Separation of variables and chain rule

Separation of variables works because we can use the chain rule in reverse.

Chain rule

Chain rule is used when we are differentiating a function of a function.

For example:

$$\mathrm{z} = \mathrm{z}(\mathrm{y}(\mathrm{t})),$$

would differentiate to:

$$\frac{dz}{d\mathrm{t}} = \frac{dz}{dy}\frac{dy}{d\mathrm{t}},$$

as a result of chain rule.

Separation of variables

When we are integrating, we can sometimes use chain rule in reverse to make things easier. This is the aim with separation of variables.

For example, let's say we would like to solve:

$$\frac{dy}{dt} = 2y,$$

which is the example from the differential equation lecture.

We can rearrange this to get:

$$\frac{dy}{dt}\frac{1}{\mathrm{y}} = 2,$$

which we want to integrate with respect to t.

Now, looking at the left hand side only we can start comparing it to the chain rule:

$$\frac{dy}{dt}\frac{1}{y} = \frac{dy}{dt}\frac{dz}{dy} = \frac{dz}{dt},$$

where we have said $\frac{dz}{dy} = \frac{1}{y}$. So here we have used a substitution to make things easier and due to this substitution we can say that $z = ln(y)$.

Now we have the left hand side, we can put it all together:

$$\int \frac{1}{y}\frac{dy}{dt}dt = \int 2\,dt,$$

$$\ln(\mathrm{y}) = 2\mathrm{t} + \mathrm{C}.$$