. Sonego, Int. J. Mod. Phys. D**10** (2001) 33.