# The ZABR family

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The underlying's dynamic is dictated by the following SDE:

$$dX_t = X_t z_t \sigma\left(X_t\right) dW_t^X$$
$$dz_t = \varepsilon\left(z\right) dW_t^z$$
$$\left\langle dW_t^z, dW_t^x \right\rangle = \rho dt$$
$$\varepsilon\left(z\right) = \eta z^{\gamma}$$

note that $\eta$ is considered to be the vol of vol.

the normal implied volatility equals:

$$\Sigma\left(X_0, K, T\right) = \frac{X_0 - K}{\delta^{1-\gamma} u\left(\delta^{\gamma-2} \int_K^{X_0} \frac{1}{\sigma(s)} ds\right)}$$

with:

$$\delta = \frac{\sigma_{ATM}}{\sigma\left(X_0\right)}$$

For sake of simplicity we choose $\gamma = 1$, this helps to find:

$$u\left(x\right) = \frac{1}{\eta} \ln\left(\frac{\sqrt{1 - 2\rho\eta x + \eta^2 x^2} + \eta x - \rho}{1 - \rho}\right)$$

The different shapes of implied volatility functions are:

**simple ZABR:**

$$\sigma\left(x\right) = x^{\beta}$$
$$\int_K^{X_0} \frac{1}{\sigma\left(s\right)} ds = \frac{X_0^{1-\beta} - K^{1-\beta}}{1 - \beta}$$

**Zabr Double Beta:**

$$\sigma\left(x\right) = \frac{1}{e^{-\lambda x} x^{-\beta_1} + \left(1 - e^{-\lambda x}\right) x^{-\beta_2}}$$

**Double Zabr**

$$\sigma\left(x\right) = x^{\beta_1}\mathbf{1}_{x\leq\phi_0} + \frac{\phi_0^{\beta_1}}{\delta f}\left(x-\phi_0+\delta f\right)^{\beta_2}\mathbf{1}_{x>\phi_0}$$

$$\delta f = \phi_0\frac{\beta_2}{\beta_1}e^{-d}$$