The extended SSVI volatility surface

Sebas Hendriks[1,2], Claude Martini[1]
[1]Zeliade Systems, Paris [2]Delft University of Technology

Abstract

We extend Gatheral and Jacquier SSVI volatility surface parameterisation by making the correlation maturity-dependent, obtaining necessary and sufficient conditions for no calendar-spread arbitrage. Parametric families for the correlation are provided for which those conditions are explicit. This extension of SSVI typically increases the calibration accuracy for short maturities, and may also be more robust in stressed market conditions.

Keywords and phrases: volatility smile, volatility surface, no arbitrage.

1 Introduction

Explicit formulas for implied volatility are of utmost practical interest: practitioners are able to *explain* the market with few parameters and, most importantly, they do so without the need to go through intricate mathematical models, complex pricing algorithms and the additional layer of an algorithm to compute the implied volatility from the price. Implied volatility surfaces are used directly by risk teams to store historical implied volatility data in a compact way which enables the design of forward-looking risk measures from the history of the calibrated parameters, and also by traders as pre-calibration smoothers.

One of the most successful volatility slice (meaning: only a given maturity is looked at) models is the Stochastic Inspired Volatility model (SVI, [1]) used internally at Merrill Lynch and publicly disclosed by Jim Gatheral in 2004. Several equivalent parameterisations of SVI are available which differ by the geometrical interpretation of the parameters. SVI is just a five-parameters formula, yet despite its simplicity, no necessary and sufficient conditions are known ensuring the absence of arbitrage. In practice SVI is massively used on Equity markets.

Numerous applied research teams have attempted to extend SVI to a whole *surface* model. This has been achieved in 2012 by Gatheral and Jacquier with the Surface SVI (SSVI, [2]) model. SSVI is parameterized with the ATM total variance θ_t, in such a way that SSVI slices at a given maturity t are SVI slices with only 3 parameters (so a sub-family of SVI). This restriction allows to get explicit sufficient conditions for absence of arbitrage, while allowing enough flexibility for calibration, at least on liquid Equity Indexes (most academic papers on SSVI test calibration on SPX 500). Besides the curve θ_t, which is assumed to be read on the market, SSVI has a constant leverage parameter ρ which drives the skew, and a curvature function $\varphi(\theta_t)$. Parametric families have been proposed for φ, like the power law $\eta\theta_t^{-\lambda}$.

SSVI is a major achievement: it provides a tractable arbitrage-free parameterisation of the volaility surface, which calibrates quite well. It has a few shortcomings though. Firstly the ATM volatility is taken directly as a parameter, which is quite unusual in the modeling litterature. To our knowledge, SSVI is the only model to do this. For instance Bergomi's last generation of volatility models takes the Variance Swap curve as a parameter instead, since Variance Swap quotes are directly available on the market. Secondly, SSVI slices depend only on 3 parameters, which is probably too few in some situations and certainly too few for a very accurate fit. Thirdly, the leverage parameter ρ is constant for the whole surface (in other words, among their 3 respective parameters, the same ρ will be shared by 2 slices at different maturities).

An attempt to address the second point has been made in [3], where a general non-parametric shape generalizing the SVI shape at a given maturity has been investigated and necessary and sufficient conditions for no arbitrage have been obtained.

The goal of this paper is to address the third point and investigate an extension of SSVI, which we will call eSSVI, where the correlation parameter ρ may also depend on the maturity through θ_t to allow an even greater accuracy of the surface calibration.

In section 2, we recall Gatheral and Jacquier results on SSVI and motivate the introduction of eSSVI. We start then by studying the consistency of two SSVI slices (so, in discrete time) with different correlation parameters ρ_1 and ρ_2 and obtain an explicit necessary and sufficient condition for the absence of calendar-spread arbitrage. Passing formally to the continuous-time limit, we obtain a set of no (calendar-spread) arbitrage conditions and prove rigorously that they are necessary and sufficient in section 4. In the last section we illustrate the calibration of eSSVI on Equity market data.

It should be noted that we make use of the word *formula* and not of the word *model*: volatility surfaces are to some extent model-free in the sense that they don't prescribe in general a dynamic for the underlying: they are by essence of a purely *static* nature. In principle, a local volatility model can always be associated to a volatility surface, so that American options and path-dependent claims can be priced in a consistent way if needed. In practice, some care is needed with the parameters to ensure that the corresponding local volatility function has no (or only mild) singularities and satisfies the required growth behaviour so that the solution to the associated SDE is a true martingale.

We would like to thank Antoine Jacquier, Stefano De Marco, Ismail Laachir and Cornelis Oosterlee for fruitful discussions at Zeliade or at Delft University. All errors are ours.

2 From the SSVI to the eSSVI surface

Gatheral and Jaquier derived SSVI as an extension of the well known SVI slice formula for a whole volatility surface with fewer parameters at a given maturity. SSVI is specified as a function of the log-forward-moneyness $k := \log\left(\frac{K}{F_t}\right)$, with K the the option strike and F_t the forward price and of the at-the-money (ATM) implied total variance $\theta_t := t\sigma^2_{BS}(0,t)$.

It is assumed that the ATM implied total variance is a function in time of at least class $\mathcal{C}^1$ on $\mathbb{R}_+^*$, and that: $\lim_{t\to 0}\theta_t = 0$. For a smooth *curvature* function: $\varphi: \mathbb{R}_+^* \mapsto \mathbb{R}_+^*$ such that $\lim_{t\to 0}\theta_t\varphi(\theta_t)$ exists in $\mathbb{R}$, SSVI consists in the following formula for the total implied variance $w(k,\theta_t) := t\sigma_{BS}^2(k,t)$:

$$w(k,\theta_t) = \frac{\theta_t}{2}\left(1+\rho\varphi(\theta_t)k+\sqrt{(\varphi(\theta_t)k+\rho)^2+1-\rho^2}\right) \tag{2.1}$$

where ρ is a constant with $|\rho| \leq 1$ representing the correlation between the stock price and its instantaneous volatility.

The main attractive feature of SSVI is the availability of tractable conditions for no arbitrage: a necessary and sufficient condition for no calendar-spread arbitrage, and a sufficient condition for no butterfly arbitrage:

Theorem 2.1 (Theorem 4.1 in [2]). *The SSVI surface is free of calendar-spread arbitrage if and only if:*

1. $\partial_t\theta_t \geq 0$, *for all* $t \geq 0$*;*
2. $0 \leq \partial_\theta(\theta\varphi(\theta)) \leq \frac{1}{\rho^2}\left(1+\sqrt{1-\rho^2}\right)\varphi(\theta)$, *for all* $\theta > 0$.

Theorem 2.2 (Theorem 4.2 in [2]). *The SSVI surface is free of butterfly arbitrage if for all* $\theta > 0$*:*

1. $\theta\varphi(\theta)(1+|\rho|) < 4$*;*
2. $\theta\varphi^2(\theta)(1+|\rho|) \leq 4$.

One of the shortcomings of the SSVI surface is that the correlation ρ is assumed to remain constant across maturities. We would like instead to allow it to depend on the maturity, or, equivalently, on θ_t: this will be the eSSVI, or *extended* SSVI surface, where the constant ρ is replaced by a function: $\rho: \mathbb{R}_+^* \mapsto (-1,1)$, such that:

$$w(k,\theta_t) = \frac{\theta_t}{2}\left(1+\rho(\theta_t)\varphi(\theta_t)k+\sqrt{(\varphi(\theta_t)k+\rho(\theta_t))^2+1-\rho(\theta_t)^2}\right) \tag{2.2}$$

To motivate this extension, it is instructive to go through the following basic experiment: on a set of liquid vanilla option quotes at a given time, calibrate an SSVI *slice* on each available maturity so that the only difference with the classical SSVI is that the parameter ρ is not constant giving a different correlation value for each slice. The typical time-dependency pattern for the calibrated ρ, on Equity indices, is as shown in Figure 1.

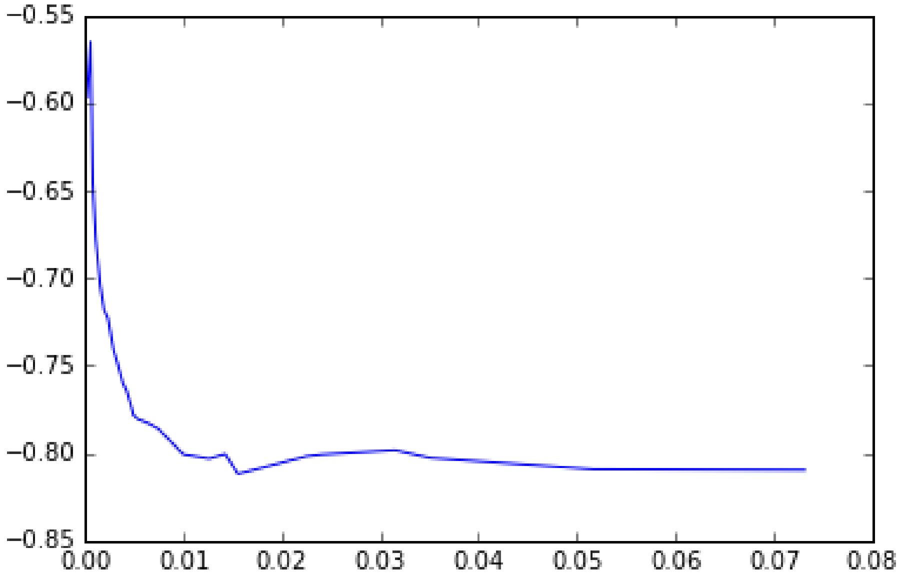

Figure 1: Typical pattern for a slicewise calibrated ρ.

We can see that the correlation parameter does exhibit some sort of constant behaviour for longer maturities, however for shorter maturities it appears that letting ρ vary will yield an enhanced calibration accuracy.

The issue at hand is to obtain no-arbitrage conditions for eSSVI. For butterfly arbitrage, since the two surfaces have similar structures for fixed maturities, nothing has changed and we can use the SSVI condition. For calendar-spread arbitrage conditions the story is more complicated. To determine these conditions we will start by investigating the case of two (e)SSVI slices with two different correlation parameters attached to two different maturities.

3 A tale of two SSVI slices

In this section we consider two SSVI slices: $w(k, \theta_{t_1})$ and $w(k, \theta_{t_2})$ with maturities $t_1 < t_2$, defined by parameters $(\theta_{t_i}, \rho(\theta_{t_i}), \varphi(\theta_{t_i}))$. We simplify notation by defining the following: $w_i := w(k, \theta_{t_i}), \theta_i := \theta_{t_i}$, $\varphi_i := \varphi(\theta_i)$ and $\rho_i := \rho(\theta_i)$. So we have:

$$
\begin{aligned}
w_1 &= \frac{\theta_1}{2}(1 + \rho_1\varphi_1 k + \sqrt{\varphi_1^2 k^2 + 2\rho_1\varphi_1 k + 1}), \\
w_2 &= \frac{\theta_2}{2}(1 + \rho_2\varphi_2 k + \sqrt{\varphi_2^2 k^2 + 2\rho_2\varphi_2 k + 1}).
\end{aligned}
$$

We look for conditions on the parameters ensuring there is no calendar-spread arbitrage, that is $\forall k, w_2 \geq w_1$, i.e.:

$$\theta_2(1+\rho_2\varphi_2 k+\sqrt{\varphi_2^2k^2+2\rho_2\varphi_2 k+1})\geq\theta_1(1+\rho_1\varphi_1 k+\sqrt{\varphi_1^2k^2+2\rho_1\varphi_1 k+1}). \tag{3.1}$$

Remark 3.1. *The same problem is addressed in [2], for two SVI slices, and leads to the study of the roots of a quartic polynomial. The corresponding conditions on the parameters in this case were judged to be too intricate by the authors. Here we consider an easier problem within the sub-family of SSVI slices.*

In particular taking $k=0$ and $k=\pm\infty$ gives:

$$\begin{cases}\theta_2\geq\theta_1 & \text{when } k=0,\\ \theta_2\varphi_2(1+\rho_2)\geq\theta_1\varphi_1(1+\rho_1) & \text{when } k=\infty,\\ \theta_2\varphi_2(1-\rho_2)\geq\theta_1\varphi_1(1-\rho_1) & \text{when } k=-\infty.\end{cases} \tag{3.2}$$

Since it is assumed that all $\rho_i\in(-1,1)$, we get: $\frac{\theta_2}{\theta_1}\geq 1$, $\frac{\theta_2\varphi_2}{\theta_1\varphi_1}\geq\frac{1+\rho_1}{1+\rho_2}$ and $\frac{\theta_2\varphi_2}{\theta_1\varphi_1}\geq\frac{1-\rho_1}{1-\rho_2}$.

Writing: $x:=\varphi_1 k$, $\theta:=\frac{\theta_2}{\theta_1}$, $\varphi:=\frac{\varphi_2}{\varphi_1}$, we get:

$$\theta(1+\rho_2\varphi x+\sqrt{\varphi^2x^2+2\rho_2\varphi x+1})\geq(1+\rho_1 x+\sqrt{x^2+2\rho_1 x+1}), \tag{3.3}$$

and our conditions read:

$$\theta\geq 1,\ \ \theta\varphi\geq\frac{1+\rho_1}{1+\rho_2}\text{ and }\theta\varphi\geq\frac{1-\rho_1}{1-\rho_2}. \tag{3.4}$$

So the question is: assuming the previous necessary conditions hold, is the inequality (3.3) in force, or do we need additional conditions?

3.1 From a quartic to a quadratic polynomial Q

The inequality (3.3) is rewritten as $\alpha+\theta z_2\geq z_1$ where:

$$\begin{aligned}\alpha&:=\theta(1+\rho_2\varphi x)-(1+\rho_1 x)=\theta-1+(\theta\rho_2\varphi-\rho_1)x,\\ z_1&:=\sqrt{x^2+2\rho_1 x+1}\text{ and }z_2:=\sqrt{\varphi^2x^2+2\rho_2\varphi x+1}.\end{aligned}$$

To analyse if the opposite can be true we look at the roots of the equation:

$$\alpha+\theta z_2=z_1. \tag{3.5}$$

Squaring twice, we get the following function $P(z)$:

$$P:=4\alpha^2\theta^2z_2^2-(z_1^2-\alpha^2-\theta^2z_2^2)^2=0, \tag{3.6}$$

where any root of (3.5) is also a root of P and if at least two of these roots exist, the slices intersect and calendar-spread arbitrage exists. For any root of P we again have the same three possibilities:

$$\begin{cases} 2\alpha\theta z_2 = -(z_1^2 - \alpha^2 - \theta^2 z_2^2) & \text{or,} \\ 2\alpha\theta z_2 = z_1^2 - \alpha^2 - \theta^2 z_2^2 \text{ and } \alpha + \theta z_2 = -z_1 & \text{or,} \\ 2\alpha\theta z_2 = z_1^2 - \alpha^2 - \theta^2 z_2^2 \text{ and } \alpha + \theta z_2 = z_1, \end{cases} \tag{3.7}$$

where only the last case corresponds to an intersection of slices. Filling in the values for α, z_1 and z_2 the polynomial P becomes:

$$\begin{aligned} P(x) \;=\; & -\left((\rho_1 - \theta\varphi\rho_2)^2 - (\theta\varphi - 1)^2\right)\left((\rho_1 - \theta\varphi\rho_2)^2 - (\theta\varphi + 1)^2\right) x^4 \\ & +4\theta\left((\rho_1 + \varphi\rho_2)((\rho_1 - \theta\varphi\rho_2)^2 - \theta^2\varphi^2 - 1) + 2\theta\varphi(\varphi\rho_1 + \rho_2)\right) x^3 \\ & +4\theta(\theta - 1)(\rho_1^2 - \theta\varphi^2\rho_2^2 + \theta\varphi^2 - 1)x^2. \end{aligned}$$

Here x^2 is a factor of P, which can again be dropped as $x = 0$ is not a root of (3.5). Hence, a polynomial Q of degree 2 remains, where:

$$P = x^2 Q. \tag{3.8}$$

3.2 Study of the roots of Q

The last stage is to investigate how the roots of the quadratic polynomial Q depend upon the parameters. The discriminant of Q reads:

$$D = 16\theta(\rho_1^2\theta^2\varphi^2\rho_2^2 + \theta^2\varphi^2 1)^2((\rho_1\theta\varphi\rho_2)^2 - (\theta - 1)(\theta\varphi^2 - 1)). \tag{3.9}$$

It is positive and so two real roots exist if and only if:

$$(\rho_1 - \theta\varphi\rho_2)^2 - (\theta - 1)(\theta\varphi^2 - 1) > 0. \tag{3.10}$$

If we define: $Z(x) = z_1^2 - \theta z_2^2 - \alpha^2$, we can see that non zero roots x_1, x_2 of $Q(x)$ also correspond to roots of (3.5) if $Z(x)\alpha < 0$, or when $\alpha \geq 0$ and $\theta z_2 + \alpha > 0$.

3.2.1 Roots as a function of ρ_1 and ρ_2

Clearly, Q only has two real roots on the domain I of all $\boldsymbol{\rho} = (\rho_1, \rho_2)$ with $\rho_1, \rho_2 \in (-1, 1)$ such that (3.10) holds. We define $\boldsymbol{\rho}^* = \left\{\boldsymbol{\rho} : ((\rho_1 - \theta\varphi\rho_2)^2 - (\theta - 1)(\theta\varphi^2 - 1) = 0\right\}$. If we let x be a root of Q, with $\boldsymbol{\rho}$ in this domain, this reads:

$$\begin{aligned} x_{\boldsymbol{\rho}} \;=\; & \frac{2\theta\left((\rho_1 + \varphi\rho_2)((\rho_1 - \theta\varphi\rho_2)^2 - \theta^2\varphi^2 - 1) + 2\theta\varphi(\varphi\rho_1 + \rho_2)\right)}{((\rho_1 - \theta\varphi\rho_2)^2 - (\theta\varphi - 1)^2)((\rho_1 - \theta\varphi\rho_2)^2 - (\theta\varphi + 1)^2)} \\ & \pm \frac{2(\rho_1^2 - \theta^2\varphi^2\rho_2^2 - \theta^2\varphi^2 + 1)\sqrt{\theta\left((\rho_1 - \theta\varphi\rho_2)^2 - (\theta - 1)(\theta\varphi^2 - 1)\right)}}{((\rho_1 - \theta\varphi\rho_2)^2 - (\theta\varphi - 1)^2)\left((\rho_1 - \theta\varphi\rho_2)^2 - (\theta\varphi + 1)^2\right)}. \end{aligned}$$

Using this value of $x_{\boldsymbol{\rho}}$ explicit expressions for α and Z can be obtained, namely $\alpha(x_{\boldsymbol{\rho}})$ and $Z(x_{\boldsymbol{\rho}})$. Furthermore, we also see that, since under (3.4) $(\rho_1 - \theta\varphi\rho_2)^2 < (\theta\varphi - 1)^2, (\theta\varphi + 1)^2$ they are continuous functions of $\boldsymbol{\rho}$, we have the following:

Lemma 3.2. *$(\rho_1, \rho_2) \mapsto \alpha(x_{\boldsymbol{\rho}})Z(x_{\boldsymbol{\rho}})$ does not change sign for $\rho_1, \rho_2 \in (-1, 1)$.*

Proof. Assume the opposite, meaning that for some $\alpha(x_{\boldsymbol{\rho}})Z(x_{\boldsymbol{\rho}}) = 0$ for some $\boldsymbol{\rho}$. Since: $2\alpha\theta z_2 = \pm Z$, we see that $Z(x_{\boldsymbol{\rho}}) = 0 \Leftrightarrow \alpha(x_{\boldsymbol{\rho}}) = 0$, so if $\alpha(x_{\boldsymbol{\rho}}) = Z(x_{\boldsymbol{\rho}}) = 0$:

$$\begin{aligned}
&\begin{cases} z_1^2 - \theta^2 z_2^2 = 0 \\ \theta(1 + \rho_2\varphi x) = 1 + \rho_1 x \end{cases} &&\Rightarrow \quad \begin{cases} \theta^2(\varphi^2 x^2 + 2\rho_2\varphi x + 1) = x^2 + 2\rho_1 x + 1 \\ \theta^2(\rho_2^2\varphi^2 x^2 + 2\rho_2\varphi x + 1) = \rho_1^2 x^2 + 2\rho_1 x + 1 \end{cases} \\
& &&\Rightarrow \quad (1-\rho_2^2)\theta^2\varphi^2 x^2 = (1-\rho_1^2)x^2 \qquad \text{(subtracting the two equations)} \\
& &&\Leftrightarrow \quad \theta^2\varphi^2 = \frac{1-\rho_1^2}{1-\rho_2^2}.
\end{aligned}$$

Under condition (3.4) this means that: $\theta\varphi = \frac{1+\rho_1}{1+\rho_2} = \frac{1-\rho_1}{1-\rho_2}$. Hence: $\rho_1 = \rho_2 = 1$, which contradicts their definition. □

Lemma 3.3. *For $\boldsymbol{\rho}^*$ as defined above, $Z(x_{\boldsymbol{\rho}^*})$ is strictly negative, and since $\alpha(x_{\boldsymbol{\rho}^*}) \geq 0 \Leftrightarrow 0 < \varphi \leq 1$, it holds that: $\alpha(x_{\boldsymbol{\rho}^*})Z(x_{\boldsymbol{\rho}^*}) < 0 \Leftrightarrow 0 < \varphi \leq 1$.*

Proof. First off, note that when $\varphi = 1$ the slices do not cross as we never have two roots since this requires $(\rho_1 - \theta\rho_2)^2 > (\theta - 1)^2$, which does not hold. So we can assume $\varphi \neq 1$. When $\varphi < 1$ it is possible that $\theta\varphi^2 - 1 < 0$, meaning that $(\rho_1 - \theta\varphi\rho_2)^2 - (\theta-1)(\theta\varphi^2 - 1) > 0$, thus we always have two roots for any values of $\rho_1, \rho_2 \in (-1, 1)$ and $\boldsymbol{\rho}^*$ does not exist. If we set in this case $\boldsymbol{\rho}^* = (0, 0)$, we get $\alpha(x_{\boldsymbol{\rho}^*}) = \theta - 1$, $z_1 = \sqrt{x_{\boldsymbol{\rho}^*}^2 + 1}$ and $z_2 = \sqrt{\varphi^2 x_{\boldsymbol{\rho}^*}^2 + 1}$. Hence: $\alpha(x_{\boldsymbol{\rho}^*}) > 0$ always holds and:

$$Z(x_{\boldsymbol{\rho}^*}) = x_{\boldsymbol{\rho}^*}^2(\theta\varphi^2 - 1) - \theta(\theta - 1) < 0, \tag{3.11}$$

also always holds. If however for $\varphi < 1$ it holds that $\theta\varphi^2 - 1 > 0$, then $\boldsymbol{\rho}^*$ always exists, and if we now define $y^2 := (\rho_1 - \theta\varphi^2) = (\theta - 1)(\theta\varphi^2 - 1)$ we can find:

$$\begin{aligned}
x_{\boldsymbol{\rho}^*} &= \frac{2\theta\left((\rho_1 + \varphi\rho_2)((\rho_1 - \theta\varphi\rho_2)^2 - \theta^2\varphi^2 - 1) + 2\theta\varphi(\varphi\rho_1 + \rho_2)\right)}{((\rho_1 - \theta\varphi\rho_2)^2 - (\theta\varphi - 1)^2)((\rho_1 - \theta\varphi\rho_2)^2 - (\theta\varphi + 1)^2)} \\
&= \frac{2(\theta\varphi\rho_2 - \varphi\rho_2 + y)}{(\varphi - 1)(\varphi + 1)}.
\end{aligned}$$

For our function $Z(x) = z_1^2 - \alpha^2 - \theta^2 z_2^2$, we can see that:

$$\begin{aligned}
Z(x) &= x^2 + 2\rho_1 x + 1 - (\theta - 1 - (\rho_1 - \theta\varphi\rho_2)x)^2 - \theta^2(\varphi^2 x^2 + 2\varphi\rho_2 x + 1) \\
&= x^2\left(-(\rho_1 - \theta\varphi\rho_2)^2 - \theta^2\varphi^2 + 1\right) + 2\theta x\left((\rho_1 - \theta\varphi\rho_2) - \rho_2\varphi(\theta - 1)\right) - 2\theta(\theta - 1).
\end{aligned}$$

Now, if we fill $x_{\boldsymbol{\rho}^*}$ into this equation, we get:

$$Z(x_{\boldsymbol{\rho}^*}) = -\frac{2\theta(\theta-1)}{(\varphi-1)^2(\varphi+1)^2}\left((2\varphi y + \rho_2(2\theta\varphi^2 - \varphi^2 - 1))^2 + (1-\rho_2^2)(\varphi-1)^2(\varphi+1)^2\right), \tag{3.12}$$

which is always negative regardless of the sign of y or ρ_2.

Now, we will look at $\alpha(x_{\boldsymbol{\rho}^*})$, which can be written as $\theta - 1 - xy$. If we fill in our value for $x_{\boldsymbol{\rho}^*}$ we get:

$$\alpha(x_{\boldsymbol{\rho}^*}) = -\frac{\theta-1}{(\varphi-1)(\varphi+1)}(2\varphi\rho_2 y + 2\theta\varphi^2 - \varphi^2 - 1). \tag{3.13}$$

It follows that if $\theta > 1$ and $0 < \varphi < 1$, we have $\alpha(x_{\boldsymbol{\rho}^*}) > 0 \Leftrightarrow \frac{2\varphi\rho_2 y + 2\theta\varphi^2 - \varphi^2 - 1}{\varphi - 1} < 0$. The numerator is always strictly positive, since this is equivalent to:

$$(\theta\varphi^2 - 1) + \varphi^2(\theta - 1) > -2\varphi\rho_2 y, \tag{3.14}$$

where we know the LHS of this inequality is always positive. So either the RHS is negative, and it is true, or it is positive. In the latter case, we have:

$$\begin{gathered}4\theta^2\varphi^4 - 4\theta\varphi^4 - 4\theta\varphi^2 + \varphi^4 + 2\varphi^2 + 1 > 4\varphi^2\rho_2^2(\theta^2\varphi^2 - \theta\varphi^2 - \theta + 1) \\ \Leftrightarrow 4\varphi^2(\theta-1)(\theta\varphi^2 - 1)(1-\rho_2^2) + (\varphi^2 - 1)^2 > 0\end{gathered}$$

Since this last inequality always holds, we can conclude that $\alpha(x_{\boldsymbol{\rho}^*}) > 0 \Leftrightarrow \varphi < 1$. Combining this with the result for $Z(x_{\boldsymbol{\rho}^*})$ allows to conclude. $\square$

By combining Lemmas 3.2 and 3.3, we see that if conditions (3.4) hold with $\varphi \leq 1$, then $\alpha(x)Z(x) < 0$. This would mean that in this case the slices never intersect. Next we look at what happens when $\varphi > 1$.

Lemma 3.4. *If P has a root $x_{\boldsymbol{\rho}}$, then:* $(\alpha + \theta z_2)(x_{\boldsymbol{\rho}}) > 0 \Leftrightarrow \varphi > 1$.

Proof. We start by showing that the function $x_{\boldsymbol{\rho}} \mapsto (\alpha + \theta z_2)(x_{\boldsymbol{\rho}})$ never changes sign. This is clear, since if the opposite were true, by continuity there would exist ρ_1, ρ_2 such that $(\alpha + \theta z_2)(x_{\rho_1,\rho_2}) = 0$. However, (3.5) then implies that: $z_1(x_{\rho_1,\rho_2}) = 0$, which is impossible since $\rho_1, \rho_2 \in (-1, 1)$.

Knowing this, we look at the value of $(\alpha + \theta z_2)(x_{\boldsymbol{\rho}^*})$ to see whether this function is always positive or negative. We find that:

$$\begin{aligned}(\alpha + \theta z_2)(x_{\boldsymbol{\rho}^*}) &= \theta\left(\frac{\sqrt{(2\varphi y + \rho_2(2\theta\varphi^2 - \varphi^2 - 1))^2 + (\varphi^2-1)^2(1-\rho_2^2)} - 2\varphi\rho_2 y - 2\theta\varphi^2 + \varphi^2 + 1}{\varphi^2 - 1}\right) \\ &\quad + \frac{2\varphi\rho_2 y + 2\theta\varphi^2 - \varphi^2 - 1}{\varphi^2 - 1},\end{aligned}$$

From the proof of Lemma 3.3 it follows that $\frac{2\varphi\rho_2 y+2\theta\varphi^2-\varphi^2-1}{\varphi^2-1} > 0 \Leftrightarrow \varphi > 1$. Moreover $\sqrt{(2\varphi y + \rho_2(2\theta\varphi^2-\varphi^2-1))^2+(\varphi^2-1)^2(1-\rho_2^2)} - 2\varphi\rho_2 y - 2\theta\varphi^2+\varphi^2+1 = 0$, since:

$$\begin{aligned}
\sqrt{(2\varphi y + \rho_2(2\theta\varphi^2-\varphi^2-1))^2+(\varphi^2-1)^2(1-\rho_2^2)} &= 2\varphi\rho_2 y + 2\theta\varphi^2-\varphi^2-1 \\
\Leftrightarrow \ (2\varphi y + \rho_2(2\theta\varphi^2-\varphi^2-1))^2+(\varphi^2-1)^2(1-\rho_2^2) &= (2\varphi\rho_2 y + 2\theta\varphi^2-\varphi^2-1)^2 \\
\Leftrightarrow \ 4\varphi^2(y^2-\theta^2\varphi^2+\theta\varphi^2+\theta-1)(1-\rho_2^2) &= 0,
\end{aligned}$$

where the second step of squaring both sides is allowed since it is known from the previous results that both sides are always positive. Since $y^2 = \theta^2\varphi^2-\theta\varphi^2-\theta+1$, we see that this final equality holds. Combining this gives: $(\alpha+\theta z_2)(x_{\boldsymbol{\rho}^*}) > 0 \Leftrightarrow \varphi > 1$ and the result follows. □

As a conclusion, we have investigated all the cases in which the roots of Q are also roots of (3.5). We can state the following necessary and sufficient conditions:

Proposition 3.5. *Let $\theta = \frac{\theta_2}{\theta_1}$ and $\varphi = \frac{\varphi_2}{\varphi_1}$. Then, for the 2 slices w_1 and w_2 to satisfy $w_2 \geq w_1$, it is necessary that $\theta > 1$ and $\theta\varphi > \max\left(\frac{1+\rho_1}{1+\rho_2}, \frac{1-\rho_1}{1-\rho_2}\right)$*
and sufficient that

$$\varphi \leq 1 \ \text{or} \ (\rho_1-\theta\varphi\rho_2)^2 \leq (\theta-1)(\theta\varphi^2-1). \tag{3.15}$$

This result is already of practical interest: those consistency constraints can be used to calibrate jointly several maturity slices, and the discrete set of calibrated slices can be extended continuously to an arbitrage-free volatility surface using for instance the method described in [2].

4 Going to continuous time

To obtain continuous-time conditions, we informally write $\theta_1 = \theta$ and $\theta_2 - \theta_1 = d\theta$, and compute the 1st order expansions of the previous conditions in $d\theta$. Likewise, we write $\varphi_2 = \varphi_1 + \varphi\prime(\theta)d\theta$ so that $\frac{\varphi_2}{\varphi_1} = 1 + \frac{\varphi\prime(\theta)d\theta}{\varphi}$. Let us set:

$$\gamma := \frac{1}{\varphi}\frac{d(\theta\varphi)}{d\theta} = 1 + \theta\frac{\varphi\prime(\theta)}{\varphi}. \tag{4.1}$$

4.1 Necessary conditions

We rewrite the necessary condition $\frac{\theta_2}{\theta_1} \geq 1$ as $\frac{d\theta}{dt} \geq 0$. Regarding the second necessary condition:

$$\begin{cases} \theta\varphi \geq \frac{1-\rho_1}{1-\rho_2}, \\ \theta\varphi \geq \frac{1+\rho_1}{1+\rho_2} \end{cases}. \tag{4.2}$$

we set: $\rho_1 = \rho$, and $\rho_2 = \rho + \rho\prime(\theta)d\theta$.
Using $\theta\varphi = \left(1 + \frac{d\theta}{\theta}\right)\left(1 + \frac{\varphi\prime(\theta)}{\varphi}d\theta\right) = 1 + \gamma\frac{d\theta}{\theta}$ we get:

$$\begin{cases} 1 + \gamma\frac{d\theta}{\theta} \geq \frac{1-\rho}{1-\rho-\rho\prime(\theta)d\theta}, \\ 1 + \gamma\frac{d\theta}{\theta} \geq \frac{1+\rho}{1+\rho+\rho\prime(\theta)d\theta}, \end{cases} \tag{4.3}$$

or:

$$\begin{cases} 1 - \rho - \rho\prime(\theta)d\theta + (1-\rho)\gamma\frac{d\theta}{\theta} + \mathcal{O}(d\theta^2) \geq 1 - \rho, \\ 1 + \rho + \rho\prime(\theta)d\theta + (1+\rho)\gamma\frac{d\theta}{\theta} + \mathcal{O}(d\theta^2) \geq 1 + \rho. \end{cases} \tag{4.4}$$

This is equivalent to

$$\begin{cases} (-\theta\rho\prime(\theta) + (1-\rho)\gamma)\frac{d\theta}{\theta} \geq 0, \\ (\theta\rho\prime(\theta) + (1+\rho)\gamma)\frac{d\theta}{\theta} \geq 0. \end{cases} \tag{4.5}$$

Now, given that $d\theta/\theta \geq 0$, setting:

$$\delta := \theta\rho\prime(\theta). \tag{4.6}$$

we can write:

$$\begin{cases} \delta \leq (1-\rho)\gamma, \\ \delta \geq -(1+\rho)\gamma. \end{cases} \tag{4.7}$$

So the temptative continuous time necessary conditions are:

$$\frac{d\theta}{dt} \geq 0 \text{ and } -\gamma \leq \delta + \rho\gamma \leq \gamma.$$

4.2 Sufficient conditions

Besides these necessary conditions, we saw from Proposition 3.5 that the slices did not cross if either: $\varphi \leq 1$ or $(\rho_1 - \theta\varphi\rho_2)^2 \leq (\theta - 1)(\theta\varphi^2 - 1)$. So in this infinitesimal case we get:

$$\begin{aligned} 1 + \frac{\varphi\prime(\theta)}{\varphi}d\theta \leq 1 \quad &\Leftrightarrow \quad \frac{\varphi\prime(\theta)}{\varphi} \leq 0 \\ &\Leftrightarrow \quad 1 + \theta\frac{\varphi\prime(\theta)}{\varphi} \leq 1 \\ &\Leftrightarrow \quad \gamma \leq 1. \end{aligned}$$

Setting $\rho_1 = \rho$ and $\rho_2 = \rho + \rho\prime(\theta)d\theta$ we get:

$$\begin{aligned} (\rho - (\rho + \rho\prime(\theta)d\theta)\theta\varphi)^2 \quad &\leq \quad (\theta - 1)(\theta\varphi^2 - 1) \\ \Leftrightarrow (\rho(1 - \theta\varphi) - \theta\varphi\rho\prime(\theta))^2 \quad &\leq \quad (\theta - 1)(\theta\varphi^2 - 1). \end{aligned}$$

Like in the previous cases, when we substitute θ and φ with $(1+\frac{d\theta}{\theta})$ and $(1+\frac{\varphi\prime(\theta)}{\varphi}d\theta)$ respectively, we get that: $(1-\theta\varphi) = -\frac{d\theta}{\theta}\gamma$, $(\theta-1) = \frac{d\theta}{\theta}$ and $(\theta\varphi^2-1) = \frac{d\theta}{\theta}(2\gamma-1)$. Filling this into the equation above results in:

$$\begin{aligned}
(-\rho\gamma\frac{d\theta}{\theta} - \left(\gamma\frac{d\theta}{\theta}+1\right)\rho\prime(\theta)d\theta)^2 &\leq \left(\frac{d\theta}{\theta}\right)^2(2\gamma-1) \\
\Leftrightarrow \left(\frac{d\theta}{\theta}\right)^2(-\rho\gamma-\theta\rho\prime(\theta))^2 + \mathcal{O}((d\theta)^4) &\leq \left(\frac{d\theta}{\theta}\right)^2(2\gamma-1).
\end{aligned}$$

By eliminating the higher order term of $d\theta$ and by cancelling out $(d\theta/\theta)^2$, using $\delta : \theta\rho\prime(\theta)$, we get:

$$\delta^2 + 2\rho\gamma\delta + (\rho^2\gamma^2 - 2\gamma + 1) \leq 0, \tag{4.8}$$

where this inequality is only true when δ lies between the roots of this polynomial, so that:

$$\delta_{1,2} = \frac{-2\rho\gamma \pm \sqrt{4\rho^2\gamma^2 - 4\rho^2\gamma^2 + 8\gamma - 4}}{2} = -\rho\gamma \pm \sqrt{2\gamma-1}. \tag{4.9}$$

Here the condition becomes $-\sqrt{2\gamma-1} \leq \delta + \rho\gamma \leq \sqrt{2\gamma-1}$.

4.3 eSSVI continuous calendar-spread conditions

We get the conditions: $(\delta+\rho\gamma)^2 \leq \gamma^2$ and either: $(\delta+\rho\gamma)^2 \leq \gamma$ or $(\delta+\rho\gamma)^2 \leq 2\gamma-1$. Observe that when $\gamma \leq 1$, $(\delta+\rho\gamma)^2 \leq \gamma^2$ implies $(\delta+\rho\gamma)^2 \leq \gamma$ so that the second set of conditions is automatically fulfilled. When $\gamma > 1$, the conditions are equivalent to $(\delta+\rho\gamma)^2 \leq 2\gamma-1$. The following result is therefore suggested from the previous small $d\theta$ expansions:

Theorem 4.1. *Define $\gamma := \frac{1}{\varphi(\theta)}$ and $\delta := \theta\partial_\theta(\rho(\theta))$. The eSSVI surface is free of calendar-spread arbitrage if and only if $\partial_t\theta_t > 0$ and*

$$(\delta+\rho\gamma)^2 \leq \begin{cases} \gamma^2 & \text{if } 0 \leq \gamma \leq 1, \\ 2\gamma-1 & \text{if } \gamma > 1. \end{cases} \tag{4.10}$$

It remains to prove this result rigorously. This is done in the Appendix, using the same kind of arguments as the discrete time.

5 Parametric families for $\rho(\theta)$

In this section we obtain a general expression for $\rho(\theta)$ in term of $\varphi(\theta)$, and use it to build parametric families for $\rho(\theta)$ for the two types of power law families for $\varphi(\theta)$ proposed by Gatheral and Jacquier which will fulfill the no calendar-spread conditions up to some maximum maturity corresponding to a total ATM variance θ_{max}. Note that this restriction is not important in practice: typically on Equity markets, liquid vanilla options are available up to 2 years only.

5.1 General function $\varphi(\theta)$

In most practical cases it will be found that $\gamma \leq 1$. Therefore, for the calendar-spread abitrage conditions, looking at the condition $|\delta + \rho\gamma| \leq \gamma$, which can be written as:

$$|\theta\partial_\theta(\rho(\theta)) + \rho(\theta)\partial_\theta(\theta\varphi(\theta))| \leq \frac{\partial_\theta(\theta\varphi(\theta))}{\varphi(\theta)}, \tag{5.1}$$

$$\frac{\partial(\theta\varphi(\theta)\rho(\theta))}{\partial\theta} = \theta\varphi(\theta)\frac{\partial(\rho(\theta))}{\partial\theta} + \rho(\theta)\frac{\partial(\theta\varphi(\theta))}{\partial\theta}. \tag{5.2}$$

Dividing (5.1) by $\varphi(\theta)$ our condition becomes:

$$|\partial_\theta(\theta\varphi\rho)| \leq \partial_\theta(\theta\varphi). \tag{5.3}$$

Therefore there exists a function $u(\theta)$ with values in $[-1,1]$ such that: $\partial_\theta(\theta\varphi(\theta)\rho(\theta)) = u(\theta)\partial_\theta(\theta\varphi(\theta))$. Integrating both sides gives us, using $\theta\varphi(\theta) \to 0$ as $\theta \to 0$:

$$\theta\varphi(\theta)\rho(\theta) = \int_0^\theta u(\tau)\partial_\tau(\tau\varphi(\tau))d\tau. \tag{5.4}$$

If both sides are divided by $\theta\varphi(\theta)$ and the following change of variable is performed: $\tau = r\theta$, with $r \in [0,1]$ the following proposition can be found:

Proposition 5.1. *Let u any measurable function with values in $[-1,1]$. Then the function ρ defined by*

$$\rho(\theta) = \frac{1}{\varphi(\theta)}\int_0^1 u(r\theta)\partial_r(r\varphi(r\theta))dr \tag{5.5}$$

fulfills the no calendar-spread arbitrage conditions.

From this we see that given a curvature function $\varphi(\theta)$ choosing $u(\theta)$ such that for all $\theta > 0$: $\left|\frac{1}{\varphi(\theta)}\int_0^1 u(r\theta)\partial_r(r\varphi(r\theta))dr\right| \leq 1$, produces an admissible $\rho(\theta)$.

From (5.5) and the fact that we can write $u(\theta) = \frac{\partial_\theta(\theta\varphi(\theta)\rho(\theta))}{\partial_\theta(\theta\varphi(\theta))}$, we can either choose a function for $\rho(\theta)$ and check whether the resulting function $u(\theta)$ lies in $[-1,1]$, or choose such a function $u(\theta)$ and obtain $\rho(\theta)$.

Note that we should check that $\rho(\theta)$ lives in $(-1,1)$. To this end defining: $u^*(\theta) = \sup_{r\in[0,1]}|u(r\theta)|$ and using $\frac{\partial_\theta(\theta\varphi(\theta))}{\varphi(\theta)} \geq 0$, we obtain:

$$\begin{aligned}
\left|\frac{1}{\varphi(\theta)}\int_0^1 u(r\theta)\partial_r(r\varphi(r\theta))dr\right| &\leq \frac{1}{\varphi(\theta)}\int_0^1 |u(r\theta)|\partial_r(r\varphi(r\theta))dr \\
&\leq \frac{u^*(\theta)}{\varphi(\theta)}\int_0^1 \partial_r(r\varphi(r\theta))dr \\
&= u^*(\theta)\frac{[r\varphi(r\theta)]_0^1}{\varphi(\theta)} \\
&= u^*(\theta)
\end{aligned}$$

From the assumptions on $u(\theta)$ we know that $u^*(\theta) \leq 1$, and therefore we see that $\rho(\theta) \in (-1, 1)$ always holds.

5.2 The first power-law family

Consider the case $\varphi(\theta) = \eta\theta^{-\lambda}$ so that the condition $\partial_\theta(\theta\varphi(\theta))/\varphi(\theta) \leq 1$ is equivalent to $0 < \lambda \leq 1$. Then it is readily checked from the general expression above that the choice

$$u(\tau) = \rho_0 + \frac{1 + a - \lambda}{1 - \lambda}(\rho_m - \rho_0)\left(\frac{\tau}{\theta_{\max}}\right)^a,$$

leads to:

$$\rho(\theta) = \rho_0 + (\rho_m - \rho_0)\left(\frac{\theta}{\theta_{\max}}\right)^a. \tag{5.6}$$

Under the assumption $a \geq 0$ it is a monotonous function from $(0, \rho_0)$ to (θ_{max}, ρ_m). The requirement $|u(\tau)| \leq 1$ reads:

$$a \leq \begin{cases} \frac{(1-\lambda)(1-\rho_m)}{\rho_m - \rho_0} & \text{if } \rho_0 < \rho_m \\ \frac{(1-\lambda)(1+\rho_m)}{\rho_0 - \rho_m} & \text{if } \rho_0 > \rho_m \end{cases} \tag{5.7}$$

5.3 The second power-law family

Now consider the family $\varphi(\theta) = \eta\theta^{-\lambda}(1+\theta)^{\lambda-1}$. We get then:

$$\begin{aligned} \rho(\theta) &= \frac{1}{\eta\theta^{-\lambda}(1+\theta)^{\lambda-1}} \int_0^1 u(r\theta)\partial_r\left(r\eta(r\theta)^{-\lambda}(1+r\theta)^{\lambda-1}\right) dr \\ &= \frac{1-\lambda}{(1+\theta)^{\lambda-1}} \int_0^1 u(r\theta) r^{-\lambda}(1+r\theta)^{\lambda-2} dr. \end{aligned}$$

The choice

$$u(\tau) = \rho_0 + (\rho_m - \rho_0)\left(1 + \frac{a(1+\tau)}{1-\lambda}\right)\left(\frac{\tau}{\theta_{max}}\right)^a,$$

leads to

$$\rho(\theta) = \rho_0 + (\rho_m - \rho_0)\left(\frac{\theta}{\theta_{\max}}\right)^a.$$

As above it is a monotonous function from $(0, \rho_0)$ to (θ_{max}, ρ_m) assuming $a \geq 0$, and the requirement $|u(\tau)| \leq 1$ reads:

$$a \leq \begin{cases} \frac{(1-\lambda)(1-\rho_m)}{(1+\theta_{max})(\rho_m - \rho_0)} & \text{if } \rho_0 < \rho_m \\ \frac{(1-\lambda)(1+\rho_m)}{(1+\theta_{max})(\rho_0 - \rho_m)} & \text{if } \rho_0 > \rho_m \end{cases}$$

5.3.1 Another example of parametric ρ

We can also go the reverse way to find whether a chosen function of $\rho(\theta)$ satisfies the arbitrage-free conditions. A possible option is to choose the exponential family $\rho(\theta) = \rho_m + (\rho_0 - \rho_m)a^{-\theta}$. It will be a monotonous function for positive a from $\rho(0) = \rho_0$ to $\lim_{\theta\to\infty} \rho(\theta) = \rho_m$.

If we work with the first power-law form for φ, so: $\varphi(\theta) = \eta\theta^{-\lambda}$, this corresponds to the following choice for u:

$$u(\theta) = \rho_m + \left(1 - \frac{\theta \log(a)}{1-\lambda}\right)(\rho_0 - \rho_m)a^{-\theta}$$

Solving this shows that $u(\theta) \in [-1, 1]$ requires, besides the obvious conditions $|\rho_0| \leq 1$ and $|\rho_m| \leq 1$:

$$|\rho_m - \frac{\rho_0 - \rho_m}{1-\lambda} \exp(\lambda - 2)| \leq 1$$

This non linear condition on λ can be solved numerically.

5.4 No butterfly arbitrage

Since eSSVI slices are SSVI slices, we have the same *sufficient* conditions for no butterfly arbitrage as in SSVI. Yet we must pay attention to the fact that ρ is now a function of θ, so that we get a family of conditions to fulfill for the parameters driving the function $\rho(\theta)$. Let us illustrate this in the case of the first power-law family, with the parametric family for ρ designed in section 5.2.

The sufficient condition for no butterfly arbitrage stated in [2] can now be altered to read for all $\theta \leq \theta_{max}$:

$$\eta \leq \min\left(\frac{4\theta^{\lambda-1}}{1+|\rho(\theta)|}, \frac{2\theta^{\lambda-1/2}}{\sqrt{1+|\rho(\theta)|}}\right)$$

Since ρ is monotonous, the condition above is equivalent to

$$\eta \leq \min\left(\frac{4\theta^{\lambda-1}}{1+\max(|\rho_m|, |\rho_0|)}, \frac{2\theta^{\lambda-1/2}}{\sqrt{1+\max(|\rho_m|, |\rho_0|)}}\right).$$

and using the assumption $0 < \lambda < 1/2$, it is still equivalent to

$$\eta \leq \min\left(\frac{4\theta_{\max}^{\lambda-1}}{1+\max(|\rho_m|, |\rho_0|)}, \frac{2\theta_{\max}^{\lambda-1/2}}{\sqrt{1+\max(|\rho_m|, |\rho_0|)}}\right).$$

6 Calibration results

We now go through the steps required to calibrate the eSSVI model to real market data and compare the results with a calibrated plain SSVI surface. We will use the power-law form for φ, so: $\varphi(\theta) = \eta\theta^{-\lambda}$, and the parametric shape computed above for the correlation: $\rho(\theta) = \rho_0 - (\rho_0 - \rho_m)\left(\frac{\theta}{\theta_{max}}\right)^a$.

6.1 eSSVI parameters versus SSVI ones

In this section, we use DJX vanilla option quotes on the 29/01/2016, obtained from the freely available delayed option quotes on the CBOE web site.

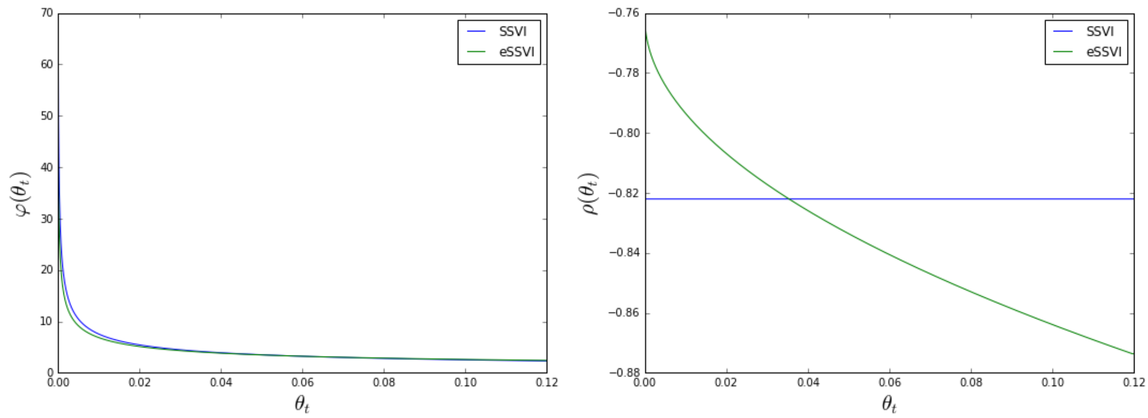

Figure 2: Comparison of the calibrated functions of $\varphi(\theta_t)$ and $\rho(\theta_t)$ for SSVI and eSSVI.

First, the difference in the functions of φ and ρ is shown in Figure 2.We have the two functions for φ, which we found were $0.84\theta_t^{-0.48}$ for SSVI and $0.98\theta_t^{-0.42}$ for eSSVI. The plot shows that there is hardly any difference here. For ρ, we found the constant value in SSVI of: $\rho = -0.82$ and the function found for eSSVI: $\rho(\theta_t) = -0.76 - 0.11\,(\theta_t/2.9)^{0.51}$. From (5.2) it can be seen that these found parameters ensure a surface free of both calendar-spread and butterfly arbitrage. It can be seen from Figure 2 that this constant value is somewhat of a weighted average of our function over the given values of θ_t.

6.2 Calibration accuracy: eSSVI, SSVI and slicewise SVI

Next, from the found parameters we plot the volatilities implied by both the SSVI and eSSVI model and those implied by the market in Figure 3.

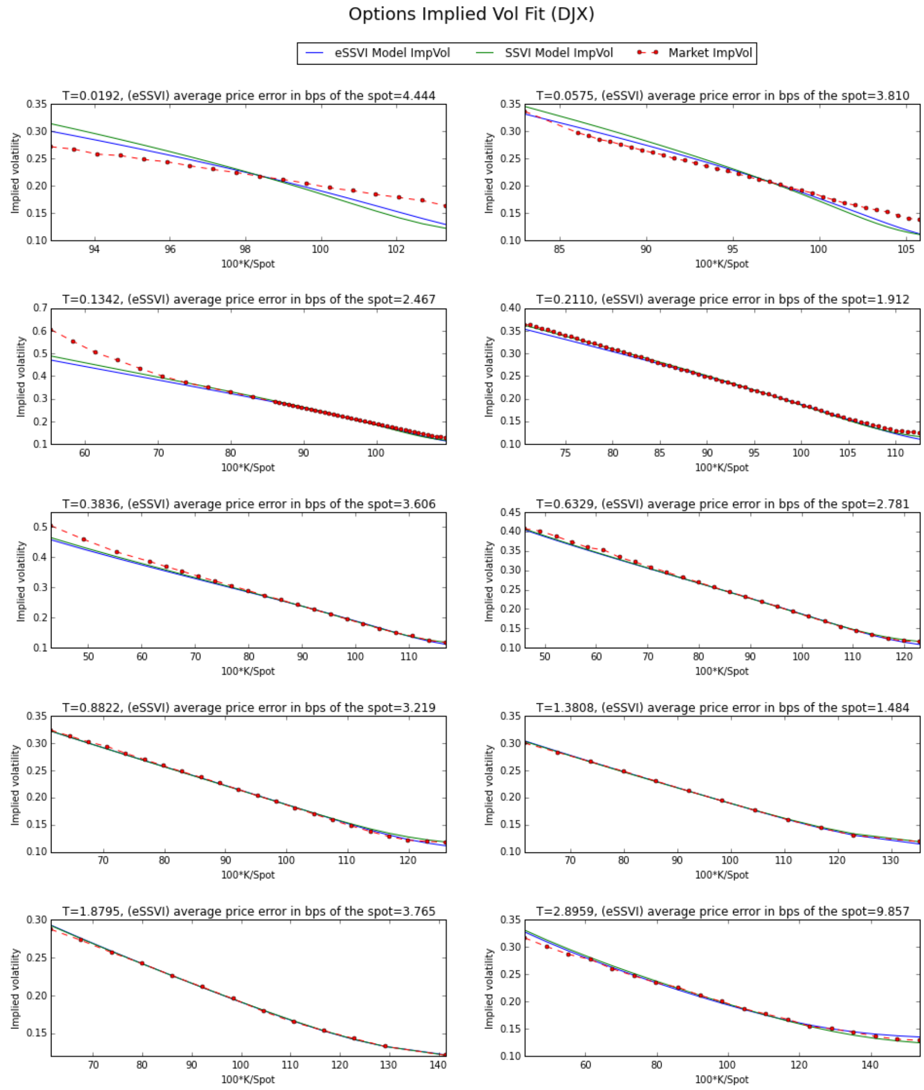

Figure 3: Market bid and ask and SSVI and eSSVI volatility slices for the first, middle and last maturity.

Lastly, the average fitting error, for which we took the average 2-norm difference between model and market price in bps of the spot price per maturity, is displayed in Figure 4 for SSVI, eSSVI and SVI (for each slice). As expected, SVI gives by far the most accurate fit, however it can also be seen that for short maturities eSSVI outperforms SSVI.

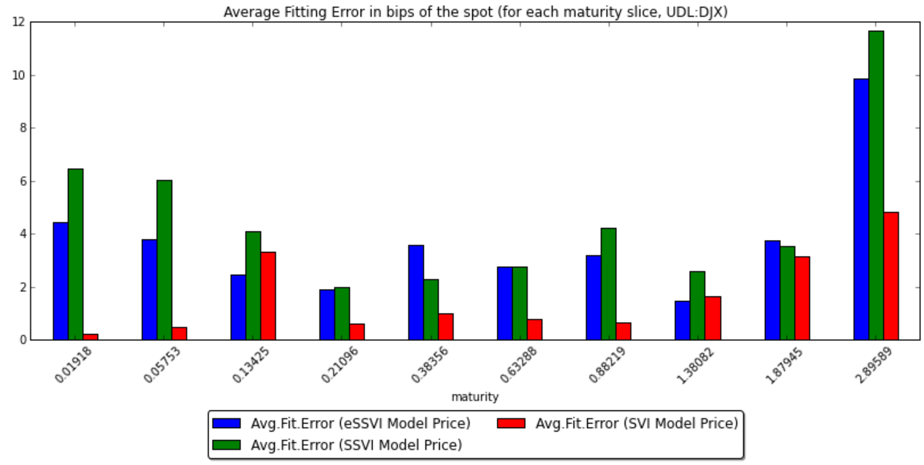

Figure 4: Average fitting error per maturity

7 Conclusion

In this paper we have designed an extension of Gatheral and Jacquier SSVI surface where the leverage parameter ρ is allowed to depend upon the maturity. We have obtained necessary and sufficient conditions for the absence of calendar-spread arbitrage, an explicit generic parameterisation for the correlation parameter, and explicit tractable families for the usual choices of curvature functions $\varphi(\theta)$. Our calibration experiments on Equity Indexes suggest that the gain in fitting quality of eSSVI is small in normal market conditions for long maturities, but that the fitting quality is practically doubled for short maturities. The eSSVI surface might also better fit in stressed market conditions.

References

[1] J. Gatheral, *The Volatility Surface: A Practitioner's Guide*, Wiley Finance, 2006.

[2] J. Gatheral and A. Jacquier, *Arbitrage-free svi volatility surfaces*, Quantitative Finance, 14 (2014), pp. 59–71.

[3] G. Guo, A. Jacquier, C. Martini, and L. Neufcourt, *Generalized arbitrage-free svi volatility surfaces*, SIAM Journal on Financial Mathematics, 7 (2016), pp. 619–641.

A Appendix

In this appendix we provide a direct proof of the no calendar-spread conditions in continous time (section 4.3). We start by stating the following lemma:

Lemma A.1. *For any $x \in \mathbb{R}$ and fixed ζ, γ such that: $\zeta^2 > 2\gamma - 1$ and $\zeta^2 \leq \gamma^2$, the function:*

$$\rho \mapsto (x\zeta + 1)(x^2\gamma + x(\zeta + \rho) + 1), \tag{A.1}$$

where $x^2\gamma + x(\zeta + \rho) + 1 = f(x, \rho)(x\zeta + 1)$ with $f(x, \rho)$ a strictly positive function, does not change sign for any $\rho \in (-1, 1)$.

Proof. If the opposite were true, the continuity of the function entails that: $(x\zeta + 1) = (x^2\gamma + x(\zeta + \rho) + 1) = 0$, meaning that:

$$\begin{cases} x = -1/\zeta, \\ \gamma - \zeta\rho = 0. \end{cases} \tag{A.2}$$

However, since it is required that $|\zeta| \leq \gamma$ and $\rho \in (-1, 1)$, the second equation of (A.2) cannot hold, meaning that (A.1) has no zeros and therefore does not change sign. $\square$

We turn now to the proof of Theorem 4.1, that we recall for convenience:

Proof. The eSSVI surface is free of calendar-spread arbitrage if and only if: $\partial_t w(k, \theta_t) \geq 0$ which is equivalent to the fact that $\partial_t \theta_t \geq 0$ and:

$$x^2\gamma + x(1 + \sqrt{x^2 + 2x\rho + 1})\zeta + x\rho + 1 + \sqrt{x^2 + 2x\rho + 1} \geq 0, \tag{A.3}$$

where $x := k\varphi(\theta)$, and γ and ζ as defined above. When evaluating (A.3) at $x \to \pm\infty$, the following condition is found:

$$|\zeta| \leq \gamma. \tag{A.4}$$

Knowing this, it can be asserted that (A.3) holds when the following equation has at most one real root:

$$x^2\gamma + x(1 + \sqrt{x^2 + 2x\rho + 1})\zeta + x\rho + 1 + \sqrt{x^2 + 2x\rho + 1} = 0. \tag{A.5}$$

In order to evaluate the roots of (A.3), the roots of the following equation are inspected:

$$\begin{aligned} & \left(x^2\gamma + x(\zeta + \rho) + 1\right)^2 = (x\zeta + 1)^2(x^2 + 2x\rho + 1) \\ \Leftrightarrow \quad & x^2\left(x^2(\zeta^2 - \gamma^2) + 2x(\rho\zeta^2 - \zeta\gamma + \zeta - \rho\gamma) + 2\rho\zeta - 2\gamma + 1 - \rho^2\right) = 0. \end{aligned}$$

As it is clear that $x = 0$ is not a real root of this equation, this factor can be dropped and we are left with the roots of the following equation:

$$x^2(\zeta^2-\gamma^2)+2x(\rho\zeta^2-\zeta\gamma+\zeta-\rho\gamma)+2\rho\zeta-2\gamma+1-\rho^2=0. \tag{A.6}$$

There are now two cases to ensure that (A.5) has at most a single real root: either (A.6) has at most a single root, meaning that its discriminant is non-positive, or the two roots of (A.6) are not roots of (A.5) as they arose from the squaring process.

For the first case, the discriminant of (A.6) can be computed expicitly:

$$D=4(\rho\zeta-\gamma)^2(\zeta^2-2\gamma+1), \tag{A.7}$$

which is non-positive if and only if: $\zeta^2\leq 2\gamma-1$.

If this does not hold, (A.6) has two real roots, $x_{1,2}$. However, these roots do not neccesarily correspond to roots of (A.5). This is the case if and only if $(1+x\zeta)(x^2\gamma+x(\zeta+\rho)+1)<0$. Using Lemma A.1, it is sufficient to show that this holds for $\rho=0$, as then the result follows for all other possible values of ρ. In this case we have the following values for $x_{1,2}$:

$$x_{1,2}=\frac{\zeta(1-\gamma)\pm\gamma\sqrt{\zeta^2-2\gamma+1}}{\gamma^2-\zeta^2}.$$

Now, under the conditions $\zeta^2\leq\gamma^2$ and $\zeta^2\geq 2\gamma-1$, the following needs to hold:

$$(1+x_{1,2}\zeta)(x_{1,2}^2\gamma+x_{1,2}\zeta+1)\leq 0. \tag{A.8}$$

Clearly, from the fact that $\gamma\geq 0$, this requires: $1+x_{1,2}\zeta\leq 0$ and therefore also: $x_{1,2}^2\gamma+x_{1,2}\zeta+1\geq 0$. Firstly, $1+x_{1,2}\zeta\leq 0$ can be written as:

$$1+\zeta\frac{\zeta(1-\gamma)+\gamma y}{\gamma^2-\zeta^2}\leq 0\Leftrightarrow\frac{\gamma-\zeta^2+\zeta y}{\gamma^2-\zeta^2}\Leftrightarrow\gamma-\zeta^2+\zeta y\leq 0, \tag{A.9}$$

with $y:=\pm\sqrt{\zeta^2-2\gamma+1}$. As this needs to hold for both the positive and negative value of y, this in turn requires $\gamma-\zeta^2<0$, which yields the following:

$$(\gamma-\zeta^2)^2\geq\zeta^2(\zeta^2-2\gamma+1)\Leftrightarrow\gamma^2\geq\zeta^2, \tag{A.10}$$

which indeed holds.

If now also $\zeta^2>\gamma$ is required, combined with the condition $\gamma^2\geq\zeta^2$, this implies that $\gamma>1$. The following can now be written:

$$x_{1,2}^2\gamma+x_{1,2}\zeta+1\geq 0\Leftrightarrow\frac{\gamma}{(\gamma^2-\zeta^2)^2}(\zeta^4+\zeta^2\gamma^2-3\zeta^2\gamma+\zeta^2-\gamma^3+\gamma^2+(\zeta^3+\zeta\gamma^2-2\zeta\gamma)y)\geq 0. \tag{A.11}$$

This now requires:

$$\zeta^4 + \zeta^2\gamma^2 - 3\zeta^2\gamma + \zeta^2 - \gamma^3 + \gamma^2 \geq 0. \tag{A.12}$$

If this holds, from the fact that (A.11) needs to hold for both values of y, it can be squared in such a sense that it can be written as:

$$\gamma^2 + \zeta^2 - 2\gamma + 1 \geq 0. \tag{A.13}$$

which always holds under the stated requirements. However, (A.12) can also be written as:

$$\zeta^2(\zeta^2 - 2\gamma + 1) + \gamma(\zeta^2 - \gamma)(\gamma - 1) \geq 0, \tag{A.14}$$

which also holds under the stated requirements.

In summary, we can conclude that the statement in the theorem holds if besides $\partial_t\theta_t \geq 0$ it is required that $\zeta^2 \leq \gamma^2$ and either: $\zeta^2 \leq \gamma$ or $\zeta^2 \leq 2\gamma - 1$. Observe that when $\gamma \leq 1$, $\zeta^2 \leq \gamma^2$ implies $\zeta^2 \leq \gamma$ so that the second set of conditions is automatically fulfilled. When $\gamma > 1$, the conditions are equivalent to $\zeta^2 \leq 2\gamma - 1$, so that we get eventually:

$$\zeta^2 \leq \begin{cases} \gamma^2 & \text{if } 0 \leq \gamma \leq 1, \\ 2\gamma - 1 & \text{if } \gamma > 1. \end{cases} \tag{A.15}$$

The result of the theorem then follows. Note that from its definition, ζ can be written as $\delta + \rho\gamma$. From this the result in section 4.3 is proven. $\square$