Matrices

K. A. M. Gaythorpe

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Eigenvalues and eigenvectors

You have already come across a quick method for calculating the eigenvalues of a 2×2 matrix. Now we look at this in a more general way, still using a 2×2 matrix for examples, that can be extended to any square matrix.

Eigenvalues and eigenvectors satisfy the following:

$$\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$$

where $\mathbf{M}$ is our square matrix, λ is an eigenvalue and $\mathbf{v}$ is the related eigenvector. We can rearrange this equation so we may calculate our eigenvalues and eigenvectors:

$$(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}.$$

This has a non-zero solution if the determinant of $(\mathbf{M} - \lambda\mathbf{I})$ is zero.

Calculating the eigenvalues of a matrix

Let us take $\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ as our 2×2 matrix. For any $n \times n$ matrix we would expect n eigenvalues, so in this case we expect to find 2 eigenvalues. Therefore, in order to find our eigenvalues, we take the determinant of $(\mathbf{M} - \lambda\mathbf{I})$ and set this to zero. The determinant of a 2×2 matrix is relatively quick to write down; however, if you have a larger matrix, but still want to work symbolically, it may be worth using a service such as Wolfram to calculate your determinant. In our case, the determinant is the following:

$$|\mathbf{M} - \lambda\mathbf{I}| = \begin{vmatrix} M_{11} - \lambda & M_{12} \\ M_{21} & M_{22} - \lambda \end{vmatrix} = (M_{11} - \lambda)(M_{22} - \lambda) - M_{12}M_{21}.$$

If we set this equal to zero, we can rearrange and solve for our eigenvalues:

$$\lambda = \frac{(M_{11} + M_{22}) \pm \sqrt{(M_{11} + M_{22})^2 - 4(M_{11}M_{22} - M_{12}M_{21})}}{2}.$$

Therefore, if our matrix $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$, our eigenvalues are $\lambda = \dfrac{(2+3) \pm \sqrt{(2+3)^2 - 4(2 \times 3 - 0 \times 1)}}{2}$ $= \dfrac{5 \pm 1}{2}$ ie. either 3 or 2.

Calculating the eigenvectors of a matrix

Now we have our eigenvalues, we calculate the corresponding eigenvectors. The largest eigenvalue is called the **leading eigenvalue**and the associated eigenvector is called the **leading eigenvector**. We calculate these by substituting each of our eigenvalues back into the equation $(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ and we will have one eigenvector for every eigenvalue. A general example of this is as follows:

$$
\begin{aligned}
&(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \\
\implies &\begin{pmatrix} M_{11} - \lambda & M_{12} \\ M_{21} & M_{22} - \lambda \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Therefore, we can write the above as a pair of simultaneous equations and solve up to a scalar multiple:

$$
\begin{aligned}
(M_{11} - \lambda)v_1 + M_{12}v_2 = 0 \\
M_{21}v_1 + (M_{22} - \lambda)v_2 = 0
\end{aligned}
$$

Let us take our numerical example from before and calculate the eigenvector for the leading eigenvalue, 3:

$$
(\mathbf{M} - \lambda\mathbf{I}) = \begin{pmatrix} 2-3 & 0 \\ 1 & 3-3 \end{pmatrix} \mathbf{v} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{v} = \mathbf{0}.
$$

The vector $\mathbf{v}$ has two elements, v_1 and v_2, and the only way the above equation will make sense is if $v_1 = 0$. However, we do not have any information on v_2 so we set it to a convenient value, 1. As such, the leading eigenvector, is $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We may repeat these steps to calculate the eigenvector corresponding to the eigenvalue equal to 2. In this case, $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Transforming a vector into a unit vector

A unit vector has **length** =1, where length is calculated as the **square root of the sum of the squared elements of the vector**. For example, the length of the vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is $\sqrt{4^2 + 3^2} = \sqrt{25} = 5$. In order to transform a vector to a unit vector, we divide each element by the length of the whole vector. For example, take the eigenvector in the previous example: $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The length of $\mathbf{v} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Therefore our unit eigenvector is $\hat{\mathbf{v}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$, where the hat denotes unit vector of $\mathbf{v}$.

Transforming a vector into a sum unit vector

This is very similar to the previous method; however, we divide by the **sum of the vector elements**. This means that the sum of the elements in the sum unit vector equal one. For example, let $\mathbf{v}$= $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Then the sum of the elements are $2 + 1 = 3$ and our sum unit vector is $\tilde{v} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$.