Characterization of plastic yielding as an absorbing phase transition

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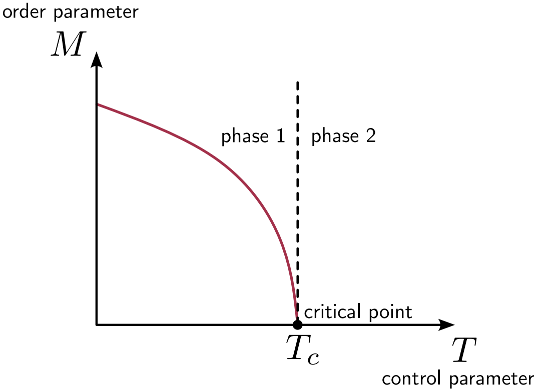
PSM group meeting

Continuous phase transitions

Definition

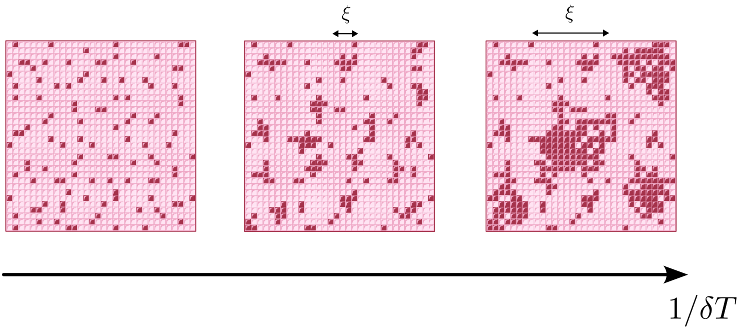
Physical transition between two phases characterized by an order parameter, under the variation of a control parameter.



Examples

- Ferromagnetism
- Superconductivity
- Superfluidity

Critical behavior



The scaling hypothesis

ξ is the only relevant length scale.

- scale-free problem

Critical exponents

$$M \sim \delta T^{\beta}, \quad \xi \sim \delta T^{-\nu}$$

$$N \times (\Delta M)^2 \sim \delta T^{-\gamma'}$$

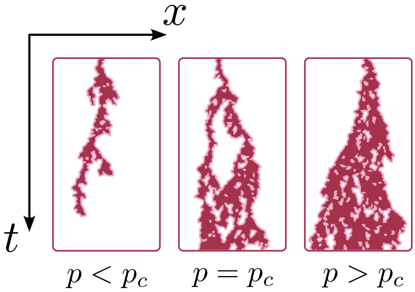
Absorbing phase transitions

- between an **active phase** and an **inactive phase** (absorbing state)

Absorbing state

can be reached by the dynamics but cannot be escaped

- e.g. epidemics models
- highly non-equilibrium



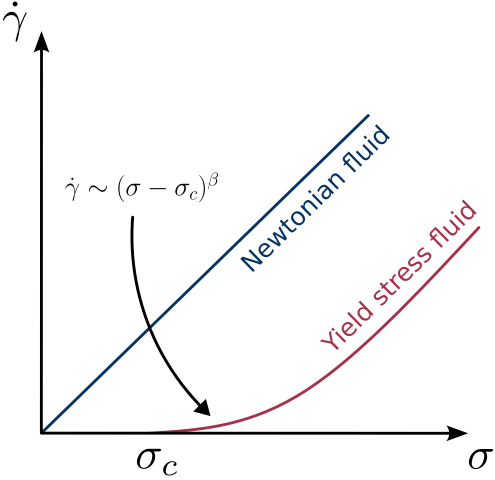
Directed percolation

- Unique absorbing state
- No special symmetry

Conserved directed percolation

- Conserved field
- Infinity of absorbing states

What are amorphous solids?



Exemples

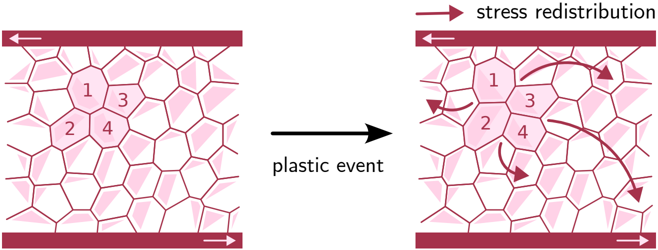
- clay
- sand
- toothpaste
- mayonnaise

Yield stress

Below which the material does not flow

Phenomenology of plastic flows

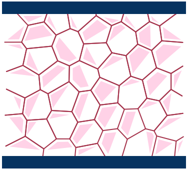
- A plastic flow is a succession of local plastic events



1. Local **stress accumulation**
2. **Stress relaxation** and **local displacement** induced by plastic events
3. **Non-local redistribution** of the locally relaxed stress
4. **Triggering** of new plastic events

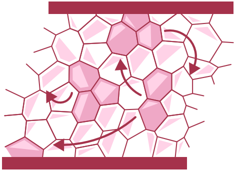
Arrested absorbing state

System eventually reaches a state with no plastic events



Flowing active state

Plastic events continuously trigger one another



σ_c σ

Order parameter : $\langle \dot{\gamma} \rangle$

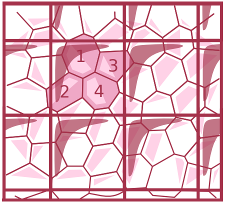
Control parameter : σ

Critical exponents

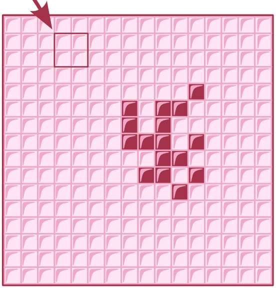
$\langle \dot{\gamma} \rangle \sim \delta\sigma^{\beta}$ $\quad \xi \sim \delta\sigma^{-\nu_\perp}$

$N(\Delta\dot{\gamma})^2 \sim \delta\sigma^{-\gamma'}$

Elastoplastic models



each block contains a few particles
(mesoscale)



 elastic state

 plastic state

Three local variables $(\sigma_i, \epsilon_{\mathrm{pl},i}, n_i)$ with a model-dependent co-evolution

Picard model

Mechanical evolution

$$\partial_t \sigma_i = \sum_j G_{ij} \partial_t \epsilon_{\mathrm{pl},j}, \quad \partial_t \epsilon_{\mathrm{pl},i} = n_i \sigma_i$$

$$\text{avec} \quad G(|\mathbf{r} - \mathbf{r}'|) = \frac{\cos(4\theta)}{\pi |\mathbf{r} - \mathbf{r}'|^2}$$

State evolution

$$\begin{cases} n_i : \quad 0 \xrightarrow{\tau} 1 & |\sigma_i| > \sigma_Y \\ n_i : \quad 0 \xleftarrow{\tau} 1 & \forall \sigma_i \end{cases}$$

Picard model

Mechanical evolution

$$\partial_t \sigma_i = \sum_j G_{ij} \partial_t \epsilon_{\mathrm{pl},j}, \quad \partial_t \epsilon_{\mathrm{pl},i} = n_i \sigma_i$$

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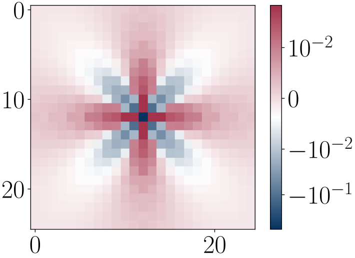

Picard model

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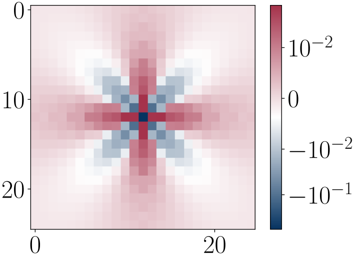
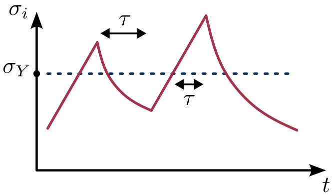
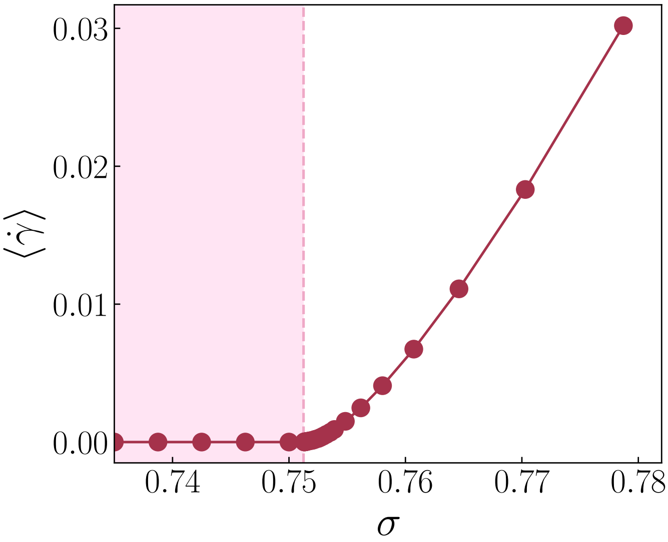


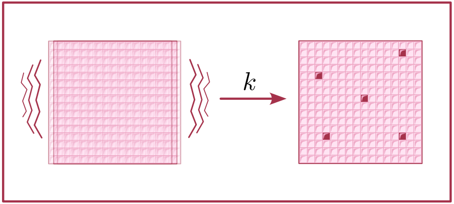


Picard model



Insights from APT: getting close to the critical point

- Finite-size systems are absorbed even for $\sigma > \sigma_c$.



Activation field

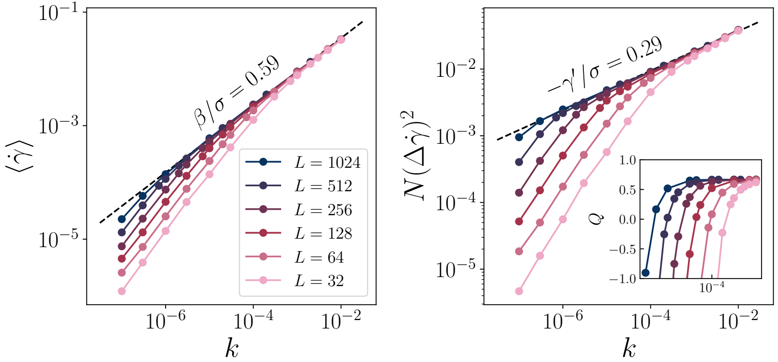
$$n_i : 0 \xrightarrow{k} 1 \quad \forall \sigma_i$$

New critical exponent

$$\dot{\gamma}(\sigma = \sigma_c) \sim k^{\beta/\sigma}$$
$$N \times (\Delta\dot{\gamma})^2(\sigma = \sigma_c) \sim k^{-\gamma'/\sigma}$$

- Enables to probe the critical region

Plastic flows under an activation field



- Universal power-law until $\xi \sim L$

Finite size scaling analysis

- FSE can be understood in the framework of the scaling hypothesis

Universal functions

$$\lambda^{\beta}\dot{\gamma} = F(\lambda\delta\sigma) \quad \forall\lambda$$

Finite size scaling analysis

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Universal functions

$$\frac{1}{\delta\sigma}^{\beta}\dot{\gamma} = F(\frac{1}{\delta\sigma}\delta\sigma)$$

Finite size scaling analysis

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Universal functions

$$\dot{\gamma} = F(1)\delta\sigma^{\beta}$$

Finite size scaling analysis

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Universal functions

$$\lambda^{\beta}\dot{\gamma} = F(\lambda\delta\sigma, \lambda^{\sigma}k) \quad \forall\lambda$$

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Universal functions

$$\lambda^{\beta}\dot{\gamma} = F(0, \lambda^{\sigma} h) \quad \forall\lambda$$

Finite size scaling analysis

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Universal functions

$$(k^{-1/\sigma})^{\beta}\dot{\gamma} = F(0,(k^{-1/\sigma})^{\sigma}h)$$

Finite size scaling analysis

- FSE can be understood in the framework of the scaling hypothesis

Universal functions

$$\dot{\gamma} = F(0,1)k^{\beta/\sigma}$$

Finite size scaling analysis

- FSE can be understood in the framework of the scaling hypothesis
  - **Idea**: introduce L as a parameter scaling as ξ

Universal functions

$$\lambda^{\beta}\dot{\gamma} = F(\lambda\delta\sigma, \lambda^{\sigma}k, \lambda^{-\nu_{\perp}}L) \quad \forall\lambda$$

Finite size scaling analysis

- FSE can be understood in the framework of the scaling hypothesis
  - **Idea**: introduce L as a parameter scaling as ξ

Universal functions

$$\lambda^{\beta}\dot{\gamma} = F(0, \lambda^{\sigma} k, \lambda^{-\nu_{\perp}} L) \quad \forall \lambda$$

Finite size scaling analysis

- FSE can be understood in the framework of the scaling hypothesis
  - **Idea**: introduce L as a parameter scaling as ξ

Universal functions

$$(L^{1/\nu_\perp})^{\beta}\dot{\gamma} = F(0, (L^{1/\nu_\perp})^{\sigma} k, (L^{1/\nu_\perp})^{-\nu_\perp} L)$$

Finite size scaling analysis

- FSE can be understood in the framework of the scaling hypothesis
  - **Idea**: introduce L as a parameter scaling as ξ

Universal functions

$$L^{\beta/\nu_\perp}\dot{\gamma} = F(0, L^{\sigma/\nu_\perp}k, 1)$$

- Curves for all L should collapse for the right set $(\beta, \nu_\perp, \sigma)$

Finite size scaling analysis

- FSE can be understood in the framework of the scaling hypothesis
  - **Idea**: introduce L as a parameter scaling as ξ

Universal functions

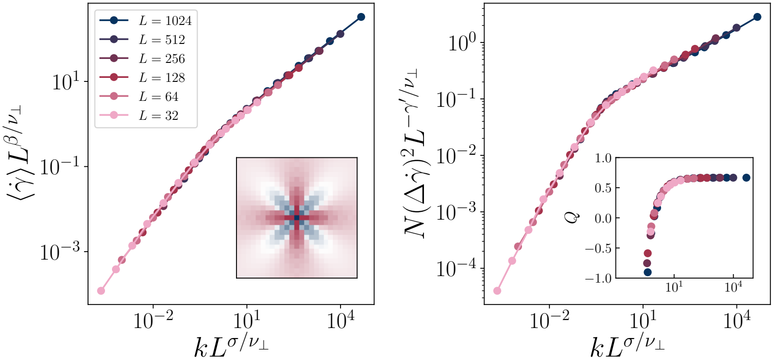
$$L^{\beta/\nu_\perp}\dot{\gamma} = F(0, L^{\sigma/\nu_\perp}k, 1)$$

$$L^{-\gamma'/\nu_\perp}N \times (\Delta\dot{\gamma})^2 = G(0, L^{\sigma/\nu_\perp}k, 1)$$

$$Q = 1 - \frac{\langle\dot{\gamma}^4\rangle}{3\langle\dot{\gamma}^2\rangle^2} = H(0, L^{\sigma/\nu_\perp}k, 1)$$

- Curves for all L should collapse for the right set $(\beta, \nu_\perp, \sigma, \gamma')$

Determination of the critical exponents



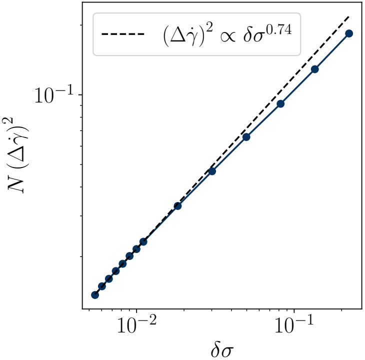
Estimated exponents

$$\beta = 1.5 \quad \nu = 1.1 \quad \gamma' = -0.70$$

2D-CDP exponents

$$\beta = 0.64 \quad \nu = 0.80 \quad \gamma' = 0.37$$

Vanishing critical fluctuations at the yielding transition



- Usually critical fluctuations diverge at the transition

Hyperscaling relation

$$2\beta + \gamma' = \nu_\perp d$$

- here $\frac{2\beta+\gamma'}{\nu_\perp d} \approx 1.02$

Why doesn't yielding belongs to CDP?

The CDP conjecture

all stochastic models with an infinite number of absorbing states in which the order parameter evolution is coupled to a non-diffusive conserved field fall into the universality class of CDP.

Elasticity induces **long range interactions**.

Non-locality of the Eshelby response

$$G(|\mathbf{r}-\mathbf{r}'|) = \frac{\cos(4\theta)}{\pi|\mathbf{r}-\mathbf{r}'|^2}$$

- LRI usually tend to mean field

MF-CDP exponents

$$\beta = 1 \quad \nu = 0.5 \quad \gamma' = 0$$

Short-ranged Picard model

- The idea is to **reduce Eshelby propagator** to a short-ranged one
- All other rules remain unchanged

$-\frac{1}{4}$	$+\frac{1}{2}$	$-\frac{1}{4}$
$+\frac{1}{2}$	-1	$+\frac{1}{2}$
$-\frac{1}{4}$	$+\frac{1}{2}$	$-\frac{1}{4}$

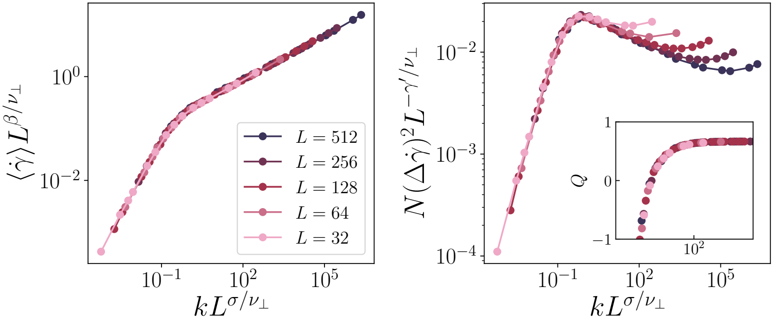
Global stress conservation

$$\sum_{\Delta x, \Delta y} G(\Delta x, \Delta y) = 0$$

Quadrupolar symmetry

$$G(r, \theta + \pi/4) = G(r, \theta)$$

Critical behavior of the SRP model



Estimated exponents

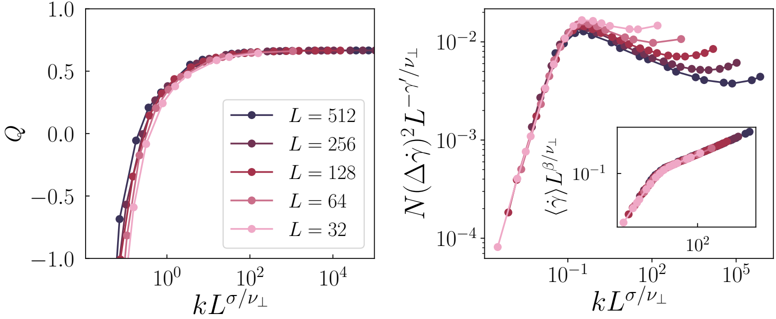
$\beta = 0.59 \quad \nu = 0.70 \quad \gamma' = 0.26$

2D-CDP exponents

$\beta = 0.64 \quad \nu = 0.80 \quad \gamma' = 0.37$

SRP-CDP discrepancy

Using 2D-CDP exponents for the rescalings we get:



- FSSA is precise enough to reject CDP
  - What could be the reason of this discrepancy ?

Kernel specificity: zero mode

SRP model

$$\begin{array}{ccc} -\frac{1}{4} & +\frac{1}{2} & -\frac{1}{4} \\ +\frac{1}{2} & -1 & +\frac{1}{2} \\ -\frac{1}{4} & +\frac{1}{2} & -\frac{1}{4} \end{array} = 0$$

$$= 0$$

Fourier point of view

$$\sum_{\Delta x} \mathcal{G}(\Delta x, \Delta y) = \Delta x \leftrightarrow \Delta y = 0$$

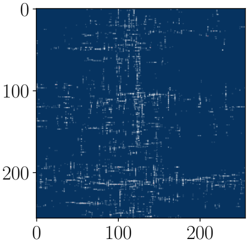
$$\hat{\mathcal{G}}(0, q_y) = \hat{\mathcal{G}}(q_x, 0) = 0$$

Picard model

Eshelby propagator in Fourier space

$$\hat{\mathcal{G}}(q_x, q_y) = -4 \frac{q_x^2 q_y^2}{q^4}$$

- comes from mechanical eq.
- lines of plasticity have zero cost



Kernel specificity: zero mode

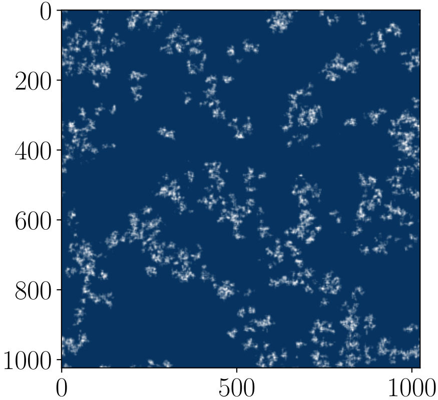
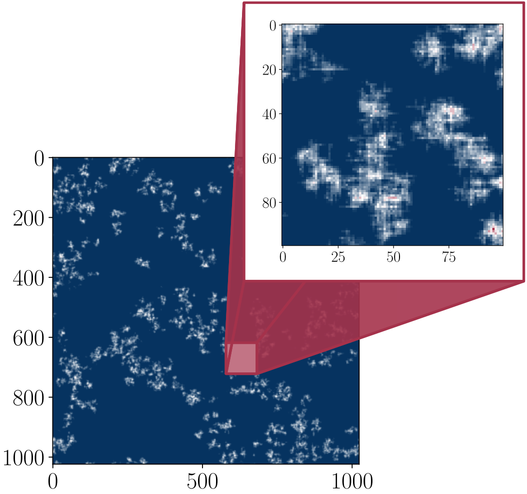


Figure: Map of the cumulated plasticity in the SRP model

Kernel specificity: zero mode



- It seems that the zero mode remains relevant at large scales

Non conservative short-ranged Picard model

$-\frac{1}{12}$	$+\frac{1}{3}$	$-\frac{1}{12}$
$+\frac{1}{3}$	-1	$+\frac{1}{3}$
$-\frac{1}{12}$	$+\frac{1}{3}$	$-\frac{1}{12}$

Global stress conservation

$$\sum_{\Delta x, \Delta y} G(\Delta x, \Delta y) = 0$$

Quadrupolar symmetry

$$G(r, \theta + \pi/4) = G(r, \theta)$$

Non conservative short-ranged Picard model

$-\frac{1}{12}$	$+\frac{1}{3}$	$-\frac{1}{12}$
$+\frac{1}{3}$	-1	$+\frac{1}{3}$
$-\frac{1}{12}$	$+\frac{1}{3}$	$-\frac{1}{12}$

$\neq 0$

$\neq 0$

Global stress conservation

$$\sum_{\Delta x, \Delta y} G(\Delta x, \Delta y) = 0$$

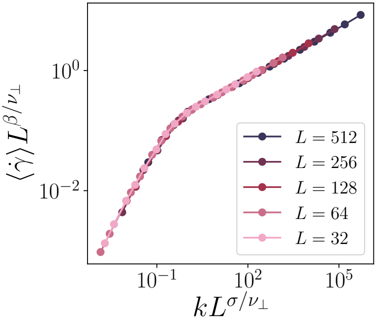
Quadrupolar symmetry

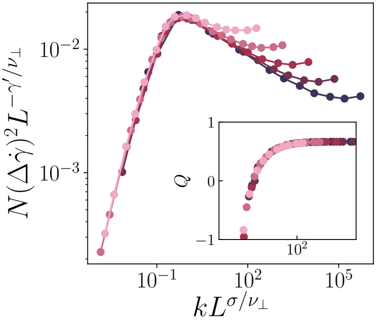
$$G(r, \theta + \pi/4) = G(r, \theta)$$

No line stress conservation

$$\sum_{\Delta x} \mathcal{G}(\Delta x, \Delta y) = \Delta x \leftrightarrow \Delta y \neq 0$$

Critical behavior of the SRPNC model





Estimated exponents

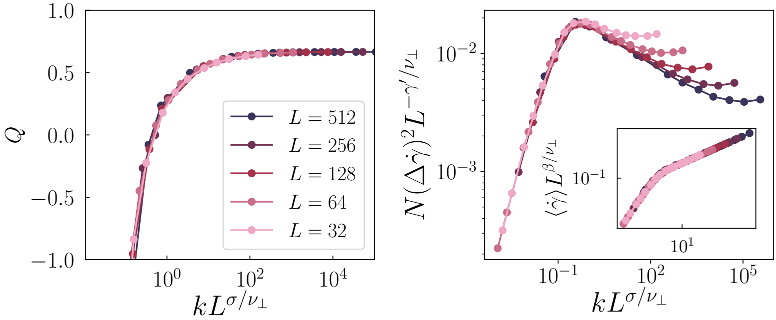
$\beta = 0.62 \quad \nu = 0.79 \quad \gamma' = 0.36$

2D-CDP exponents

$\beta = 0.64 \quad \nu = 0.80 \quad \gamma' = 0.37$

Critical behavior of the SRPNC model

Using 2D-CDP exponents for the rescalings we get:



- We fall back into **the universality class of CDP** !

Loss of the zero mode

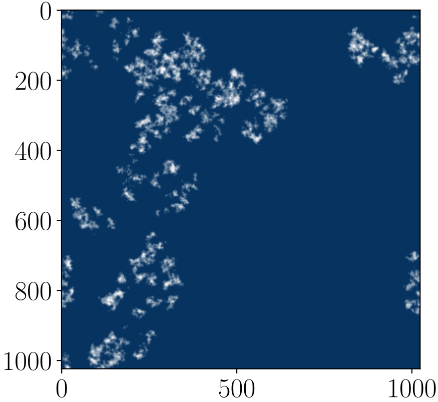
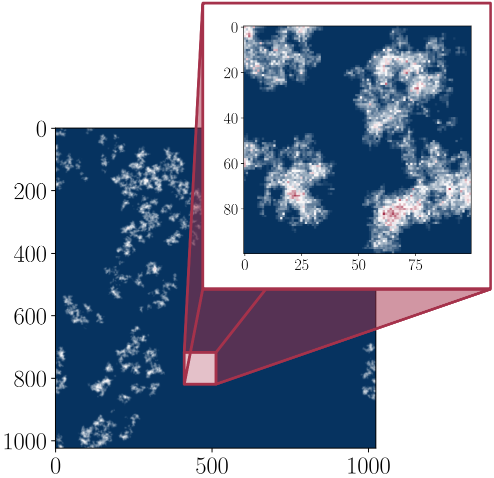


Figure: Map of the cumulated plasticity in the SRPNC model

Loss of the zero mode



- Isotropy is recovered !

Summary

CDP

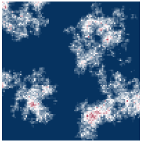
$-\frac{1}{12}$	$+\frac{1}{3}$	$-\frac{1}{12}$
$+\frac{1}{3}$	-1	$+\frac{1}{3}$
$-\frac{1}{12}$	$+\frac{1}{3}$	$-\frac{1}{12}$

No particular structure

Short-ranged interactions

exponents

$$\beta = 0.62$$
$$\gamma' = 0.36$$
$$\nu_\perp = 0.79$$



Summary

CDP

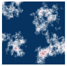
$-\frac{1}{12}$	$+$	
$+\frac{1}{3}$	-1	$+\frac{1}{3}$
$-\frac{1}{12}$	$+\frac{1}{3}$	$-\frac{1}{12}$

No particular structure

0.62

$\gamma' = 0.36$

$\nu_\perp = 0.79$



CDP-0

$-\frac{1}{4}$	$+\frac{1}{2}$	$-\frac{1}{4}$
$+\frac{1}{2}$	-1	$+\frac{1}{2}$
$-\frac{1}{4}$	$+\frac{1}{2}$	$-\frac{1}{4}$

$\hat{\mathcal{G}}(q_x, 0) = \hat{\mathcal{G}}(0, q_y) = 0$

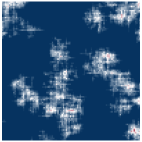
Short-ranged interactions

exponents

$\beta = 0.59$

$\gamma' = 0.26$

$\nu_\perp = 0.70$



Summary

CDP

$$\begin{bmatrix} -\frac{1}{12} & +\frac{1}{3} & -\frac{1}{12} \\ +\frac{1}{3} & -1 & +\frac{1}{3} \\ -\frac{1}{12} & +\frac{1}{3} & -\frac{1}{12} \end{bmatrix}$$

No particular structure

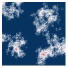
Short-ranged interactions

exponents

$\beta = 0.62$

$\gamma' = 0.36$

$\nu_\perp = 0.79$



CDP-0

$$\begin{bmatrix} -\frac{1}{4} & +\frac{1}{2} & -\frac{1}{4} \\ +\frac{1}{2} & & \\ -\frac{1}{4} & & \end{bmatrix}$$

$\hat{\mathcal{G}}(q_x, 0) = \hat{\mathcal{G}}(0, q_y) = 0$

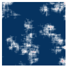
Short-ranged interactions

exponents

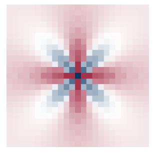
$\beta = 0.59$

6

0



Yielding



$\hat{\mathcal{G}}(q_x, 0) = \hat{\mathcal{G}}(0, q_y) = 0$

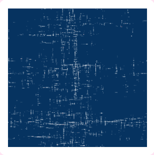
Long-ranged interactions

exponents

$\beta = 1.5$

$\gamma' = -0.70$

$\nu_\perp = 1.1$



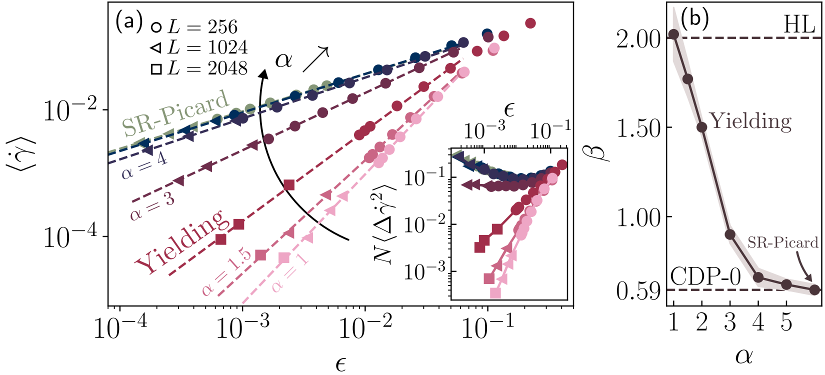
Conclusion

Results

- Plastic yielding is an absorbing phase transition from an **arrested absorbing phase** to a **flowing active phase**.
- Characterization of yielding leads to **unusual exponents** and vanishing critical fluctuations
- **Yielding differs from CDP** because of:
  - the absence of a **zero-mode** interaction
  - **non-local** elastic interactions

Perspectives

- Field theory for the CDP-0 models
- Difference in mean field formulations


(a)
$L = 256$
$L = 1024$
$L = 2048$
$\alpha \nearrow$
SR-Picard
$\alpha = 4$
$\alpha = 3$
Yielding
$\alpha = 1.5$
$\alpha = 1$
$\langle \dot{\gamma} \rangle$
ϵ
$N\langle \Delta\dot{\gamma}^2 \rangle$
(b)
HL
Yielding
SR-Picard
CDP-0
β
α