Why does the definite integral give the area under a graph?

Claim (as per lecture)

If $f(x)$ is a continuous function that is *non-negative* for $a \leq x \leq b$, then the area of the region bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

(where $F(x)$ is the indefinite integral of $f(x)$).

Justification (note this is just for interest)

Consider a continuous non-negative $f(x)$ on the interval $a \leq x \leq b$. Then for each value of x in the interval, let $A(x)$ denote the area under the graph of $y = f(x)$ from $x = a$ to $x = x$ (i.e. up to an unspecified general value of x), as shown below, and call this the "area so far function". Clearly $A(a) = 0$, while $A(b)$ is the area under the graph of $y = f(x)$ from $x = a$ to $x = b$.

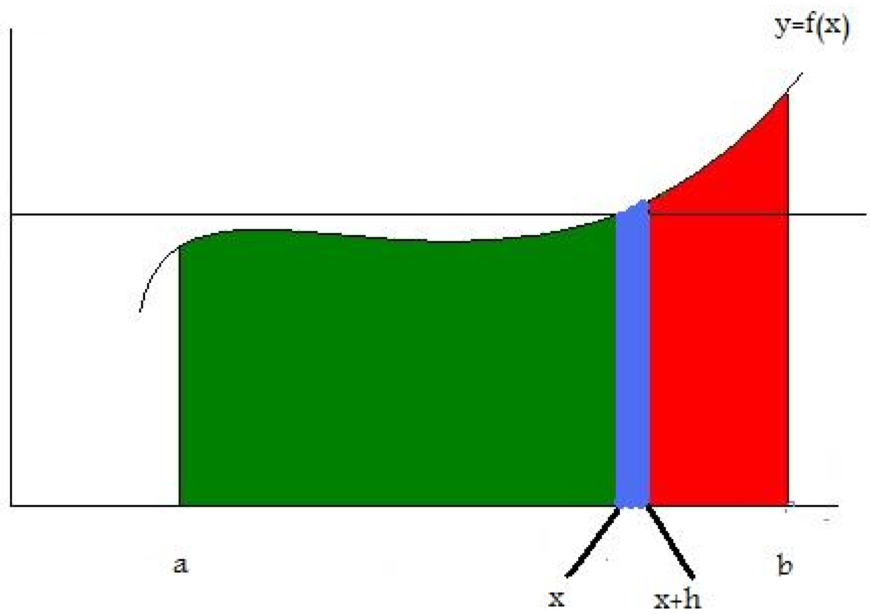

Figure 1: The area so far function $A(x)$ (the green region)

On the diagram above, the vertical line through $x + h$ is also drawn, where h is a small positive number. Consider the (blue) region under the graph between x and $x + h$. By the definition of the area so far function, this is $A(x+h) - A(x)$. However, if h is small, the area is well-approximated by that of the rectangle with base h and height $f(x)$. Therefore

$$A(x+h) - A(x) \approx hf(x).$$

Now dividing both sides by h and taking the limit as $h \to 0$, the approximation becomes ever-better, and in fact

$$\lim_{h\to 0} \left(\frac{A(x+h) - A(x)}{h} \right) = f(x),$$

which recalling the definition of a derivative means

$$\frac{dA}{dx} = f(x). \tag{1}$$

Now if $F(x)$ is *any* integral of $f(x)$, then the definition of the definite integral states

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Since $A(x)$ is one such $F(x)$ (this is the meaning of Equation (1)), it must be that

$$\begin{aligned} \int_a^b f(x)\, dx &= A(b) - A(a), \\ &= A(b) \quad (\text{since } A(a) = 0). \end{aligned}$$

Since $A(b)$ is the area under the graph of $y = f(x)$ from $x = a$ to $x = b$, this concludes the argument.