TJHSST AI/ML CLUB SEMINAR SERIES

AI-based Game Playing with Reinforcement Learning

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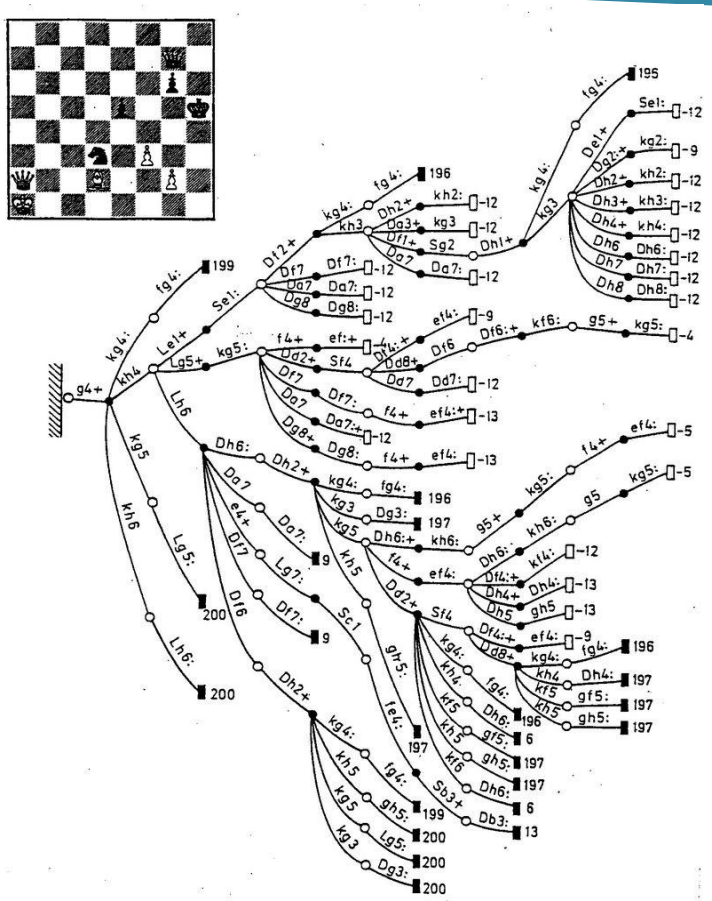
Review of Last Two Lecture Topics

- Game theory
  - Gives techniques like Nash equilibrium to determine an action for each player
- Reinforcement learning
  - Gives techniques for calculating a policy – best action that an RL agent should take at each step or step to solve a problem

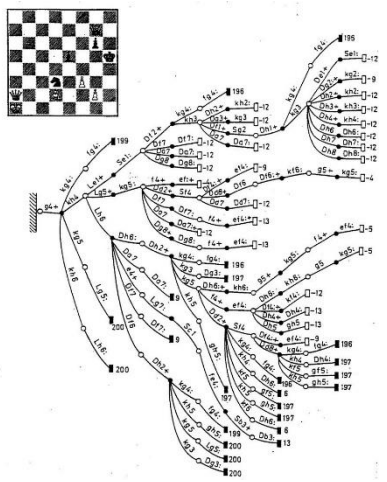
Game Trees and Tree Search



- Recall: Two classes of graph search algorithms - uninformed, informed
- Both behave deterministically
  - Evaluate the next (uninformed) or 'best' (informed) node in fringe
    - 'best' defined using evaluation or objective
- Generates the worst case tree for the algorithm used, $O(b^d)$ for IDS, etc.

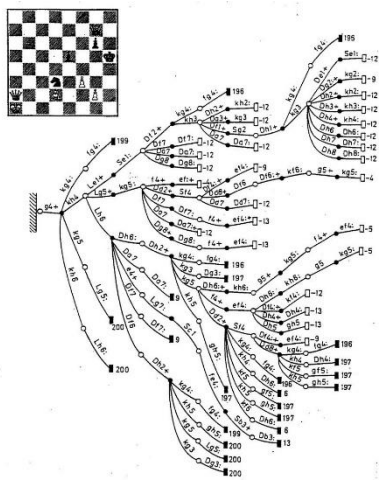
Probabilistic Tree Search

- Some state spaces are very large
  - Worst-case trees are also very large
- Two-player games - how many states in
  - tic-tac-toe ?
  - chess?



states in tic-tac-toe: https://math.stackexchange.com/questions/485752/tictactoe-state-space-choose-calculation
states in chess: https://math.stackexchange.com/questions/1406919/how-many-legal-states-of-chess-exists

Probabilistic Tree Search



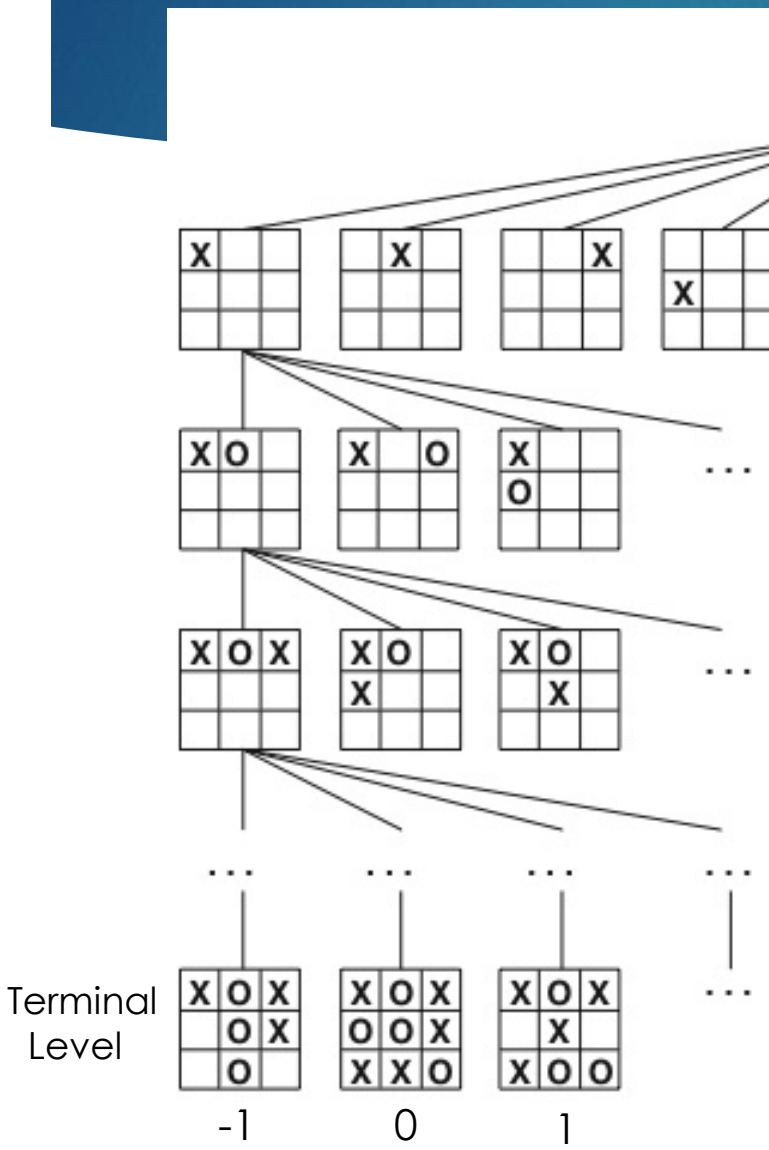
- Some state spaces are very large
  - Worst-case trees are also very large
- Two-player games - how many states in
  - tic-tac-toe ? ~6000
  - chess? ~10^{45}
- Idea: Can we still find the goal node without generating the whole tree (or worst case tree) …so that we can save time and space
- Instead of deterministic search, do stochastic search

states in tic-tac-toe: https://math.stackexchange.com/questions/485752/tictactoe-state-space-choose-calculation
states in chess: https://math.stackexchange.com/questions/1406919/how-many-legal-states-of-chess-exists

Monte Carlo Tree Search (MCTS)

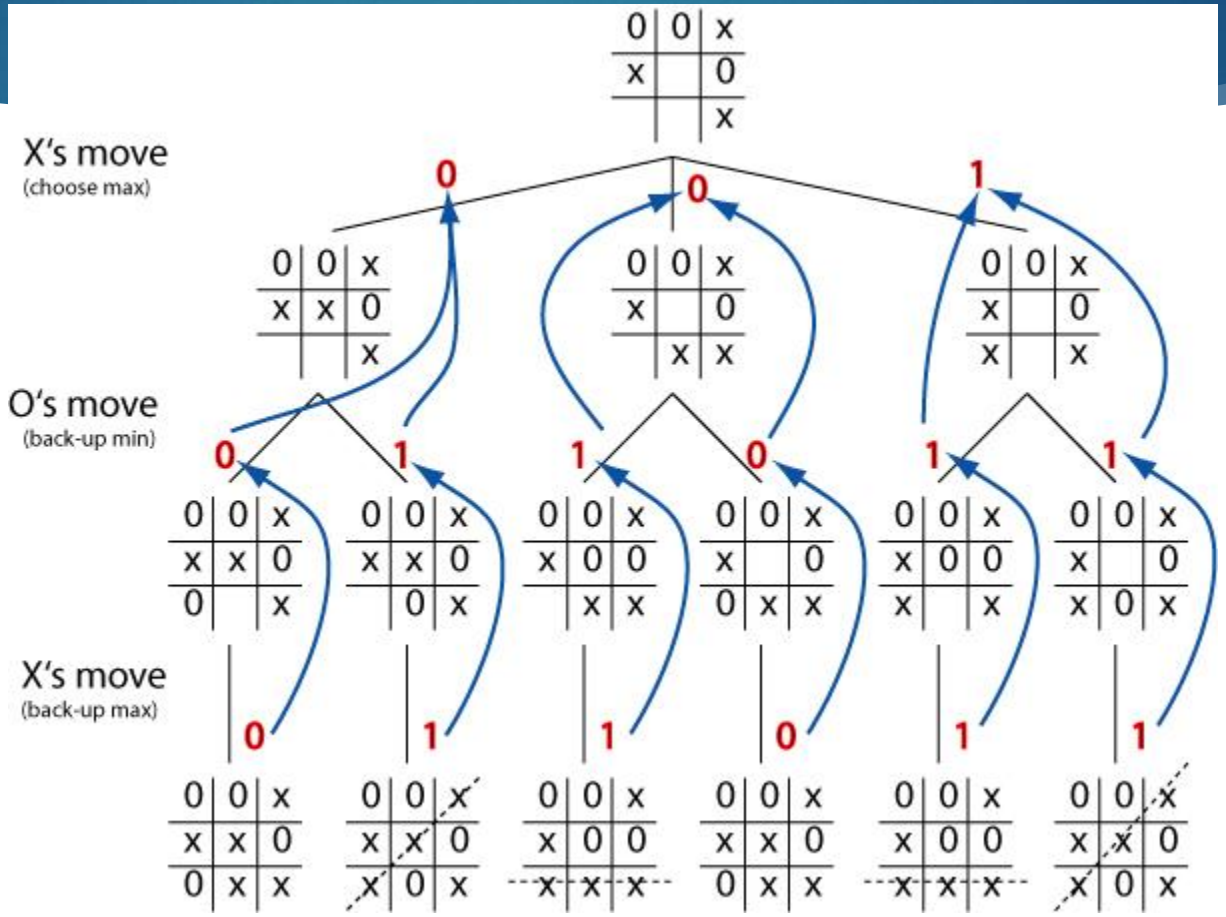
- One of the most popular stochastic tree search algorithms
  - Used by Google DeepMind to defeat human champion in AlphaGo game in 2017
    - Used combination of search and deep learning
- Several other applications
  - Games: Total War: Rome II, Poker; see https://en.wikipedia.org/wiki/Monte_Carlo_tree_search#History
  - Drug discovery, DNA sequencing
  - Autonomous vehicles

MCTS Example: Tic-Tac-Toe

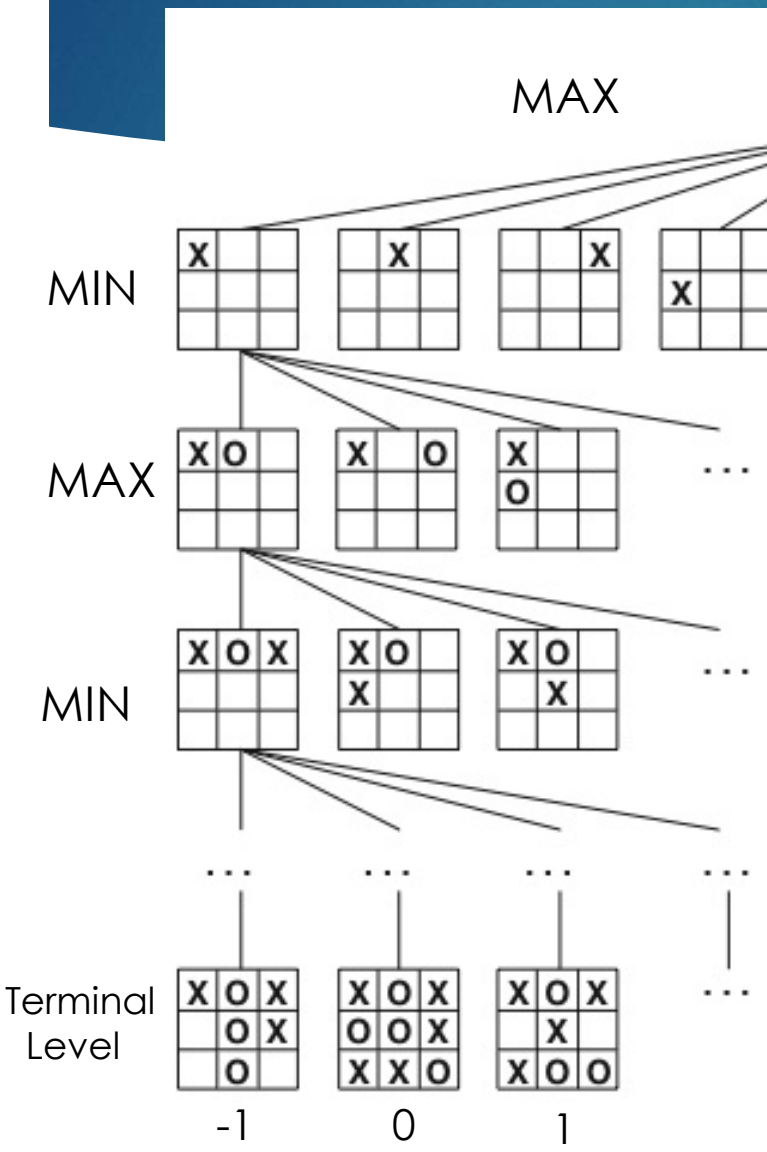


- Assign utilities to players at terminal level
- Win = +1, Draw = 0, Loss = -1
- Zero sum game: gain of player X = loss of player O, vice-versa
- Show utilities of player X only, player O's utility = - player X's utility

MCTS Example: Tic-Tac-Toe



MCTS Example: Tic-Tac-Toe

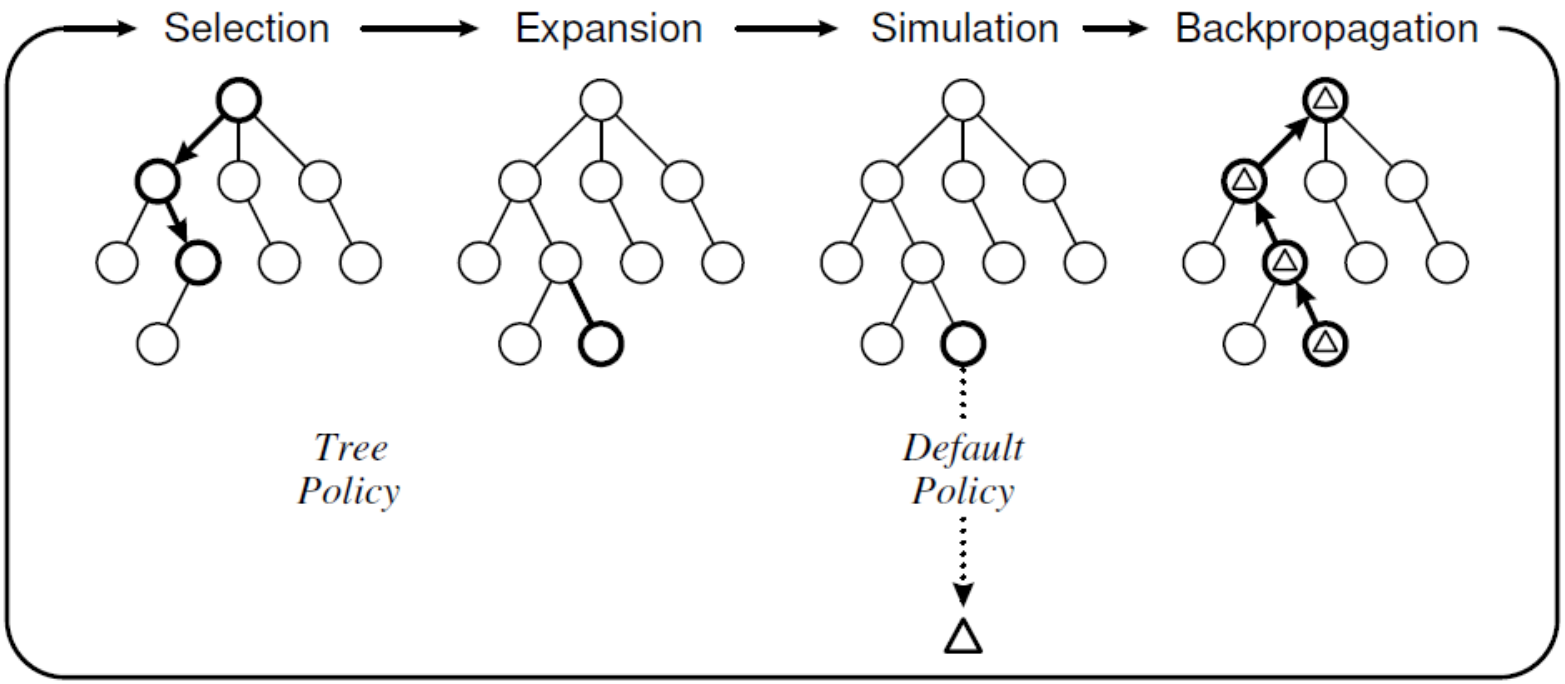


- This is an instance of a min-max game tree
- Also called zero-sum game
- Solved using technique called α–β pruning
- α–β pruning too computationally intensive for very large game trees

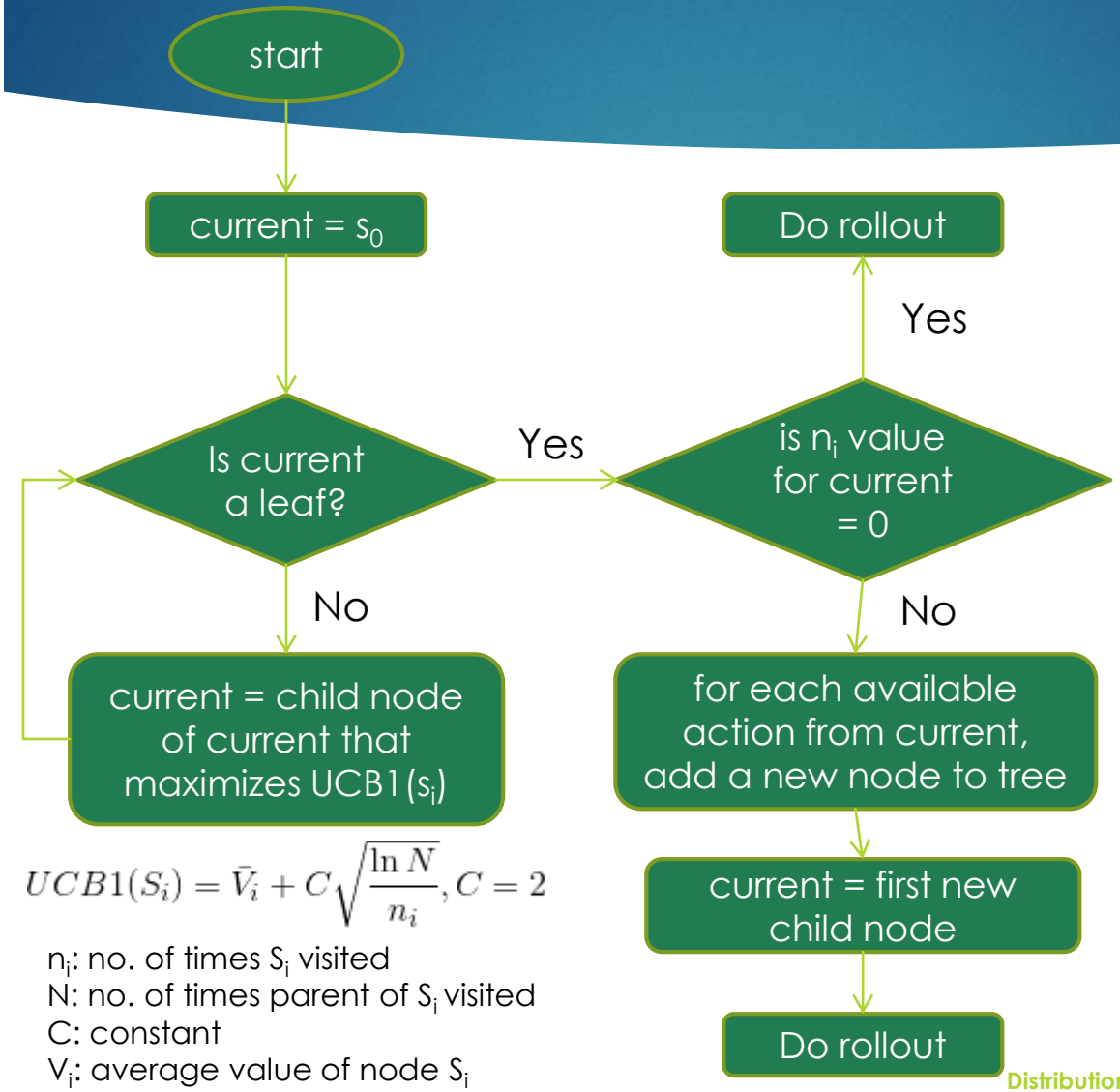
Monte Carlo Tree Search Algorithm

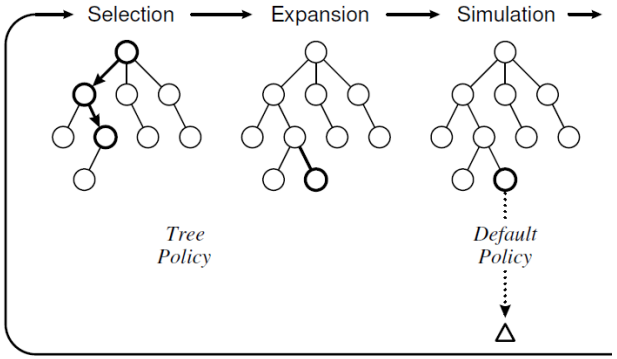
- Four steps:
  - Selection - tree traversal – going down the tree (generated thus far)
  - Expansion – add nodes to tree
  - Simulation– randomly generate one path down tree to terminal level
    - Also called rollout
    - Find values (utilities) at leaf level reached by rollout
  - Backpropagation – move values from rollout up tree

MCTS Algorithm



MCTS Tree Traversal + Node Expansion Algorithm





$$UCB1(S_i) = \bar{V}_i + C\sqrt{\frac{\ln N}{n_i}}, C = 2$$

n_i: no. of times S_i visited
N: no. of times parent of S_i visited
C: constant
V_i: average value of node S_i

MCTS Resources

- Short tutorial MCTS: https://www.youtube.com/watch?v=UXW2yZndl7U
- Wikipedia: https://en.wikipedia.org/wiki/Monte_Carlo_tree_search
- Youtube Udacity: https://youtu.be/onBYsen2_eA
- C. B. Browne *et al*., "A Survey of Monte Carlo Tree Search Methods," in IEEE Transactions on Computational Intelligence and AI in Games, vol. 4, no. 1, pp. 1-43, March 2012. doi: 10.1109/TCIAIG.2012.2186810 (Available via IEEE Xplore)
- mcts4j library: https://github.com/avianey/mcts4j

Deep Learning based Game Playing AI

- Three successively improving algorithms:
  - Alpha Go, 2016
  - Alpha Zero, 2017
  - Alpha Star, 2019

Alpha Go: Key Ideas

- Search space (no. of states and actions) is enormous! (more than no. of atoms in universe)
- Need to reduce search space: breadth + depth

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- Breadth: limit number of possible actions from a state s
  - Using how many times an action was taken from state s
  - Learn a function g(s) = p(a | s)
  - g(s) learned using a combination of supervised learning and RL
  - Using 12 layer CNN
  - Supervised part: Trained with moves from 160K games, 30 M states; 4 weeks training on 50 GPUs
  - RL part: plays with older models learned by itself during supervised learning (self-play); 1 week RL on 50 GPUs

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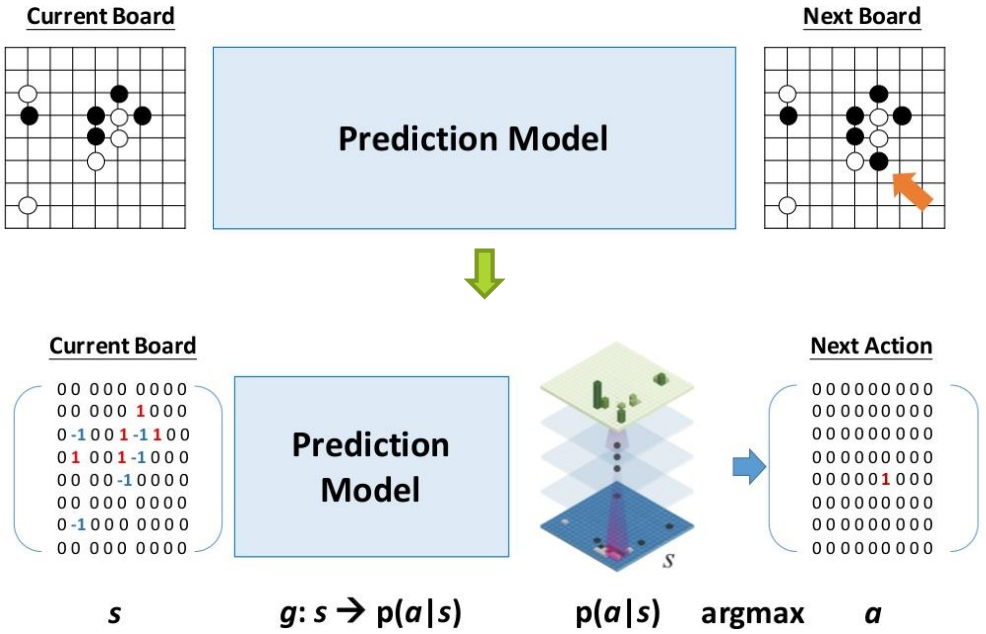
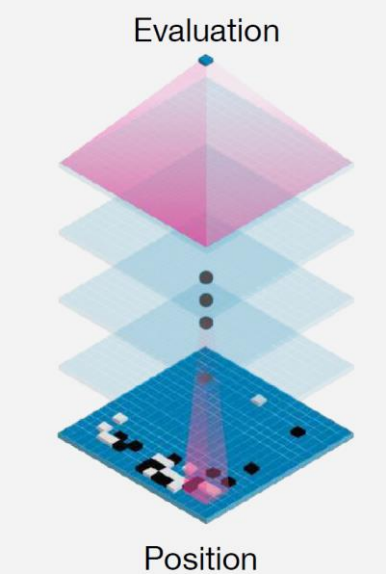


Alpha Go: Key Ideas

- Search space (no. of states and actions) is enormous! (more than no. of atoms in universe)
- Need to reduce search space: breadth + depth

- Depth: by estimating value function V(s) at state s
  - V(s) called board evaluation function
  - 0 (loss) < V(s) < 1 (win)
  - V(S) is learned using regression learning w/ stochastic gradient descent
  - Using a 12 layer CNN
  - 1 week on 50 GPUs



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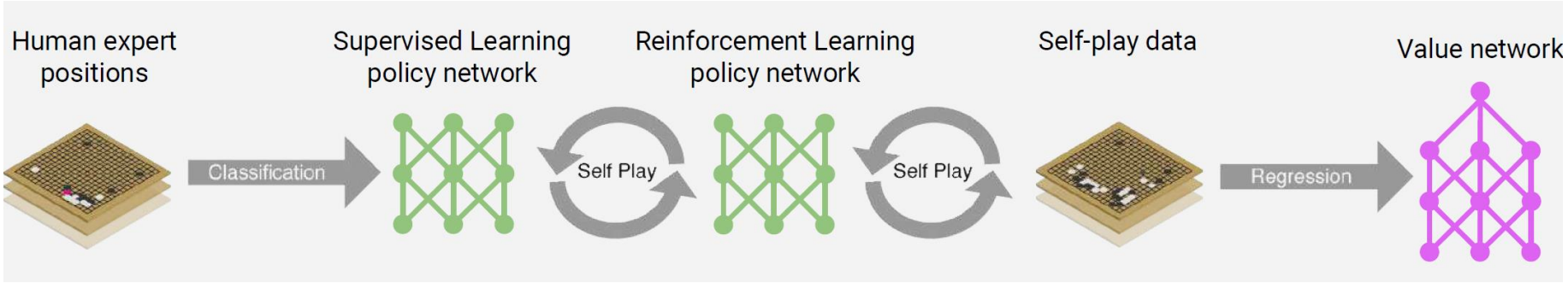

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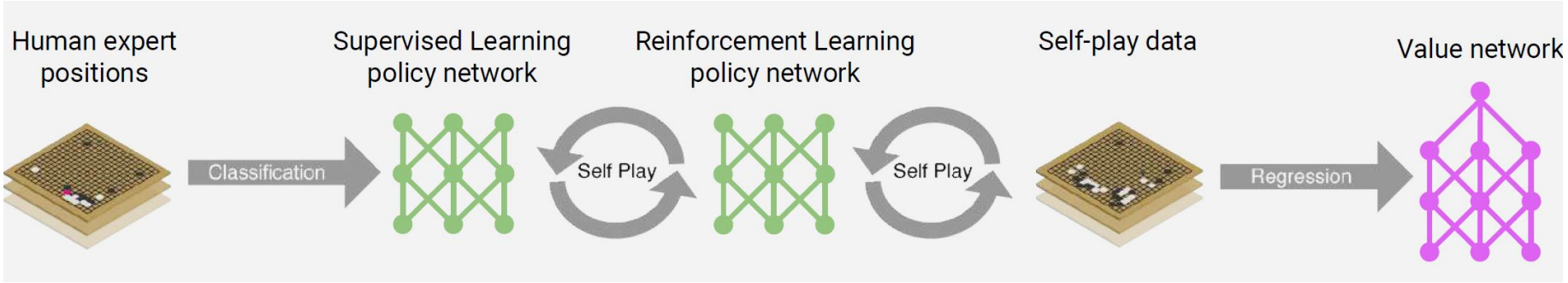
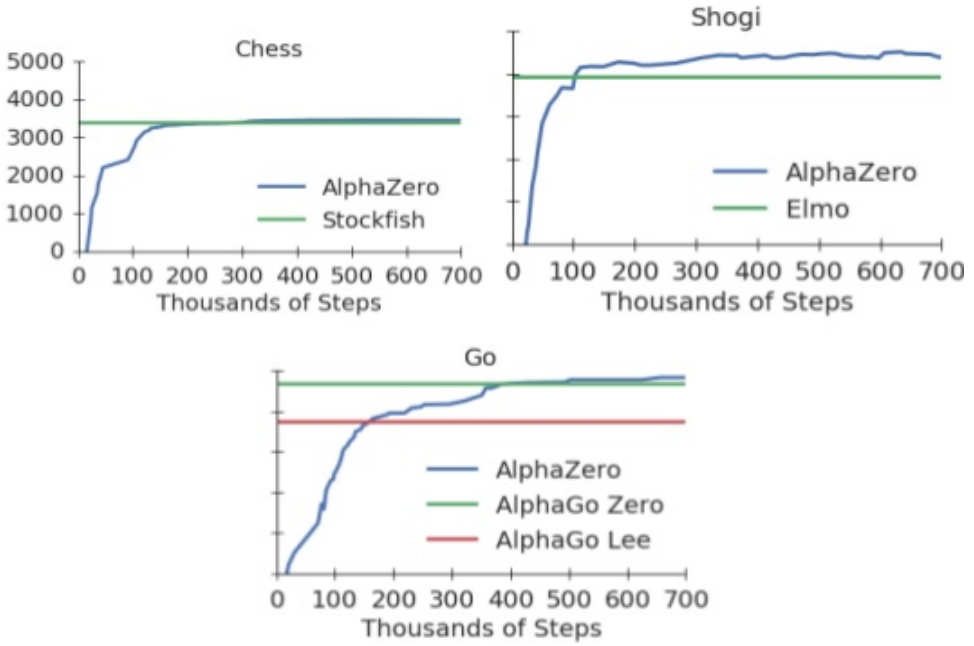
- Need to reduce search space: breadth + depth



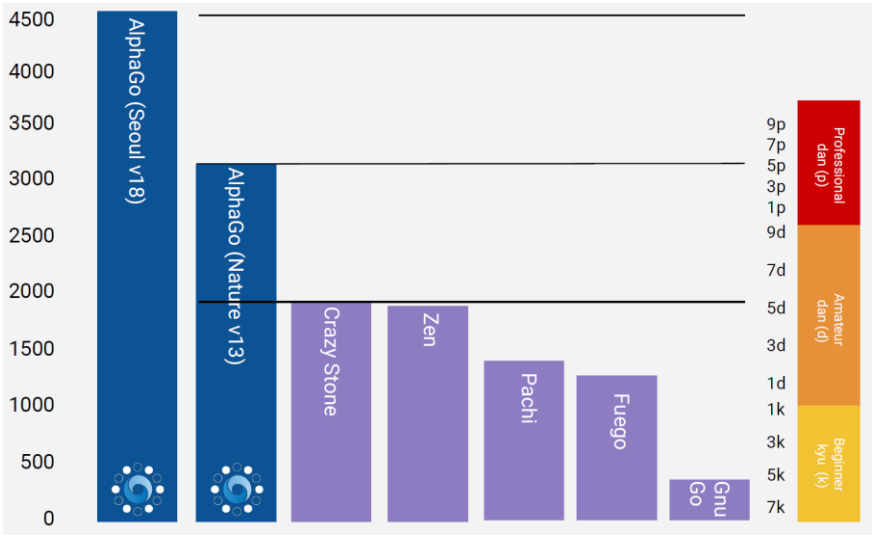
- Need to reduce search space: breadth + depth + lookahead
- Lookahead:
  - Done using MCTS on reduced search space
  - Starting from current state virtually play out the game till end

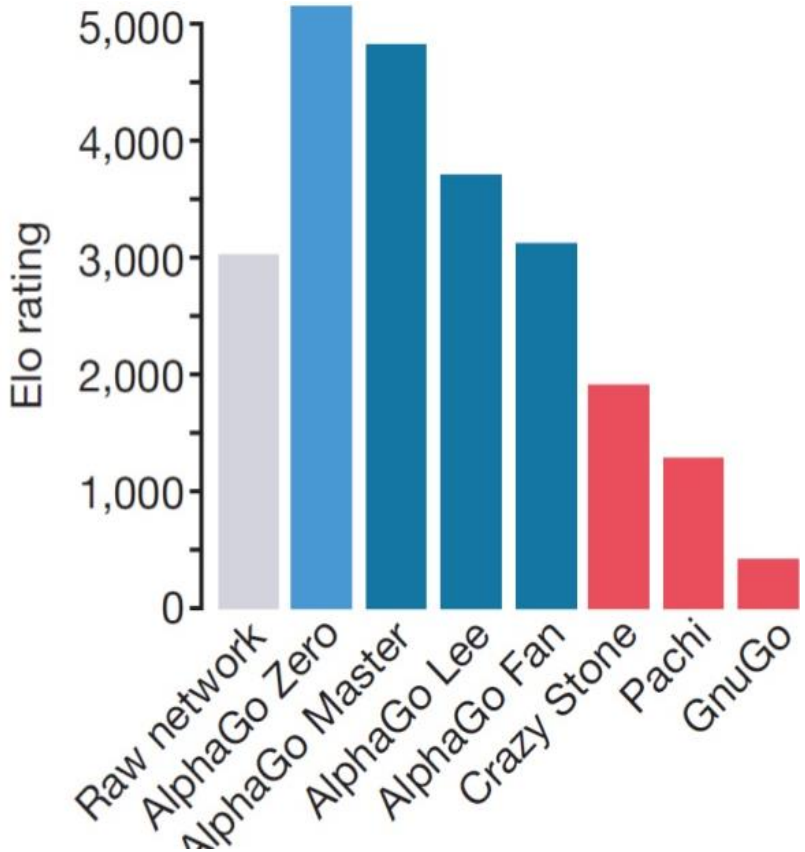
Alpha (Go) Zero: Key Ideas

- Improvement on AlphaGo
- Use self-play only (learns from scratch by playing against itself) instead of using play information form human played games
- Uses single neural network instead of value and policy networks
- Simpler tree search than MCTS rollouts
- No hyper parameter tuning (in Alpha Zero, tuning done in AlphaGo and AlphaGo Zero)



AlphaGo vs. Alpha(Go) Zero ELO comparison





AlphaStar

- Game playing AI for StarCraft-II
- Multi-player game – played by multiple agents using MARL (multi-agent RL)
- Partial Observability: each agent can see the game state (other players) only within its camera's FOV
- Imperfect information: only see opponent pieces within range of own units
- Large action space, simultaneous moves by multiple players at same time
- Counter strategies developed by human players

AlphaStar

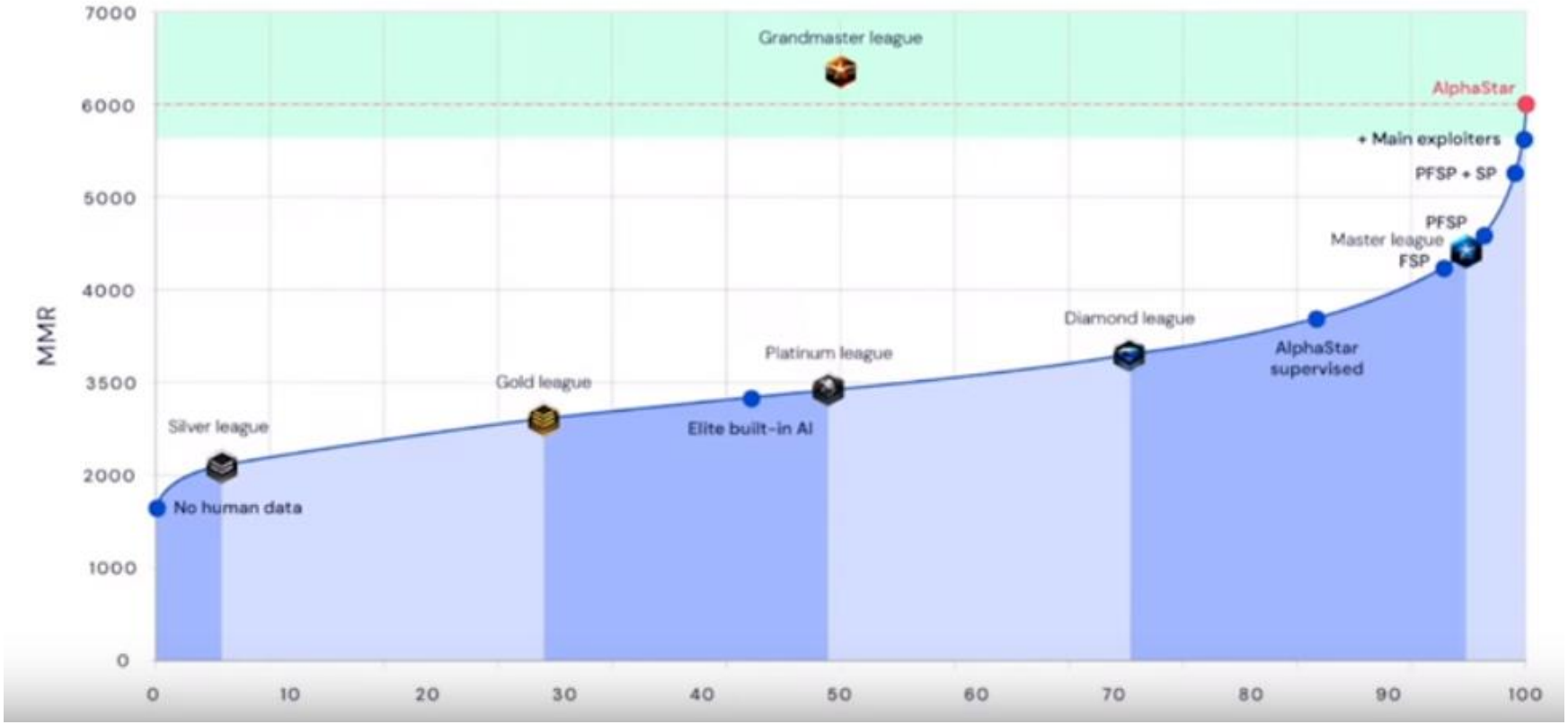
- Main algorithm: AlphaZero + league play
- League play: self-play between multiple players (two sides)
  - Teammates (agents) versus opponents (competitors)
- M agents, N competitors
- Agents learn a policy (policy update via A2C RL)
- Competitors have fixed policy
- Best agents morph into competitors periodically
  - Ensures that agents remember to playing against some of the best opponents (past selves)

AlphaStar

- Prioritized Fictitious Self-Play
  - Matchmaking between agents and competitors done to match expert level of both sides
  - Exploiters (type of competitors) identify weaknesses in agents so that agents can discover and overcome those weaknesses (via self-play)
- Uses supervised learning from human games to speed-up learning (like AlphaGo)

https://youtu.be/xP7LwZxq0ss

AlphaStar Performance

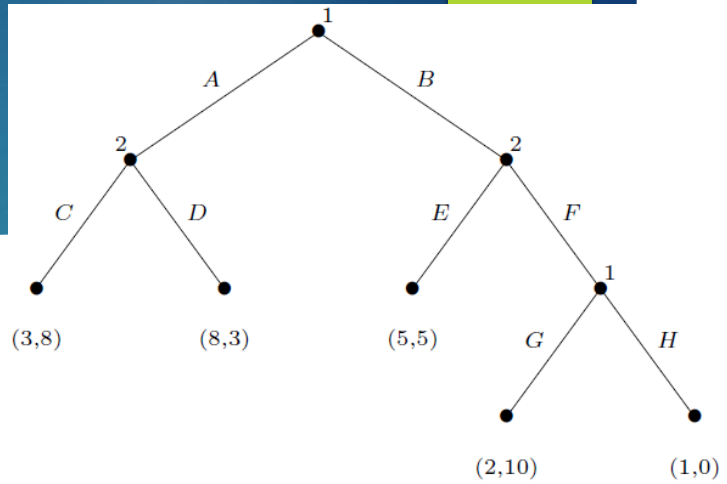


Resources

- Papers:
  - Silver, D., Huang, A., Maddison, C. *et al.* Mastering the game of Go with deep neural networks and tree search. *Nature* **529,** 484–489 (2016).
  - Silver, David, Thomas Hubert, Julian Schrittwieser, Ioannis Antonoglou, Matthew Lai, Arthur Guez, Marc Lanctot et al. "Mastering chess and shogi by self-play with a general reinforcement learning algorithm." *arXiv preprint arXiv:1712.01815* (2017).
  - Vinyals, O., Babuschkin, I., Czarnecki, W.M. *et al.* Grandmaster level in StarCraft II using multi-agent reinforcement learning. *Nature* **575,** 350–354 (2019).
- Slides:
  - Alpha Go: https://www.slideshare.net/ShaneSeungwhanMoon/how-alphago-works
  - Alpha (Go) Zero: https://www.slideshare.net/KarelHa1/alphazero

Sequential Games

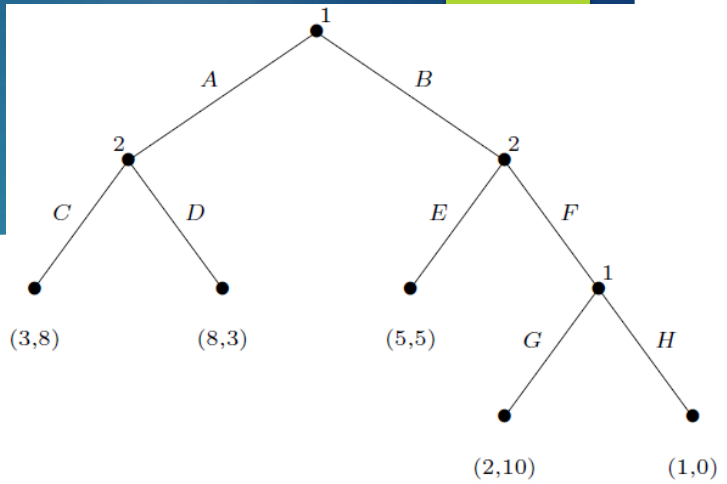
Example Sequential Game



- Pure strategies of player 1:
- (A, G) (A, H) (B,G) (B, H)
- Pure strategies for player 2:
- (C, E) (C, F) (D, E) (D, F)
- Note: when player 1 chooses A, its own choice of G vs. H has no consequence
- With above pure strategies, we can write induced normal form of the game

	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3, 8	3, 8	8, 3	8, 3
(A,H)	3, 8	3, 8	8, 3	8, 3
(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

Example Sequential Game



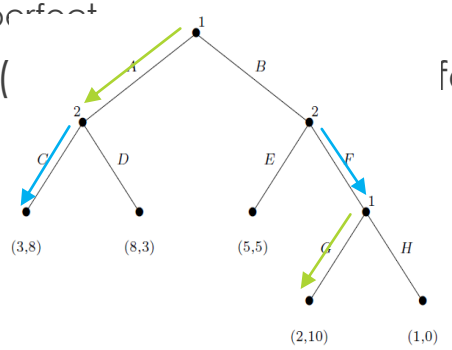
- Temporal structure of extensive form cannot be captured in induced normal form
- Redundancy in normal form
  - 16 outcomes (with duplicates) in normal form vs. 5 outcomes in extensive form
- Converting from extensive to induced normal form can lead to exponential blowup in number of outcomes

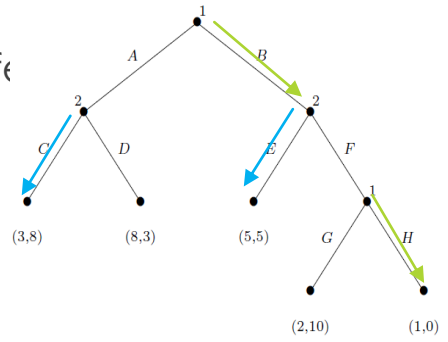
	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3, 8	3, 8	8, 3	8, 3
(A,H)	3, 8	3, 8	8, 3	8, 3
(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

Backward Induction

- Offers an easy, systematic way to check if a NE is subgame perfect
- Idea:
  - Given a NE strategy start at bottommost subtree which has terminal nodes (with joint utilities)
  - Go upwards towards root - at each decision node (branch) check if the NE strategy is the best response for the player moving at that node
    - Yes: move (copy) the utility to the player one level up the tree
    - No: stop; NE is not subgame perfect
- Check (A, G) (C, F) and (B, H) (

Backward induction can be implemented as depth first traversal with recursion





Connection between Sequential Games and Game-playing AI

- AlphaStar selects actions using Nash equilibrium in sequential games
- Nash equilibrium analysis of sequential games is useful in imperfect information games:
  - StarCraft-II
  - Texas Hold 'em Poker (DeepStack)
  - Hanabi
- Uses a technique called counter factual regret (CFR) minimization instead of Nash equilibrium

Summary and Conclusions

- Game theory gives formal, mathematical methods for taking actions by players in a game with guaranteed solutions (like Nash equilibrium outcome)
  - But for small settings – mainly 2 players, with few actions
  - Developed in mathematics and economics, does not use AI techniques inherently for solutions
- Deep RL and AI-based search techniques give methods to solve larger games with many actions
  - But mainly heuristics-based solutions
  - Learns via trial and error against humans or self-play
- Combination of game theory and RL can be used to solve large games with imperfect information with guaranteed solutions

Questions

- Dr. Raj Dasgupta
- Naval Research Laboratory, Washington D. C.
- Email: raj.Dasgupta@nrl.navy.mil

- 8 weeks of research internships high school students to participate at Department of Navy Laboratories including NRL
- Major criteria:
  - Completed Grade 9
  - Graduating seniors can apply
  - Must be 16 years or older at time of application
  - U S Citizenship (for NRL)
- NRL research areas: AI/ML, computer science, engineering, space sciences, radar, remote sensing, plasma physics, chemistry, bio-sciences, material sciences, acoustics
- Application deadline: November 30, 2020
- Website: https://seap.asee.org/

Backup Slides

MORE INFORMATION

Sequential Game Overview

- Recall: Normal form game – players move (select actions) simultaneously
- Sequential games - remove the requirement that players move (select actions) simultaneously
- Players take turns to make moves (e.g., chess, tic-tac-toe, checkers)
- Also called extensive form game
- Normal form game – matrix representation
- Extensive form game – tree representation
- Each extensive form game can be converted to a corresponding normal form game – called game's induced normal form

Sequential Game Overview

- All results related to Nash equilibrium and its calculations in normal form games also hold for extensive form games
- Extensive form game allows calculation of subgame perfect equilibrium
  - Allows to speed up Nash equilibrium calculation

Sequential Games with Perfect Information

- Perfect information: Assume that every player can see the actions and payoffs (at joint outcomes) of every other player
- Assume games are finite (finish in finite number of moves)

Definition 5.1.1 (Perfect-information game) *A (finite) perfect-information game (in extensive form) is a tuple* $G = (N, A, H, Z, \chi, \rho, \sigma, u)$*, where:*

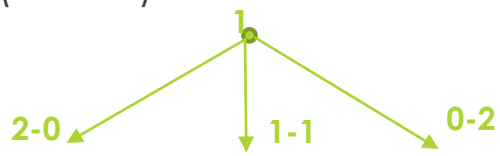
- N *is a set of* n *players;*
- A *is a (single) set of actions;*
- H *is a set of nonterminal choice nodes;*
- Z *is a set of terminal nodes, disjoint from* H*;*
- $\chi : H \mapsto 2^A$ *is the action function, which assigns to each choice node a set of possible actions;*
- $\rho : H \mapsto N$ *is the player function, which assigns to each nonterminal node a player* $i \in N$ *who chooses an action at that node;*
- $\sigma : H \times A \mapsto H \cup Z$ *is the successor function, which maps a choice node and an action to a new choice node or terminal node such that for all* $h_1, h_2 \in H$ *and* $a_1, a_2 \in A$*, if* $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ *then* $h_1 = h_2$ *and* $a_1 = a_2$*; and*
- $u = (u_1, \ldots, u_n)$*, where* $u_i : Z \mapsto \mathbb{R}$ *is a real-valued utility function for player* i *on the terminal nodes* Z*.*

Sequential Game Example: Sharing Game

- Brother and sister sharing two presents
- Presents cannot be divided
- Both siblings value presents equally, and additively
- Brother moves first, will pick one of three possible splits
  - Brother keeps both
  - Sister keeps both
  - Brother keeps one, sister keeps one

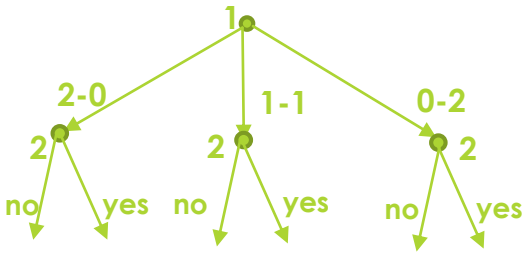
Sequential Game Example: Sharing Game

- Brother and sister sharing two presents
- Presents cannot be divided
- Both siblings value presents equally, and additively
- Brother moves first, will pick one of three possible splits (moves)
  - Brother keeps both
  - Sister keeps both
  - Brother keeps one, sister keeps one
- Start node labeled with player id 1
- Each move is labeled with the corresponding split



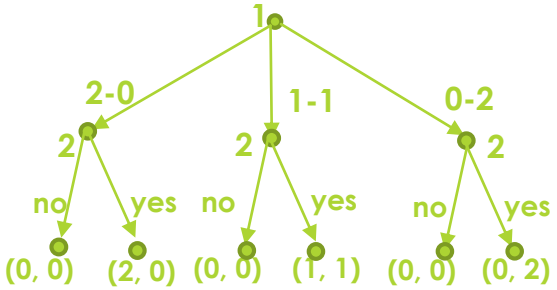
Sequential Game Example: Sharing Game

- Sister moves next, picks one of two moves
  - Accept (yes) brother's split (move)
  - Reject (no) brother's split (move)



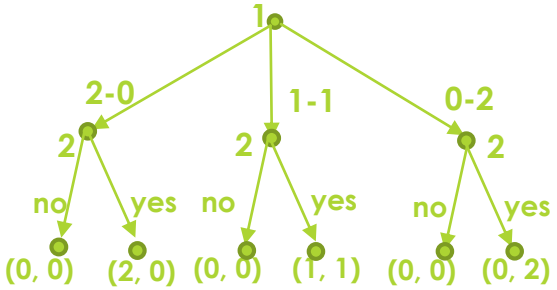
Sequential Game Example: Sharing Game

- Sister moves next, picks one of two moves
  - Accept (yes) brother's split (move)
  - Reject (no) brother's split (move)
- Game ends after sister's move
- Leaf nodes are labeled with outcome of game – joint utility of players from each of the outcomes



Sequential Game Example: Sharing Game

- Pure strategies of player 1: { 2-0, 1-1, 0-2}
- Pure strategies of player 2 – all possible combinations of moves, for each of player 1's moves
  - {(2-0-no), (2-0, yes), (1-1-no), (1-1, yes), (0-2, no), (0-2, yes)}



Extensive Form Game with Perfect Information: Pure Strategy NE

Theorem 5.1.3 *Every (finite) perfect-information game in extensive form has a pure-strategy Nash equilibrium.*

- Why do perfect information extensive form games always have a pure strategy NE?

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Theorem 5.1.3 *Every (finite) perfect-information game in extensive form has a pure-strategy Nash equilibrium.*

- Why do perfect information extensive form games always have a pure strategy NE?
- Why did we have mixed strategies in normal form game?
- Because each player did not know which action other player would select, and tried to guess by randomizing over other players possible actions
- In extensive form each player can observe other player's moves as players move sequentially… no need to guess or randomize

Sub-game Perfect Equilibria

Example Sequential Game: Induced Normal Form

	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3, 8	3, 8	8, 3	8, 3
(A,H)	3, 8	3, 8	8, 3	8, 3
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What are the pure strategy NE in the above game?

Example Sequential Game: Induced Normal Form

	(C,E)	(C,F)	(D,E)	(D,F)
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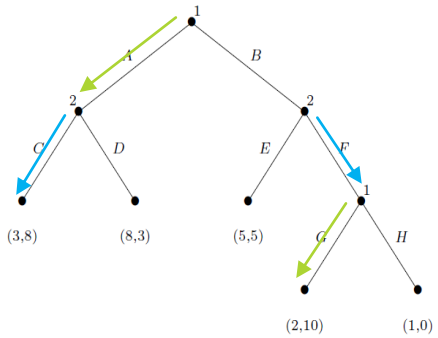
(A,H)	3, 8	3, 8	8, 3	8, 3
(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	3, 8	**3, 8**	8, 3	8, 3
(A, H)	3, 8	**3, 8**	8, 3	8, 3
(B, G)	5, 5	2, 10	5, 5	2, 10
(B, H)	**5, 5**	1, 0	5, 5	1, 0

What are the pure strategy NE in the above game?

Subgame Perfect Equilibria

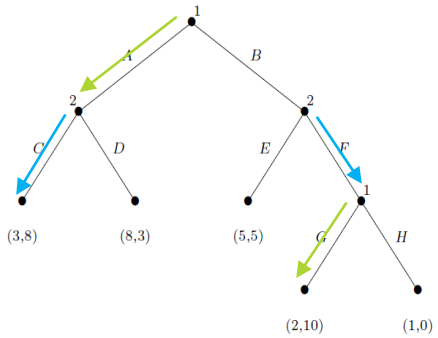
	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	3, 8	**3, 8**	8, 3	8, 3
(A, H)	3, 8	**3, 8**	8, 3	8, 3
(B, G)	5, 5	2, 10	5, 5	2, 10
(B, H)	**5, 5**	1, 0	5, 5	1, 0



- Are all pure strategy NE possible in extensive form
- Consider (A, G) (C, F)
- When P1 plays H instead of G can he improve utility?
- When P2 plays E instead of F can he improve utility?
- When P1 plays B instead of A can he improve utility?
  - If P1 plays B, due to P2's move F (remember we assume other player does not deviate) P1 will end up getting either utility of 2 or 1
  - If P1 plays A, due to P2's move C (P2 does not deviate), P1 will get 3
  - So, best for P1 not to deviate to B from A
- When P2 plays D instead of C can he improve utility?

Subgame Perfect Equilibria

	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	3, 8	**3, 8**	8, 3	8, 3
(A, H)	3, 8	**3, 8**	8, 3	8, 3
(B, G)	5, 5	2, 10	5, 5	2, 10
(B, H)	**5, 5**	1, 0	5, 5	1, 0



- Are all pure strategy NE possible in extensive form
- Consider (A, G) (C, F)
- When P1 plays H instead of G can he improve utility?
- When P2 plays E instead of F can he improve utility?
- When P1 plays B instead of A can he improve utility?

No

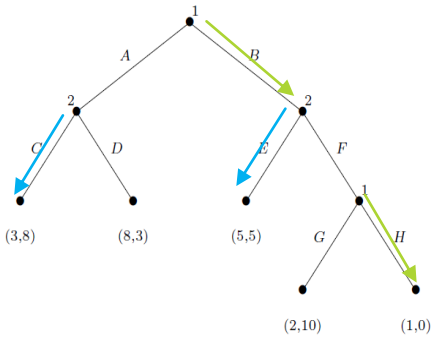
  - If P1 plays B, due to P2's move F (remember we assume other player does not deviate) P1 will end up getting either utility of 2 or 1
  - If P1 plays A, due to P2's move C (P2 does not deviate), P1 will get 3
  - So, best for P1 not to deviate to B from A

No incentive to deviate: (A, G) (C, F) is NE

When P2 plays D instead of C can he improve utility?

Subgame Perfect Equilibria

	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	3, 8	**3, 8**	8, 3	8, 3
(A, H)	3, 8	**3, 8**	8, 3	8, 3
(B, G)	5, 5	2, 10	5, 5	2, 10
(B, H)	**5, 5**	1, 0	5, 5	1, 0



- Are all pure strategy NE possible in extensive form
- Consider (B, H) (C, E)
- When P1 plays G instead of H can he improve utility?
- When P2 plays F instead of E can he improve utility?
  - P2 could get utility 10 if P1 played G
  - But P1 gives "threat" to P2: if you play F, I will punish you by playing H (and giving you 0 utility)
  - But, is this threat credible?

Subgame Perfect Equilibria

	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	3, 8	**3, 8**	8, 3	8, 3
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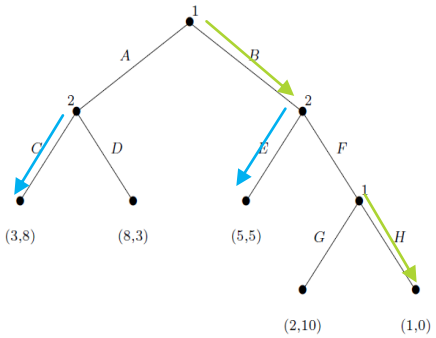

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  - But, is this threat credible?
  - P2 sees that if P1 plays H, P1 will get utility of 1 vs. getting utility of 2 by playing G
  - So, P2 reasons that the threat is not credible

Subgame Perfect Equilibria

	(C, E)	(C, F)	(D, E)	(D, F)
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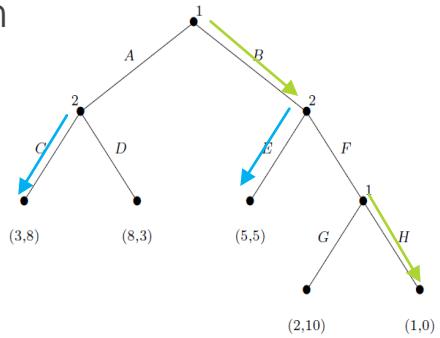

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  - P2 sees that if P1 plays H, P1 will get utility of 1 vs. getting utility of 2 by playing G
  - So, P2 reasons that the threat is not credible…which means P1 will play G, and P2 will get utility of 10…incentive to deviate for P2 from E (util. 5) to F (util. 10)

P2 has incentive to deviate:
(B, H) (C, E) is not NE

Backward Induction

ALGORITHM DETAILS

Subgame Perfect Equilibria

	(C, E)	(C, F)	(D, E)	(D, F)
(A, G)	3, 8	**3, 8**	8, 3	8, 3
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(B, G)	5, 5	2, 10	5, 5	2, 10
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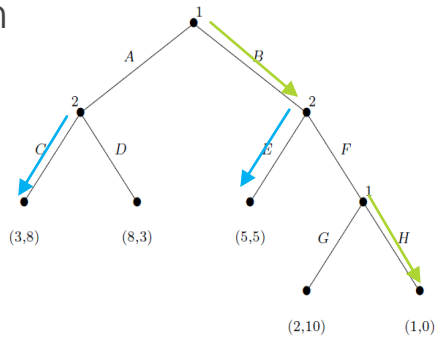

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P2 has incentive to deviate:
(B, H) (C, E) is not NE

In such scenarios, we say the NE (of induced normal form) is not subgame perfect

Backward Induction

- Start with equilibria at bottommost level of subtrees (in subgames)
- Assume that those equilibria will be played as we move upwards to larger sub-trees
- Can be implemented as depth first traversal of game tree
  - Good! … because extensive form (tree representation) is almost always smaller (exponentially) than induced normal form

Recall: Converting from extensive to induced normal form can lead to exponential blowup in number of outcomes

function BACKWARDINDUCTION (node h) **returns** $u(h)$
if $h \in Z$ **then**
  return $u(h)$ // h is a terminal node
$best\_util \leftarrow -\infty$
forall $a \in \chi(h)$ **do**
  $util_at_child \leftarrow$BACKWARDINDUCTION$(\sigma(h, a))$
  if $util\_at\_child_{\rho(h)} > best\_util_{\rho(h)}$ **then**
    $best\_util \leftarrow util\_at\_child$
return $best\_util$

Backward Induction

Symbols (from Def 5.1.1)
- Z: terminal nodes
- $\chi(h)$: action set at node h
- $\sigma(h,a)$: next node reached by playing action a at node h
- $\rho(h)$: player at node h

```
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if h ∈ Z then
    return u(h)                                   // h is a terminal node
best_util ← −∞
forall a ∈ χ(h) do
    util_at_child ← BACKWARDINDUCTION(σ(h, a))
    if util_at_child_ρ(h) > best_util_ρ(h) then
        best_util ← util_at_child
return best_util
```

- $util\_at\_child_{\rho(h)}$ is the utility received by player ρ(h) - element of joint utility vector corresponding to player ρ(h)'s utility

If (utility returned from subtree for player ρ(h) > best utility for player ρ(h) player till now) then
update best utility for ρ(h)

Backward Induction

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- Returns *u(h)* – utility 'label' for each node *h* (not the equilibrium strategy)
- Player $\rho(h)$ can then use this label to find its best (equilibrium) action at node h given by arg max $_{a\_i \, \varepsilon \, \chi(h)}$ $u_i(\sigma(h, a_i))$

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Utility label of the next node reached by playing a_i at h

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Utility label of the next node reached by playing a_i at h

Select action that will maximize utility in subgame (recursively)

Backward Induction

- Backward induction is an instance of depth first traversal
- Recall, depth first traversal has time complexity $O(b^m)$
  - b: branch factor
  - m: maximum depth
- For chess, b=10, m =150…backward induction still takes order of 10^{150} steps…still a problem