

No arbitrage SVI

Claude Martini^{*1} and Arianna Mingone^{†1,2}

¹Zeliade Systems, 56 rue Jean-Jacques Rousseau, Paris, France

²Università degli Studi di Udine, Udine, Italy

May 25, 2021

Abstract

We fully characterize the absence of Butterfly arbitrage in the SVI formula for implied total variance proposed by Gatheral in 2004. The main ingredient is an intermediate characterization of the necessary condition for no arbitrage obtained for any model by Fukasawa in 2012 that the inverse functions of the $-d_1$ and $-d_2$ of the Black-Scholes formula, viewed as functions of the log-forward moneyness, should be increasing. A natural rescaling of the SVI parameters and a meticulous analysis of the Durrleman condition allow then to obtain simple range conditions on the parameters. This leads to a straightforward implementation of a least-squares calibration algorithm on the no arbitrage domain, which yields an excellent fit on the market data we used for our tests, with the guarantee to yield smiles with no Butterfly arbitrage.

1 Introduction

Jim Gatheral proposed in 2004 the following *Stochastic Volatility Inspired* model for the implied total variance (meaning: the square of the implied volatility times the time-to-maturity):

$$SVI(k) = a + b(\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2})$$

where k is the log-forward moneyness, and (a, b, ρ, m, σ) parameters.

This formula quickly became the benchmark at least on Equity markets, due to its ability to produce very good fits. Fabien Le Floch (head of research at Calypso) has a blog article on a situation where SVI *does not fit*, which is a good indicator of how rare such a situation is in practice. The practitioner literature on SVI and its variants is plentiful ([3], [14], [18], [1], [15]), and SVI is now part of every reference textbook on volatility models ([9], [11]).

In 2009, the whitepaper on the *Quasi-explicit calibration of Gatheral's SVI* ([4], also part of Stefano De Marco PHD thesis) proposed a simple trick to disambiguate the calibration of SVI, and became itself a reference calibration algorithm.

SVI has been extended by Gatheral and Jacquier in a seminal paper to *surfaces* in [10], which provides the first explicit family of implied volatility surfaces with explicit and tractable no arbitrage

^{*}cmartini@zeliade.com

[†]arianna.mingone@gmail.com

conditions, both for the Butterfly and Calendar Spread arbitrages. SSVI has been extended further in [12] to other smile shapes, and in [13] to correlation parameters functions of the time-to-maturity. A quick and robust calibration algorithm for the latter is provided in [2].

SSVI smiles (at a fixed time to maturity) are a subset of SVI smiles with 3 parameters instead of 5, and so, for them, an explicit sufficient condition for no (Butterfly) arbitrage is available (cf [10]). [16] discusses also partial necessary and sufficient conditions for SSVI smiles.

A remarkable fact is that, despite the simplicity of the formula, no Butterfly arbitrage conditions for a SVI smile remained up to now too intricate. So for instance in the algorithm [4] there is no guarantee that the calibrated parameter will be arbitrage-free. An interesting practical approach is provided in [6], where the no arbitrage constraints are expressed as a discretized set of Durrleman conditions and encoded as non-linear constraints in the optimizer; *stricto sensu* there also, there is no guarantee though that the calibrated parameter will be arbitrage-free. In this paper, we solve this long-standing issue.

We start in section 2 with a precise discussion of the meaning of no Butterfly arbitrage, which is based on [20] and on [19] for the corresponding statements in terms of volatility.

We proceed in section 3 with a slight generalization of the beautiful result by Fukasawa in [8], which states that the inverse functions of the $-d_1$ and $-d_2$ coefficients of the Black-Scholes formula should be increasing under no Butterfly arbitrage. We need this generalization to handle all the configurations of SVI parameters. In this section we also clarify when and how Call and Put SVI option prices, given by the Black-Scholes formula with the SVI formula as argument, can be represented as expectations, using the results in [20].

The main ingredient (section 4) is then to use a natural rescaling of SVI: we work with the parameters α, μ where $a = \sigma\alpha$ and $m = \sigma\mu$, and the dummy variable $l = \frac{k-m}{\sigma}$ instead of k . It turns out that the Fukasawa conditions for SVI in the new parameters do not involve σ . An interesting property of those conditions is that they provide the positivity of the 1st term of the Durrleman condition; based on the fact that in our case the complementing 2nd term reads $\frac{1}{2\sigma}G_2(l)$ where G_2 does not depend on σ , ensuring the Durrleman condition yields a simple condition on σ . In section 5 we give the full characterization of the Fukasawa conditions for SVI, and section 6 finishes the work with the full characterization of no Butterfly arbitrage.

It should be noted that we do not impose the condition $a \geq 0$, as is often done without justification; we work out the necessary and sufficient conditions in the full domain of the SVI parameters.

At this stage, we have made fully explicit the domain of the SVI parameters for which no Butterfly arbitrage holds. It is straightforward to code, resorting to root finding numerical routines (like the Brent algorithm) for the evaluation of the thresholds we characterized in our computations. There are then 2 byproducts of this parametrization of the domain of high practical interest:

- a quick check routine that a given SVI parameter lies in the domain or not, which disentangles between 4 possible situations of arbitrage;
- a calibration algorithm, using any least-squares type objective function and a minimizer able to handle bounds.

We provide in the last section (section 7) numerical tests performed on data on index options purchased from the CBOE.

SVI models a volatility smile, not a volatility surface, so without ambiguity when we use the no arbitrage wording for SVI, we mean the absence of Butterfly arbitrage.

We thank Antoine Jacquier and Stefano De Marco for useful discussions and comments.

1.1 Domain of SVI parameters

The SVI model is defined when $a, m \in \mathbb{R}$, $b \geq 0$, $|\rho| \leq 1$, $\sigma \geq 0$. We recall that SVI is a convex function, with a minimum value given by $a + b\sigma\sqrt{1-\rho^2}$ (possibly attained at infinity if $|\rho| = 1$) and which goes to infinity as k goes to $\pm\infty$ (for $|\rho| < 1$). Since SVI models total variances, it is therefore required that $a + b\sigma\sqrt{1-\rho^2} \geq 0$.

If $\rho = -1$ the SVI smile decreases from ∞ to a , and if $\rho = +1$ the SVI smile increases from a to ∞ .

2 The Durrleman condition and no arbitrage for SVI

Let S_0 denote the underlying asset value of standard Call options with a fixed maturity $t > 0$. Without loss of generality we assume that there is no interest rates nor dividend rates. In case of deterministic interest r and dividend rates δ , all the statements in this section still hold once S_0 is replaced by the Forward corresponding to the option maturity $F_t = S_0 \exp \int_0^t (r_s - \delta_s) ds$ and working with the numéraire of the option maturity.

2.1 Axiom of no Butterfly arbitrage

The condition of no Butterfly arbitrage is achieved when the Call price function with respect to the strike is (we follow the very careful treatment in [20]):

1. convex;
2. non-increasing;
3. with value in the range $[(S_0 - K)^+, S_0]$.

These properties assume only that there is a perfect market for the underlying asset and for the Call options, with short-selling allowed, and that there is no static buy-sell strategy involving the underlying asset and a finite set of Call options with a Profit and Loss which is strictly positive.

We recall in particular that the *large moneyness behaviour* stating that the Call price function should go to zero at ∞ is an additional assumption, and does not strictly follow from the no arbitrage axiom.

In the case of a Call price function specified through an implied volatility: $C(K) = C_{BS}(k, \sqrt{w(k)})$ where w is the implied total variance $\sigma^2 t$ and $C_{BS}(k, a)$ is the Black-Scholes formula expressed as a function of the log-forward moneyness $k = \log \frac{K}{S_0}$ and the implied total volatility, the 3rd property is automatically granted since the C_{BS} function is increasing with respect to its 2nd argument and since the range bounds correspond to the limit when a goes to 0 and ∞ .

Observe now that if the 3rd property is satisfied, then the 1st one implies the 2nd one because an increasing convex function cannot be bounded.

2.2 Smiles vanishing at some point

Can a volatility smile reach 0 at some (finite) point? Assume that it is the case, so $w(k_m) = 0$ at the log-forward moneyness k_m corresponding to some strike K_m . Then it means that the Call price with this strike is equal to its intrinsic value $(S_0 - K_m)_+$. If K_m lies on the right of S_0 , the price is therefore 0, and by the property 2 above all the Call prices with $K > K_m$ will also be 0. If K_m lies on the left of S_0 , the option price is $S_0 - K_m$; as the option price with a strike 0 is equal

to $S_0 = S_0 - 0$, the convexity property implies that all the Call prices with $K < K_m$ are smaller than $S_0 - K$ which is the value of the chord between the points 0 and K_m . Since this value $S_0 - K$ is also lower bound for the Call prices, they are eventually equal to this value. So, in the implied volatility space, this means that $w = 0$ for $K \geq K_m$ in the 1st case, and $w = 0$ for $K \leq K_m$ in the second case.

This means that no arbitrage implies that smiles reaching 0 above (respectively below) the At-The-Money (forward) point will vanish above (respectively below) this point. In the case of SVI, smiles reach zero at most at a single strike, and only if $a + b\sigma\sqrt{1-\rho^2} = 0$ and $|\rho| < 1$, in which case they are strictly positive for other strike values, and there is a Butterfly arbitrage. So we can discard this case and assume $a + b\sigma\sqrt{1-\rho^2} > 0$ when $|\rho| < 1$.

2.3 No Butterfly arbitrage criterion for SVI

At this stage we know that SVI smiles with no Butterfly arbitrage are positive, and that the 3rd property above is automatically satisfied. So there is no Butterfly arbitrage if and only if the 1st property holds. Now for positive smiles, as recalled in [10] after Lemma 2.2, with $w(k) = SVI(k)$:

$$\begin{aligned} p(K) &:= \frac{\partial^2 C_{BS}}{\partial K^2} \Big|_{K=S_0 e^k} = \frac{\partial^2 C_{BS}(k, \sqrt{w(k)})}{\partial K^2} \Big|_{K=S_0 e^k} \\ &= \frac{g(k)}{S_0 e^k \sqrt{2\pi w(k)}} \exp\left(-\frac{d_2(k, \sqrt{w(k)})^2}{2}\right) \end{aligned} \quad (1)$$

where d_2 is the standard coefficient of the Black-Scholes formula:

$$d_{1,2}(k, \sigma) = -\frac{k}{\sigma} \pm \frac{\sigma}{2}.$$

So convexity is equivalent to ask the function $g(k)$ ([10], equation 2.1)

$$g(k) := \left(1 - \frac{kSVI'(k)}{2SVI(k)}\right)^2 - \frac{SVI'(k)^2}{4} \left(\frac{1}{SVI(k)} + \frac{1}{4}\right) + \frac{SVI''(k)}{2} \quad (2)$$

to be non-negative, which is usually called the *Durrleman condition* (cf. Theorem 2.9, condition (IV3) of [19]).

Note that the first derivative of the Call function with respect to the strike necessarily goes to zero as K goes to ∞ , and to a finite limit between -1 and 0 as K goes to 0 , which means that the total mass of p is less than one, but not necessarily one, meaning that there could be a non-zero mass at zero. It will sum to one if and only if the limit is -1 ; in this case, p can be interpreted as a probability measure; the expectation of the underlying asset under this measure will be strictly less than the underlying asset value, unless the additional property that the Call price vanishes at infinity holds, in which case it will be exactly the underlying asset value (cf. Theorem 2.1.2 of [20]).

The above discussion can be translated in properties of the smile: we know from Theorem 2.9 in [19] that the large moneyness behaviour is one-to-one with the fact that $d_1(k)$ goes to $-\infty$ at infinity:

$$\lim_{k \rightarrow \infty} d_1(k, \sqrt{w(k)}) = -\infty.$$

The fact that there is no mass at zero, or, equivalently, that the derivative of the Call price with respect to the strike goes to -1 when the strike goes to zero, is equivalent to (cf. [7], Proposition 2.4):

$$\lim_{k \rightarrow -\infty} d_2(k, \sqrt{w(k)}) = +\infty.$$

In the case of SVI, the 1st condition translates to $b(1+\rho) < 2$ and the second one to $b(1-\rho) < 2$. In particular the following Lemma holds:

Lemma 2.1. *In SVI, the limit of $d_1(k, \sqrt{w(k)})$ for k going to ∞ , is*

- $-\infty$ if $b(1+\rho) < 2$;
- 0 if $b(1+\rho) = 2$;
- ∞ if $b(1+\rho) > 2$.

The proof is simple and it is omitted. An important consequence to this result is that when $b(1+\rho) = 2$, the Call prices do not go to zero when the strike goes to infinity and so they are not given as the expectation of the payoff; we will come back to this situation in detail in section 3. In such a case, this does not necessarily lead to an arbitrage and so the request $b(1+\rho) < 2$ is not a necessary condition for the absence of arbitrage. We can summarize the previous discussion as follows:

Proposition 2.2 (No Butterfly arbitrage criterion for SVI). *A necessary condition for no Butterfly arbitrage to hold in SVI is that $SVI(k) > 0$ for all k . Under this condition, there is no arbitrage in SVI if and only if the function g in eq. (2) is non-negative. In this case, the function $p(K)$ in eq. (1) where $K = S_0 e^k$, and S_0 is the underlying asset value, defines a positive density on \mathbb{R}_+ such that $\int p(x)dx \leq 1$.*

Moreover, the Call prices in SVI go to zero when the strike goes to infinity if and only if $b(1+\rho) < 2$, and the derivative of the Call price (expressed in numéraire of the maturity) with respect to the strike goes to -1 if and only if $b(1-\rho) < 2$. In the first case $\int xp(x)dx = S_0$ and in the second case $\int p(x)dx = 1$.

Note that the two conditions $b(1+\rho) = 2$ and $b(1-\rho) = 2$ can occur simultaneously if and only if $b = 2$ and $\rho = 0$. We turn now to the weak no Butterfly condition obtained by Fukasawa. Characterizing this intermediary condition will eventually lead us to our full characterization result.

3 Fukasawa necessary condition for no Butterfly arbitrage

We recall the beautiful model-free necessary no Butterfly arbitrage condition obtained by Fukasawa in [8]. Following Fukasawa, let us denote the Black-Scholes prices as $C_{BS}(k, \sigma) = S_0 \Phi(d_1(k, \sigma)) - S_0 e^k \Phi(d_2(k, \sigma))$ for Calls and $P_{BS}(k, \sigma) = S_0 e^k \Phi(-d_2(k, \sigma)) - S_0 \Phi(-d_1(k, \sigma))$ for Puts; the implied total volatility is $\sqrt{w(k)} = \sigma(k)$; for a given total implied volatility let us set

$$f_{1,2}(k) = -d_{1,2}(k, \sigma(k)).$$

Fukasawa proved in Theorem 2.8 of [8] that (under the hypothesis that option prices are given by the expectation of their payoff) if a total variance smile w , expressed as a function of the log-forward moneyness, has no Butterfly arbitrage, then the two functions f_1 and f_2 are necessarily strictly increasing with $f'_{1,2} > 0$.

3.1 A slight generalization of Fukasawa result

In the following, we generalize Fukasawa's result to the case where the only request on the Put prices is their convexity and differentiability, without requiring that they are given by the expectation of the payoff. Note that the proof is essentially Fukasawa's one. This will allow us to cover the boundary case $b(1 + \rho) = 2$.

Lemma 3.1. *Let Put prices be defined as the Black-Scholes Put prices with a total volatility $\sigma(k)$: $P(K) = P_{BS}(k, \sigma(k))$, where $K = S_0 e^k$. If the function P is convex and the total volatility is differentiable, then the functions $f_{1,2}$ are increasing.*

Proof. Because the total volatility is differentiable, then also the Put prices are differentiable. Define $D_{BS}(K) := \frac{1}{K} \frac{\partial P_{BS}}{\partial k}(k, \sigma) |_{\sigma=\sigma(k)} = \Phi(f_2(k))$ and $D(K) := \frac{dP}{dK}(K)$. Note that here we do not use the equality $D(K) = E[I_{K > S_T}]$ whose proof requires that the Put prices are the expectation of their payoff. Since the Put prices are given by the Black-Scholes formula, they lie between $(K - S_0)^+$ and K , and since the function $K \rightarrow P(K)$ is convex, then its derivative lies necessarily between 0 and 1: $0 \leq D(K) \leq 1$. It holds

$$\begin{aligned} D(K) &= \frac{d}{dK} P_{BS}(\log(K/S_0), \sigma(\log(K/S_0))) \\ &= D_{BS}(K) + \frac{1}{K} \frac{\partial P_{BS}}{\partial \sigma}(\log(K/S_0), \sigma(\log(K/S_0))) \frac{d\sigma}{dk}(\log(K/S_0)) \\ &= D_{BS}(K) + \phi(f_2(\log(K/S_0))) \frac{d\sigma}{dk}(\log(K/S_0)). \end{aligned} \quad (3)$$

We now check that

$$f_2(k) \frac{d\sigma}{dk}(k) < 1.$$

From the previous equations, $\frac{d\sigma}{dk}(k) = \frac{D(S_0 e^k) - D_{BS}(S_0 e^k)}{\phi(f_2(k))}$ and because of the bounds for $D(K)$, this quantity lies in $\left[-\frac{1-\Phi(-f_2(k))}{\phi(-f_2(k))}, \frac{1-\Phi(f_2(k))}{\phi(f_2(k))}\right]$. So when $f_2(k)$ is non-negative, $f_2(k) \frac{d\sigma}{dk}(k) \leq f_2(k) \frac{1-\Phi(f_2(k))}{\phi(f_2(k))}$. Otherwise, $f_2(k) \frac{d\sigma}{dk}(k) \leq -f_2(k) \frac{1-\Phi(-f_2(k))}{\phi(-f_2(k))}$. Both quantities are less than 1.

At this point, we verify that

$$f_1(k) \frac{d\sigma}{dk}(k) < 1. \quad (4)$$

It holds $KD(K) \geq P(K)$. Indeed, since P is convex, its tangent at K lies below the function P itself, so for any $x \geq 0$ one has $P(K) + (x - K)D(K) \leq P(x)$ and evaluating at $x = 0$ we obtain the target inequality since $P(0) = 0$. From this inequality, using eq. (3) and writing the explicit formula of $P(K)$, one gets $0 \leq S_0 \Phi(f_1(k)) + S_0 e^k \phi(f_2(k)) \frac{d\sigma}{dk}(k) = S_0 \Phi(f_1(k)) + S_0 \phi(f_1(k)) \frac{d\sigma}{dk}(k)$. If f_1 is non-positive, then $f_1(k) \frac{d\sigma}{dk}(k) \leq -f_1(k) \frac{1-\Phi(-f_1(k))}{\phi(-f_1(k))} < 1$. Otherwise, eq. (4) is automatically satisfied when $\frac{d\sigma}{dk}(k)$ is non-positive, while when it is positive, we notice that $f_1(k) \frac{d\sigma}{dk}(k) = f_2 \frac{d\sigma}{dk}(k) - \sigma \frac{d\sigma}{dk}(k) < 1 - \sigma \frac{d\sigma}{dk}(k) < 1$.

Finally, we show that f_1 and f_2 are increasing. Indeed, from their definition, $\frac{df_{1,2}}{dk}(k) = \frac{1}{\sigma(k)} \left(1 - \frac{d\sigma}{dk}(k) \left(\frac{k}{\sigma(k)} \pm \frac{\sigma(k)}{2}\right)\right) = \frac{1}{\sigma(k)} \left(1 - \frac{d\sigma}{dk}(k) f_{2,1}(k)\right) > 0$. \square

What if we start from convex Call prices defined by the Black-Scholes Call prices instead of Put prices? In such case, it would be enough to prove that also the Put prices come from the

Black-Scholes Put prices and they are convex. Indeed, synthetizing the strategy of selling a Call with strike K , buying the underlying and selling a quantity K of cash at time 0 yields a payoff $(X_T - K)^+ - X_T + K = (K - X_T)^+$, where X_T is the realized value of the underlying at maturity, so that the assumption of no arbitrage leads to $P(K) = C(K) - S_0 + K$ which is the Put-Call parity. Since the equality $P_{BS}(k, \sigma(k)) = C_{BS}(k, \sigma(k)) - S_0 + K$ also holds from the definition of the functions C_{BS} and P_{BS} , it follows firstly that $P(K) = P_{BS}(k, \sigma(k))$, so the Calls and Puts with the same strike have the same implied volatility. Secondly, looking at the Put-Call parity, one notices that the Call prices are convex iff the Put prices are convex. Applying the previous Lemma one finds again $f'_{1,2}(k) > 0$.

3.2 Expectation-based representation of the Calls and Put prices in SVI

The issue of the representation of the Call and Put price functions by an expectation under our purely analytical assumptions has been settled by Tehranchi in [20]. Indeed re-starting from the assumption that the implied volatility function is such that the Call price function $K \rightarrow C(K)$ is convex, it follows from the above discussion that we are exactly in the situation of Theorem 2.1.2 in [20]. So there exists a non-negative random variable S_T such that $E[S_T] \leq S_0$ and $C(K) = S_0 - E[K \wedge S_T]$. We have then from the above discussion that $P(K) = C(K) + K - S_0 = K - E[K \wedge S_T]$. It is interesting to note that:

$$C(K) = S_0 - E[K \wedge S_T] = E[(S_T - K)^+] + S_0 - E[S_T],$$

whereas the usual expectation for the Put formula still holds:

$$P(K) = K - E[K \wedge S_T] = E[(K - S_T)^+].$$

Going back to SVI, when $b(1 + \rho) < 2$ we are in the situation where $C(K) \rightarrow 0$ when $K \rightarrow \infty$, so that $E[S_T] = S_0$ in the above representation. In the case $b(1 + \rho) = 2$ one has $\lim_{k \rightarrow \infty} f_1(k) = 0$, which plugged into the Black-Scholes formula gives $\lim_{K \rightarrow \infty} C(K) = \frac{S_0}{2}$; in turns this gives $E[S_T] = \frac{S_0}{2}$. We can state the following:

Proposition 3.2. *Let $C(K) := C_{BS}(k, \sqrt{SVI(k)})$ and $P(K) := P_{BS}(k, \sqrt{SVI(k)})$ be the Call and Put prices in SVI. Then there exists a positive random variable S_T such that:*

1. $P(K) = K - E[K \wedge S_T] = E[(K - S_T)^+]$ and $C(K) = S_0 - E[K \wedge S_T] = E[(S_T - K)^+] + S_0 - E[S_T]$;
2. if $b(1 + \rho) < 2$, $C(K) \rightarrow 0$ when $K \rightarrow \infty$, $E[S_T] = S_0$ and $C(K) = E[(S_T - K)^+]$;
3. if $b(1 + \rho) = 2$, $C(K) \rightarrow \frac{S_0}{2}$ when $K \rightarrow \infty$, $E[S_T] = \frac{S_0}{2}$ and $C(K) = E[(S_T - K)^+] + \frac{S_0}{2}$.

The last ingredient we will require is a natural change of parameters in SVI, that we describe in the next section altogether with the main argument of our full characterization.

4 Normalizing SVI

We now rescale SVI in the following way, which is natural:

$$\begin{aligned} SVI(k) &= \alpha\sigma + b\sigma \left(\rho \frac{k-m}{\sigma} + \sqrt{\left(\frac{k-m}{\sigma} \right)^2 + 1} \right) \\ &= \sigma N \left(\frac{k-m}{\sigma} \right) \end{aligned}$$

with $\alpha := a/\sigma$ and $N(l) := \alpha + b(\rho l + \sqrt{l^2 + 1})$. With this rewriting, the derivatives of the SVI model become

$$\begin{aligned} SVI'(k) &= N' \left(\frac{k-m}{\sigma} \right), \\ SVI''(k) &= \frac{1}{\sigma} N'' \left(\frac{k-m}{\sigma} \right). \end{aligned}$$

Observe that the second derivative N'' is positive so N is strictly convex. Its only critical point is a minimum that we call $l^* = -\frac{\rho}{\sqrt{1-\rho^2}}$. We gather the important properties of N in the following:

Lemma 4.1 (Normalized SVI). *Let $N(l) := \alpha + b(\rho l + \sqrt{l^2 + 1})$ where $a = \alpha\sigma$. Then N is strictly convex with a minimum at $l^* = -\frac{\rho}{\sqrt{1-\rho^2}}$, where $N(l^*) = \alpha + b\sqrt{1-\rho^2}$. Also:*

$$\begin{aligned} N'(l) &= b \left(\rho + \frac{l}{\sqrt{l^2 + 1}} \right), \\ N''(l) &= \frac{b}{(l^2 + 1)^{\frac{3}{2}}}. \end{aligned}$$

In particular as $l \rightarrow \pm\infty$:

$$N(l) \sim \alpha + b(\rho \pm 1)l, \quad N'(l) \rightarrow b(\rho \pm 1), \quad N''(l) \rightarrow 0,$$

and for every k

$$SVI(k) = \sigma N \left(\frac{k-m}{\sigma} \right).$$

In the above Lemma, note that the statement $N(l) \sim a' + b(\rho \pm 1)l$ covers the cases $b = 0$ and $b \neq 0$.

Hereafter we will also put $m = \mu\sigma$, so that $k = \sigma(l + \mu)$ and

$$SVI_{a,b,\rho,m,\sigma}(k) = \sigma N_{\alpha,b,\rho} \left(\frac{k}{\sigma} - \mu \right)$$

where the parameters have the following constraints:

$$b \geq 0, \quad |\rho| \leq 1, \quad \mu \in \mathbb{R}, \quad \sigma \geq 0, \quad \alpha + b\sqrt{1-\rho^2} \geq 0.$$

4.1 Expressing g with $f_{1,2}$ in rescaled parameters, and our main argument

There is a nice expression of g involving the functions $f_{1,2}$; indeed as shown e.g. in [5] (Eq. 55 p. 25):

$$\left. \frac{\partial^2 C}{\partial K^2} \right|_{K=S_0 e^k} = \phi(f_2(k)) \left(f_1'(k) f_2'(k) \sqrt{w(k)} + (\sqrt{w})''(k) \right) \frac{1}{S_0 e^k}$$

where ϕ is the standard Gaussian density. By identification this yields

$$g(k) = \left(f_1'(k) f_2'(k) \sqrt{w(k)} + (\sqrt{w})''(k) \right) \sqrt{w(k)}.$$

With our rescaled parameters, we have

$$g(k) = \left(1 - \frac{k N'(\frac{k}{\sigma} - \mu)}{2\sigma N(\frac{k}{\sigma} - \mu)} \right)^2 - \frac{N'(\frac{k}{\sigma} - \mu)^2}{4} \left(\frac{1}{\sigma N(\frac{k}{\sigma} - \mu)} + \frac{1}{4} \right) + \frac{N''(\frac{k}{\sigma} - \mu)}{2\sigma}$$

and writing $G(l) := g(\sigma(l + \mu))$ we find

$$G(l) = \left(1 - \frac{(l + \mu) N'(l)}{2N(l)} \right)^2 - \frac{N'(l)^2}{4} \left(\frac{1}{\sigma N(l)} + \frac{1}{4} \right) + \frac{N''(l)}{2\sigma}.$$

We can rewrite G as

$$\begin{aligned} G(l) &= \left(1 - N'(l) \left(\frac{(l + \mu)}{2N(l)} + \frac{1}{4} \right) \right) \left(1 - N'(l) \left(\frac{(l + \mu)}{2N(l)} - \frac{1}{4} \right) \right) + \frac{1}{2\sigma} \left(N''(l) - \frac{N'(l)^2}{2N} \right) \\ &= G_1(l) + \frac{1}{2\sigma} G_2(l) \end{aligned}$$

where

$$\begin{aligned} G_1(l) &:= \left(1 - N'(l) \left(\frac{(l + \mu)}{2N(l)} + \frac{1}{4} \right) \right) \left(1 - N'(l) \left(\frac{(l + \mu)}{2N(l)} - \frac{1}{4} \right) \right), \\ G_2(l) &:= N''(l) - \frac{N'(l)^2}{2N(l)}. \end{aligned}$$

Call G_{1+} the first factor of G_1 and G_{1-} the second one. Then $f'_{1,2}(\sigma(l + \mu)) = f'_{1,2}(k) = \frac{1}{\sqrt{\sigma N(\frac{k}{\sigma} - \mu)}} \left(1 - N'(\frac{k}{\sigma} - \mu) \left(\frac{k}{2\sigma N(\frac{k}{\sigma} - \mu)} \pm \frac{1}{4} \right) \right) = \frac{G_{1\pm}(l)}{\sqrt{\sigma N(l)}}$ and the Fukasawa conditions correspond to $G_{1\pm} > 0$, which entails that $G_1 > 0$.

Completing the identification yields $G_1(\frac{k}{\sigma} - \mu) = f_1'(k) f_2'(k) w(k)$ and $\frac{1}{2\sigma} G_2(\frac{k}{\sigma} - \mu) = (\sqrt{w})''(k) \sqrt{w(k)}$. It is now instrumental to observe that:

$$g(k) = G(l) = G_1(l) + \frac{1}{2\sigma} G_2(l)$$

where

- G_1 depends only on α, b, ρ, μ ,
- G_2 depends only on α, b, ρ ,

so that the dependency of G in σ is particularly simple; this is the main benefit of our rescaling of SVI.

Our main argument is now as follows: the Fukasawa conditions yield that it is necessary that $G_1 > 0$; for a given choice of b, ρ, α , this will give a condition on μ , which therefore characterizes the Fukasawa conditions in SVI. Given then a parameter μ satisfying this condition, the positivity of g (or G) can be casted as a simple condition on σ :

$$\sigma \geq \sup_l \frac{G_2(l)}{G_1(l)}$$

which yields the full characterization of no Butterfly arbitrage in SVI.

In section 5 we investigate the conditions on G_1 related to the Fukasawa conditions, and in section 6 this latter condition on σ .

4.2 Classifying the normalized SVI parameters

We will use the following notations to clarify the assumptions made on the SVI parameters:

- (A1) $\alpha + b\sqrt{1 - \rho^2} > 0$ and $|\rho| < 1$,
- (A2) $\alpha \geq 0$ and $|\rho| = 1$,
- (B1) $b(1 \pm \rho) < 2$,
- (B2) $b(1 + \rho) < 2$ and $b(1 - \rho) = 2$,
- (B3) $b(1 + \rho) = 2$ and $b(1 - \rho) < 2$,
- (B4) $b(1 + \rho) = 2$ and $b(1 - \rho) = 2$, which is equivalent to $b = 2, \rho = 0$.

In the sequel, to avoid singularities in our computations, we will assume b positive since the case $b = 0$ is the Black-Scholes case, which is a trivial case of no arbitrage, and exclude the boundary cases $|\rho| = 1$, so work under assumption (A1). We revisit those boundary cases in section 6.4 where we will assume (A2).

5 Investigating Fukasawa necessary no arbitrage conditions

5.1 Limits at infinity

We have the following:

Lemma 5.1 (Limits of G_1).

$$\lim_{\pm\infty} G_1(l) = \left(\frac{1}{2} - \frac{b(\rho \pm 1)}{4}\right) \left(\frac{1}{2} + \frac{b(\rho \pm 1)}{4}\right).$$

In particular, $G_1(\infty) \geq 0$ and $G_1(-\infty) \geq 0$ iff simultaneously $b(1 \pm \rho) \leq 2$.

These conditions are conditions on the asymptotic slopes of the total variance smile, and are therefore related to the Roger Lee Moment formula [17]; this is a general fact for the Fukasawa conditions: [8] contains several asymptotic statement on f_1 and f_2 which are directly related to the asymptotic behaviour of $\frac{w(k)}{k}$.

5.2 The conditions as an interval for μ

Let us investigate the corresponding Fukasawa conditions of positivity of G_{1+} and G_{1-} in terms of SVI parameters. We start with the following:

Lemma 5.2. *Let*

$$L_{\pm}(l; \alpha, b, \rho) := 2N(l) \left(\frac{1}{N'(l)} \mp \frac{1}{4} \right) - l \quad (5)$$

where L_+ is defined on $]l^*, +\infty[$ and L_- on $] -\infty, l^*[$. Then $G_{1\pm} > 0$ if and only if $\sup_{l < l^*} L_-(l) < \inf_{l > l^*} L_+(l)$ and

$$\mu \in I_{\alpha, b, \rho} :=] \sup_{l < l^*} L_-(l), \inf_{l > l^*} L_+(l) [.$$

Proof. In order to have $G_{1\pm} > 0$, we need $\sup_{l < l^*} L_{\pm}(l) < \mu < \inf_{l > l^*} L_{\pm}(l)$. Indeed $G_{1\pm}(l) = 1 - N'(l) \left(\frac{(l+\mu)}{2N(l)} \pm \frac{1}{4} \right)$ so that $G_{1\pm}(l) > 0$ iff $1 \mp \frac{N'(l)}{4} > N'(l) \frac{(l+\mu)}{2N(l)}$. Since $L_+(l) < L_-(l)$ for every l , we obtain an interval for μ given by $\sup_{l < l^*} L_-(l) < \mu < \inf_{l > l^*} L_+(l)$. \square

Remark 5.3. *In order to alleviate the notations, we will often suppress the list of parameters in L_{\pm} , or when we need it just denote the dependency in α , (b, ρ) being fixed.*

What are the basic properties of L_- and L_+ ?

Note that $L_-(l^{*-}) = -\infty$ and, under $b(1 - \rho) < 2$, $L_-(-\infty) = -\infty$. It follows that l_- such that $L_-(l_-) = \sup_{l < l^*} L_-(l)$ lays in $] -\infty, l^*[$. Similarly, $L_+(l^{*+}) = +\infty$ and $L_+(+\infty) = +\infty$ when $b(1 + \rho) < 2$, so l_+ such that $L_+(l_+) = \inf_{l > l^*} L_+(l)$ lays in $]l^*, +\infty[$. When $b(1 - \rho) = 2$ then $L_-(-\infty) = -\frac{\alpha}{2}$ while when $b(1 + \rho) = 2$ then $L_+(+\infty) = \frac{\alpha}{2}$. Indeed at infinity L_- behaves as $2\alpha \left(\frac{1}{b(\rho-1)} + \frac{1}{4} \right) + \frac{2+b(\rho-1)}{2}l$ while L_+ as $2\alpha \left(\frac{1}{b(\rho+1)} - \frac{1}{4} \right) + \frac{2-b(\rho+1)}{2}l$. In these cases the supremum of L_- (or the infimum of L_+), could be reached at $-\infty$ (or $+\infty$).

Experiments show that not every choice of (α, b, ρ) leads to $L_-(l) < -\epsilon < 0$ for all $l < l^*$ and $L_+(l) > \epsilon > 0$ for all $l > l^*$, so the interval for μ could be empty: for example, for $\alpha = -0.8, b = 1$ and $\rho = 0.5$, we have $L_-(l_-) > L_+(l_+)$. This suggests that the situation is intricate; we show below that when $\alpha \geq 0$, the interval is non-empty.

5.2.1 The case $\alpha \geq 0$

In the case $\alpha \geq 0$, we can indeed demonstrate that the interval for μ is non-empty, with the following easy argument:

L_- is negative for $l < l^*$ iff $\frac{N}{2N'}(4 + N') - l$ is negative. In this domain N' is negative, so the previous condition is equivalent to ask $N(4 + N') - 2lN' > 0$, or equivalently $2(N - lN') + N(2 + N') > 0$. Let us consider the first term. We have $N - lN' = \alpha + b\sqrt{l^2 + 1} - \frac{bl^2}{\sqrt{l^2 + 1}}$ which is greater than 0 iff, multiplying by $\sqrt{l^2 + 1}$, also $\alpha\sqrt{l^2 + 1} + b > 0$ or equivalently $\alpha > -\frac{b}{\sqrt{l^2 + 1}}$. This holds for $\alpha \geq 0$ (note that the latter quantity reaches its maximum at $-\infty$ where it equals 0, so this proof cannot handle the case $\alpha < 0$).

We can now consider the second term. We want $2 + N' > 0$. Since $N' > b(\rho - 1)$, then $2 + N' > 2 + b(\rho - 1) \geq 0$. So L_- is always strictly negative for $l < l^*$ and $\alpha \geq 0$.

Similarly, L_+ is positive for $l > l^*$ iff $2(N - lN') + N(2 - N') > 0$. With the same arguments as before we obtain that L_+ is strictly positive for $l > l^*$ and $\alpha \geq 0$.

Under (B1), we showed $L_-(-\infty) = -\infty$ and $L_+(\infty) = \infty$, so the interval I is non-empty.

When $b(1 - \rho) = 2$ or $b(1 + \rho) = 2$ this result is still valid. Since in such cases $L_-(-\infty) = -\frac{\alpha}{2}$ and $L_+(+\infty) = \frac{\alpha}{2}$ respectively, then L_- is negative in $[-\infty, l^*[$ while L_+ is positive in $]l^*, +\infty]$ for $\alpha > 0$. Otherwise if $\alpha = 0$, $\sup_{l < l^*} L_-(l) = L_-(-\infty) = 0$ and $\inf_{l > l^*} L_+(l) = L_+(+\infty) = 0$ respectively.

We have proven the following:

Lemma 5.4 (SVI parameters fulfilling Fukasawa necessary no arbitrage conditions: case $\alpha \geq 0$). *Assume (A1). For every (α, b, ρ) with $\alpha \geq 0$:*

- under (B1), the interval $I_{\alpha, b, \rho}$ is non-empty and contains 0;
- under (B2),
 - if $\alpha > 0$, the interval $I_{\alpha, b, \rho}$ is non-empty and contains 0;
 - if $\alpha = 0$, the interval $I_{0, b, \rho}$ is non-empty and has 0 as left boundary;
- under (B3),
 - if $\alpha > 0$, the interval $I_{\alpha, b, \rho}$ is non-empty and contains 0;
 - if $\alpha = 0$, the interval $I_{0, b, \rho}$ is non-empty and has 0 as right boundary;
- under (B4),
 - if $\alpha > 0$, the interval $I_{\alpha, 2, 0}$ is non-empty and contains 0;
 - if $\alpha = 0$, the interval $I_{0, 2, 0}$ is empty.

5.2.2 Computation of the interval for μ under (B1)

We tackle now the computation of the interval for μ in the general case where α is not necessarily positive, which is less straightforward. In this section we will assume (B1); we deal with the other cases in the dedicated section 5.2.5.

We consider the function L_- for $l < l^*$ and L_+ for $l > l^*$. We have $L'_\pm(l) = 1 \mp \frac{N'}{2} - \frac{2NN''}{N'^2}$ and it follows that $L'_-(l_-) = L'_+(l_+) = 0$.

The corresponding equations in l are:

$$1 \mp \frac{b}{2} \left(\rho + \frac{l}{\sqrt{l^2 + 1}} \right) - \frac{2(\alpha + b(\rho l + \sqrt{l^2 + 1}))}{b\sqrt{l^2 + 1}(\rho\sqrt{l^2 + 1} + l)^2} = 0.$$

Actually, we don't need to solve these equations. Accordingly, we set:

$$g_{\pm(b, \rho)}(l) = \left(\rho\sqrt{l^2 + 1} + l \right)^2 \left(\sqrt{l^2 + 1} \left(\frac{1}{2} \mp \frac{b\rho}{4} \right) \mp \frac{bl}{4} \right) - \left(\rho l + \sqrt{l^2 + 1} \right) \quad (6)$$

where $g_{+(b, \rho)}$ is defined on $[l^*, \infty[$ and $g_{-(b, \rho)}$ on $] -\infty, l^*]$. Then $L'_\pm(l) = 0$ iff $g_{\pm(b, \rho)}(l) = \frac{\alpha}{b}$.

The following technical result turns to be a key one:

Proposition 5.5. *Assume (A1) and (B1), and let $g_{\pm(b, \rho)}$ defined by eq. (6). Then $g_{\pm(b, \rho)}(l^*) = -\sqrt{1 - \rho^2}$, $g_{\pm(b, \rho)}(\pm\infty) = \infty$, and $g_{\pm(b, \rho)}$ is either monotonous or with a single minimum. Let $s_\pm = l^*$ if $g_{\pm(b, \rho)}$ is monotonous and $s_\pm \neq l^*$ such that $g_{\pm(b, \rho)}(s_\pm) = -\sqrt{1 - \rho^2}$ otherwise. Then:*

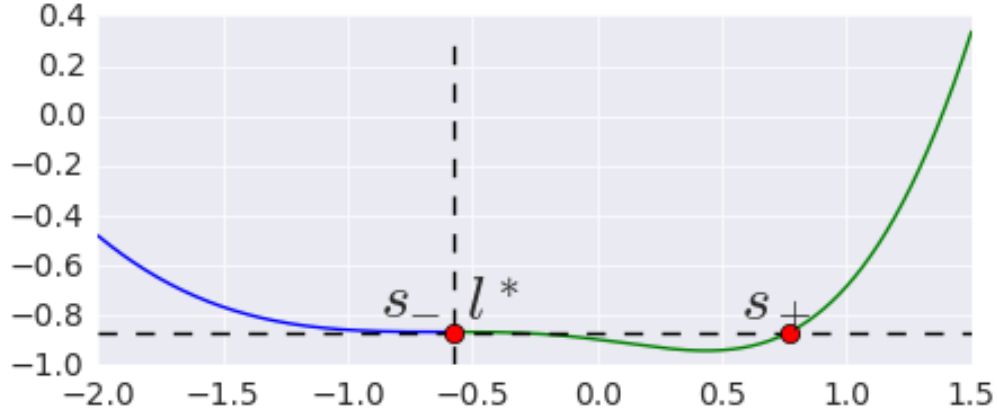


Figure 1: Typical plot of the functions $g_{\pm(b,\rho)}$ with $b = \frac{2}{3}$ and $\rho = \frac{1}{2}$. The horizontal dotted line corresponds to the level $-b\sqrt{1-\rho^2}$.

- $L_-(x; bg_{-(b,\rho)}(x)) = \sup_{l < l^*} L_-(l; bg_{-(b,\rho)}(x))$ for any $x < s_-$, $L_-(x; bg_{-(b,\rho)}(x)) \rightarrow -\infty$ when $x \rightarrow -\infty$, and the function $L_-(x; bg_{-(b,\rho)}(x))$ is increasing iff $g_{-(b,\rho)}$ is decreasing;
- $L_+(x; bg_{+(b,\rho)}(x)) = \inf_{l > l^*} L_+(l; bg_{+(b,\rho)}(x))$ for any $x > s_+$, $L_+(x; bg_{+(b,\rho)}(x)) \rightarrow +\infty$ when $x \rightarrow +\infty$, and the function $L_+(x; bg_{+(b,\rho)}(x))$ is increasing iff $g_{+(b,\rho)}$ is increasing.

The proof is provided in Appendix A. We display a typical plot of $g_{-(b,\rho)}$ and $g_{+(b,\rho)}$ in Figure 1. This proposition has in turn two important corollaries:

Corollary 5.6. Assume (A1) and (B1). There is a unique (l_-, l_+) such that $l_- < s_-$, $l_+ > s_+$ and $\alpha = bg_{-(b,\rho)}(l_-) = bg_{+(b,\rho)}(l_+)$. The interval $I_{\alpha,b,\rho}$ is non-empty iff $L_-(l_-; \alpha) < L_+(l_+; \alpha)$. In this case the distance between $L_+(l_+; \alpha)$ and $L_-(l_-; \alpha)$ increases with α .

Proof. This follows directly from the previous analysis: increasing α , the functions $g_{\pm(b,\rho)}$ increase so the corresponding $l_- < s_-$ decreases while $l_+ > s_+$ increases. In turn, the function $L_+(l_+; bg_{+(b,\rho)}(l_+))$ increases and the function $L_-(l_-; bg_{-(b,\rho)}(l_-))$ decreases. Note that $l_- < s_-$ and $l_+ > s_+$ because $\alpha > -b\sqrt{1-\rho^2}$ from (A1). We can also use the fact that

$$\begin{aligned} \frac{d}{d\alpha}(L_+(l_+; \alpha) - L_-(l_-; \alpha)) &= L'_+(l_+) \frac{d}{d\alpha} l_+ - L'_-(l_-) \frac{d}{d\alpha} l_- + \partial_\alpha L_+(l_+; \alpha) - \partial_\alpha L_-(l_-; \alpha) \\ &= \partial_\alpha L_+(l_+; \alpha) - \partial_\alpha L_-(l_-; \alpha) \end{aligned}$$

where l_+ and l_- are functions of α given by $\alpha = bg_{+(b,\rho)}(l_+) = bg_{-(b,\rho)}(l_-)$. Now, the RHS is equal to $2(\frac{1}{N'(l_+)} - \frac{1}{N'(l_-)} - \frac{1}{2})$ and since $\frac{1}{N'(l_+)} > \frac{1}{2}$ and $-\frac{1}{N'(l_-)} > \frac{1}{2}$, the previous quantity is greater than 1. \square

Let $F(b, \rho)$ denote the unique value of α such that $L_+(l_+; \alpha) = L_-(l_-; \alpha)$ if there exists such a value for $\alpha > -b\sqrt{1-\rho^2}$, otherwise let $F(b, \rho) = -b\sqrt{1-\rho^2}$. Then $L_+(l_+; \alpha) > L_-(l_-; \alpha)$ if and only if $\alpha > F(b, \rho)$. In other words we define $F(b, \rho)$ as:

$$F(b, \rho) := \inf\{\alpha \mid L_+(l_+; \alpha) > L_-(l_-; \alpha)\} \vee -b\sqrt{1-\rho^2}$$

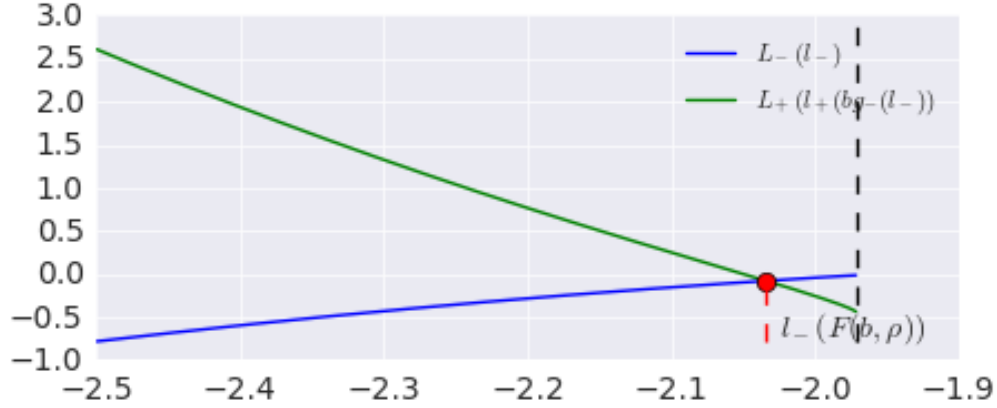


Figure 2: Plot of $L_-(l_-)$ and $L_+(l_+(bg_{-(b, \rho)}(l_-)))$ as functions of l_- , with $b = \frac{25}{21}$ and $\rho = \frac{2}{5}$. The vertical dotted line corresponds to the level $-b\sqrt{1 - \rho^2}$.

under the assumptions (A1) and (B1). We name F the Fukasawa threshold of SVI.

Figure 2 shows:

- in blue the function $l_- \rightarrow L_-(l_-; bg_{-(b, \rho)}(l_-))$ with $l_- < s_-$ where s_- is the point at which $g_{-(b, \rho)}$ is equal to $-\sqrt{1 - \rho^2}$;
- in green the corresponding value of $l_- \rightarrow L_+(l_+(bg_{-(b, \rho)}(l_-)); bg_{-(b, \rho)}(l_-))$.

The following corollary gives an easy criterion of existence of a Butterfly arbitrage:

Corollary 5.7. *Assume (A1) and (B1). If $\alpha \leq F(b, \rho)$ then for every choice of μ and σ , the SVI model does not satisfy the Fukasawa conditions.*

5.2.3 Study of the Fukasawa threshold under (B1)

In the previous section we showed that the difference $L_+(l_+; bg_{+(b, \rho)}(l_+)) - L_-(l_-; bg_{-(b, \rho)}(l_-))$ goes to infinity when increasing $\alpha = bg_{+(b, \rho)}(l_+) = bg_{-(b, \rho)}(l_-)$ to infinity, so there exists $\bar{\alpha}$ such that the interval for μ is non-empty; from the previous corollaries for each $\alpha > \bar{\alpha}$ the interval for μ is also in non-empty. Decreasing α , we could bump into two situations:

- α reaches the value $F(b, \rho) > -b\sqrt{1 - \rho^2}$ for which $L_+(l_+; F(b, \rho)) = L_-(l_-; F(b, \rho))$;
- α reaches the value $F(b, \rho) = -b\sqrt{1 - \rho^2}$. In such case $l_{\pm} = s_{\pm}$.

Our simulations suggest that the first scenario always occurs.

Could we prove this? In this respect we can observe the following: it is equivalent to prove that $L_+(s_+; -b\sqrt{1 - \rho^2}) < L_-(s_-; -b\sqrt{1 - \rho^2})$.

If $s_+ = l^*$ then $L_+(s_+; -b\sqrt{1 - \rho^2}) = -l^*$ and the function $L_+(l_+; bg_{+(b, \rho)}(l_+))$ is increasing. It follows that the function $L_-(l_-; bg_{-(b, \rho)}(l_-))$ cannot be increasing and $s_- < l^*$. We should show that $L_-(s_-; -b\sqrt{1 - \rho^2}) > -l^*$.

When $s_- = l^*$ then $L_-(s_-; -b\sqrt{1-\rho^2}) = -l^*$ and the function $L_-(l_-; bg_{-(b,\rho)}(l_-))$ is increasing. Again, the function $L_+(l_+; bg_{+(b,\rho)}(l_+))$ cannot be increasing so $s_+ > l^*$. In this case we should prove that $L_+(s_+; -b\sqrt{1-\rho^2}) < -l^*$.

In the final case when both $g_{\pm(b,\rho)}$ have a minimum, it is enough to prove $L_-(s_-; -b\sqrt{1-\rho^2}) > -l^*$ and $L_+(s_+; -b\sqrt{1-\rho^2}) < -l^*$.

So to sum up, it would remain to prove that when $g_{-(b,\rho)}$ (or $g_{+(b,\rho)}$) has a minimum, it holds $L_-(s_-; -b\sqrt{1-\rho^2}) > -l^*$ (or $L_+(s_+; -b\sqrt{1-\rho^2}) < -l^*$) to obtain the result in each case. We did not manage to conclude along those lines though.

Remark 5.8. *We don't know whether $F(b, \rho) > -b\sqrt{1-\rho^2}$ but we conjecture it. Indeed we prove in Appendix B that there is a closed formula for $F(b, 0)$ which satisfies $F(b, 0) > -b$; the statement $F(b, \rho) > -b\sqrt{1-\rho^2}$ can be also assessed numerically.*

5.2.4 Symmetries

We can exploit the symmetry property of N with respect to ρ in order to restrict the required computations to the function L_- only.

Indeed $N(l; \alpha, b, \rho) = N(-l; \alpha, b, -\rho)$, $N'(l; b, \rho) = -N'(-l; b, -\rho)$ and $N''(l; b) = N''(-l; b)$. This brings to the consideration that

$$L_-(l; \alpha, b, \rho) = -L_+(-l; \alpha, b, -\rho), \quad L_+(l; \alpha, b, \rho) = -L_-(-l; \alpha, b, -\rho),$$

so that

$$\begin{aligned} \inf_{l > l^*(\rho)} L_+(l; \alpha, b, \rho) &= - \sup_{l > l^*(\rho)} L_-(-l; \alpha, b, -\rho) \\ &= - \sup_{l < -l^*(\rho)} L_-(l; \alpha, b, -\rho) \\ &= - \sup_{l < l^*(-\rho)} L_-(l; \alpha, b, -\rho) \end{aligned}$$

so $L_+(l_+(\alpha, b, \rho); \alpha, b, \rho) = -L_-(l_-(\alpha, b, -\rho); \alpha, b, -\rho)$.

Note that $l_+(\alpha, b, \rho)$ is the unique $l > l^*(\rho)$ such that $L'_+(l; \alpha, b, \rho) = 0$ while $l_-(\alpha, b, -\rho)$ is the unique $l < l^*(-\rho)$ such that $L'_-(l; \alpha, b, -\rho) = 0$. Since $L'_+(l; \alpha, b, \rho) = L'_-(-l; \alpha, b, -\rho)$ and $-l_-(\alpha, b, -\rho) > -l^*(-\rho) = l^*(\rho)$, then $l_+(\alpha, b, \rho) = -l_-(\alpha, b, -\rho)$.

Lemma 5.9. *Assume (A1) and (B1). Then:*

- $L_+(l_+(\alpha, b, \rho); \alpha, b, \rho) = -L_-(l_-(\alpha, b, -\rho); \alpha, b, -\rho)$;
- $l_+(\alpha, b, \rho) = -l_-(\alpha, b, -\rho)$;
- $I_{\alpha, b, \rho} =]L_-(l_-(\alpha, b, \rho); \alpha, b, \rho), -L_-(l_-(\alpha, b, -\rho); \alpha, b, -\rho)[$.

From the above equations we also have $g_{+(b,\rho)}(l) = g_{-(b,-\rho)}(-l)$ so with easy arguments one gets $s_+(b, \rho) = -s_-(b, -\rho)$.

5.2.5 The cases (B2), (B3) and (B4)

Assume (B2) or (B4). Using the same definitions and following the proof of Proposition 5.5, we obtain that $g_{-(b,\rho)}(l)$ is increasing. Now since $g_{-(b,\rho)}$ is increasing on $] -\infty, l^*]$ and since $g_{-(b,\rho)}(l^*) = -\sqrt{1-\rho^2}$, it follows that there is no solution to the equation $g_{-(b,\rho)}(l_-) = \frac{\alpha}{b}$. In this case so, the supremum of L_- is attained at $-\infty$ and it is $-\frac{\alpha}{2}$. Under (B3) or (B4), for symmetrical reasons L_+ attains its infimum $\frac{\alpha}{2}$ at ∞ .

Under (B2), L_+ reaches its infimum in $]l^*, +\infty[$ while under (B3), L_- reaches its supremum in $] -\infty, l^*[$. Finally under (B4), $I_{\alpha,2,0} =]-\frac{\alpha}{2}, \frac{\alpha}{2}[$.

We can extend the definition of the Fukasawa threshold to the cases (B2), (B3) and (B4):

- under (B2), $F(b, \rho)$ denotes the unique value of α such that $L_-(l_-(\alpha, b, -\rho); \alpha, b, -\rho) = \frac{\alpha}{2}$ if there exists such a value for $\alpha > -b\sqrt{1-\rho^2}$, otherwise $F(b, \rho) = -b\sqrt{1-\rho^2}$: $F(b, \rho) := \inf\{\alpha \mid L_+(l_+; \alpha) > -\frac{\alpha}{2}\} \vee -b\sqrt{1-\rho^2}$;
- under (B3), $F(b, \rho)$ denotes the unique value of α such that $L_-(l_-(\alpha, b, \rho); \alpha, b, \rho) = \frac{\alpha}{2}$ if there exists such a value for $\alpha > -b\sqrt{1-\rho^2}$, otherwise $F(b, \rho) = -b\sqrt{1-\rho^2}$: $F(b, \rho) := \inf\{\alpha \mid \frac{\alpha}{2} > L_-(l_-; \alpha)\} \vee -b\sqrt{1-\rho^2}$;
- under (B4), $F(2, 0) := 0$.

From Lemma 5.4, under cases (B2) and (B3) it holds $F(b, \rho) < 0$.

5.3 Conclusion

We can now state the full characterization of the Fukasawa necessary no arbitrage conditions for SVI:

Theorem 5.10 (SVI parameters $(\alpha, b, \rho, \mu, \sigma)$ fulfilling Fukasawa necessary no arbitrage conditions). *Assume (A1). Then:*

- under (B1), $F(b, \rho) < 0$ and the interval $I_{\alpha,b,\rho} =]L_-(l_-(\alpha, b, \rho); \alpha, b, \rho), -L_-(l_-(\alpha, b, -\rho); \alpha, b, -\rho)[$ is non-empty iff $\alpha > F(b, \rho)$;
- under (B2) (resp. (B3)), $F(b, \rho) < 0$ and the interval $I_{\alpha,b,\rho} =]-\frac{\alpha}{2}, -L_-(l_-(\alpha, b, -\rho); \alpha, b, -\rho)[$ (resp. $I_{\alpha,b,\rho} =]L_-(l_-(\alpha, b, \rho); \alpha, b, \rho), \frac{\alpha}{2}[$) is non-empty iff $\alpha > F(b, \rho)$;
- under (B4), the interval $I_{\alpha,2,0} =]-\frac{\alpha}{2}, \frac{\alpha}{2}[$ is non-empty iff $\alpha > F(2, 0) = 0$.

In every case, the Fukasawa conditions are satisfied iff $\mu \in I_{\alpha,b,\rho}$.

Except for $F(2, 0)$, the result $F(b, \rho)$ negative holds even in the case $F(b, \rho) > -b\sqrt{1-\rho^2}$ because we have proven that for $\alpha \geq 0$ the interval for μ is always non-empty. In terms of the usual SVI parameters the conditions translate into $\frac{\alpha}{\sigma} > F(b, \rho)$ and $\frac{m}{\sigma} \in I_{\frac{\alpha}{\sigma}, b, \rho}$.

Is the existence of the Fukasawa threshold surprising? We would say no: indeed the values of α too close to the lower bound $-b\sqrt{1-\rho^2}$ correspond to values of the smile too close to zero, and this will lead to an arbitrage as discussed in section 2.2, so that one even expects that $F(b, \rho) > -b\sqrt{1-\rho^2}$.

The explanation of the range constraint for μ is less intuitive to us; we would say that it results from the geometrical constraint that the Fukasawa conditions impose on the shape of SVI, as follows from our computations.

5.4 Numerics

$F(b, \rho)$ at a fixed b We plot in Figure 3 the Fukasawa threshold at fixed $b = \frac{1}{2}$ as a function of ρ .

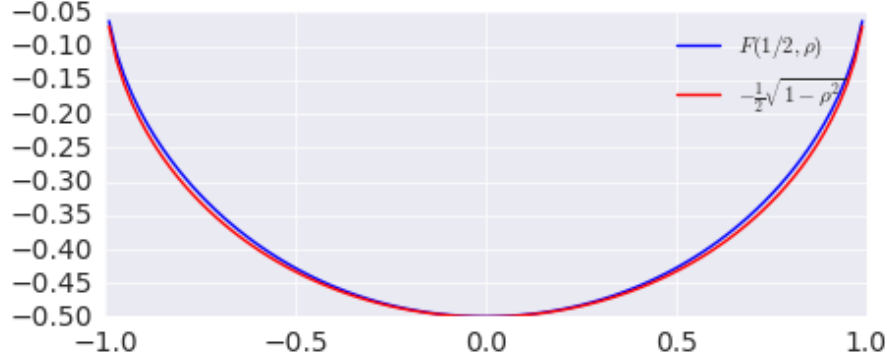


Figure 3: Plot of $F(b, \rho)$ as a function of ρ , with $b = \frac{1}{2}$.

The graph is symmetric with respect to ρ because $F(b, \rho)$ is the value of α such that the difference between $L_+(l_+(\alpha, b, \rho); \alpha, b, \rho)$ and $L_-(l_-(\alpha, b, \rho); \alpha, b, \rho)$ is null, where $bg_{\pm(b, \rho)}(l_{\pm}(\alpha, b, \rho)) = \alpha$. But $L_+(l_+(\alpha, b, \rho); \alpha, b, \rho) = -L_-(l_-(\alpha, b, -\rho); \alpha, b, -\rho)$ so we look for α such that

$$L_-(l_-(\alpha, b, -\rho); \alpha, b, -\rho) + L_-(l_-(\alpha, b, \rho); \alpha, b, \rho) = 0$$

and this is symmetric with respect to ρ .

The red line is the level $\alpha = -b\sqrt{1-\rho^2}$ and it again confirms our hypothesis that $F(b, \rho) > -b\sqrt{1-\rho^2}$.

From the previous graph, it seems that $F(b, \rho)$ has monotonicity of the same sign as ρ .

$F(b, \rho)$ at fixed ρ as a function of b In Figure 4 we plot the Fukasawa threshold at fixed $\rho = \frac{1}{5}$ as a function of b .

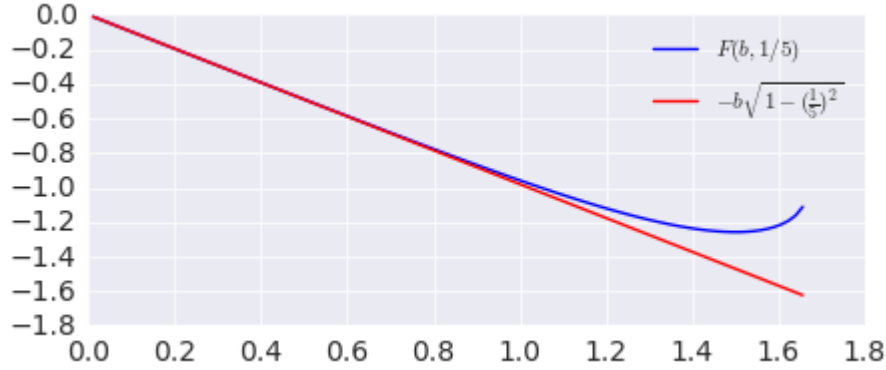


Figure 4: Plot of $F(b, \rho)$ as a function of b , with $b = \frac{1}{5}$.

$L_-(l_-(F(b, \rho), b, \rho); F(b, \rho), b, \rho)$ and $L_-(l_-(F(b, \rho), b, -\rho); F(b, \rho), b, -\rho)$ as functions of ρ The following Figure 5 shows in blue the function $L_-(l_-(F(b, \rho), b, \rho); F(b, \rho), b, \rho)$ (denoted for brevity as $L_-(F(b, \rho), \rho)$) with respect to ρ while in green the function $L_-(l_-(F(b, \rho), b, -\rho); F(b, \rho), b, -\rho)$ (or $L_-(F(b, \rho), -\rho)$) with respect to ρ . The fixed value for b is $\frac{3}{5}$.

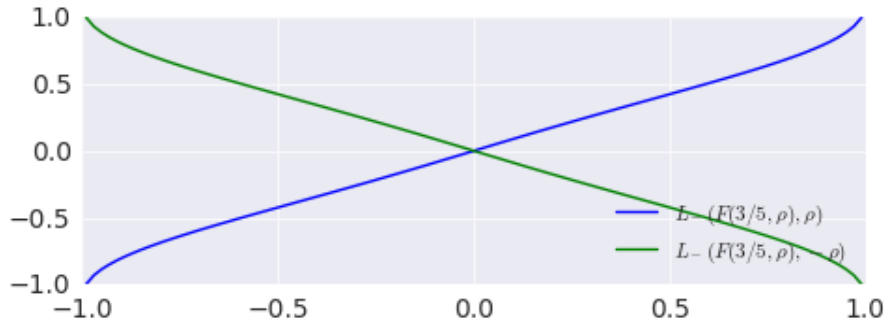


Figure 5: Plot of $L_-(l_-(F(b, \rho), \rho)$ and $L_-(F(b, \rho), -\rho)$ as functions of ρ , with $b = \frac{3}{5}$.

This graph also shows in blue the value of the two bounds for μ when they shrink to one point. Note that for $\rho = 0$ this is 0 for every b , while it depends on b for the other values of ρ .

The function $\rho \rightarrow L_-(l_-(F(b, \rho), b, \rho); F(b, \rho), b, \rho)$ is odd due to the symmetry of $\rho \rightarrow F(b, \rho)$. Furthermore, from the graph it seems that ρ and $L_-(l_-(F(b, \rho), b, \rho); F(b, \rho), b, \rho)$ have the same sign.

$L_-(l_-(F(b, \rho), b, \rho); F(b, \rho), b, \rho)$ as a function of b Figure 6 shows the function $L_-(l_-(F(b, \rho), b, \rho); F(b, \rho), b, \rho)$ (denoted as $L_-(F(b, \rho), \rho)$) with respect to b . Here we fix $\rho = \frac{1}{2}$.

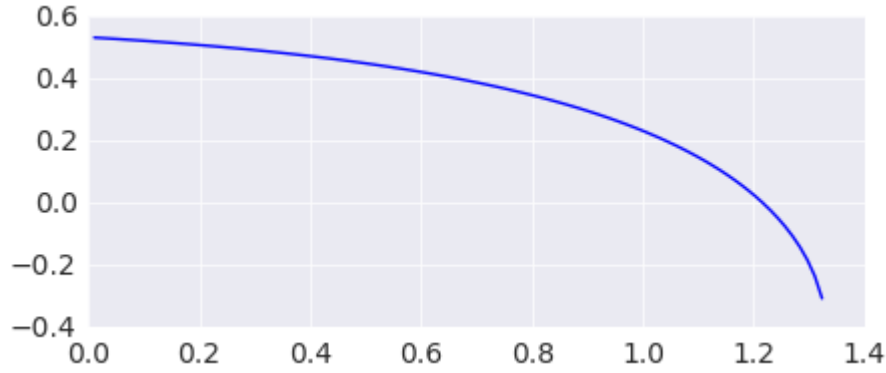


Figure 6: Plot of $L_-(l_-(F(b, \rho), \rho))$ as a function of b , with $\rho = \frac{1}{2}$.

5.5 Algorithm

We can *parametrize* the normalized SVI parameters satisfying the Fukasawa conditions as follows:

1. choose $\rho \in]-1, 1[$ and b positive such that $b(1 \pm \rho) \leq 2$ by choosing $b' \in]0, 1]$ and setting $b = b' \frac{2}{1+|\rho|}$;
2. compute numerically $F(b, \rho)$, and parametrize α by setting $\alpha = F(b, \rho) + u$ for positive u ;
3. compute numerically (L_-, L_+) for this value of u , and parametrize μ by setting $\mu = \frac{(1+q)}{2} L_+ + \frac{(1-q)}{2} L_-$ for $q \in]-1, 1[$.

The values in point 3 can be computed using the same functions employed to find $F(b, \rho)$, indeed it is sufficient to evaluate $L_-(l_-(\alpha, b, \rho); \alpha, b, \rho)$ and $-L_-(l_-(\alpha, b, -\rho); \alpha, b, -\rho)$.

If we are interested only by a test that a given parameter satisfies the Fukasawa conditions, we have the corresponding waterfall of failure possibilities that we define as follows:

1. $b(1 - \rho) > 2$ or $b(1 + \rho) > 2$: *failure of type 1*; otherwise:
2. $\alpha \leq F(b, \rho)$: *failure of type 2*; otherwise:
3. μ not in $I_{\alpha, b, \rho}$: *failure of type 3*.

5.5.1 Application to Axel Vogt parameters

The so-called *Axel Vogt example* (cf [10]) became the archetypal example of a smile with arbitrage. The SVI parameters are $(a, b, \rho, m, \sigma) = (-0.041, 0.1331, 0.3060, 0.3586, 0.4153)$, and they are known to lead to a Butterfly arbitrage. Do they satisfy the Fukasawa conditions?

No, since the respective value for μ is 0.86347, while its arbitrage free interval is $] -0.72407, 0.82939[$.

The Fukasawa conditions are not satisfied because of μ . However $\alpha = -0.09872$ and $F(b, \rho) = -0.12663$, so $\alpha > F(b, \rho)$ and the interval for μ is non-empty. The problem here is due to μ , which is too large: we face a *failure of type 3*.

6 No arbitrage domain for SVI

6.1 Behaviour of the function G_2

Recall that the function G_2 is defined as

$$G_2(l) := N''(l) - \frac{N'(l)^2}{2N(l)} \quad (7)$$

and that it depends only on (α, b, ρ) . As discussed in section 4.1, G_2 is positively proportional to the second derivative of the *volatility* smile, meaning of $\sqrt{SVI(k)}$. Since the *variance* smile is convex and asymptotically linear on both sides, it is expected that G_2 will be asymptotically *negative*, while it is positive around the minimum of the smile. In particular it is expected that it will have zeros, on both sides of the minimum of the smile.

6.1.1 The zeros of G_2

In this section we prove the following:

Lemma 6.1 (Zeros of G_2). *G_2 has exactly two zeros l_1, l_2 which satisfy $l_1 < l^* \wedge 0$ and $l_2 > l^* \vee 0$ such that $G_2(l) > 0 \iff l \in]l_1, l_2[$. Furthermore, $G_2(l) \rightarrow 0^-$ for $l \rightarrow \pm\infty$.*

Proof. For $l \rightarrow \pm\infty$ we have that the first addend behaves as bl^{-3} while the second as $-\frac{b(\rho\pm 1)}{2}l^{-1}$, so G_2 behaves as $-\frac{b(\rho\pm 1)}{2}l^{-1}$. This means that G_2 goes to 0^- as $l \rightarrow \pm\infty$. Since $G_2(l^*) = N''(l^*) > 0$ and G_2 is continuous, then there exists an interval $]l_1, l_2[$ containing l^* such that for every l in this interval, G_2 is positive. It follows that G_2 has at least two zeros. Deriving, we find the following interesting relationship between G_2' and G_2 :

$$G_2'(l) = N'''(l) - \frac{N'(l)}{N(l)}G_2(l).$$

We will prove now that this relationship entails that the first zero of G_2 is negative. Indeed if $l_1 > 0$ is the first zero of G_2 , since

$$N'''(l) = -\frac{3bl}{(l^2 + 1)^{\frac{5}{2}}} \quad (8)$$

we have $G_2'(l_1) < 0$, which is not possible because $G_2(l)$ is negative for every $l < l_1$. If $l_1 = 0$, then $G_2'(0) = 0$ but 0 cannot be a point of local maximum for G_2 , otherwise there would be a following zero $l_2 > 0$. In such case, $G_2'(l_2) < 0$ for eq. (8) but having G_2 so far negative, it should be increasing in l_2 . Then 0 could at most be an inflection point. However,

$$G_2''(l) = N^{iv}(l) - \frac{N'(l)N'''(l)}{N(l)} + \left(2\frac{N'(l)^2}{N(l)^2} - \frac{N''(l)}{N(l)}\right)G_2(l)$$

so $G_2''(0) = N^{iv}(0) = -3b$, which is negative since $b > 0$. Therefore, the first zero l_1 of G_2 is necessarily negative. With similar arguments we obtain that the next zero l_2 must be non-negative. Suppose $l_2 = 0$. Then, as before, $G_2'(0) = 0$ and $G_2''(0) = -3b < 0$, so it would be a point of local maximum, which is not possible. Then l_2 must be positive.

Moreover, there cannot be other zeros for G_2 . Indeed, suppose l_3 was the first zero after l_2 . Then $l_3 > 0$ and from eq. (8) it should be $G_2'(l_2) < 0$ but this cannot be true since G_2 is negative in the left neighborhood of l_3 .

This leads to the conclusion that G_2 has exactly two zeros, one positive and the other one negative. As a consequence, $G_2(0) = b(1 - \frac{b\rho^2}{2(\alpha+b)}) > 0$. This could have been obtained also from the fact that $\alpha + b\sqrt{1-\rho^2} \geq 0$ due to the positivity of N .

Then, we find that $G_2 > 0$ in $[l^*, 0]$ when $\rho \geq 0$ or in $[0, l^*]$ when $\rho < 0$. \square

Substituting the explicit formulas for N, N' and N'' in eq. (7), we obtain

$$G_2(l) = \frac{b}{(l^2 + 1)^{\frac{3}{2}}} - \frac{b^2(\rho\sqrt{l^2 + 1} + l)^2}{2(l^2 + 1)(\alpha + b(\rho l + \sqrt{l^2 + 1}))}$$

which leads to the remark that $\frac{G_2(l)}{b} = \tilde{G}_{2, \frac{\alpha}{b}, \rho}(l)$ where $\tilde{G}_{2, x, \rho}(l) := \frac{1}{(l^2 + 1)^{\frac{3}{2}}} - \frac{(\rho\sqrt{l^2 + 1} + l)^2}{2(l^2 + 1)(x + (\rho l + \sqrt{l^2 + 1}))}$, which reduces in general the study of G_2 to the study of a 2-parameters function.

In order to find the zeros of G_2 we should solve $2\frac{\alpha}{b} + b(2 - l^2)\sqrt{l^2 + 1} - \rho^2(l^2 + 1)^{\frac{3}{2}} - 2\rho l^3 = 0$ or equivalently $2\frac{\alpha}{b} - 2\rho l^3 = ((\rho^2 + 1)l^2 + \rho^2 - 2)\sqrt{l^2 + 1}$.

Note that when $\rho = 0$ this equation is explicitly solvable.

6.1.2 Plot of a typical G_2 function

We plot in Figure 7 the function G_2 for the parameters $\alpha = \frac{1}{10}$, $b = \frac{1}{2}$, $\rho = -\frac{3}{10}$.

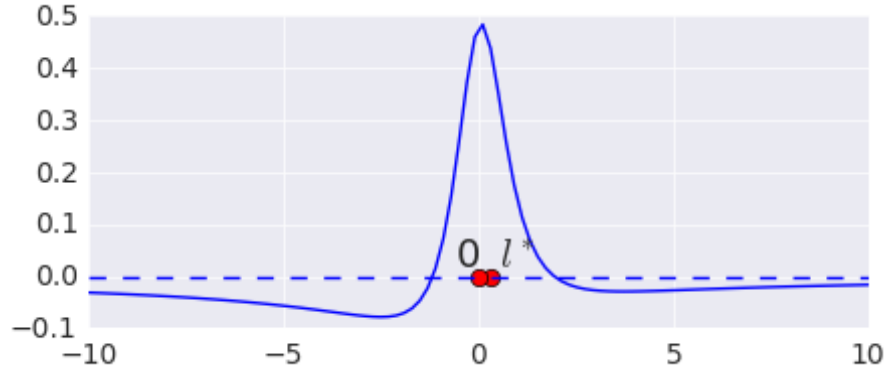


Figure 7: Plot of G_2 with $\alpha = \frac{1}{10}$, $b = \frac{1}{2}$ and $\rho = -\frac{3}{10}$.

6.2 The final condition on σ under (A1)

We recall that the non-negativity of the Durrleman condition in the case of SVI amounts to the non-negativity of the function

$$G(l) = G_1(l) + \frac{1}{2\sigma} G_2(l) \quad (9)$$

where G_1 and G_2 do not depend on σ .

We have proven that:

1. for every (α, b, ρ) with $b(1 \pm \rho) \leq 2$ and $\alpha > F(b, \rho)$, where $F(b, \rho) \leq 0$, there exists an interval for μ such that G_1 is positive on \mathbb{R} (in fact each factor of G_1 is positive on \mathbb{R}). Moreover it is necessary that the conditions on (α, b, ρ) hold and that μ lies in this interval under no arbitrage.
2. for every (α, b, ρ) with $b(1 \pm \rho) \leq 2$ there exists an interval $]l_1, l_2[$ containing 0 and l^* such that $G_2(l) > 0$ iff $l \in]l_1, l_2[$.

We insist here again on the key property brought by the Fukasawa condition that it is *necessary* that G_1 is positive. This structures a lot the picture; previous to Fukasawa's observation, people investigating the positivity of G could not assume this. Another consequence is that under the Fukasawa conditions of section 3, G is granted to be positive on $[l_1, l_2]$.

The last step is to exploit the fact that thanks to our re-parametrization, the dependency of G in σ is very simple. Let $_$ stand for a fixed set of parameters (α, b, ρ, μ) fulfilling the Fukasawa conditions. Then given the fact that $G_2(l) < 0$ for some l , it follows that if G is non-negative everywhere for $(_, \sigma)$, then G is also non-negative everywhere for every $(_, \tau)$ with $\tau > \sigma$. As a consequence, there exists a function $_ \rightarrow \sigma^*(_)$ such that G is non-negative everywhere for $(_, \tau)$ iff $\tau \geq \sigma^*(_)$.

The value of σ^* can be obtained asking the RHS of eq. (9) to be non-negative, which holds for $\sigma \geq \sup_l -\frac{G_2(l)}{2G_1(l)}$. Then

$$\sigma^*(\alpha, b, \rho, \mu) := \sup_{l < l_1 \vee l > l_2} -\frac{G_2(l)}{2G_1(l)}.$$

Since $G_2(l_1^-) = G_2(l_2^+) = 0^-$ and $G_2(\pm\infty) = 0^-$, the maximum of $-\frac{G_2(l)}{2G_1(l)}$ for $l < l_1 \vee l > l_2$ is reached for a finite real value in $] -\infty, l_1] \cup [l_2, +\infty[$.

We have therefore proven the following:

Theorem 6.2 (Necessary and sufficient no Butterfly arbitrage conditions for SVI under (A1)). *No Butterfly arbitrage in SVI entails that G_1 is positive, which requires $b(1 \pm \rho) \leq 2$. Under this condition:*

- each of the factors of the function G_1 is positive on \mathbb{R} if and only if $\alpha > F(b, \rho)$ and $\mu \in I_{\alpha, b, \rho}$;
- for such μ 's, calling $l_1 < 0 < l_2$ the only zeros of G_2 , the function G is positive in $]l_1, l_2[$ for every $\sigma \geq 0$ and the function G is non-negative on \mathbb{R} if and only if $\sigma \geq \sigma^*(\alpha, b, \rho, \mu)$.

6.2.1 Practical computation of σ^*

Computationally, it would be easier to implement an algorithm with bounded intervals for l . It is enough to substitute $h = \frac{1}{l}$ to obtain

$$\sigma^*(\alpha, b, \rho, \mu) := \sup_{\frac{1}{l_1} < h < \frac{1}{l_2}} -\frac{G_2(\frac{1}{h})}{2G_1(\frac{1}{h})}.$$

For h which goes to 0^\pm , the function G_2 goes to 0^- while G_1 is always positive under the Fukasawa conditions. So the function $f(\frac{1}{h}) = -\frac{G_2(\frac{1}{h})}{2G_1(\frac{1}{h})}$ goes to 0^+ . This is a point of minimum for f in the interval $] \frac{1}{l_1}, \frac{1}{l_2} [$ because here the function is always positive.

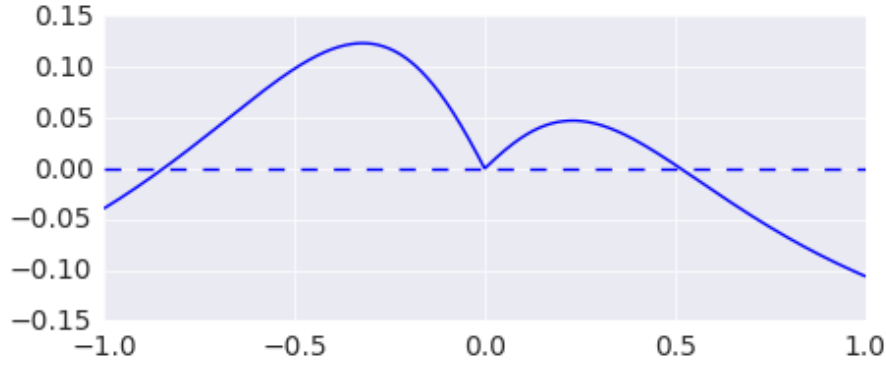


Figure 8: Plot of $f(\frac{1}{h})$ as a function of h , with $b = \frac{1}{2}$, $\rho = -\frac{3}{10}$, $\alpha = \frac{1}{10}$ and $\mu = \frac{1}{10}$.

To numerically compute σ^* we can use an algorithm which finds the maximum of f in $]\frac{1}{l_1}, 0[$ and in $]0, \frac{1}{l_2}[$ and then compares the two maxima.

It can be shown that $f'(\frac{1}{h})$ goes to $\frac{4b(\rho-1)}{(2-b(\rho-1))(2+b(\rho-1))} < 0$ when h goes to 0^- while it goes to $\frac{4b(\rho+1)}{(2-b(\rho+1))(2+b(\rho+1))} > 0$ when h goes to 0^+ . Furthermore, $f'(\frac{1}{l_1}) > 0$ and $f'(\frac{1}{l_2}) < 0$.

We plot in Figure 8 the function $f(\frac{1}{h})$ with $b = \frac{1}{2}$, $\rho = -\frac{3}{10}$, $\alpha = \frac{1}{10}$ and $\mu = \frac{1}{10}$.

The function $f(\frac{1}{h})$ seems to have always three extrema: two points of maximum (one in each interval $]\frac{1}{l_1}, 0[$ and $]0, \frac{1}{l_2}[$) and one point of minimum at 0. The sign of ρ does not imply in which of the two intervals the maximum lies.

For $\rho = 0$ and $\mu = 0$ the maxima have the same height, furthermore the two points of maximum are symmetrical with respect to 0, this last one is also the point of minimum. This follows from the fact that G_2 is symmetric for $\rho = 0$ and that when $\mu = 0$, also G_1 is symmetric.

Note that we have not proven that there is a single maximum on each side of 0. So a strict implementation should take into account the possibility that there are several ones, and use a global optimizer on each side. We strongly conjecture that there is in fact a single maximum on each side.

6.3 Algorithm under (A1)

We can now complete the algorithms stated for the Fukasawa conditions. For the parametrization of the no arbitrage domain, we just need to add the final step which specifies the range of σ :

1. choose $\rho \in]-1, 1[$ and b positive such that $b(1 \pm \rho) \leq 2$ by choosing $b' \in]0, 1[$ and setting $b = b' \frac{2}{1+|\rho|}$;
2. compute numerically $F(b, \rho)$, and parametrize α by setting $\alpha = F(b, \rho) + u$ for positive u ;
3. compute numerically (L_-, L_+) for this value of u , and parametrize μ by setting $\mu = \frac{(1+q)}{2} L_+ + \frac{(1-q)}{2} L_-$ for $q \in]-1, 1[$;
4. compute numerically $\sigma^*(\alpha, b, \rho, \mu)$, and parametrize σ by setting $\sigma = \sigma^* + v$ where $v \geq 0$.

The main benefit of this parametrization is that it is eventually a simple product of intervals:

$$(\rho, b', u, q, v) \in]-1, 1[\times]0, 1[\times]0, \infty[\times]-1, 1[\times [0, \infty[$$

and this is perfectly suitable to feed optimization algorithms working with bounds, like the standard ones in the `scipy.optimize` scientific library.

A drawback to keep in mind is that sampling this product sub-space in a uniform way corresponds to a distorted sampling in the initial space.

There again, we can specify an algorithm which decides whether a SVI parameter lies or not in the no arbitrage domain:

1. $b(1 - \rho) > 2$ or $b(1 + \rho) > 2$: failure of type 1; otherwise:
2. $\alpha \leq F(b, \rho)$: failure of type 2; otherwise:
3. μ not in $I_{\alpha, b, \rho}$: failure of type 3; otherwise:
4. $\sigma < \sigma^*$: failure of type 4.

6.4 The monotonous case (A2)

In all the previous discussion, we have assumed $|\rho| < 1$ to avoid singular cases in our computations. What happens when $|\rho| = 1$? We discuss below the case $\rho = -1$, the case $\rho = 1$ follows by symmetry.

In this case the SVI smile is (convex) decreasing, and reaches its minimum α at infinity, so the domain of α is now $\alpha \geq 0$. Note that the boundary value 0 is allowed, unlike in the regular case, because the implied volatility does not vanish at any finite strike. The negative slope condition requires $b \leq 1$, and the positive (rightmost) one is automatically fulfilled.

Regarding the Fukasawa conditions, the proofs in section 3 still hold with the convention that $l^* = +\infty$ so that N is decreasing. The interval for μ becomes $I_{\alpha, b, -1} =]L_-(l_-(\alpha, b, -1); \alpha, b, -1), +\infty[$, so exactly equal to $I_{\alpha, b, \rho}$ with the convention $L_-(l_-(\alpha, b, 1); \alpha, b, 1) = -\infty$. For $\alpha \geq 0$, we have $L_-(l) < 0$ for every l so $L_-(l_-(\alpha, b, -1); \alpha, b, -1) < 0$ also, and this interval always contains $[0, \infty[$. We can then extend the definition of the Fukasawa threshold to the case $\rho = -1$, putting $F(b, -1) = 0$. This implies that the interval for μ is non-degenerate even when $\alpha = F(b, -1) = 0$.

The function G_2 has only one negative zero l_1 , above which it is always positive with $G_2(+\infty) = 0^+$ while $G_2(-\infty) = 0^-$. So $\sigma^* = \sup_{l < l_1} -\frac{G_2(l)}{2G_1(l)}$.

Theorem 6.3 (Necessary and sufficient no Butterfly arbitrage conditions for SVI, $\rho = -1$). *No Butterfly arbitrage in SVI entails that G_1 is positive, which requires $b \leq 1$ and $\alpha \geq 0$. Under these conditions:*

- each of the factors of the function G_1 is positive on \mathbb{R} if and only if $\mu > L_-(l_-; \alpha, b, -1)$;
- for such μ 's, calling $l_1 < 0$ the only zero of G_2 , the function G is positive on $]l_1, \infty[$ for every $\sigma \geq 0$ and the function G is non-negative on \mathbb{R} if and only if $\sigma \geq \sigma^*(\alpha, b, -1, \mu)$ where $\sigma^*(\alpha, b, -1, \mu) = \sup_{l < l_1} -\frac{G_2(l)}{2G_1(l)}$.

6.4.1 Application: SVI decreasing to zero

Let us consider the case $\rho = -1$ and $a = 0$, so SVI is given by the formula $SVI(k) = b(-(k - \mu\sigma) + \sqrt{(k - \mu\sigma)^2 + \sigma^2})$ with $b \leq 1$.

Can we compute the lower bound for μ ? Consider the equation $g_{-(b, -1)}(l) = 0$ or equivalently from eq. (6), $(-\sqrt{l^2 + 1} + l)(\sqrt{l^2 + 1}(\frac{1}{2} - \frac{b}{4}) + \frac{bl}{4}) + 1 = 0$. Simplifying, we obtain $2l(1 - b)\sqrt{l^2 + 1} =$

$2(1-b)l^2 - (b+2)$ and squaring we find the two solutions $l = \pm \frac{b+2}{2\sqrt{3(1-b)}}$ when $b < 1$. The positive one does not solve the initial equation, so with the notations used in section 3, we finally find $s_- = -\frac{b+2}{2\sqrt{3(1-b)}}$. If $b = 1$, then $s_- = +\infty$. Note that s_- corresponds to l_- when $\alpha = 0$, and we get that $L_-(l_-(0, b, -1); 0, b, -1) = -\sqrt{3(1-b)}$.

So for $\alpha = 0$:

- the Fukasawa conditions are satisfied if and only if $\mu > -\sqrt{3(1-b)}$;
- the unique zero of G_2 does not depend on b and is given by $l_1 = -\frac{1}{\sqrt{3}}$, and the parameters with no arbitrage are eventually given by $b \leq 1$, $\mu > -\sqrt{3(1-b)}$, $\sigma \geq \sigma^*(0, b, -1, \mu) = \sup_{l < -1/\sqrt{3}} -\frac{G_2(l)}{2G_1(l)}$.

7 Calibration experiments

Now that we have parametrized the no arbitrage domain, the design of a calibration algorithm is straightforward:

1. choose an objective function;
2. choose a starting point policy;
3. for the chosen starting points (possibly several of them), run a minimization algorithm of the objective function over the no arbitrage domain;
4. pick up the optimal parameters.

As objective function, we choose the classical least squares criterion, which takes as input the differences of the data and model total variances on the available set of log-forward moneyneess. This will give equal weights to far-from the money points, where the precise value of the implied volatility, and so the accuracy of the calibration, matters less, and to close-to the money ones, which is not a desirable feature: it can be easily patched by adding weights given by the Vegas (computed once for all with the data points), so that the errors are more in line with losses, unit-wise. This would moreover stabilize the calibration from one day to another one, especially on illiquid markets, as discussed in detail in [18].

Now the big question for us is rather whether or not the no arbitrage constraint will deteriorate the quality of the fit, and we will also work on model generated data or on index options data which are liquid ones, whence our choice of a standard non-weighted objective function.

Regarding the starting point policy, we are not big fans of *smart guess strategies* which try to compute the best starting point from the data. Such strategies can work brilliantly in many favorable situations, yet they might fail heavily on data with low quality (e.g. due to a dubious treatment by an internal department), or when faced with new market behavior and configurations. There is a clear risk of over-engineering here also. We would be more confident by using a *set* (with small cardinality) of starting points, possibly produced by a machine learning algorithm duly trained on the markets in scope. We implement a very basic version of this idea, which picks up uniformly generated points within the hyperrectangle of the no arbitrage domain, irrespective of the data.

The scipy function here used is the ‘least_squares’ which lies in the optimize library. The method used is the ‘dogbox’, which handles bounds. The tolerances regarding the change of the cost function (‘ftol’), the change of the independent variables (‘xtol’) and the norm of the gradient (‘gtol’) are

all set at the Python numpy machine epsilon. The maximum number of function evaluations ('max_nfev') is set at 1000.

Even though the arbitrage region does not impose an upper bound for α and σ , we choose arbitrary ones. In particular, we ask

$$\sigma \leq \max \left(\frac{|k_0|}{r}, \frac{|k_N|}{r}, 1.5\sigma^* \right)$$

with r as parameter to be chosen by the user (default value equal to 0.1). This bound is related to the fact that when $\frac{|k_i|}{\sigma}$ is below a threshold r , then the smile is almost flat and this causes uncertainty on the parameters to be chosen.

The upper bound for α is left to be chosen by the user. For the index option data we set $\alpha < 1$ since it is enough to achieve a very good fit, while for the model generated data, in order to have an almost perfect calibration, the upper bound actually depends on the α parameter used to generate data. We set in every case $\alpha < 3$, since we know a priori that all the data are generated with α lower than 3.

We provide below our calibration results on model generated data and then on market data.

7.1 On model data

To check the robustness of the algorithm we firstly run it on data generated by arbitrary SVI parameters with no arbitrage, and on the Axel Vogt parameters. We take a vector of 13 log-forward strikes taken from Table 3.2 of [6].

The parameters chosen for each of the graphs in Figure 9 are arbitrage-free. The red and the blue lines, which represent the total variances generated from the arbitrary parameters and the total variances obtained from the calibrated parameters respectively, overlap.

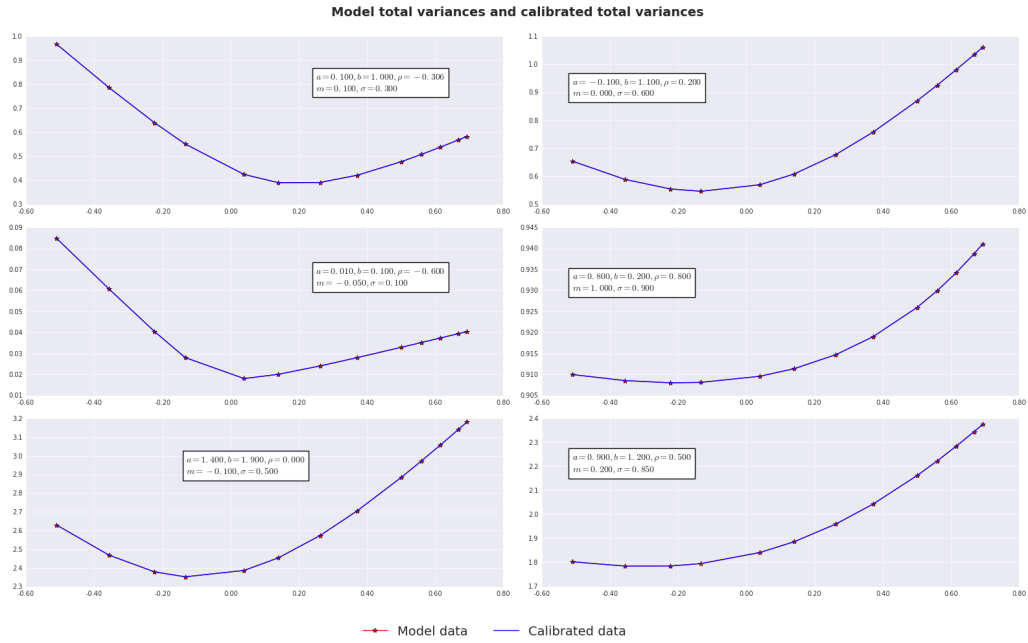


Figure 9: Model total variances generated by arbitrage-free parameters (in red) and calibrated total variances (in blue).

The fact that the fit is excellent can be seen by the Frobenius relative errors in Table 1.

Table 1: Frobenius relative errors for the total variances with arbitrage-free parameters calibrated on model total variances.

	a	b	ρ	m	σ	Relative error ($\times 10^{-16}$)
0	0.10	1.0	-0.306	0.10	0.30	1.48
1	-0.10	1.1	0.200	0.00	0.60	1.63
2	0.01	0.1	-0.600	-0.05	0.10	2.30
3	0.80	0.2	0.800	1.00	0.90	1.77
4	1.40	1.9	0.000	-0.10	0.50	2.35
5	0.90	1.2	0.500	0.20	0.85	2.25

Furthermore, also the Frobenius relative error on the parameters is low (Table 2). This means that the algorithm is robust and recovers the original data.

Table 2: Frobenius relative errors for the parameters calibrated on model total variances.

	a	b	ρ	m	σ	Relative error ($\times 10^{-14}$)
0	0.10	1.0	-0.306	0.10	0.30	0.10
1	-0.10	1.1	0.200	0.00	0.60	0.30
2	0.01	0.1	-0.600	-0.05	0.10	0.06
3	0.80	0.2	0.800	1.00	0.90	20.00
4	1.40	1.9	0.000	-0.10	0.50	0.10
5	0.90	1.2	0.500	0.20	0.85	3.00

7.1.1 Axel Vogt parameters

For a matter of completeness we run our algorithm on the notorious Axel Vogt parameters, which lead to an arbitrage SVI. The original and the calibrated parameters are reported in Table 3 while the graphs of the original and arbitrage-free total variances are shown in Figure 10.

Table 3: Axel Vogt parameters vs best fitting no arbitrage.

	a	b	ρ	m	σ
Original	-0.041	0.1331	0.306	0.3586	0.4153
Calibrated	-0.0198444	0.102745	0.180754	0.266125	0.310459

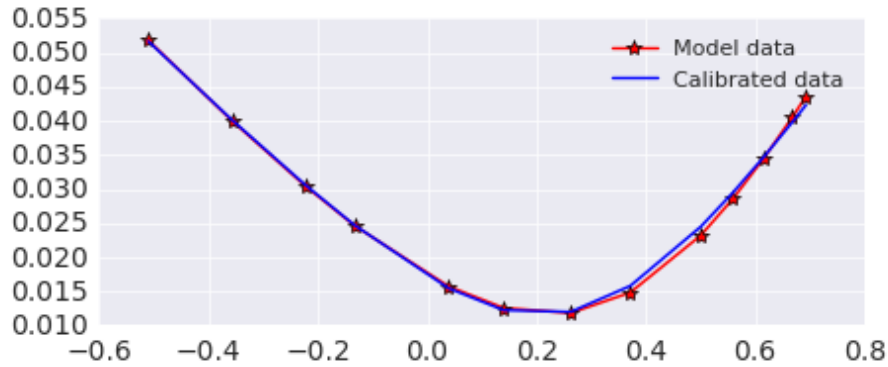


Figure 10: Total variances generated by the Axel Vogt parameters (in red) and total variances with arbitrage-free parameters (in blue).

Of course, the calibration is not perfect as in the previous case and the Frobenius error between the Axel Vogt total variances and the non arbitrage SVI corresponding total variances is 2.15%.

We compare the function g defined in eq. (2) with the original Axel Vogt parameters and the same function with the new arbitrage-free parameters in Figure 11.

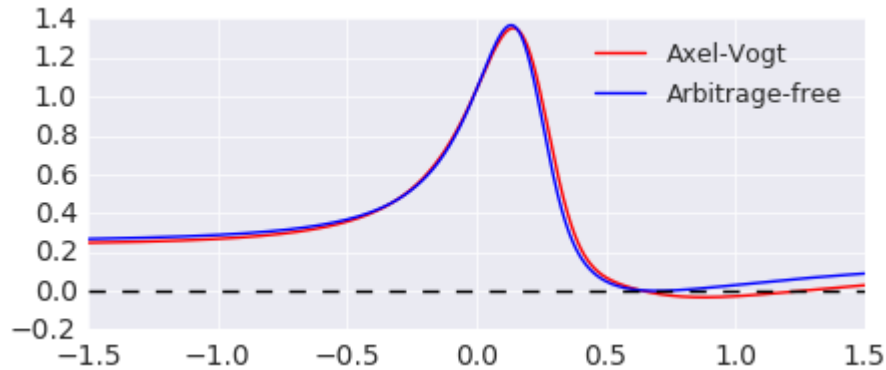


Figure 11: Plot of the functions g with the the Axel Vogt parameters (in red) and with the arbitrage-free parameters (in blue).

From the plot it can be seen that the function g with the new arbitrage-free parameters can be very close to zero, but it is always positive.

In the following study, we compare the results obtained with the new arbitrage-free parameters and the ones with the parameters described in Example 5.1 of [10], which are also arbitrage free. Figure 12 shows that the fit of our new parameters is better than the one of Gatheral and Jacquier.

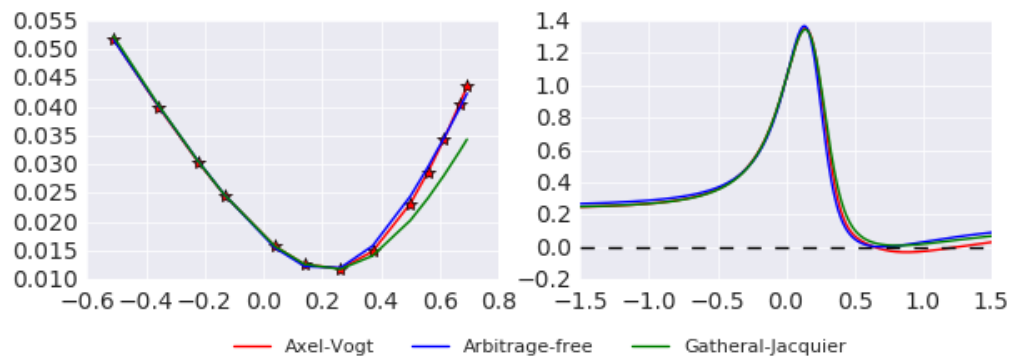


Figure 12: On the left, plot of the total variances generated by the Axel Vogt parameters (in red), the total variances with the arbitrage-free parameters (in blue) and the total variances with the Gatheral-Jacquier parameters (in green). On the right, plot of the function g with the the Axel Vogt parameters (in red), with the arbitrage-free parameters (in blue) and with the Gatheral-Jacquier parameters (in green).

In Table 4 we compare the relative errors on the total variances for the two sets of arbitrage-free parameters.

Table 4: Frobenius relative errors for the total variances with arbitrage-free parameters vs Gatheral-Jacquier parameters.

	a	b	ρ	m	σ	Relative error
Arbitrage-Free	-0.0198444	0.102745	0.180754	0.266125	0.310459	0.022
Gatheral-Jacquier	-0.0305199	0.102717	0.100718	0.272344	0.412398	0.133

7.2 On data from CBOE

We now turn to market data. We work with market data of good quality bought from the CBOE datastore by Zeliade. They cover daily files for the DJX, SPX500 and NDX equity indices, with bid and ask prices.

To obtain implied total variances from the prices, we operate the classical treatment of inferring the discount factor and forward values at each option maturity by performing a linear regression of the (mid) Call minus Put prices with respect to the strike. Since the markets under study are very liquid, the fit is excellent and the residual error is extremely small.

Then, given the discount factor and forward values for each maturity, we are able (after working out the exact maturity of each contract from its code, if not provided explicitly) to compute the implied volatilities, for the Bid and Ask prices.

We feed the objective function with the implied volatility corresponding to the mid price, and plot below the implied volatilities for the calibrated model and the bid and ask market data. Results are reported in Figures 13 to 15.

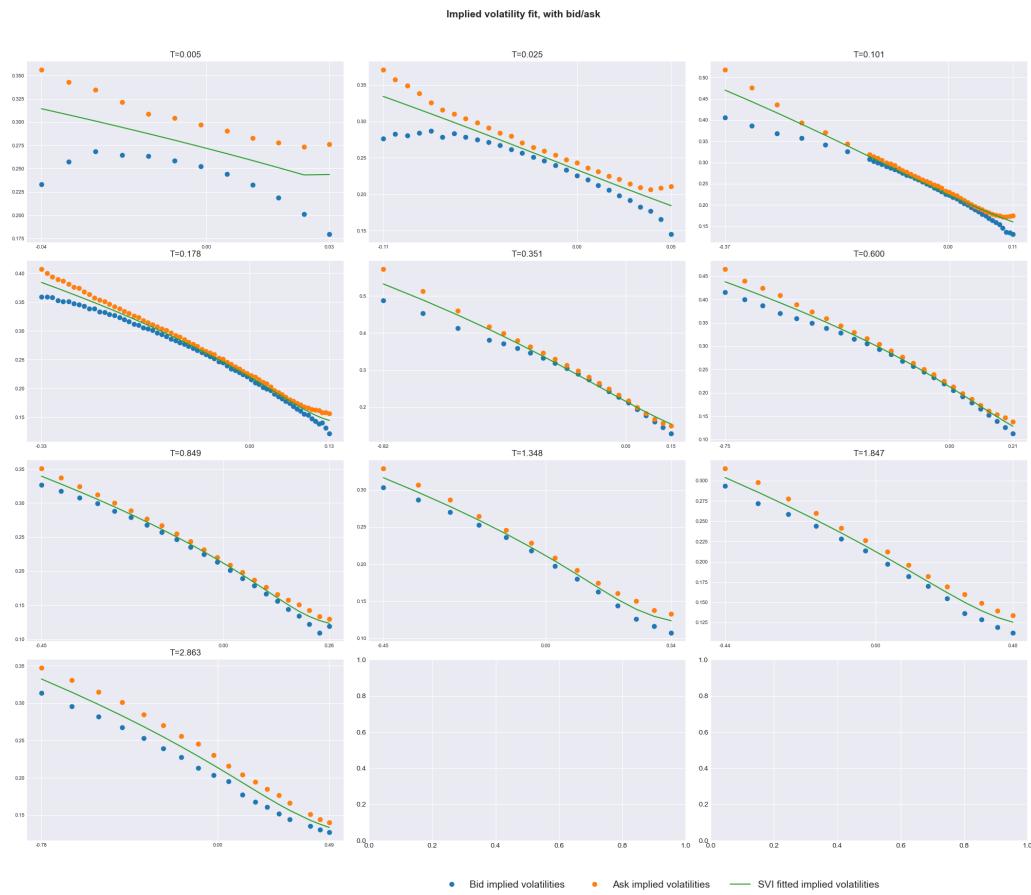


Figure 13: Fitted implied volatilities with arbitrage-free parameters (in red) and bid (in blue) and ask (in green) implied volatilities for the DJX index.

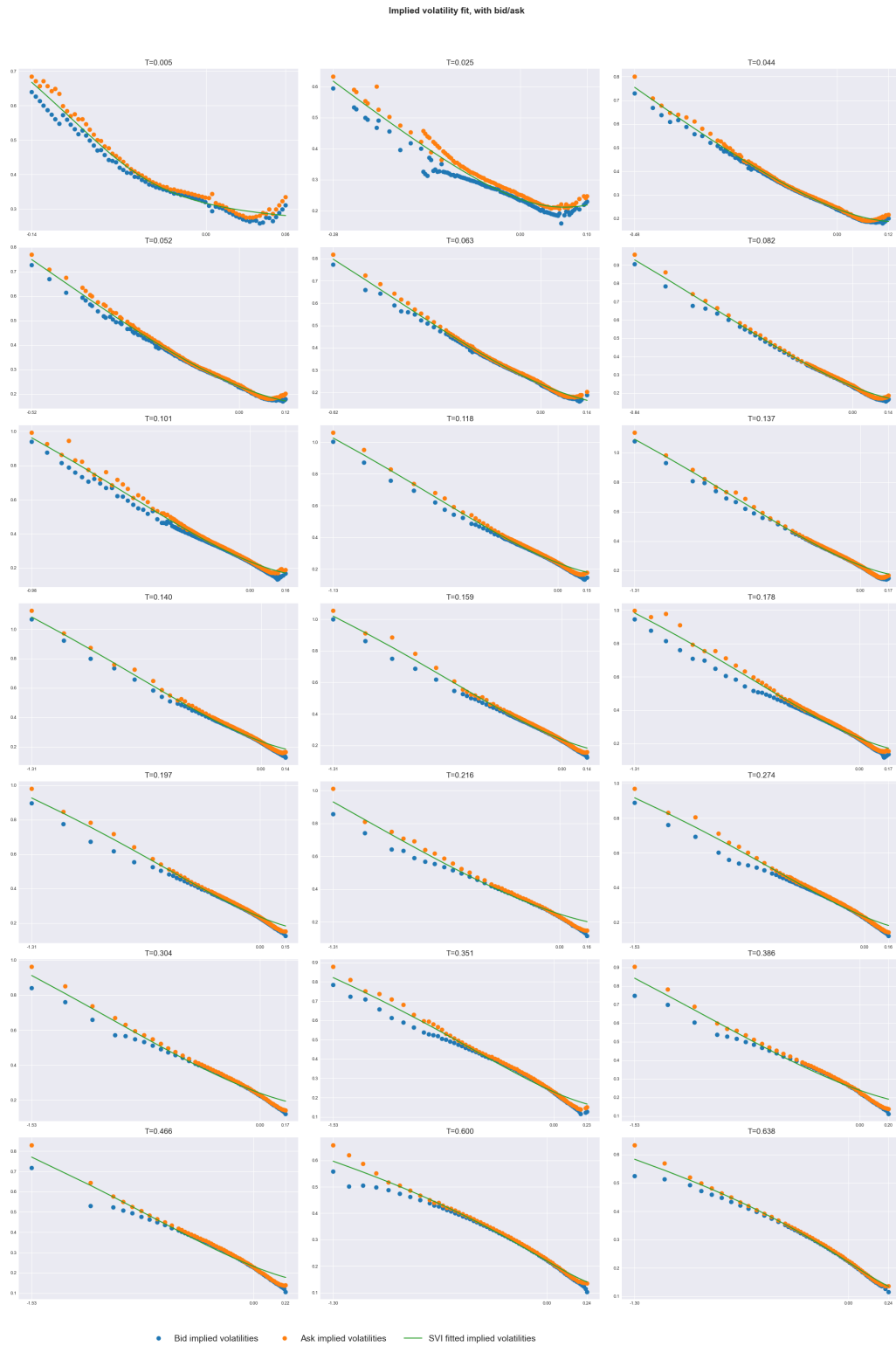


Figure 14: Fitted implied volatilities with arbitrage-free parameters (in red) and bid (in blue) and ask (in green) implied volatilities for the SPX500 index.

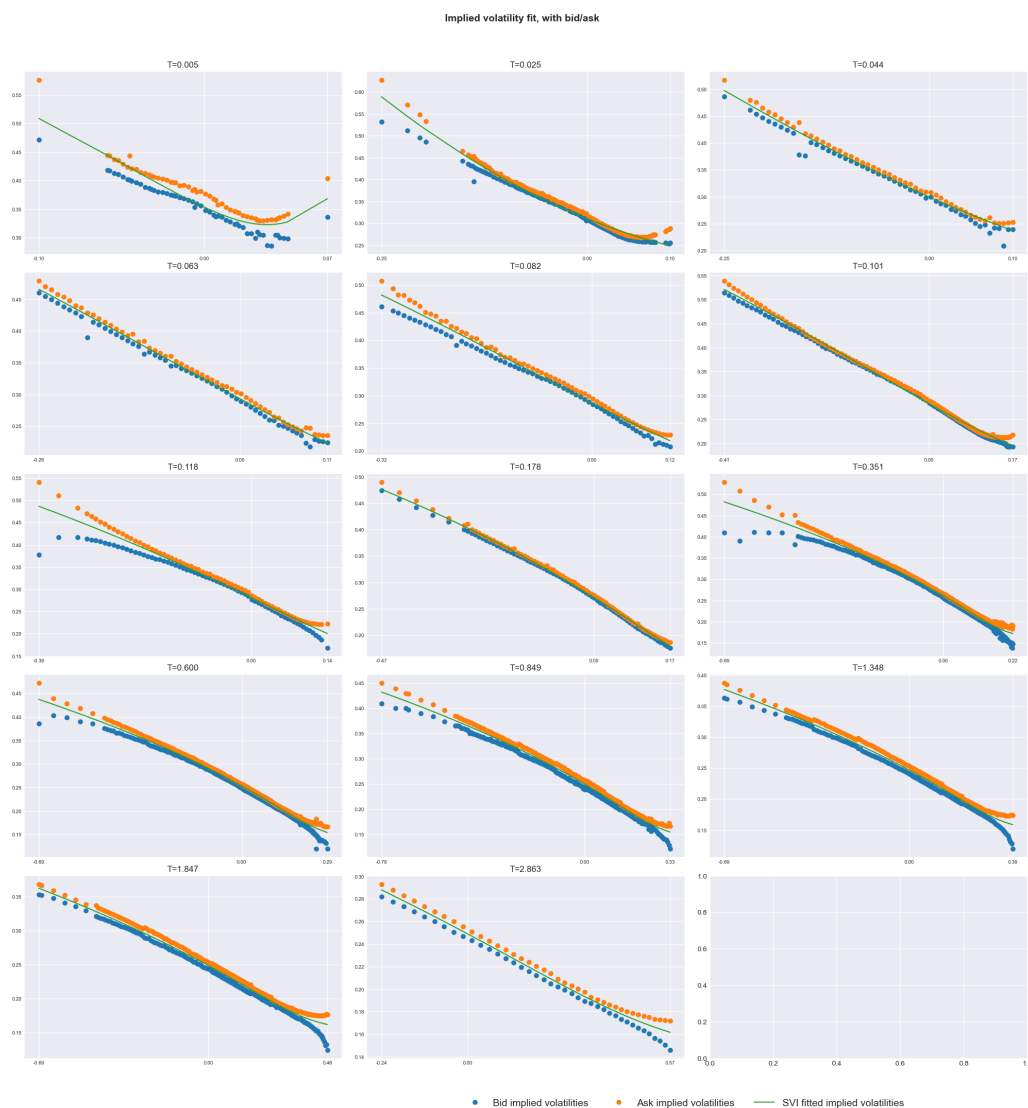


Figure 15: Fitted implied volatilities with arbitrage-free parameters (in red) and bid (in blue) and ask (in green) implied volatilities for the NDX index.

7.3 Discussion

From our experiments we draw several positive conclusions:

- the quality fit is excellent, and there is no big loss resulting from the no arbitrage constraint;
- the implementation we have designed seems sufficiently robust in practice; of course such a statement should be re-assessed continuously;
- the payload of the root finding algorithms used to compute the Fukasawa threshold and the

bounds for μ and σ is not an issue, the calibration is still reasonably fast on a basic chip; the average for each maturity for the DJX data is 51.598 seconds, for the SPX data 36.490 seconds and for the NDX data 44.900 seconds.

Of course, there is room for improvement, at least at the level of the starting point strategy. One could also think of pre-computing the numerical functions computed on the fly, or to design once for all explicit proxies for them, which would speed massively the execution of the algorithm.

8 Conclusion

Fukasawa's remark that the inverse of d_1 and d_2 functions of the Black-Scholes formula have to be increasing under no Butterfly arbitrage, paired with the natural rescaling of the SVI parameters which consists in scaling a and m by σ , allow us to fully describe the domain of no Butterfly arbitrage for SVI.

The no Butterfly arbitrage domain can be parametrized as an hyperrectangle, with 2 downstream algorithms of practical importance: one for checking that a SVI parameter lies or not in the no arbitrage domain, and the other one to effectively perform a calibration. Three functions have to be computed numerically by resorting to root-finding type algorithms; due to the fact that our careful mathematical analysis provided safe bracketing intervals for those functions, this can be achieved in a very quick manner. We provide calibration results on model and market data, the latter showing that there is no loss of fit quality due to imposing the no arbitrage constraint.

This analysis settles one important issue in the SVI saga. Other ones are still pending, like the study of sub-SVI parametrizations with 4 parameters instead of 5, in the spirit of SSVI (which has 3 parameters slice-wise), which could display more parameter stability than SVI and a better fit quality than SSVI; and also the question of the characterization of no Calendar Spread arbitrage for two SVI slices corresponding to different maturities.

A Proof of Proposition 5.5

Proof. Observe that at the point l^* , $\rho\sqrt{l^2+1}+l=0$ and also after computations, $\rho l + \sqrt{l^2+1} = \sqrt{1-\rho^2}$, so we have $g_{\pm(b,\rho)}(l^*) = -\sqrt{1-\rho^2}$.

We have $\frac{d}{d\alpha}L_{\pm}(l_{\pm}) = L'_{\pm}(l_{\pm})\frac{d}{d\alpha}l_{\pm} + \partial_{\alpha}L_{\pm}(l_{\pm}) = \partial_{\alpha}L_{\pm}(l_{\pm})$. Deriving eq. (5) with respect to α , we find $\partial_{\alpha}L_{\pm}(l_{\pm}) = 2\left(\frac{1}{N'(l_{\pm})} \mp \frac{1}{4}\right)$.

Since $N'(l) > 0$ iff $l > l^*$ and $4 \mp N' > 0$, then $\partial_{\alpha}L_{-}(l_{-}) < 0$ and $\partial_{\alpha}L_{+}(l_{+}) > 0$. So the function $\alpha \rightarrow L_{-}(l_{-}, \alpha)$ is decreasing while $\alpha \rightarrow L_{+}(l_{+}, \alpha)$ is increasing. It means that the bounds for μ are an increasing family of sets (possibly empty) parametrized by α . Consider the lower bound, so $l < l^*$. We can write the expression for $g_{-(b,\rho)}$ in another way. We have

$$L'_{-}(l) = 1 + \frac{N'(l)}{2} - \frac{2N''(l)}{N'(l)^2}(\alpha + lN'(l) + N''(l)(l^2 + 1)).$$

Evaluating this in l_{-} , the LHS becomes 0 and we can isolate α , obtaining

$$g_{-(b,\rho)}(l) = \frac{1}{b} \left(\frac{N'(l)^2}{2N''(l)} \left(1 + \frac{N'(l)}{2} \right) - lN'(l) - N''(l)(l^2 + 1) \right). \quad (10)$$

From this expression, we get the derivative of $g_{-(b,\rho)}$ such as $g'_{-(b,\rho)}(l) = \frac{N'(l)^2}{4b} \left(3 - \frac{N'''(l)}{N''(l)^2} (N'(l) + 2) \right)$, which is positive iff the second factor is positive. Substituting with the explicit expressions, we find that this holds iff $\frac{3}{b}\sqrt{l^2+1}(l(2+b\rho) + b\sqrt{l^2+1}) > 0$ or equivalently $-l(2+b\rho) < b\sqrt{l^2+1}$.

Note that since $b(1-\rho) \leq 2$, then $2+b\rho > 0$. If $\rho < 0$ then the equation is true for $l \geq 0$. For negative l s we can square, obtaining that it holds iff $(b^2(\rho^2-1) + 4b\rho + 4)l^2 < b^2$. For $b(1-\rho) < 2$, the coefficient of l^2 is positive, so the inequality holds iff $l > -\frac{b}{\sqrt{b^2(\rho^2-1)+4b\rho+4}} := m_{-}$. Since ρ is negative, $m_{-} < l^*$. So in this case $g_{-(b,\rho)}(l)$ is increasing iff $l > m_{-}$.

If $\rho \geq 0$, we proceed in a similar way taking the square and obtaining that, if $b(1-\rho) < 2$, the inequality holds iff $l > -\frac{b}{\sqrt{b^2(\rho^2-1)+4b\rho+4}} := m_{-}$. If $b \leq \frac{2\rho}{1-\rho^2}$, then $m_{-} \geq l^*$ and g_{-} is always decreasing. Otherwise if $b > \frac{2\rho}{1-\rho^2}$, then $m_{-} < l^*$ and $g_{-(b,\rho)}$ is increasing iff $l > m_{-}$. We can write α as a function of l_{-} , indeed $\alpha = bg_{-(b,\rho)}(l_{-})$. This function has the same monotonicity as $g_{-(b,\rho)}$.

We obtain from the previous analysis that the function $x \rightarrow L_{-}(x; bg_{-(b,\rho)}(x))$ is:

- increasing iff $x < m_{-}$ when $b > \frac{2\rho}{1-\rho^2}$;
- increasing for every $x < l^*$ when $b \leq \frac{2\rho}{1-\rho^2}$.

Note that $N(l) = \alpha + lN'(l) + N''(l)(l^2 + 1)$. Using eq. (5) and substituting α with $bg_{-(b,\rho)}(x)$ considered as in eq. (10), we obtain

$$L_{-}(x; bg_{-(b,\rho)}(x)) = \frac{N'(x)^2}{N''(x)} \left(1 + \frac{N'(x)}{2} \right) \left(\frac{1}{N'(x)} + \frac{1}{4} \right) - x. \quad (11)$$

From here it can be seen that $L_{-}(x; bg_{-(b,\rho)}(x))$ goes to $-l^*$ when x goes to l^{*-} . Similarly, we can do all the equivalent computations for L_{+} . First, the function $g_{+(b,\rho)}$ can be re-written as

$$g_{+(b,\rho)}(l) = \frac{1}{b} \left(\frac{N'(l)^2}{2N''(l)} \left(1 - \frac{N'(l)}{2} \right) - lN'(l) - N''(l)(l^2 + 1) \right)$$

while

$$L_+(x; bg_{+(b,\rho)}(x)) = \frac{N'(x)^2}{N''(x)} \left(1 - \frac{N'(x)}{2}\right) \left(\frac{1}{N'(x)} - \frac{1}{4}\right) - x$$

and even in this case $L_+(x; bg_{+(b,\rho)}(x))$ goes to $-l^*$ when x goes to l^{*+} . We can study the monotonicity of $g_{+(b,\rho)}$, obtaining $g'_{+(b,\rho)}(l) = \frac{N'(l)^2}{4b}(-3 + \frac{N'''(l)}{N''(l)^2}(N'(l) - 2))$.

Considering the second factor and substituting with the explicit expressions, the latter quantity is positive iff $-\frac{3}{b}\sqrt{l^2+1}(-l(2-b\rho) + b\sqrt{l^2+1}) > 0$ or equivalently $l(2-b\rho) > b\sqrt{l^2+1}$.

Here, since $b(1+\rho) \leq 2$, then $2-b\rho > 0$. If $\rho > 0$ then the equation is false for $l \leq 0$. For positive l s we can square, obtaining that it holds iff $(b^2(\rho^2-1) - 4b\rho + 4)l^2 > b^2$. For $b(1+\rho) < 2$, the coefficient of l^2 is positive, so the inequality holds iff $l > \frac{b}{\sqrt{b^2(\rho^2-1)-4b\rho+4}} := m_+$. Since ρ is positive, $m_+ > l^*$. So in this case $g_{+(b,\rho)}(l)$ is increasing iff $l > m_+$.

If $\rho \leq 0$, we proceed in a similar way taking the square and obtaining that, if $b(1+\rho) < 2$, the inequality holds iff $l > \frac{b}{\sqrt{b^2(\rho^2-1)-4b\rho+4}} := m_+$. If $b \leq -\frac{2\rho}{1-\rho^2}$, then $m_+ \leq l^*$ and $g_{+(b,\rho)}$ is always increasing. Otherwise if $b > -\frac{2\rho}{1-\rho^2}$, then $m_+ > l^*$ and $g_{+(b,\rho)}$ is increasing iff $l > m_+$. Remember that the function $\alpha \rightarrow L_+(l_+, \alpha)$ is increasing. To recap, the function $x \rightarrow L_+(x; bg_{+(b,\rho)}(x))$ is:

- increasing iff $x > m_+$ when $b > -\frac{2\rho}{1-\rho^2}$;
- increasing for every $x > l^*$ when $b \leq -\frac{2\rho}{1-\rho^2}$.

If $b \leq -\frac{2\rho}{1-\rho^2}$ then $\rho < 0$ and $b > \frac{2\rho}{1-\rho^2}$ while if $b \leq \frac{2\rho}{1-\rho^2}$ then $\rho > 0$ and $b > -\frac{2\rho}{1-\rho^2}$. This means that $L_+(x; bg_{+(b,\rho)}(x))$ and $L_-(x; bg_{-(b,\rho)}(x))$ cannot be both monotonous.

The last statement of the proposition is a direct consequence to the fact that $\frac{d}{dx}L_{\pm}(x; bg_{\pm(b,\rho)}(x)) = \partial_{\alpha}L_{\pm}(x; bg_{\pm(b,\rho)}(x))bg'_{\pm(b,\rho)}(x)$ where $\partial_{\alpha}L_-(x) < 0$ and $\partial_{\alpha}L_+(x) > 0$. \square

B Computation of $F(b, 0)$

In this appendix we compute $F(b, 0)$ and prove that $F(b, 0) > -b$.

With $\rho = 0$ we have $l^* = 0$ and

$$\begin{aligned} N &= \alpha + b\sqrt{l^2+1}, \\ N' &= \frac{bl}{\sqrt{l^2+1}}, \\ N'' &= \frac{b}{(l^2+1)^{\frac{3}{2}}}. \end{aligned}$$

Consider the particular case $b = 2$. Then we have already shown $F(2, 0) = 0$, which is greater than -2 .

Consider $b \neq 2$. Since $b > \frac{2\rho}{1-\rho^2} = 0$, then the function $l_- \rightarrow L_-(l_-; bg_{-(b,0)}(l_-))$ is increasing iff $l_- < m_-$ where $m_- = -\frac{b}{\sqrt{4-b^2}}$. Furthermore the Fukasawa interval for μ is equal to $I_{\alpha,b,0} =]L_-(l_-(\alpha, b, 0); \alpha, b, 0), -L_-(l_-(\alpha, b, 0); \alpha, b, 0)[$ so it is symmetrical with respect to 0. The Fukasawa threshold $F(b, 0)$ is then the solution to $L_-(l_-(F(b, 0), b, 0); F(b, 0), b, 0) = 0$.

From equation eq. (11) we obtain

$$L_-(l_-; bg_{-(b,0)}(l_-)) = b\frac{l_-^2}{2} \left(2\sqrt{l_-^2+1} + bl_-\right) \left(\frac{\sqrt{l_-^2+1}}{bl_-} + \frac{1}{4}\right) - l_-.$$

For $l_- < 0$, this expression is equal to 0 iff $(8 + b^2)l = -6b\sqrt{l^2 + 1}$ and so iff l_- equals $l_-^* := -\frac{6b}{\sqrt{b^4 - 20b^2 + 64}}$. Then

$$F(b, 0) = bg_{-(b,0)}\left(-\frac{6b}{\sqrt{b^4 - 20b^2 + 64}}\right)$$

where $g_{-(b,0)}(l) = \frac{l^2}{4}(2\sqrt{l^2 + 1} + bl) - \sqrt{l^2 + 1}$.

We now need to prove $g_{-(b,0)}(l_-^*) > -1$ or equivalently $l_-^* < s_-$. From the expression of $g_{-(b,0)}$, we immediately find that s_- satisfies $2(l^2 - 2)\sqrt{l^2 + 1} = -bl^3 - 4$, so we look for a negative root such that $\frac{-bl^3 - 4}{l^2 - 2} > 0$. This happens iff l lies outside the interval $\left[\left(-\frac{4}{b}\right)^{\frac{1}{3}}, -\sqrt{2}\right]$ if $b \leq \sqrt{2}$, or outside the interval $\left[-\sqrt{2}, \left(-\frac{4}{b}\right)^{\frac{1}{3}}\right]$ if $b > \sqrt{2}$. Squaring the previous equation and simplifying by l^3 we find $(4 - b^2)l^3 - 12l - 8b = 0$. Call $P_b(l)$ the LHS.

At 0, this polynomial and its derivative are negative. Its local maximum is at $-\frac{2}{\sqrt{4-b^2}}$ and its value at this point is $\frac{16}{\sqrt{4-b^2}} - 8b$ which is always positive. So the polynomial has two negative roots and a positive one.

We can observe that $P_{\sqrt{2}}(-\sqrt{2}) = P_{\sqrt{2}}\left(\left(-\frac{4}{b}\right)^{\frac{1}{3}}\right) = 0$ with

- $P_b(-\sqrt{2}) = 2\sqrt{2}(b - \sqrt{2})^2 > 0$,
- $P_b\left(\left(-\frac{4}{b}\right)^{\frac{1}{3}}\right) = -\frac{4}{b}\left(b^2 - 3(2b)^{\frac{2}{3}} + 4\right) < 0$.

Then if $b < \sqrt{2}$ the root of interest s_- is the second negative root of the polynomial while if $b \geq \sqrt{2}$ it is the first negative root.

The value of the polynomial in l_-^* is

$$-\frac{8b((b^2 - 16)^2 - 36\sqrt{b^4 - 20b^2 + 64})}{(b^2 - 16)^2}$$

which is positive iff $b < \tilde{b}$ where $\sqrt{\frac{8}{5}} < \tilde{b} < \sqrt{2}$. The derivative of the polynomial evaluated in l_-^* is $\frac{24(5b^2 - 8)}{16 - b^2}$, which is positive iff $b > \sqrt{\frac{8}{5}}$.

Then:

- if $b \leq \tilde{b}$ the polynomial is positive in l_-^* and s_- is its second root, so $l_-^* < s_-$;
- if $\tilde{b} < b < \sqrt{2}$ the polynomial is negative in l_-^* while its derivative is positive and s_- is its second root, so $l_-^* < s_-$;
- finally if $b \geq \sqrt{2}$ the polynomial is negative with a positive derivative in l_-^* so even if s_- is now its first root we have $l_-^* < s_-$.

References

- [1] S. Bossu. *Advanced Equity Derivatives: Volatility and Correlation*. John Wiley & Sons, 2014.
- [2] P. Cohort, J. Corbetta, I. Laachir, and C. Martini. Robust calibration and arbitrage-free interpolation of SSVI slices. *Decisions in Economics and Finance*, 42(2):665–677, 2019.

- [3] B. M. Damghani and A. Kos. De-arbitraging With a Weak Smile: Application to Skew Risk. *Wilmott*, 2013(64):40–49, 2013.
- [4] S. De Marco and C. Martini. Quasi-explicit calibration of Gatheral SVI model. *Zeliade White Paper*, pages 1–15, 2009.
- [5] S. De Marco and C. Martini. Moment generating functions and normalized implied volatilities: unification and extension via Fukasawa pricing formula. *Quantitative Finance*, 18(4):609–622, 2018.
- [6] T. Ferhati. Robust Calibration For SVI Model Arbitrage Free. 2020.
- [7] M. Fukasawa. Normalization for Implied Volatility. 2010.
- [8] M. Fukasawa. The normalizing transformation of the implied volatility smile. *Mathematical Finance*, 22(4):753–762, 2012.
- [9] J. Gatheral. The volatility surface: a practitioner’s guide. *John Wiley & Sons*, 357, 2011.
- [10] J. Gatheral and A. Jacquier. Arbitrage-free SVI volatility surfaces. *Quantitative Finance*, 14(1):59–71, 2014.
- [11] A. Gulisashvili. *Analytically Tractable Stochastic Stock Price Models*. Springer Science & Business Media, 2012.
- [12] G. Guo, A. Jacquier, C. Martini, and L. Neufcourt. Generalized arbitrage-free SVI volatility surfaces. *SIAM Journal on Financial Mathematics*, 7(1):619–641, 2016.
- [13] S. Hendriks and C. Martini. The Extended SSVI Volatility Surface. *Journal of Computational Finance*, 22(5), 2019.
- [14] A. Itkin. One more no-arbitrage parametric fit of the volatility smile. *North Am. J. Econ. Financ*, 2014.
- [15] A. Itkin et al. Fitting Local Volatility: Analytic and Numerical Approaches in Black-Scholes and Local Variance Gamma Models. *World Scientific Books*, 2020.
- [16] T. R. Klassen. Necessary and Sufficient No-Arbitrage Conditions for the SSVI/S3 Volatility Curve. 2016.
- [17] R. W. Lee. The Moment Formula for Implied Volatility at Extreme Strikes. *Mathematical Finance*, 14(3):469–480, 2004.
- [18] L. Nagy and M. Ormos. Volatility Surface Calibration to Illiquid Options. *The Journal of Derivatives*, 26(3):87–96, 2019.
- [19] M. Roper. Arbitrage Free Implied Volatility Surfaces. *preprint*, 2010.
- [20] M. R. Tehranchi. A Black-Scholes inequality: applications and generalisations. *Finance and Stochastics*, 24(1):1–38, 2020.