Bernoulli's problem $x^y = y^x$ and Maple

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Abstract. We study the problem of $x^y = y^x$, first proposed by Daniel Bernoulli in 1728. We present MAPLE's parametric solution and a solution using the Lambert W function. This leads us to consider an implementation in Maple of new simplifications of the Lambert W function. The method uses a mixture of exact and floating-point computation.

Keywords: Lambert W function · Simplification · Algorithms.

1 Bernoulli's problem

On 29 June 1728, Daniel Bernoulli wrote to Christian Goldbach. Bernoulli was working at the new St Petersburg Academy of Sciences[1], and Goldbach had recently left the same Academy for Moscow. In his letter, Bernoulli considered the equation

$$x^y = y^x \ . \tag{1}$$

Obviously, it has the trivial solution $x = y$, but Bernoulli wrote that he had found the non-trivial solution $x = 2, y = 4$ (and, of course, $y = 2, x = 4$), and further that he had proved that there are no other integer solutions [1].
Goldbach wrote to Bernoulli on 31 January 1729 giving a solution to (1) in parametric form. Goldbach's expressions can be obtained from MAPLE's `solve` command, using the syntax (see MAPLE help for `solve/parametrized`)

```
> solve(x^y=y^x,[x(t),y(t)])
```

$$\left[\left[x = \mathrm{e}^{-\frac{\ln\left(\frac{1}{t}\right)}{t-1}}, y = t\,\mathrm{e}^{-\frac{\ln\left(\frac{1}{t}\right)}{t-1}}\right]\right] \tag{2}$$

```
> simplify(%)
```

$$\left[\left[x = \left(\frac{1}{t}\right)^{-\frac{1}{t-1}}, y = t\left(\frac{1}{t}\right)^{-\frac{1}{t-1}}\right]\right] \tag{3}$$

[1] The Saint Petersburg Academy of Sciences opened its doors in 1725, shortly after the death of its founder, Peter the Great.

Solving using a parameter is a useful MAPLE command that deserves to be more widely known. Goldbach would certainly have been assuming $t > 0$, and so the solution becomes

```
> simplify(%) assuming t > 0
```

$$\left[\left[x = t^{\frac{1}{t-1}}, y = t^{\frac{t}{t-1}}\right]\right] \tag{4}$$

Leonard Euler was in St Petersburg at that time, also working at the Academy, and 20 years later in 1748 (having moved to Berlin) he included this problem and its solution in his famous textbook on analysis [3].
A more conventional use of solve is to ask for y as a function of x. MAPLE returns the expression (using the `alias` command to abbreviate `LambertW` to W)

```
> alias(W= LambertW):
> solve(x^y = y^x,y)
```

$$y = -\frac{x}{\ln x} W\left(-\frac{\ln x}{x}\right) \ . \tag{5}$$

We recall that the Lambert W function obeys

$$W_k(z)\exp\left(W_k(z)\right) = z \ , \tag{6}$$

where $k \in \mathbb{Z}$ is the branch label [2]. Its real values are plotted in Figure 1.

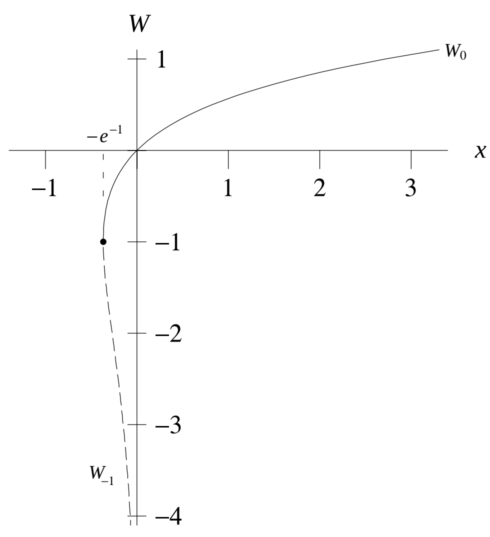

Fig. 1. The real values of W. The solid line is the principal branch $k = 0$ and the dashed line is the $k = -1$ branch.

1.1 Comparing solutions

Having the two solutions (4) and (5) to Bernoulli's problem invites a comparison between them. Substituting various values for t into (4) produces some typical solutions as (x, y) pairs; see Table 1.

t	x	y
2	2	4
3	$\sqrt{3}$	$\sqrt{27}$
7	$7^{\frac{1}{6}}$	$7^{\frac{7}{6}}$
3/2	9/4	27/8
4/3	64/27	256/81
5/4	625/256	3125/1024

Table 1. Goldbach's solution evaluated for some particular values of t. Bernoulli's own solution is the first entry.

Substituting one member of an (x, y) pair into (5) should produce the other. We can start with Bernoulli's solution $(2, 4)$. Substituting $x = 2$ in (5), we obtain

$$y = -\frac{2}{\ln 2} W\left(-\frac{\ln 2}{2}\right) .$$

MAPLE can simplify this, and returns 2, the trivial solution, because MAPLE assumes, by default, branch $k = 0$. To obtain the non-trivial solution, we must ask for $k = -1$.

$$y = -\frac{2}{\ln 2} W_{-1}\left(-\frac{\ln 2}{2}\right) = 4 , \tag{7}$$

which simplification MAPLE can also do. What happens if we start with $x = 4$?

$$y = -\frac{4}{\ln 4} W\left(-\frac{\ln 4}{4}\right) = -\frac{2}{\ln 2} W\left(-\frac{\ln 2}{2}\right) = 2 .$$

Notice that whether one starts with $x = 2$ or $x = 4$, one arrives at the same expression in terms of W. This underlines the importance of the branch index in the analysis: *different data* can lead to the *same argument* for W, meaning that *only* the branch index can differentiate cases.
Now we try another entry from Table 1 and substitute $x = \sqrt{3}$ in (5) to obtain

$$y = -\frac{\sqrt{3}}{\ln\sqrt{3}} W_k\left(-\frac{\ln\sqrt{3}}{\sqrt{3}}\right) = -\frac{2\sqrt{3}}{\ln 3} W_k\left(-\frac{1}{6}\sqrt{3}\ln 3\right) .$$

MAPLE cannot simplify this expression, for either branch, but we can approximate it using `evalf`.

$$y = -\frac{2\sqrt{3}}{\ln 3} W_0 \left(-\frac{1}{6}\sqrt{3} \ln 3 \right) \approx 1.732050812 \ .$$

The MAPLE `identify` command[2] gives this as $\sqrt{3}$, and it also identifies the $k = -1$ branch as giving $\sqrt{27}$. We are thus led to conjecture the simplifications

$$W_0 \left(-\frac{1}{6}\sqrt{3}\ \ln 3 \right) = -\tfrac{1}{2} \ln 3 \ , \tag{8}$$

$$W_{-1} \left(-\frac{1}{6}\sqrt{3}\ \ln 3 \right) = -\tfrac{3}{2} \ln 3 \ , \tag{9}$$

and we would like MAPLE to incorporate these and similar simplifications into its library.

2 A class of simplifications

Guided by the above observations, we develop a class of simplifications for MAPLE's implementation of Lambert W. We first generalize Bernoulli's problem to a problem which has appeared, in various disguises, in a number of mathematical contests. Solve for x

$$x^a b^x = c \ , \tag{10}$$

with the parameters $\{a, b, c\}$ chosen to give x a suitably simple form. The equation has the real solution

$$x = \frac{a}{\ln b} W_k \left(\frac{1}{a}\, c^{1/a} \ln b \right) \ , \tag{11}$$

with Bernoulli's problem corresponding to $a = y$, $b = 1/y$, and $c = 1$. Our ambition, then, is to detect cases in which (11) can be simplified. Having observed (7), (8) and (9), we aim to decide whether a rational number r exists such that for some branch k

$$W_k \left(\frac{1}{a}\, c^{1/a} \ln b \right) = r \ln b \ . \tag{12}$$

We limit the domains of the parameters so that the problem is tractable. We restrict a to be an integer n, and assume $c, b, r \in \mathbb{Q}$. This is sufficient to cover solutions to (1) obtained by substituting rational values of t into Goldbach's expressions.

We focus, therefore, on deciding whether r exists, and on calculating it. A first strategy could be to return to the `identify` command we used successfully above. By rewriting (12) as

$$r = \frac{W_k \left(\frac{1}{n}\, c^{1/n} \ln b \right)}{\ln b} \ ,$$

[2] Another interesting, under-appreciated, feature.

we can evaluate the right side to a floating-point number, and then use `identify` to find the rational number. This conceptually simple approach has, however, practical difficulties. First, the correct identification of a rational number is sensitive to the precision of the floating-point evaluation. With multi-digit fractions, `identify` can struggle, and MAPLE users can be relied upon to pose, sooner or later[3], problems containing large numbers. For example, consider

```
> evalf(1234567/678912)
```

$$1.818449225$$

```
> identify(%)
```

$$\frac{\sqrt{3}}{5} + \frac{5\,\zeta(5)}{8} + \frac{3\,\ln(3)}{4}$$

```
> evalf[15](1234567/678912)
```

$$1.81844922464178$$

```
> identify(%)
```

$$1.81844922464178$$

```
> evalf[20](1234567/678912)
```

$$1.8184492246417797888$$

```
> identify(%)
```

$$\frac{1234567}{678912}$$

A second problem concerns rounding errors during evaluation [4], which can mean that the number given to `identify` is not accurate.

```
> evalf( W(-1,-189/256*ln(4/3)*sqrt(3))/ln(4/3) );
```

$$-3.500000258$$

```
> identify(%)
```

$$-3.500000258$$

```
> evalf[15]( W(-1,-189/256*ln(4/3)*sqrt(3))/ln(4/3) );
```

$$-3.50000000000461$$

```
> identify(%)
```

$$-3.50000000000461$$

[3] and usually sooner!

The human will exclaim "But it's obvious", and ignore the erroneous digits.

$$W_{-1}\left(-\frac{189}{256}\sqrt{3}\,\ln\frac{4}{3}\right) = -\frac{7}{2}\ln\frac{4}{3}\ . \tag{13}$$

Again, as the numbers grow larger identification will become more difficult. It is possible to modify the approach so that we need only identify an integer. As a step to an improved algorithm, we consider a numerical example: solve for x

$$x^3\left(\frac{9}{4}\right)^x = \frac{3}{16}\ . \tag{14}$$

The solution is $x = \frac{1}{2}$, and this works because $(9/4)^x = (9/4)^{1/2} = 3/2$. Note that we chose $9/4$ because it appears in Table 1, and rational numbers containing pure powers must be expected in simplification problems.

Now we can see that since (12) is equivalent to solving the problem

$$x^n b^x = c\ , \tag{15}$$

and we want all quantities in this equation to be rational, then b^x will have to be rational. Thus we must be able to extract, if necessary, a root from b. Suppose b can be written $b = B^p$, with $B \in \mathbb{Q}$ and $p \in \mathbb{Z}$, with p maximal. Then we can write

$$x^n b^x = x^n B^{px} = c\ ,$$

and all terms will be rational, provided $px \in \mathbb{Z}$. This means we can change our search from finding a rational x to finding an integer px.

The first step in our algorithm, therefore, must be to find p. This can be done using MAPLE's `iperfpow` function, which takes a positive integer m, and computes integers q and j, such that $m = q^j$. We extend its functionality in two ways. First, `iperfpow` does not find the maximal power. Thus, `iperfpow(256)` returns 16^2, whereas we need $256 = 2^8$. Secondly, `iperfpow` accepts only integers, whereas we need rational arguments. The calculation is then

$$X = px = \frac{pn}{\ln b}W_k\left(\frac{1}{n}\,c^{1/n}\ln b\right) = \frac{n}{\ln B}W_k\left(\frac{1}{n}\,c^{1/n}\,p\ln B\right)\ . \tag{16}$$

The right-hand side is evaluated in floating point to give a *candidate* integer for X. We return to the numerical example in (13). We convert it to the standard form just given:

$$\begin{aligned} X &= \frac{2}{\ln(4/3)}W_{-1}\left(\frac{1}{2}\left(\frac{107163}{16384}\right)^{1/2}\ln\frac{4}{3}\right) \\ &\approx -7.00000000000922\ . \end{aligned} \tag{17}$$

From this we test the candidate $X = -7$ using the exact computation

$$(-7)^2\left(\frac{4}{3}\right)^{-7} = \frac{107163}{16384}\ .$$

With this exact verification, we can accept the simplification proposed in (13).

2.1 Limitations to simplification

A careful study of the concept and the definition of simplification has been made in [5]. Pragmatically, we are following one particular idea from that paper, which is an ordered list of sub-expressions. For the present paper, simplification (12) consists of expressing an instance of W in terms of simpler functions appearing lower in the ordered list.

In the previous section, we restricted ourselves to branches $k = 0, -1$ and real values. The question naturally arises whether complex simplifications are also possible. We briefly consider the option here, but are not tempted to generalize our algorithm. Substituting negative values in (4) gives the pairs $(i, -i)$ and $\left(2^{-1/3}(-1)^{1/3}, -2^{2/3}(-1)^{1/3}\right)$. For the first pair, we substitute $x = i$ into (5), and get

$$y = -\frac{i}{\ln i} W_k \left(-\frac{\ln i}{i} \right) = -\frac{2W_k(-\pi/2)}{\pi} \ .$$

The branch behaviour is reversed from previous examples. Now it is the $k = 0$ branch that gives the non-trivial solution,

$$y = -\frac{2W_0(-\pi/2)}{\pi} = -i \ ,$$

which current Maple gets, and it is the $k = -1$ branch that gives the trivial solution, which Maple also gets.

For the second pair, many people will immediately simplify $(-1)^{1/3} = -1$. In Maple, this is obtained using the `surd` command: `surd(-1,3)=-1`. In a Maple worksheet, `surd(x,3)` is printed as $\sqrt[3]{x}$, as opposed to $x^{1/3}$ for the principal value. Thus, they would arrive at the pair

$$(x, y) = \left(\frac{-2^{2/3}}{2}, 2^{2/3} \right) \ .$$

Sadly, this tidy solution does not work.

$$x^y = \left(-2^{2/3}/2 \right)^{2^{2/3}} \ ; y^x = \left(2^{2/3} \right)^{-2^{2/3}/2} \ .$$

Among the infinite complex values of x^y, we compute

$$-0.69297,\ 0.5911 \pm 0.3617i,\ -0.3154 \pm 0.6170i, \ldots$$

and for y^x, we compute

$$1.44306,\ 0.3913 \pm 1.389i,\ -1.231 \pm 0.7532i, \ldots$$

Not even Carathéodory could discover an equality there.

In contrast, the principal value of $(-1)^{1/3} = \frac{1}{2}(1 + i\sqrt{3})$ works a treat. Thus

$$\left(\frac{2^{2/3}}{4}(1 + i\sqrt{3}) \right)^{2^{2/3}(-1-i\sqrt{3})/2} = 4.4145302 - 2.48981i$$

$$\left(\frac{2^{2/3}}{2}(-1 - i\sqrt{3}) \right)^{2^{2/3}(1+i\sqrt{3})/4} = 4.4145302 - 2.48981i$$

An interesting new feature with the W solution, is that now $k = +1$ gives the non-trivial solution:

$$-\frac{(-1)^{1/3}W_1\left(\ln\left((-1)^{1/3}2^{-1/3}\right)(-1)^{\frac{2}{3}}\sqrt[3]{2}\right)}{2^{1/3}\ln\left((-1)^{1/3}2^{-1/3}\right)} = -(-1)^{1/3}2^{\frac{2}{3}} = -0.7937 - 1.3747i$$

and it is the principal branch that gives the trivial solution.
Simplifications such as these can be expanded indefinitely, and a software system has to draw the line somewhere, and we draw it before complex values.

3 Implementation

The implementation of the simplification proceeds in 3 stages. First, we process the argument to W to write it in the standard form seen in (16). Then there is the computation of the simplification, and finally a new output form that guides users in their requests.

3.1 The argument

Maple's simplification routines use standard forms which are different from the form needed for simplification. For example, suppose the standard form of the argument for simplification is $\frac{1}{3}(3/2)^{1/3}\ln(9/8)$. The following Maple session shows how this might be altered.

```
> simplify( (1/3)*(3/2)^(1/3)*ln(9/8) )
```

$$-\frac{\sqrt[3]{3}\,2^{\frac{2}{3}}\left(-2\ln(3) + 3\ln(2)\right)}{6}$$

The analysis of the argument assumes a form

$$Arg = a^b c^d \ldots (r\ln s + u\ln v \ldots)\ .$$

This is converted to a list using `convert(Arg, 'list', '‘*‘')`, and then we use `selectremove(hastype, ArgList, specfunc(rational, ln))` to separate the powers and the logarithms. After that, it is straightforward Maple programming to restore the form $(1/n)C^{1/n}\ln B$. If the argument cannot be converted, then simplification fails.

3.2 Testing simplification

Once the argument is established, X is computed using (16), and the candidate integer obtained using `intX:=round(X)`. If the magnitude of `intX` is large, then the working precision of `evalf` is increased (to mitigate rounding errors) and the evaluation repeated. We check the fractional part of X to ensure the candidate is plausible (`abs(X-intX)` $< 10^{-5}$) and then conduct the test (15). An alternative is to check the definition (6) directly.

3.3 Option parametric

A long-standing challenge for computer algebra systems has been the question of how to help users obtain useful results. A user who is not familiar with Lambert W, or not used to thinking in terms of branches, might miss a simplification. For example, the principal branch $k = 0$ of $W\left(-\frac{189}{256}\ln(4/3)\sqrt{3}\right)$ does not simplify, but $k = -1$ does. A simple call to simplify, therefore, might not produce a result useful to the user.

Our implementation includes a parametric option, in which both $k = 0$ and $k = -1$ are checked, and the user is given results for all branches. This option is activated by asking for simplification leaving the branch k unassigned. For the current example, this mode produces

$$W_k\left(-\frac{189}{256}\ln(4/3)\sqrt{3}\right) = \begin{cases} W_0\left(-\frac{189}{256}\ln(4/3)\sqrt{3}\right) , & \text{if } k = 0 , \\ -\frac{7}{2}\ln\frac{4}{3} , & \text{if } k = -1 . \end{cases}$$

This is to be understood to mean that for $k = 0$ there is no simpler expression for the number being represented than the value of W_0, which can be obtained as a floating-point approximation using `evalf`. For $k = -1$, on the other hand, there is a simpler expression. The user is then in possession of useful information with which to proceed.

4 Concluding remarks

To repeat a comment made above, the simplifications explored here do not exhaust the list of all possible simplifications. We have concentrated here on covering those that are likely to arise in practice. Any person who wishes to expose gaps in the coverage of simplifications can take inspiration and encouragement from Richardson's famous theorem that zero-recognition is undecidable [6]. For the Bernoulli-Goldbach problem, we considered only real, rational values of the parameter. We have shown in §2.1 problems that escape the present approach.

In addition to practicality, there are also æsthetic considerations. The equations here have already become complicated. To cover more detailed cases would require even greater length and complexity in the function arguments. We feel we have chosen a good density of coverage without undue complexity.

The basic principal of the simplifications is the fact that W is the inverse of the function $V = ze^z$. A common source of confusion in this regard is the apparent equation $W(xe^x) = x$. This is the result of the standard mathematical education, which teaches that a function f and its inverse f^{-1} obey $f^{-1}(f(x)) = f\left(f^{-1}(x)\right) = x$. For W, however, this is wrong. The correct statement is

$$W_k(xe^x) = y \text{ , } \mathbf{where} \;\; ye^y = xe^x \ .$$

The choice of y is determined by the branch k as much as by the value of x.

We anticipate that the simplifications described here will be available for all users in Maple 2021.

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