Calibration Model Scoring with new market induced narrower arbitrability bounds for the implied volatility skew

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Abstract

A new measurement of the calibration ability of a stochastic volatility pricing model is presented. It is focused on the central concept of arbitrage rather than on a simple pricing error measure that gives no clue to know whether the calibrated model is arbitrage free or not when its pricing ability is confronted to the market of vanilla options. To achieve the building of a new test specifically based on the ability of a calibrated model to avoid arbitrage, new abritrability bounds are identified for the implied volatility skew by exploiting the inherent Russian Dolls structure of the market of vanilla options. The good new is these new bounds for the implied volatility skew are much narrower than previously identified bounds in [7]. Then a new calibration test, the **SharkJaw**[1] **test**, is defined on the ground of a simple trading strategy exploiting the potential arbitrage existing between the prices of european binaries given by the tested model and the existing market quotes of vanilla options. This test comes down to test whether a half-butterfly is valued positively for each quoted strike.

Then running this test for the whole set of existing market quotes of vanilla allows an easy calculus of two new calibration scoring that are based on the concept of arbitrage and are ranging from 0 to 10. The first one measures the overall arbitrage freeness of the model while the second grades the exotic added value provided by a model over the use of a Black Scholes for which the volatility is constant. Last, a graphic instrument, the **arbitrogram** allows the modeller to have an immediate view of where his stochastic volatility model fails to be arbitrage free. Said otherwise, what matters the most in your calibration is not to be wrong - apart from the theoretical case of Local Volatility and Local Stochastic Volatility, you will always be - but to be arbitrage free.

By running the Sharkjaw test on all quoted strikes of an example market data sample for a set of classic stochastic volatility models ranging from the Black Scholes model to pure stochastic volatility models (like the Heston model [10], the Bates [1], the Double Heston [5]) and Local Volatility (LV) and Local Stochastic Volatility models (LSV Bergomi one and two factors [3]), a clear objective view of the calibration performance of each model is obtained. Moreover it helps to show that models which calibrates perfectly in theory the market like the LV and the LSV may endure serious difficulties to do so because their calibration methods are hard to master and control. Indeed the relative precision and complexity of the LSV calibration methods tested here (the solving of the related Fokker-Planck 2D PDE with an ADI scheme and the Guyon-Labordère [9] method of particles) shows the "smile problem is not solved" as both methods fail to deliver a perfect fit to the market and exhibit arbitrages.

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[1] The illustration of the test makes two rows of shark-like tooth to appear.

Introduction

The Stochastic Volatility models are the main quantitative tools used to price and hedge exotic derivative product since several years as the constant volatility Black Scholes has proven since 1987 its incapacity to cope with the dynamic of the market of vanilla options. The main feature of the market is the existence of the implied volatility smile which means the Black Scholes implied volatility is not constant and exhibits a strong dependance on the spot value of the underlying as well as a strong and random variability with the time being. To deal with these issues, the industry and the academic world has developed since forty years an important number of different stochastic volatility models ([10], [1], [5] [2], [11], [16], [17], [19, 20], [6], etc.), each one trying 1/ to capture as best as possible all the statistic features of the volatility dynamic 2/ and trying to obtain the best possible fit to the market quotes of vanilla options.

In relation to this second objective, to appreciate the quality of the fit provided by a stochastic volatility model, several classic statistic measures have been extensively used like the Mean Absolute Error (MAE), the Mean Squared Error (MSE) or the RMSE (Relative Mean Squared Error). To define such a statistic, one needs to specify:

1. how to measure the calibration error : for example either like a relative or absolute spread between the market and model prices or like a related or absolute spread between the market and model implied volatilities. While these ones gives a view of the average error of fitting in terms of price, they ignore the question of arbitrage as on the basis of a value of MAE or RMSE no one can say exactly whether the calibrated model is arbitrable or not. This is due to the fact this kind of measure completely ignores the intricated nature of the quoted options ;

2. how to weight each error in the statistic measure as the modeller may want to match specifically more precisely some specific parts of the market. For example, in the equity market of vanilla options, the modellers tend to give more weight to below the money strikes as the volatility smile is usually more pronounced there. The issue here is there is no common admitted practice to define such weights. Note however that as pointed out in [4], the definition of the loss function may have important impact on the quality of the fit.

The common drawback of the loss function is that it completely ignores the main nature of the market of vanilla options. Indeed in all these statistics, the errors are added in a way all prices are independant the ones from the others except that each error is related to a specific weight different for each strike.

In the first section of the paper, the market vanilla is shown to have a Russian Dolls structure that interlinks each quoted strike with its direct quoted neighbors. This implies an arbitrage strategy may be set up to confront the model price of european binaries against the existing market quotes of two vanilla options among which one has the same strike than the binary. This arbitrage is based upon the evaluation of a call triangle (or a put triangle depending of the sense of the strategy) which the pay-off equals $(X_T - K_1)\,\mathbf{1}_{K_1 \leq X_T \leq K_2}$ with K_1 and K_2 two successive quoted strikes such that $K_1 < K_2$(or respectively $(K_2 - X_T)\,\mathbf{1}_{K_1 \leq X_T \leq K_2}$). It comes down to know whether this market triangle has a positive price or not. Due to the asymetric nature of this arbitrage, the test may be run two times on each quoted strike.

In the second section, the link between the market triangles and the implied volatility skew is established. It exhibits the fact market triangles are not instruments replicable by current quoted market call and puts. Then, due to the Russian Dolls structure, the skew is shown to be comprised to be comprised minium and maximum value defined for each quoted strike. This new bounds are narrower than the ones known until now as they are defined in the theorems 3.3 and 3.6 in the paper of [7]. This results allows us to define a new desarbitraging technic exploiting the intricated nature of the Russian Dolls structure of the market of vanilla options.

In the third section, all the preceding results leads us to define two new arbitrage tests: the hard sharkjaw test and the soft shark jaw test. For each quoted strike, the latter test relies on the market bid and ask quotes whereas the former one relies on the market mid quotes.

By relying on these new tests, two new calibration scoring are proposed in the fourth section to measure whether a calibrated is arbitrage free or not. The first scoring is the Absolute Calibration Arbitrability score (ACA) that ranges from 0 (the model is arbitrable for each quoted strike) to 10 (no arbitrage). The second scoring is the Exotic Calibration Score (ECA) that ranges from 0 to 10 and measures the added value of a stochastic volatility over the Black Schole model where the volatility is constant. It helps to know whether the used stochastic volatility is really more arbitrage free than the simpler constant volatility model.

In the ultimate section, two sets of tests are provided to illustrate the new calibration scoring measures over a wide range of stochastic volatility models. A first run of tests is done with a simulated set of data. Then a second run is done using real market coming from the vanilla market of the SP500 index of the 23^{th} on may 2022. To exhibit graphically the scattering of the arbitrages of each model, a new graphic tool, the Arbitrogam is presented.

It helps to see arbitrages are not necessarily appearing only on the ends of the implied volatility surface but arise potentially everywhere especially near the money where the liquidity is assumed to be better and occurrence of arbitrages lower.

In the following presentation, the dynamics of the underlying price and its related processes (mainly its variance) will have their SDE written under the risk neutral probability $\mathbb{Q}$. Except when explicitly mentioned, all standard Brownian motions and jump process are defined as such under the risk neutral probability measure $\mathbb{Q}$. Moreover, without loss of generality, the underlying X_T is assumed to be a pure exponential martingale under $\mathbb{Q}$ such that: $E_{\mathbb{Q}}\left(X_T \left| \mathcal{F}_t\right.\right) = X_t$ with $X_{t_0} = X_0 = 1$ and $\mathcal{F}_t$ the filtration generated by the $\mathbb{Q}$-brownian motions and jump processes driving the SDE of X_t under $\mathbb{Q}$.

1 The Russian Dolls structure of the vanilla market

While calibrating a model to a set of vanilla options, the common approach is to minimize a weighted sum of spread between each market quote and the corresponding model price. Using our notations, if the set of model parameters is noted θ, for any one quoted maturity T, with the L^2-norm and given weights w_j, the calibration problem to solve is to find θ so that:

$$\theta = \arg\min_{\theta} \sum_{j=1}^{n} w_j \left[C_{mkt}\left(K_j, T\right) - C_{mod}\left(K_j, T\right)\right]^2$$

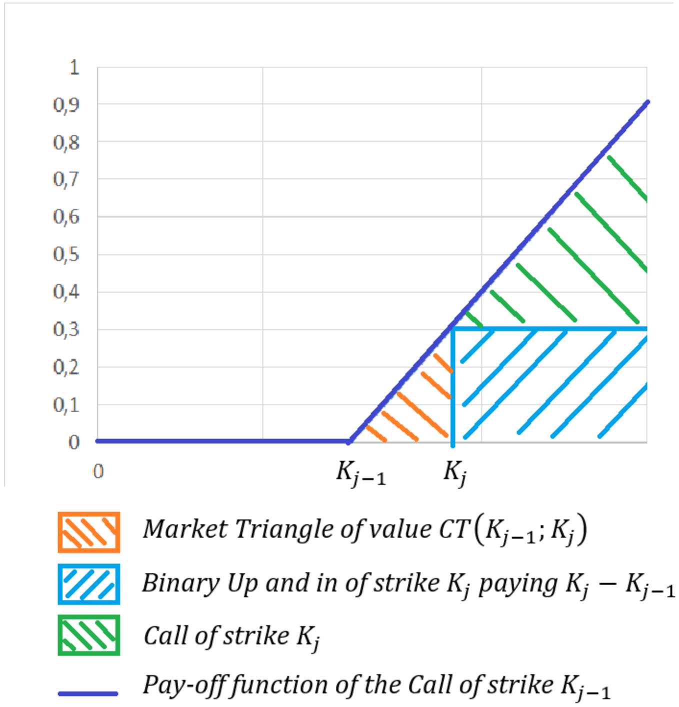
This kind of minimization problem has been viewed and used a countless number of times in every technical paper dealing with the topic of the calibration of a Stochastic Volatility model to market data. Other minimization schemes are also possible that are based on implied volatilities especially when the model is focus on the modelling of the implied volatility.

Then, once the model is calibrated, the usual way to measure the calibration performance is to rely on measure like the RMSE and to classify the models from the one with the smallest RMSE to the one with the biggest one. The main issue of this approach is that it is a “best in class” approach: it tells you which model is the best depending on the chosen measure but it won’t give any clear clue of whether your calibrated model is arbitrage free or not.

However in the two cases, the market structure of the calls [2] is not fully exploited. In particular, the fact calls of successive strike - or correspondingly puts of successive strikes - have their values intertwined the ones in the others meaning the vanilla call market has a kind of russian dolls structure. Taking two successive strikes K_{j-1} and K_j, the following schema shows us how the value of the call of strike K_{j-1} is built upon:

- the value of the call K_j ;
- the value of the Binary Up and in which the strike also equals K_j.

[2] or of the puts. we may define the problem conversely in terms of puts and obtain exactly the same results



So once you calibrate the call of strike K_i, you implicitly make a choice for the Binary Up and In of the same strike, i.e. the in-the-money probability $P(X_T \geq K)$ associated to the bank account numeraire.

Then the only remaining unknown to match the market quote of the nearest call of strike K_{i-1} is the Market Triangle appearing in orange on the schema.

1.1 The Market triangles

More formally, the two calls are linked by the following equation:

$$C(K_{j-1}, T) = C(K_j, T) + (K_j - K_{j-1}) BUI(K_j, T) + CT(K_{j-1}, K_j T) \tag{1.1}$$

with:

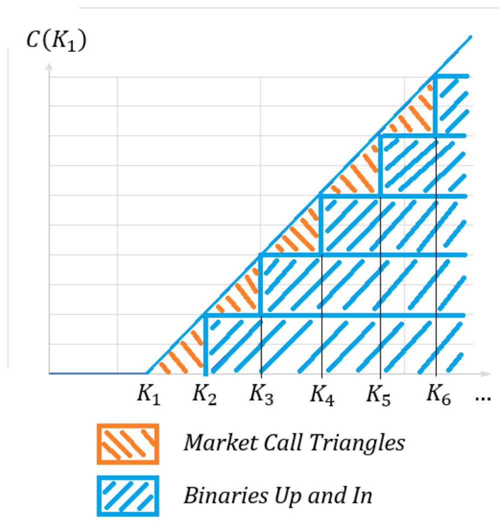
- $C(K, T)$: the undiscounted price of the call of strike K;
- $BUI(K, T)$: the undiscounted price of the up and in binary of strike K paying 1 at the maturity date T (aka the probability $P(X_T \geq K)$);
- $CT(K_{j-1}, K_j, T)$: the undiscounted price of the market triangle depending on the strikes K_j and K_{j-1}. Its pay-off equals:

$$\begin{aligned} PO\_CT(K_{j-1}; K_j, T) &= (X_T - K_{j-1}) \left(\mathbf{1}_{K_j > X_T > K_{j-1}}\right) \\ &= (X_T - K_{j-1}) \left(\mathbf{1}_{X_T > K_{j-1}} - \mathbf{1}_{X_T > K_j}\right) \end{aligned}$$

This last formula shows us again that the market triangle is depending on the pseudo probabilities $P_x(X_T > K) \equiv E_{\mathbb{Q}}(X_T \mathbf{1}_{S_T > K})$ and $P(X_T > K) \equiv E_{\mathbb{Q}}(\mathbf{1}_{X_T > K})$ for $K = K_j$ and $K = K_{j-1}$. Summing the equation 1.1 from $j = 1$ to $j = n_{i,K}$ yields:

$$C(K_1, T) = C(K_n, T) + \sum_{j=1}^{n} (K_j - K_{j-1}) BUI(K_j, T) + \sum_{j=1}^{n} CT(K_{j-1}, K_j T)$$

Graphically the intricate nature of the structure of the call's market appears clearly with the call triangles and the Up and in Binaries being the links of the chain of quotations:



Of course, a similar structure exists for the puts. Noting $PT\left(K_{j-1},K_j,T\right)$ to be the undiscounted price of the market put triangle depending on the strikes K_j and K_{j-1} with its pay-off equalling:

$$PO\_PT\left(K_{j-1};K_j,T\right)=\left(K_j-X_T\right)\left(\mathbf{1}_{X_T>K_{j-1}}-\mathbf{1}_{X_T>K_j}\right)$$

The put triangle is then linked to the Down and In Binary and the two successive puts $P\left(K_{j-1},T\right)$ and $P\left(K_j,T\right)$ by the following arbitrage equation:

$$P\left(K_j,T\right)=P\left(K_{j-1},T\right)+\left(K_j-K_{j-1}\right)BDI\left(K_{j-1},T\right)+PT\left(K_{j-1},K_jT\right) \tag{1.2}$$

$$\Leftrightarrow PT\left(K_{j-1},K_jT\right)=P\left(K_j,T\right)-P\left(K_{j-1},T\right)-\left(K_j-K_{j-1}\right)BDI\left(K_{j-1},T\right)$$

The put triangle may never be of negative value and so:

$$P\left(K_j,T\right)-P\left(K_{j-1},T\right)-\left(K_j-K_{j-1}\right)BDI\left(K_{j-1},T\right)\geq 0$$

$$\Leftrightarrow \frac{P\left(K_j,T\right)-P\left(K_{j-1},T\right)}{\left(K_j-K_{j-1}\right)}\geq BDI\left(K_{j-1},T\right)$$

$$\Leftrightarrow \frac{P\left(K_j,T\right)-P\left(K_{j-1},T\right)}{\left(K_j-K_{j-1}\right)}\geq P\left(X_T<K_{j-1},T\right)$$

This last condition is nothing than the classic put spread arbitrage that tells us the down and in binary has a strictly superior price given by the super-replication strategy coming down to buy $\frac{1}{\left(K_j-K_{j-1}\right)}$ puts of strike K_j and to sell $\frac{1}{\left(K_j-K_{j-1}\right)}$ puts of strike K_{j-1}. Ultimately, by chaining the expression (1.2), we get

$$P\left(K_n,T\right)=\sum_{j=2}^{n}\left(K_j-K_{j-1}\right)BDI\left(K_{j-1},T\right)+\sum_{i=2}^{n}PT\left(K_j-K_{j-1}\right)+P\left(K_1,T\right)$$

and also:

$$P\left(K_j,T\right)=P\left(K_{j-1},T\right)+\left(K_j-K_{j-1}\right)BDI\left(K_{j-1},T\right)+PT\left(K_{j-1},K_jT\right)$$

In the end, for any one given a set of two successive quoted strikes $\{K_{j-1};K_j\}$, the corresponding call triangle $CT\left(K_{j-1},K_jT\right)$ and put triangle $PT\left(K_{j-1},K_j;T\right)$ must have positive values which implies the two following arbitrage free conditions

$$\begin{cases} P\left(K_{j+1},T\right)-P\left(K_j,T\right)-\left(K_{j+1}-K_j\right)BDI\left(K_j,T\right) & \geq 0 \\ C\left(K_{j-1},T\right)-C\left(K_j,T\right)-\left(K_j-K_{j-1}\right)BUI\left(K_j,T\right) & \geq 0 \end{cases} \tag{1.3}$$

The set of conditions 1.3 links for any one pair of quoted strike the prices of the corresponding calls, puts and binaries. Usually, for Equity/index derivatives markets, market quotes are available for the calls and puts but not for the european up and in and down and in binaries $BDI\left(K_{j-1},T\right)$ and $BUI\left(K_j,T\right)$.

1.2 The pseudo probabilities $I(x)$ and $I_x(x)$

These pseudo probabilities take two kind of shape that are the quantities $I(j)$ and $I_x(j)$ defined like:

$$\begin{cases} I(j) = \int_{K_j}^{K_{j+1}} f(x)\,dx \\ I_x(j) = \frac{1}{X_0}\int_{K_j}^{K_{j+1}} xf(x)\,dx \end{cases} \quad \forall i = 0;1;...;n_i$$

with

- $f(x)$: the underlying's density ;
- $K_0 = 0$ so that $I(0) = \int_0^{K_1} f(x)\,dx$ and $I_x(0) = \frac{1}{X_0}\int_0^{K_1} xf(x)\,dx$
- $K_{n_i+1} = +\infty$;
- n_i being the number of quoted strikes for the quoted maturity $T_i \in Cal_T$.

The main advantage of focusing on the pseudo probabilities is they are not depending on a specific choice of density. Indeed, an infinite number of densities may perfectly cope with any set of values of the pseudo probabilities. This helps to define in an unique manner the calibration problem independantly from the density.

Given these last conclusions, the ignition of all this development is to find a way to estimate arbitrage free values of $I(j)$ which is done in the next subsection.

2 The market triangle and the volatility skew

The dependance of the values of the European Binaries on the implied volatility skew is a well known fact. It means the existence of arbitrage due negative values of market triangles is directly linked to the evaluation of this skew by the stochastic volatility model chosen to calibrate the market data.

2.1 What matters is the skew

The quantities $I(j)$ are differences of the cumulative distribution function computed at the quoted strikes for $T \in Cal_T$:

$$I(j) = \int_{K_j}^{K_{j+1}} f(x)\,dx = P(X_{T_i} < K_{j+1}) - P(X_{T_i} < K_j) \quad \forall j = 1;...;n-1 \tag{2.1}$$

$$I(j=0) = P(X_{T_i} < K_1)$$

$$I(j=n) = 1 - P(X_{T_i} < K_n) \tag{2.2}$$

So, a correct estimation of the cdf is the key to get a solution to the calibration problem. A first guess of this cdf may be computed using the Black Scholes Formula. Indeed for any positive maturity T, in a Black Scholes model, the cdf equals one plus the first derivative of the Call price in the strike K:

$$P(X_T < K) = \frac{dC^{BS}(K;\sigma(K))}{dK} + 1 \tag{2.3}$$

with

$$C^{BS}(K,\sigma) = X_0 N(d_1) - KN(d_2)$$

$$d_1 = \frac{\ln(X_0/K)}{\sigma\sqrt{(T-t_0)}} + \frac{\sigma\sqrt{(T-t_0)}}{2}$$

$$d_1 = d2 - \sigma\sqrt{(T-t_0)}$$

$$N(x) = \int_{-\infty}^{x} n(x)\,dx$$

$$n(x) = \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}}$$

and $\sigma(K)$ being the implied volatility.

As the market doesn't react following the BS model, rough estimates of the shape of the volatility smile must be taken into account. That's one of the main critical information provided by the stochastic volatility model chosen to calibrate the market data. Provided the stochastic volatility model is the true model of the market, it comes down to estimate the real market cdf like

$$P\left(X_T < K\right) = \frac{\partial C^{BS}\left(K;\sigma_{SV}\left(K\right)\right)}{\partial K} + \frac{\partial C^{BS}\left(K;\sigma_{SV}\left(K\right)\right)}{\partial \sigma}\frac{\partial \sigma_{SV}\left(K\right)}{\partial K} + 1$$

with $\sigma_{SV}\left(K\right)$ and $\frac{\partial \sigma_{SV}(K)}{\partial K}$ being respectively the implied volatility model and the implied volatility skew calculated by the stochastic volatility model for the pair $\{K;T\}$ of strike and maturity

$$\Leftrightarrow P_{SV}\left(X_T < K\right) = \delta\left(K\right) + \nu\left(K\right)\frac{\partial \sigma_{SV}\left(K\right)}{\partial K} + 1 \tag{2.4}$$

with:

$$\delta = DeltaK^{BS}\left(K;\sigma_{SV}\left(K\right)\right) = \frac{\partial C^{BS}\left(K;\sigma_{SV}\left(K\right)\right)}{\partial K} = -N\left(d_2\left(K;\sigma_{SV}\left(K\right)\right)\right)$$

$$\nu = Vega^{BS}\left(K;\sigma_{SV}\left(K\right)\right) = X_0\sqrt{T}n\left(d_1\left(K;\sigma_{SV}\left(K\right)\right)\right)$$

$$d_1\left(K;\sigma\right) = \frac{\ln\left(X_0/K\right)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$d_2\left(K;\sigma\right) = d_1\left(K;\sigma\right) - \sigma\sqrt{T}$$

The good new with this last formula is the Black Scholes DeltaK and Vega are quantities which the formulas are known in closed formulas. The only remaining needed component is the level $\sigma_{SV}\left(K\right)$ and skew $s\left(K\right) = \frac{\partial \sigma(K)}{\partial K}$. In most of the case, these two quantities are not known explicitly for all K. For any quoted strike K (i.e. $K \in \mathbb{K}$), the only unknown is then only the skew $s\left(K\right)$ as the market implied volatility is retrieved from the quoted prices of vanilla options using the Black Scholes formula. As long as there is no additionnal market information, it means the information $s\left(K\right)$ may only be provided like a guess by the choice of a relevant stochastic volatility model.

Once the skew is estimated for every quoted strike, the fact the cdf (2.4) values of the stochastic volatility model at the quoted strikes is a set of ascending values along the ascending strike abcissa (i.e. the estimated cdf values doesn't contradict the growing nature of the cdf function along the strike) is not sufficient to ensure the data to be arbitrage free. The arbitrage free condition (1.3) must be respected by the estimated values of $\widehat{P}\left(X_T < K_j\right)$. In practice, nothing forces the values of $\widehat{P}\left(X_T < K_j\right)$ to respect this condition.

Crossing the definition of the pseudo probabilities (2.1) with the condition (1.3) links the set of pseudo probabilities $I\left(j\right)$ to the market quotes of vanilla options. As the market triangle also have maximum values equalling the spacing of the two related strikes, we have:

$$\begin{cases} K_j - K_{j-1} \geq C_{mkt}\left(K_{j-1}\right) - C_{mkt}\left(K_j\right) - \left(K_j - K_{j-1}\right)\left(\sum_{k>=j} I\left(k\right)\right) \geq 0 & \forall j = 2;...;n_{i,K} \\ K_{j+1} - K_j \geq P_{mkt}\left(K_{j+1}\right) - P_{mkt}\left(K_j\right) - \left(K_{j+1} - K_j\right)\left(\sum_{k<j} I\left(k\right)\right) \geq 0 & \forall j = 1;...;n_{i,K} - 1 \end{cases}$$

Then the following proposition gives arbitrability bounds for the skews of the model volatility for all the quoted strikes. These bounds are depending sequentially the one from the others like russian dolls are. It means a way to avoid arbitrages is to take into account this "Russian Dolls" structure.

Now, more generally for any triplet of strike $\{K - \Delta K; K; K + \Delta K\}$ with ΔK, the preceding arbitrage conditions on the call and put triangle become:

$$\begin{cases} C_{mkt}\left(K - \Delta K\right) - C_{mkt}\left(K\right) - \Delta K \times P_u^{mod}\left(K\right) \geq 0 \\ P_{mkt}\left(K + \Delta K\right) - P_{mkt}\left(K\right) - \left(\Delta K\right) \times P_d^{mod}\left(K\right) \geq 0 \end{cases} \quad \forall K \tag{2.5}$$

with

$$P_u^{mod}\left(K\right) = P_{SV}\left(X_T > K\right)$$

$$P_d^{mod}\left(K\right) = P_{SV}\left(X_T < K\right) = 1 - P_{SV}\left(X_T > K\right)$$

These two probabilities may be expressed in terms of the implied volatility skew, $\frac{\mathrm{d}\sigma}{\mathrm{d}K}$, the Black-Scholes delta strike for the call, $\delta = \frac{\mathrm{d}C_{BS}}{\mathrm{d}K}$, and the Black Scholes vega $\nu = \frac{\mathrm{d}C}{\mathrm{d}\sigma}$. Indeed we have:

$$P_u^{mod}\left(K\right) = -\frac{\mathrm{d}C\left(K,T\right)}{\mathrm{d}K} = -\delta - \nu\frac{\partial \sigma_{SV}\left(K\right)}{\partial K} \tag{2.6}$$

which leads to:

$$P_d^{mod}(K) = 1 + \delta + \nu \frac{\partial \sigma_{SV}(K)}{\partial K}$$

Remark 1. From (2.6), the strike delta $\frac{\mathrm{d}C(K)}{\mathrm{d}K}$ of any call may be expressed in the terms of the Black Scholes delta strike δ and vega ν like:

$$\frac{\mathrm{d}C(K)}{\mathrm{d}K} = \delta + \nu \frac{\partial \sigma_{SV}(K)}{\partial K} \tag{2.7}$$

it means the real market value of the market delta strike is directly linked to the unknown value of the skew $\frac{\partial \sigma_{SV}(K)}{\partial K}$.

Replacing (2.6) in (2.5) gives:

$$\begin{cases} C_{mkt}(K - \Delta K) - C_{mkt}(K) + \Delta K \times \left(\delta + \nu \frac{\partial \sigma_{SV}(K)}{\partial K}\right) \geq 0 \\ P_{mkt}(K + \Delta K) - P_{mkt}(K) - (\Delta K) \times \left(\delta + 1 + \nu \frac{\partial \sigma_{SV}(K)}{\partial K}\right) \geq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{\partial \sigma_{SV}(K)}{\partial K} \geq \frac{C_{mkt}(K) - C_{mkt}(K - \Delta K) - \Delta K \times \delta}{\nu \Delta K} \\ \frac{P_{mkt}(K + \Delta K) - P_{mkt}(K) - \Delta K \times (\delta + 1)}{\nu \Delta K} \geq \frac{\partial \sigma_{SV}(K)}{\partial K} \end{cases}$$

The delta strike of the put δ_P being such that $\delta_P = 1 + \delta$, we have then:

$$\frac{C_{mkt}(K) - C_{mkt}(K - \Delta K)}{\nu \Delta K} - \frac{\delta}{\nu} \leq \frac{\partial \sigma_{SV}(K)}{\partial K} \leq \frac{P_{mkt}(K + \Delta K) - P_{mkt}(K)}{\Delta K \nu} - \frac{\delta_P}{\nu} \tag{2.8}$$

if $\Delta K \to 0$, we have:

$$\lim_{\Delta K \to 0} \frac{C_{mkt}(K) - C_{mkt}(K - \Delta K)}{\Delta K} = \delta^{mkt}$$

$$\lim_{\Delta K \to 0} \frac{P_{mkt}(K + \Delta K) - P_{mkt}(K)}{\Delta K} = \delta_P^{mkt}$$

This implies 2.8 becomes

$$\frac{\delta^{mkt}}{\nu} - \frac{\delta}{\nu} \leq \frac{\partial \sigma_{SV}(K)}{\partial K} \leq \frac{\delta_P^{mkt}}{\nu} - \frac{\delta_P}{\nu}$$

$$\Leftrightarrow \frac{\delta^{mkt} - \delta}{\nu} \leq \frac{\partial \sigma_{SV}(K)}{\partial K} \leq \frac{\delta_P^{mkt} - \delta_P}{\nu}$$

As

$$\delta^{mkt} = \delta + \nu \frac{\partial \sigma_{mkt}(K)}{\partial K}$$

$$\delta_P^{mkt} = \delta_P + \nu \frac{\partial \sigma_{mkt}(K)}{\partial K}$$

we have:

$$\frac{\delta + \nu \frac{\partial \sigma_{mkt}(K)}{\partial K} - \delta}{\nu} \leq \frac{\partial \sigma_{SV}(K)}{\partial K} \leq \frac{\delta_P + \nu \frac{\partial \sigma_{mkt}(K)}{\partial K} - \delta_P}{\nu}$$

$$\Leftrightarrow \frac{\partial \sigma_{mkt}(K)}{\partial K} \leq \frac{\partial \sigma_{SV}(K)}{\partial K} \leq \frac{\partial \sigma_{mkt}(K)}{\partial K}$$

which means the only way to get this is to have $\frac{\partial \sigma_{SV}(K)}{\partial K} = \frac{\partial \sigma_{mkt}(K)}{\partial K}$. So in the case where all possible $K \in \mathbb{R}+$ correspond to quoted european vanilla option, the only way for a model to match the market everywhere is to have the implied volatility skew $\frac{\partial \sigma_{SV}(K)}{\partial K}$ computed by the model to equal the one of the market $\frac{\partial \sigma_{mkt}(K)}{\partial K}$so trat in the end, what matters is the skew!

2.2 Desarbitraging the naive probabilities

In practice, not all possible strike levels are quoted so that only a countable set of strike are quoted. However the more we have quoted strikes, the more we get information on the value of the market implied volatility skew. In particular, due the Russian Dolls structure of the market of vanilla options, the possible values for the skew must respect the arbitrage free conditions given in the following theorem:

Theorem 2. *For any given quoted maturity T_i, given a set $\mathbb{K}_i$ of quoted strikes, a necessary arbitrage free condition for the volatility skew, $s(K_j) := \frac{d\sigma(K_j,T)}{dK_j}$ for any quoted strike $K_j \in \mathbb{K}_i$ is the following inequality:*

$$s_{min}(j) < s(K_j) < s_{max}(j) \quad \forall j = 1;...;n_{iK}$$

with

$$\begin{cases} s_{min}(j) & = \dfrac{\frac{C(K_j,T)-C(K_{j-1},T)}{(K_j-K_{j-1})}-\delta_j}{\nu_j} \quad \forall j = 2;...;n_{iK} \\ s_{min}(1) & = \dfrac{\frac{C(K_1,T)-X_0}{K_1}-\delta_1}{\nu_1} \quad \text{when } j=1 \end{cases} \tag{2.9}$$

$$\begin{cases} s_{max}(j) & = \dfrac{\frac{C_{mkt}(K_{j+1})-C_{mkt}(K_j)}{K_{j+1}-K_j}-\delta_j}{\nu_j} \quad \forall j = 1;...;n_{iK}-1 \\ s_{max}(n_{i,k}) & = \frac{-\delta_j}{\nu_j} \quad \text{when } j = n_{i,K} \end{cases} \tag{2.10}$$

$$\delta_j = \frac{dC(K_j,T)}{dK_j}$$

with $s_{min}(j) < s_{max}(j)$ if and only if the market quotes $C(K_{j-1})$, $C(K_j)$ and $C(K_{j+1})$ are arbitrage free in terms of butterfly arbitrage.

Proof. See the subsection (6) in the Annex. □

In practice, as no quotations are available for binaries options at least in the Equity/Index market of vanillas, one may rely on the following finite difference scheme to get first naive estimation $\widehat{s}(K)$ of $s(K)$ for all $K \in \mathbb{K}_i$:

For the current quoted maturity T under scrutiny (i.e $T \in Cal_T$), let n_L be the index of the strike with the lowest quoted implied volatility for a total number of $n_{i,K}$ quoted vanilla options of maturity T such that $n_T \geq n_L$. Then for any quoted strike K_j, an estimation $\widehat{s}(K_j)$ of $s(K_j)$ is:

$$\widehat{s}(K_j) = \frac{\widehat{\frac{\mathrm{d}C}{\mathrm{d}K}} - \delta_j}{\nu_j} \quad \forall j = 1;...;n_{i,K}$$

$$\frac{\mathrm{d}\widehat{C(K_j)}}{\mathrm{d}K} = \frac{C_{mkt}(K_{j+1}) - C_{mkt}(K_j)}{K_{j+1}-K_j} \quad \forall j < n_L$$

$$\frac{\mathrm{d}\widehat{C(K_j)}}{\mathrm{d}K} = \frac{C_{mkt}(K_j) - C_{mkt}(K_{j-1})}{K_{j+1}-K_j} \quad \forall j > n_L \,\text{or}\, j = n_L = n_{i,K}$$

$$\frac{\mathrm{d}\widehat{C(K_j)}}{\mathrm{d}K} = \frac{C_{mkt}(K_{j+1}) - C_{mkt}(K_{j-1})}{(K_{j+1}-K_{j-1})} \quad \forall j = n_L < n_{i,K}$$

The reason for which the same finite difference scheme is not used for all the strike is to take into account the U Shape of the smile that may appear for certain strikes. In particular, the last formula, which is a centered finite difference approximation of the skew, is used for the strike where the skew of the implied volatility is suspected to be near from a change of sign. In the end, the estimated cdf for any strike belonging to the set of quoted strikes is given by the following formula:

$$\widehat{P}(X_T < K_j) = DeltaK^{BS}(K_j;\sigma(K_j)) + Vega^{BS}(K_j;\sigma(K_j))\,\widehat{s}(K_j) + 1$$

$$\widehat{P}(X_T < K_j) = \delta_j + \nu_j \widehat{s}(K_j) + 1 \tag{2.11}$$

A given probability being always comprised between 0 and 1, a natural bounding of the skew $s(K_j)$ is then also:

$$0 \leq \delta_j + \nu_j \widehat{s}(K_j) + 1 \leq 1$$

$$\Leftrightarrow \mathcal{S}_{down}^{prob}(j) \leq \widehat{s}(K_j) \leq \mathcal{S}_{up}^{prob}(j) \tag{2.12}$$

with

$$\mathcal{S}_{down}^{prob}(j) = \frac{-1-\delta_j}{\nu_j} \tag{2.13}$$

$$\mathcal{S}_{up}^{prob}(j) = \frac{-\delta_j}{\nu_j} \tag{2.14}$$

As $0 \leq \delta_j \leq 1$, the upper bound $\mathcal{S}_{up}^{prob}$ is positive while the lower bound $\mathcal{S}_{down}^{prob}$ is always negative.

Compared to the SharkJaw bounds (2.9) and (2.10), as long as the market quotes of vanillas are arbitrage free in terms of call spreads, these two bounds are wider so that we have the following result:

Proposition 3. *The SharkJaw bounds $s_{min}(j)$ and $s_{max}(j)$ defined in the theorem (2) are always narrower than the "probabilist" bounds $\mathcal{S}_{down}^{prob}(j)$ and $\mathcal{S}_{up}^{prob}(j)$ defined by the equations (2.14) and (2.13). for any $K_j \in \mathbb{K}_i$, this means we have:*

$$\mathcal{S}_{down}^{prob}(j) < s_{min}(j) < s_{max}(j) < \mathcal{S}_{up}^{prob}(j) \tag{2.15}$$

Proof. When the market quotes of vanillas calls are arbitrage free in terms of call spreads, we have for any triplet of successive quoted strike $\{K_{j-1}; K_j; K_{j+1}\}$:

$$\begin{cases} -1 < \frac{C(K_j,T)-C(K_{j-1},T)}{(K_j-K_{j-1})} < 0 \\ -1 < \frac{C_{mkt}(K_{j+1})-C_{mkt}(K_j)}{K_{j+1}-K_j} < 0 \end{cases}$$

Then as the black schole vega ν_j is always positive, it is fairly easy to obtain:

$$\begin{cases} \mathcal{S}_{down}^{prob}(j) = \frac{-1-\delta_j}{\nu_j} < \frac{\frac{C(K_j,T)-C(K_{j-1},T)}{(K_j-K_{j-1})}-\delta_j}{\nu_j} < \frac{-\delta_j}{\nu_j} = \mathcal{S}_{up}^{prob}(j) \\ \mathcal{S}_{down}^{prob}(j) = \frac{-1-\delta_j}{\nu_j} < \frac{\frac{C_{mkt}(K_{j+1})-C_{mkt}(K_j)}{K_{j+1}-K_j}-\delta_j}{\nu_j} < \frac{-\delta_j}{\nu_j} = \mathcal{S}_{up}^{prob}(j) \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathcal{S}_{down}^{prob}(j) < s_{min}(j) < \mathcal{S}_{up}^{prob}(j) \\ \mathcal{S}_{down}^{prob}(j) < s_{max}(j) < \mathcal{S}_{up}^{prob}(j) \end{cases}$$

Then the final result comes from the fact that $s_{min}(j) < s_{max}(j)$. Q.E.D. □

2.3 Relations with existing arbitrability bounds

Arbitrability bounds for the implied volatility skew is already a well debated matter and in particular the bounds of the proposition (2) must be related with the works of [7]. In particular his Lemmas 2.6 and 2.7 as well as his theorems 3.3 and 3.6. from this paper. These results give rise to several important results useful for practionneers working on the definition of arbitrage free implied volatility parametric shape (See [13] and [21] which exploit these results from Fukasawa). Below the arbitrability bounds of the volatility skew and convexity discovered by [7] are exposed. Then we will show the SharkJaw bounds are always narrower than the ones from [7]. This important theoretical result comes from the fact not only exploits the properties of the Black Scholes formula but also crosses them with the knowledge of the markets given the existing quotations of the vanilla options. In the end, up to our knowledge, the SharkJaw bounds supplant any other existing theoretical bounds [7] and [21] and so are the only bounds to consider whether there may be arbitrage in the volatility skew and convexity.

2.3.1 Fukasawa Bounds for the volatility skew

Lemma. *(Lemma 2.6 in [7]): $\forall k \in \mathbb{R}$, we have:*

$$-d_2\sqrt{\tau}\frac{d\sigma}{dk} < 1$$

This condition may be expressed in terms of log-strike k as :

$$\begin{cases} \frac{\mathrm{d}\sigma}{\mathrm{d}k} > \frac{-1}{d_2\sqrt{\tau}} & \text{if } k < -\frac{\sigma^2\tau}{2} \\ \frac{\mathrm{d}\sigma}{\mathrm{d}k} < \frac{-1}{d_2\sqrt{\tau}} & \text{if } k > -\frac{\sigma^2\tau}{2} \end{cases}$$

Note that $k = -\frac{\sigma^2\tau}{2} \Rightarrow d_2 = 0$ and the condition becomes $0 < 1$ which means, $\frac{\mathrm{d}\sigma}{\mathrm{d}k}$ has no restriction. In particular, we have $\lim\limits_{k\uparrow -\frac{\sigma^2\tau}{2}} \frac{-1}{d_2\sqrt{\tau}} = -\infty$ which means the first condition tends to disappear when k tends to $-\frac{\sigma^2\tau}{2}$ by its left. Similarly we $\lim\limits_{k\downarrow -\frac{\sigma^2\tau}{2}} \frac{-1}{d_2} = +\infty$ and the second condition also disappears as k gets closer from $-\frac{\sigma^2\tau}{2}$ by its right side.

Lemma. *(Lemma 2.7 in [7]): $\forall k$ such that $d_1(k) \geq 0$ $\left(\text{aka } k < \frac{\sigma^2\tau}{2}\right)$, we have:*

$$-d_1 \frac{d\sigma}{dk}\sqrt{\tau} < 1$$

Theorem. *(Theorem 3.3 in [7]): $\forall k > 0$, we have:*

$$\frac{d\sigma}{dk} < \frac{1}{\sqrt{2k\tau}}$$

Theorem. *(Theorem 3.6 in [7]): $\forall k < 0$, we have:*

$$\frac{d\sigma}{dk} > -\frac{1}{\sqrt{2\,|k|\,\tau}}$$

Useful to our coming developments are the Theorem 2.8 in [7] establishing the monotonicity of both d_2 and d_1:

Theorem. *(Theorem 2.8 in [7]): The first and second normalizing transformations $f_1(k)$ and $f_2(k)$ are increasing with*

$$f_1(k) = -d_1(k, \sigma(k))$$
$$f_2(k) = -d_2(k, \sigma(k))$$

A direct implication of the Theorem (2.3.1) is the fact d_1 and d_2 are monotone decreasing function of k. In particular, this means any conditions comparing k and $\frac{\sigma(k)^2\tau}{2}$ with an operator $<$, $>$, $\leq$ or $\geq$ always corresponds to the designation of simple set corresponding to half rights in $\mathbb{R}$.

Also useful to our coming results are the lemmas 3.2 and 3.5 and the theorems 3.3 and 3.6 from [7]:

Lemma. *(Lemma 3.2 in [7]) $\forall k \geq 0$, $-d_2(k) \geq \sqrt{2k}$*

Lemma. *(Lemma 3.5 in [7]) $\forall k \leq 0$, $-d_1(k) \leq -\sqrt{2\,|k|}$*

The inequalities in the lemmas 2.3.1 and 2.3.1 above provide bounds for the skew $\frac{\mathrm{d}\sigma}{\mathrm{d}k}$. However to know whether these bounds are either upper or lower bound, we need to know the sign of d_1 and d_2 depending on the value of k. It is easy to see $d_1 < 0$ when $k > \frac{\sigma(k)^2\tau}{2}$ and $d_2 < 0$ when $k > -\frac{\sigma(k)^2\tau}{2}$. Then we have the following lemma:

Theorem 4. *A necessary abritrage free condition is to have the skew $\frac{d\sigma}{dk}$ respecting the following set of conditions:*

$$\begin{cases} \frac{d\sigma}{dk} > \frac{-1}{d_1\sqrt{\tau}} & \text{if } k < -\frac{\sigma^2\tau}{2} \\ \frac{-1}{d_1\sqrt{\tau}} < \frac{d\sigma}{dk} < \frac{-1}{d_2\sqrt{\tau}} & \text{if } -\frac{\sigma^2\tau}{2} < k < \frac{\sigma^2\tau}{2} \\ \frac{d\sigma}{dk} < \frac{-1}{d_2\sqrt{\tau}} & \text{if } k > \frac{\sigma^2\tau}{2} \end{cases} \tag{2.16}$$

Proof. Gathering the Lemmas (2.3.1) and (2.3.1) and the Theorem (2.3.1), we have the the following set of conditions:

$$\begin{cases} \frac{\mathrm{d}\sigma}{\mathrm{d}k} > \frac{-1}{d_2\sqrt{\tau}} & \text{if } k < -\frac{\sigma(k)^2\tau}{2} \\ \frac{\mathrm{d}\sigma}{\mathrm{d}k} < \frac{-1}{d_2\sqrt{\tau}} & \text{if } k > -\frac{\sigma(k)^2\tau}{2} \\ \frac{\mathrm{d}\sigma}{\mathrm{d}k} > \frac{-1}{d_1\sqrt{\tau}} & \text{if } k < \frac{\sigma(k)^2\tau}{2} \\ \frac{\mathrm{d}\sigma}{\mathrm{d}k} < \frac{1}{\sqrt{2k\tau}} & \text{if } k > 0 \\ \frac{\mathrm{d}\sigma}{\mathrm{d}k} > -\frac{1}{\sqrt{2|k|\tau}} & \text{if } k < 0 \end{cases}$$

This implies four regions for k, $\left]-\infty;-\frac{\sigma^2\tau}{2}\right]$, $\left[-\frac{\sigma(k)^2\tau}{2};0\right]$, $\left[0;\frac{\sigma(k)^2\tau}{2}\right]$, and $\left[\frac{\sigma(k)^2\tau}{2};+\infty\right[$ where the following constraints apply:

$$\begin{cases} \frac{\mathrm{d}\sigma}{\mathrm{d}k} > \max\left(\frac{-1}{d_1\sqrt{\tau}};\frac{-1}{d_2\sqrt{\tau}};-\frac{1}{\sqrt{2|k|\tau}}\right) & \text{if } k < -\frac{\sigma^2\tau}{2} \\ \max\left(-\frac{1}{\sqrt{2|k|\tau}};\frac{-1}{d_1\sqrt{\tau}}\right) < \frac{\mathrm{d}\sigma}{\mathrm{d}k} < \frac{-1}{d_2\sqrt{\tau}} & \text{if } -\frac{\sigma^2\tau}{2} < k < 0 \\ \frac{-1}{d_1\sqrt{\tau}} < \frac{\mathrm{d}\sigma}{\mathrm{d}k} < \min\left(\frac{1}{\sqrt{2k\tau}};\frac{-1}{d_2\sqrt{\tau}}\right) & \text{if } 0 < k < \frac{\sigma^2\tau}{2} \\ \frac{\mathrm{d}\sigma}{\mathrm{d}k} < \min\left(\frac{1}{\sqrt{2k\tau}};\frac{-1}{d_2\sqrt{\tau}}\right) & \text{if } k > \frac{\sigma^2\tau}{2} \end{cases}$$

When $k < -\frac{\sigma^2\tau}{2}$, we have $d_1 > d_2 > 0$ and $\max\left(\frac{-1}{d_1\sqrt{\tau}};\frac{-1}{d_2\sqrt{\tau}};-\frac{1}{\sqrt{2|k|\tau}}\right) = \max\left(\frac{-1}{d_1\sqrt{\tau}};-\frac{1}{\sqrt{2|k|\tau}}\right)$. Using the Lemma (2.3.1), $\frac{-1}{d_1} > -\frac{1}{\sqrt{2|k|}}\ \forall k < 0$ so that $\max\left(\frac{-1}{d_1\sqrt{\tau}};-\frac{1}{\sqrt{2|k|\tau}}\right) = \frac{-1}{d_1\sqrt{\tau}}$ and using (2.3.1), we have $\frac{-1}{d_2\sqrt{\tau}} < \frac{1}{\sqrt{2k\tau}}\ \forall k > 0$ so that $\min\left(\frac{1}{\sqrt{2k\tau}};\frac{-1}{d_2\sqrt{\tau}}\right) = \frac{-1}{d_2\sqrt{\tau}}$. In the end, this implies the second and third inequalities collapse into one inequality for all k such $-\frac{\sigma^2\tau}{2} < k < \frac{\sigma^2\tau}{2}$:

$$\begin{cases} \frac{\mathrm{d}\sigma}{\mathrm{d}k} > \frac{-1}{d_1\sqrt{\tau}} & \text{if } k < -\frac{\sigma^2\tau}{2} \\ \frac{-1}{d_1\sqrt{\tau}} < \frac{\mathrm{d}\sigma}{\mathrm{d}k} < \frac{-1}{d_2\sqrt{\tau}} & \text{if } -\frac{\sigma^2\tau}{2} < k < \frac{\sigma^2\tau}{2} \\ \frac{\mathrm{d}\sigma}{\mathrm{d}k} < \frac{-1}{d_2\sqrt{\tau}} & \text{if } k > \frac{\sigma^2\tau}{2} \end{cases}$$

Q.E.D.

□

Last, although the convexity doesn't play any direct role into the value of a market triangle, if one practitionneer wants to use skew values coming from a model, from theorem 3.1 of [7], the corresponding related convexity must also be superior to the following lower bound:

Theorem 5. *An necessary arbitrage condition for the* $\frac{d^2\sigma(k)}{dk^2}$ *is:*

$$F_d \leq \frac{d^2\sigma(k)}{dk^2} \tag{2.17}$$

with $F_d \leq 0\ \forall k \in \mathbb{R}$ *such that:*

$$F_d = -\frac{1}{\sigma(k)}\left(\frac{1}{\sqrt{\tau}} + d_2\frac{d\sigma}{dk}\right)\left(\frac{1}{\sqrt{\tau}} + d_1\frac{d\sigma}{dk}\right)$$

Proof. The proof is in the continuation of the proof of Theorem 3.1 in section 3 of [7]. The issue with the convexity bounds this theorem 3.1 of [7] is that the upper bound depends on the quantity it is supposed to bound via the density $f_x(k)$. In truth, it is easy to see this upper condition does not depend on the convexity as:

$$\begin{aligned} \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2} &< \frac{f_x(k)}{\phi(-d_2)} \\ \Leftrightarrow \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2} &< \frac{\frac{\phi(-d_2)}{\sqrt{\tau}}\left\{\sqrt{\tau}\frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2} - \frac{\mathrm{d}d_2}{\mathrm{d}k}\left(1 + d_2\sqrt{\tau}\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right)\right\}}{\phi(-d_2)} \\ \Leftrightarrow \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2} &< \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2} - \frac{\mathrm{d}d_2}{\mathrm{d}k}\left(\frac{1}{\sqrt{\tau}} + d_2\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right) \\ \Leftrightarrow 0 &< -\frac{\mathrm{d}d_2}{\mathrm{d}k}\left(\frac{1}{\sqrt{\tau}} + d_2\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right) \end{aligned} \tag{2.18}$$

Note then this last condition is always verified: Indeed the lemma (2.3.1) implies $d_2(k)$ is a strictly monotones decreasing function so that $\frac{\mathrm{d}d_2}{\mathrm{d}k} < 0\ \forall k$ while the positivity of $\left(\frac{1}{\sqrt{\tau}} + d_2\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right)$ is guaranteed by the Lemma (2.3.1).

As the Lemma (2.3.1) implies $\frac{1}{\sqrt{\tau}} + d_2 \frac{\mathrm{d}\sigma}{\mathrm{d}k} > 0 \ \forall k \in \mathbb{R}$ and the Theorem (2.3.1)implies $\frac{\mathrm{d}d_2}{\mathrm{d}k} \leq 0 \ \forall k \in \mathbb{R}$, F_d is necessarily always non-positive (aka negative or null). Of course, the condition (2.17) on the convexity $\frac{\mathrm{d}^2\sigma(k)}{\mathrm{d}k^2}$ may be only expressed in terms of the skew $\frac{\mathrm{d}\sigma}{\mathrm{d}k}$. Indeed as $\frac{\mathrm{d}d_2}{\mathrm{d}k} = -\frac{1}{\sigma\sqrt{\tau}} - \frac{d_1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}k}$, we have:

$$\frac{\mathrm{d}^2\sigma(k)}{\mathrm{d}k^2} \geq -\frac{1}{\sigma(k)}\left(\frac{1}{\sqrt{\tau}} + d_2\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right)\left(\frac{1}{\sqrt{\tau}} + d_1\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right) \tag{2.19}$$

□

All the preceding results lead to a new no arbitrage theorem combining the conditions of the theorems (4) and (5)on both the skew and the convexity depending on the value of the log-strike k:

Theorem 6. *For any $K \in \mathbb{R}^+$, the values of the skew $\frac{d\sigma}{dK}$ and the convexity $\frac{d^2\sigma}{dK^2}$ must be such that:*

$$\frac{d^2\sigma}{dK^2} \geq L_{conv}$$

$$\begin{cases} \frac{d\sigma}{dK} > \mathcal{S}_{down}^{Fukawasa} & \text{if } K < K_{ZDS-P} \\ \mathcal{S}_{down}^{Fukawasa} < \frac{d\sigma}{dK} < \mathcal{S}_{up}^{Fukawasa} & \text{if } K_{ZDS-P} < K < X_0 K_{ZDS} \\ \frac{d\sigma}{dK} < \mathcal{S}_{up}^{Fukawasa} & \text{if } K > K_{ZDS} \end{cases}$$

with:

** L_{conv} being a non positive level such that*

$$L_{conv} = -\frac{1}{\sigma(K)}\left(\frac{1}{K\sqrt{\tau}} + d_2\frac{d\sigma}{dK}\right)\left(\frac{1}{K\sqrt{\tau}} + d_1\frac{d\sigma}{dK}\right) - \frac{1}{K}\frac{d\sigma}{dK}$$

** K_{ZDS} and K_{ZDS} being the Zero-Delta-Straddle and net Zero-Delta-Straddle levels such that:*

$$K_{ZDS} = X_0 \exp\left(\frac{\sigma^2\tau}{2}\right)$$

$$K_{ZDS-P} = X_0 \exp\left(-\frac{\sigma^2\tau}{2}\right)$$

** $\mathcal{S}_{down}^{Fukawasa}$ a stricty negative real such that:*

$$\mathcal{S}_{down}^{Fukawasa} = \frac{-1}{Kd_1\sqrt{\tau}}$$

** $\mathcal{S}_{up}^{Fukawasa}$ a strictly positive real such that:*

$$\mathcal{S}_{up}^{Fukawasa} = \frac{-1}{K\sqrt{\tau}d_2}$$

Proof. The proof of this theorem is a direct consequence of the theorems (4) and (5) with their results expressed in terms of the strike K. To proceed the link between $\frac{\mathrm{d}\sigma}{\mathrm{d}k}$ and on one side and $\frac{\mathrm{d}\sigma}{\mathrm{d}K}$ on the other side is such that:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}k} = \frac{\mathrm{d}\sigma}{\mathrm{d}K}\frac{\mathrm{d}K}{\mathrm{d}k}$$

$$\Leftrightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}k} = \frac{\mathrm{d}\sigma}{\mathrm{d}K}K \tag{2.20}$$

Similarly the link between $\frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2}$ and $\frac{\mathrm{d}^2\sigma}{\mathrm{d}K^2}$ is:

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2} = \frac{\mathrm{d}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right)}{\mathrm{d}k} = \frac{\mathrm{d}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}K}K\right)}{\mathrm{d}K}\frac{\mathrm{d}K}{\mathrm{d}k}$$

$$\Leftrightarrow \frac{\mathrm{d}^2\sigma}{\mathrm{d}k^2} = K\left(\frac{\mathrm{d}\sigma}{\mathrm{d}K} + K\frac{\mathrm{d}^2\sigma}{\mathrm{d}K^2}\right) \tag{2.21}$$

Replacing (2.20) and (2.21) in (2.19) leads to:

$$\frac{\mathrm{d}^2\sigma(k)}{\mathrm{d}k^2} \geq -\frac{1}{\sigma(k)}\left(\frac{1}{\sqrt{\tau}}+d_2\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right)\left(\frac{1}{\sqrt{\tau}}+d_1\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right)$$

$$\Leftrightarrow K\left(\frac{\mathrm{d}\sigma}{\mathrm{d}K}+K\frac{\mathrm{d}^2\sigma}{\mathrm{d}K^2}\right) \geq -\frac{1}{\sigma(k)}\left(\frac{1}{\sqrt{\tau}}+d_2\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right)\left(\frac{1}{\sqrt{\tau}}+d_1\frac{\mathrm{d}\sigma}{\mathrm{d}k}\right)$$

$$\Leftrightarrow K^2\frac{\mathrm{d}^2\sigma}{\mathrm{d}K^2} \geq -\frac{1}{\sigma(k)}\left(\frac{1}{\sqrt{\tau}}+d_2\frac{\mathrm{d}\sigma}{\mathrm{d}K}K\right)\left(\frac{1}{\sqrt{\tau}}+d_1\frac{\mathrm{d}\sigma}{\mathrm{d}K}K\right)-K\frac{\mathrm{d}\sigma}{\mathrm{d}K}$$

$$\Leftrightarrow K^2\frac{\mathrm{d}^2\sigma}{\mathrm{d}K^2} \geq -\frac{K^2}{\sigma(k)}\left(\frac{1}{K\sqrt{\tau}}+d_2\frac{\mathrm{d}\sigma}{\mathrm{d}K}\right)\left(\frac{1}{K\sqrt{\tau}}+d_1\frac{\mathrm{d}\sigma}{\mathrm{d}K}\right)-K\frac{\mathrm{d}\sigma}{\mathrm{d}K}$$

$$\Leftrightarrow \frac{\mathrm{d}^2\sigma}{\mathrm{d}K^2} \geq -\frac{1}{\sigma(k)}\left(\frac{1}{K\sqrt{\tau}}+d_2\frac{\mathrm{d}\sigma}{\mathrm{d}K}\right)\left(\frac{1}{K\sqrt{\tau}}+d_1\frac{\mathrm{d}\sigma}{\mathrm{d}K}\right)-\frac{1}{K}\frac{\mathrm{d}\sigma}{\mathrm{d}K}$$

The set of conditions (2.16) becomes:

$$\begin{cases} \frac{\mathrm{d}\sigma}{\mathrm{d}K}K > \frac{-1}{d_1\sqrt{\tau}} & \text{if } \ln(K/X_0) < -\frac{\sigma^2\tau}{2} \\ \frac{-1}{d_1\sqrt{\tau}} < \frac{\mathrm{d}\sigma}{\mathrm{d}K}K < \frac{-1}{d_2\sqrt{\tau}} & \text{if } -\frac{\sigma^2\tau}{2} < \ln(K/X_0) < \frac{\sigma^2\tau}{2} \\ \frac{\mathrm{d}\sigma}{\mathrm{d}K}K < \frac{-1}{d_2\sqrt{\tau}} & \text{if } \ln(K/X_0) > \frac{\sigma^2\tau}{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{\mathrm{d}\sigma}{\mathrm{d}K} > \frac{-1}{Kd_1\sqrt{\tau}} & \text{if } K < X_0\exp\left(-\frac{\sigma^2\tau}{2}\right) \\ \frac{-1}{Kd_1\sqrt{\tau}} < \frac{\mathrm{d}\sigma}{\mathrm{d}K} < \frac{-1}{Kd_2\sqrt{\tau}} & \text{if } X_0\exp\left(-\frac{\sigma^2\tau}{2}\right) < K < X_0\exp\left(\frac{\sigma^2\tau}{2}\right) \\ \frac{\mathrm{d}\sigma}{\mathrm{d}K} < \frac{-1}{Kd_2\sqrt{\tau}} & \text{if } K > X_0\exp\left(\frac{\sigma^2\tau}{2}\right) \end{cases}$$

□

The level $K_{ZDS} = X_0\exp\left(\frac{\sigma^2\tau}{2}\right)$ is nothing less than the Zero-Delta-Straddle [3] strike level which cancels the delta of a straddle written on the underlying (A straddle being the sum of one call and one put sharing the same maturity and the same strike). Similarly the level $K_{ZDS-P} = X_0\exp\left(-\frac{\sigma^2\tau}{2}\right)$ is nothing less than the Zero-Delta-Straddle strike level, , which cancels the net real delta cost, *Delta_Cost*, of the delta hedging strategy of the same straddle. Indeed a pure static delta hedging strategy needs you to borrow an net amount of money that is the delta straddle times the underlying spot minus the premium you cash in while selling the straddle. A pure delta hdeging strategy is funded by the premium and by borrowing if the premium is not sufficient. More formally:

$$\delta_{Str}(K) = \frac{C(K)+P(K)}{X_0} + \delta^{net}_{Str}(K)$$

with

$$Delta\_Cost = \delta^{net}_{Str}(K) \times X_0$$
$$\delta_{Str}(K) = N(d_1) - N(-d_1)$$
$$\delta^{net}_{Str}(K) = N(d_2) - N(-d_2)$$

2.3.2 The Narrowest bounds are the SharkJaw ones

Now it remains to cross the theorem (6) with the Theorem (2) to delivers our last theorem bounding both the skew $\frac{\mathrm{d}\sigma}{\mathrm{d}K}$ and the convexity $\frac{\mathrm{d}^2\sigma}{\mathrm{d}K^2}$ for any quoted strike level:

[3]The Zero-Delta-Straddle strike is famous in the community of FX traders

Theorem 7. *For any given quoted maturity T_i, given a set $\mathbb{K}_i$ of quoted strikes, the following inequality is a necessary and sufficient arbitrage free condition for the skew $s(K) := \frac{d\sigma(K,T)}{dK}$ of the implied volatility for all $K \in \mathbb{K}_i$:*

$$\begin{cases} \max\left(s_{min}(K_j); \frac{-1}{Kd_1\sqrt{\tau}}\right) < \frac{d\sigma(K_j)}{dK} < s_{max}(K_j) & \text{if } K_j < X_0 \exp\left(-\frac{\sigma^2\tau}{2}\right) \\ \max\left(s_{min}(K_j); \frac{-1}{K_jd_1\sqrt{\tau}}\right) < \frac{d\sigma(K_j)}{dK} < \min\left(\frac{-1}{K_jd_2\sqrt{\tau}}; s_{max}(K_j)\right) & \text{if } X_0 \exp\left(-\frac{\sigma^2\tau}{2}\right) < K_j < X_0 \exp\left(\frac{\sigma^2\tau}{2}\right) \\ s_{min}(K_j) < \frac{d\sigma(K_j)}{dK} < \min\left(\frac{-1}{K_jd_2\sqrt{\tau}}; s_{max}(K_j)\right) & \text{if } K_j > X_0 \exp\left(\frac{\sigma^2\tau}{2}\right) \end{cases}$$

with

$$\begin{cases} s_{min}(K_j) & = \frac{\frac{C_{mkt}(K_j,T)-C_{mkt}(K_{j-1},T)}{(K_j-K_{j-1})}-\delta_j}{\nu_j} \quad \forall j = 2; ...; n_{iK} \\ s_{min}(K_1) & = \frac{\frac{C_{mkt}(K_1,T)-X_0}{K_1}-\delta_1}{\nu_1} \quad \text{when } j = 1 \end{cases}$$

$$\begin{cases} s_{max}(K_j) & = \frac{\frac{C_{mkt}(K_{j+1})-C_{mkt}(K_j)}{K_{j+1}-K_j}-\delta_j}{\nu_j} \quad \forall j = 1; ...; n_{iK}-1 \\ s_{max}(K_{n_{i,k}}) & = \frac{-\delta_j}{\nu_j} \quad \text{when } j = n_{i,K} \end{cases}$$

while the convexity $\frac{d^2\sigma(K,T)}{dK^2}$ must respect the following necessary arbitrage free condition $\forall K$:

$$\frac{d^2\sigma}{dK^2} \geq L_{conv} = -\frac{1}{\sigma(K)}\left(\frac{1}{K\sqrt{\tau}} + d_2\frac{d\sigma}{dK}\right)\left(\frac{1}{K\sqrt{\tau}} + d_1\frac{d\sigma}{dK}\right) - \frac{1}{K}\frac{d\sigma}{dK}$$

Proof. Using (2.11), for any quoted strike $K_j \in \mathbb{K}_i$, we have:

$$\frac{-1-\delta_j}{\nu_j} \leq \frac{\mathrm{d}\sigma(K_j)}{\mathrm{d}K} \leq \frac{-\delta_j}{\nu_j}$$

Then using the theorems (6) and (2) delivers the result. □

Theorem 8. *For any given quoted maturity T_i, given a set $\mathbb{K}_i$ of quoted strikes, the following inequality is a necessary arbitrage free condition for the volatility skew $s(K) := \frac{d\sigma(K,T)}{dK}$ of the implied volatility for all $K_j \in \mathbb{K}_i$ with $j = 1; ...; n_{iK}$:*

$$\mathcal{S}_{down}^{Fukawasa} < \mathcal{S}_{down}^{prob} < s_{min}(j) < \frac{d\sigma(K_j)}{dK} < s_{max}(j) < \mathcal{S}_{up}^{Prob} < \mathcal{S}_{up}^{Fukawasa}$$

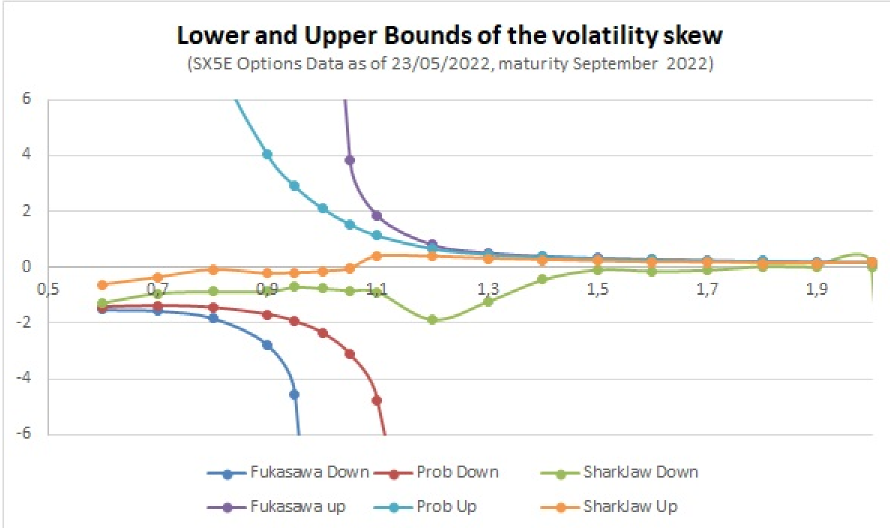
while the convexity $\frac{d^2\sigma(K,T)}{dK^2}$ must respect the following necessary arbitrage free condition $\forall K$:

$$\frac{d^2\sigma(K_j)}{dK^2} \geq L_{conv} = -\frac{1}{\sigma(K_j)}\left(\frac{1}{K\sqrt{\tau}} + d_2\frac{d\sigma(K_j)}{dK}\right)\left(\frac{1}{K\sqrt{\tau}} + d_1\frac{d\sigma(K_j)}{dK}\right) - \frac{1}{K_j}\frac{d\sigma(K_j)}{dK}$$

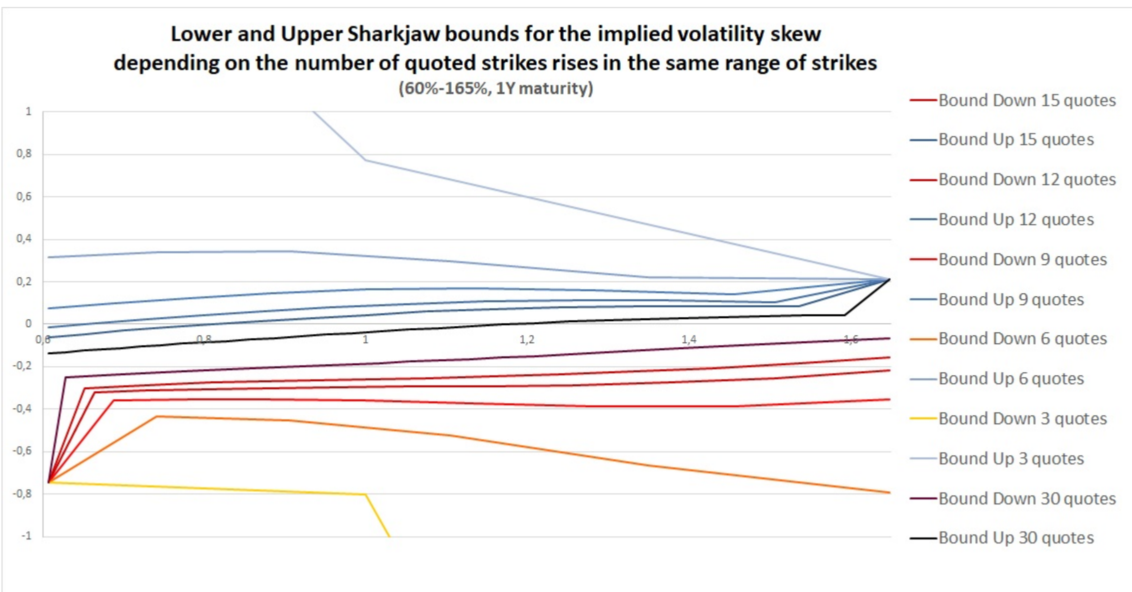
Proof. See the proof (6)in the annex □

The direct consequence of the theorem (8) is that as long market quotes exist for the vanilla options only the SharkJaw bounds matter as they are always the narrowest ones.

To give a direct example of this claim, a first graph below draws the different bounds computed using the real market quotes of vanilla options for the SX5E Equity Index as of 23/05/2022. The Sharkjaw bounds are clearly narrower than the Fukasawa ones and the "natural" ones deduced from the limit values of a probability.



Similarly below is the graph of the lower and upper sharkjaw bounds as the number of quoted strikes rises for a fixed 1Y maturity when the market is assumed to be driven by an Heston stochastic volatility model. As the number of quoted strikes in a fixed given range of strikes rises, the market becomes more and more denser and the range of possible arbitrage free values for the implied volatility skew gets more and more narrower.



One may also note that whatever the number of quoted strikes is, the lower bound stays the same for the lowest quoted strike while the upper bound is also the same for the upper quoted strike. That is due to the fact this bound depends respectively on backward and forward estimation of the delta strike that are not changing as the first one is calculated with a zero lowest strike while the second one is calculated by a passage to the limit to the infinite.

In the end, when the calibration process of a stochastic volatility model or a volatility parametric shape gives the n values of the skews $\frac{\mathrm{d}\sigma(K_j)}{\mathrm{d}K}\,\forall j = 1;...;n$, it implies values of the pseudo probabilities that may prove to be arbitrable while the related model prices of the vanilla may appear to be not so far from the market ones in terms of bps. The fact the stochastic volatility is arbitrable because of its model values of market triangles built with only quoted strikes means the model is giving values $s_{SV}(K)$ of the volatility skew that are outside the intervals specified by the theorem (8).

The existence of potential arbitrage due to negative market triangles due themselve to bad estimations of the implied volatility skew out of the Sharkjaw bounds lays the ground for new measurements of the calibration quality of a Stochastic Volatility model. Usually the quality of this calibration is mesured in terms of the spread in bps between the quoted options and their corresponding model prices. However litlle is done to settle whether the corresponding prices are arbitrage free or not. By relying on the market triangle arbitrage free conditions (1.3), a new model risk test, **the Sharkjaw test**, is designed to determine without any ambiguity if a given calibrated stochastic volatility[4] is arbitrage free. Up to our knowledge, that has never been done before in this way.

3 The Sharkjaw test for Stochastic Volatility models[5]

Usually the paremeter set θ of a stochastic volatility model is calibrated using a cost function measuring the overall distance between the set of the market prices and the set of the prices delivered by the model. A typical cost function is the following one:

$$C(\theta) = \sum_{j=1}^{n} w_j \left[C_{mkt}(K_j, T) - C_{mod}(K_j, T) \right]^2$$

with w a vector of weight generally designed to accomodate a more precise fit for near the money options. Then in practice, the quality of the fit is asserted by observing the Mean absolute Error (MAE), a weighted average of the absolute calibration errors $|C_{mkt}(K_j, T) - C_{mod}(K_j, T)|$. Alternatively one may use the squared difference $(C_{mkt}(K_j, T) - C_{mod}(K_j, T))^2$ to get the Mean Squared Error (MSE) or the Relative Mean Squared Error (RMSE).

When the arbitrability of a model is challenged along the strike dimension, most of practionneers rely on classic tests based on call spreads and butterflies. However these tests are good to check the self-consistency either of the market quotes or of the stochastic model in itself, they never really oppose model prices and market prices. That's what the new SharkJaw proposes to do: opposing quotations of vanillas options to model prices of european binaries to check the non-negativity of the call and put triangle defined in terms of quoted strikes. What this subsection proposes here is to exploit the Russian Dolls structure of the market to test, for each quoted maturity T_i and for each related quoted strike K_j, the ability of a calibrated model to deliver positive values for the related call and put triangles.

The way the SharkJaw test is run depends on whether it is run on mid-prices or on bid and ask prices. On the latter case, the test, the "soft" Sharkjaw test, is easier to pass for the stochastic volatility model while in the former case, it is harder.

3.1 The "hard" Shark Jaw test

The "hard" Shark Jaw test:

Let assume that for a quoted maturity T_i a set $\{K_1; ...; K_{n_K}\}$ of n_K market quotes of vanilla options are available:

If the tested model is arbitrage-free, it may provide prices of binaries to the other market competitors without allowing them to generate free-lunch gains by adopting one the two following strategies:

Assuming quoted strikes K_{j-1}, K_j and K_{j+1}such that $K_1 \leq K_{j-1} < K_j < K_{j+1} \leq K_{n_K}$, a competitor:

- Buy the call of quoted strike K_{j-1} in the market ;
- Sell the call of quoted strike K_j in the market ;
- Sell $(K_j - K_{j-1})$ european up and in binaries of strike K_j to the bank using the tested model that provides a binary price of $BEUI_{mod}(K_j)$.

The binaries are usually not quoted in Equity/index derivative market which means the bank need trust its model to deliver arbitrage free prices for the binaries.

The immediate position SJ_C of a competitor adopting the strategy above equals:

$$SJ_C = C_{mid}(K_{j-1}, T_i) - C_{mid}(K_j, T_i) - (K_j - K_{j-1}) BEUI_{mod}(K_j) \tag{3.1}$$

Like said before this position is the replication of the call triangle which the pay-off is:

$$CT = (X_T - K_{j-1}) \mathbf{1}_{K_{j-1} < X_T < K_j}$$

[4]Note that by stochastic volatility model, i mean also the LSV model. In theory, these ones are designed to match perfectly the market quotes but in practice, even the current best numeric methods designed to guarantee this result may fail to match market vanilla quotes by an important margins and so are prone to fail to be arbitrage free with their resulting calibration.

[5]Why "Shark Jaw" ? For two reasons: first, graphically the market triangle look like a razor-sharp teeth of a shark and second, when the test is run with bid and ask prices of market vanilla options, the number of arbitrage is doubled like a shark is reknown to have two rows of teeth...

which means the bank's model is not arbitrage free if

$$SJ_C < 0$$

Conversely using now, the quoted values of the puts, the same model used to price the same Up and In Binary may be used to set the "put" version of the strategy that is:Buy the call of quoted strike K_j ;

- Buy the put of quoted strike K_{j+1} in the market ;
- Sell the put of quoted strike K_j in the market ;
- Sell $(K_{j+1} - K_j)$ risk-free zero-coupon bonds of maturity T_i $(B(t, T_i)$ being the undiscounted price at time t of this risk-free zero coupon bond aka 1);
- buy $(K_{j+1} - K_j)$ european up and in binaries of strike K_j to the bank using the tested model.

The resulting position of the competitor equals:

$$SJ_P = P_{mid}(K_{j+1}, T_i) - P_{mid}(K_j, T_i) - (K_{j+1} - K_j) B(t_0, T_i) + (K_{j+1} - K_j) BEUI_{mod}(K_j) \tag{3.2}$$

This position comes down to the replication of the put triangle which the pay-off is

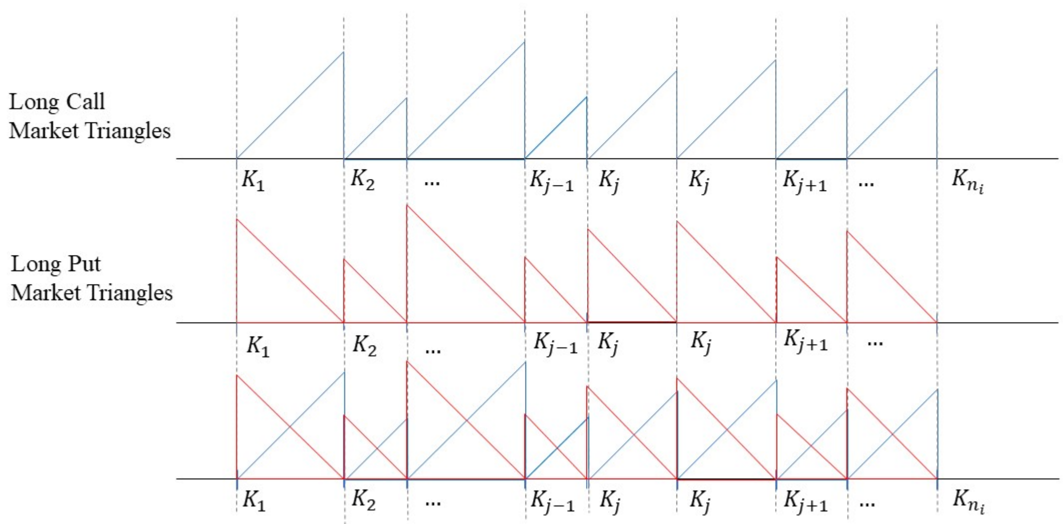
$$PT = (K_j - X_T) \mathbf{1}_{K_{j-1} < X_T < K_j}$$

which means the bank's model is not arbitrage free if

$$SJ_P < 0$$

One may note the two conditions (3.1) and (3.2) are not the same as they are not based on the same set of strike. The call version are based on vanilla quotes of strike K_{j-1} and K_j while the put version is based on K_j and K_{j+1} (See the Schema (3.1).

The rows of call and put market triangles shaping the two series of tests of the Shark Jaw test



Gathering the arbitrage based on (3.1) and (3.2) yields the following sharkjaw arbitrage free condition for a model for any strike K_j such $1 < j < n_{iK}$:

$$\begin{cases} C_{mid}(K_{j-1}, T_i) - C_{mid}(K_j, T_i) - (K_j - K_{j-1}) BEUI_{mod}(K_j) & > 0 \\ P_{mid}(K_{j+1}, T_i) - P_{mid}(K_j, T_i) - (K_{j+1} - K_j) B(t_0, T_i) + (K_{j+1} - K_j) BEUI_{mod}(K_j) & > 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{C_{mid}(K_{j-1},T_i)-C_{mid}(K_j,T_i)}{(K_j-K_{j-1})} & > BEUI_{mod}(K_j) \\ BEUI_{mod}(K_j) & > \frac{P_{mid}(K_j,T_i)-P_{mid}(K_{j+1},T_i)+(K_{j+1}-K_j)B(t_0,T_i)}{(K_{j+1}-K_j)} \end{cases}$$

$$\Leftrightarrow \frac{P_{mid}(K_j,T_i)-P_{mid}(K_{j+1},T_i)+(K_{j+1}-K_j)B(t_0,T_i)}{(K_{j+1}-K_j)} < BEUI_{mod}(K_j) < \frac{C_{mid}(K_{j-1},T_i)-C_{mid}(K_j,T_i)}{(K_j-K_{j-1})} \tag{3.3}$$

3.2 The "soft" Shark Jaw test:

Now let assume that to generate its strategy, the competitor uses bid and ask prices for market vanilla options and risk free zero coupon bonds. From the point of view of the bank buying and selling the binary, there is no need to distinguish between a model bid and a model ask as what matters is the price calculated by the model. Indeed, one may add a bid spread and remove a sell spread to this price, the overall pnL will be computed with the model price as the bid and ask spread may be considered like sales margins. Contrarily for the competitors, the call version and the put strategies he may set up are doubled as there are now two ways to implement each of these strategies depending on the fact you are buying or selling a specific market call or put.

For the call version, the competitor may:

- set up the long call strategy:
  - Buy the call of quoted strike K_{j-1} at a market price of $C_{bid}(K_{j-1},T_i)$;
  - sell the call of quoted strike K_j at a market price of $C_{ask}(K_j,T_i)$;
  - Sell (K_j-K_{j-1}) european up and in binaries of strike K_j to the bank using the tested model that provides a binary price of $BEUI_{mod}(K_j)$,
- set up the short call strategy:
  - Sell the call of quoted strike K_{j-1} at a market price of $C_{ask}(K_{j-1},T_i)$;
  - buy the call of quoted strike K_j at a market price of $C_{bid}(K_j,T_i)$;
  - buy (K_j-K_{j-1}) european up and in binaries of strike K_j to the bank using the tested model that provides a binary price of $BEUI_{mod}(K_j)$,

This leads to two arbitrage free shark jaw conditions:

$$C_{bid}(K_{j-1},T_i)-C_{ask}(K_j,T_i)-(K_j-K_{j-1})BEUI_{mod}(K_j)>0$$
$$C_{bid}(K_{j-1},T_i)-C_{ask}(K_j,T_i)+(K_j-K_{j-1})BEUI_{mod}(K_j)<0$$

Combining these two conditions leads to:

$$\begin{cases} \frac{C_{bid}(K_{j-1},T_i)-C_{ask}(K_j,T_i)}{(K_j-K_{j-1})} > BEUI_{mod}(K_j) \\ BEUI_{mod}(K_j) < \frac{C_{ask}(K_{j-1},T_i)-C_{bid}(K_j,T_i)}{(K_j-K_{j-1})} \end{cases}$$

$$\Rightarrow BEUI_{mod}(K_j) < \min\left(\frac{C_{ask}(K_{j-1},T_i)-C_{bid}(K_j,T_i)}{(K_j-K_{j-1})};\frac{C_{bid}(K_{j-1},T_i)-C_{ask}(K_j,T_i)}{(K_j-K_{j-1})}\right) \tag{3.4}$$

Conversely, the put version, the competitor may:

- set up the long put strategy:
  - Buy the put of quoted strike K_{j+1} at a market price of $P_{bid}(K_{j+1},T_i)$;
  - sell the put of quoted strike K_j at a market price of $P_{ask}(K_j,T_i)$;
  - Sell $(K_{j+1}-K_j)$ risk-free zero-coupon bonds of maturity T_i ($B_{ask}(t,T_i)$ being the undiscounted ask price at time t of this risk-free zero coupon bond aka 1);
  - buy $(K_{j+1}-K_j)$ european up and in binaries of strike K_j to the bank using the tested model that provides a binary price of $BEUI_{mod}(K_j)$,
- set up the short put strategy:

– Sell the put of quoted strike K_{j+1} at a market price of $P_{ask}(K_{j+1}, T_i)$;

– buy the put of quoted strike K_j at a market price of $P_{bid}(K_j, T_i)$;

– buy $(K_{j+1} - K_j)$ risk-free zero-coupon bonds of maturity T_i ($B_{bid}(t, T_i)$ being the undiscounted ask price at time t of this risk-free zero coupon bond aka 1);

– sell $(K_{j+1} - K_j)$ european up and in binaries of strike K_j to the bank using the tested model that provides a binary price of $BEUI_{mod}(K_j)$,

This leads to two arbitrage free shark jaw conditions:

$$\begin{cases} P_{bid}(K_{j+1}, T_i) - P_{ask}(K_j, T_i) - (K_{j+1} - K_j) B_{ask}(t_0, T_i) + (K_{j+1} - K_j) BEUI_{mod}(K_j) > 0 \\ P_{bid}(K_j, T_i) - P_{ask}(K_{j+1}, T_i) + (K_{j+1} - K_j) B_{bid}(t_0, T_i) - (K_{j+1} - K_j) BEUI_{mod}(K_j) < 0 \end{cases}$$

which finally gives the second "soft" shark jaw arbitrage free condition:

$$BEUI_{mod}(K_j) > \max\left(\frac{P_{ask}(K_j, T_i) - P_{bid}(K_{j+1}, T_i)}{(K_{j+1} - K_j)} + B_{ask}(t_0, T_i); \frac{P_{bid}(K_j, T_i) - P_{ask}(K_{j+1}, T_i)}{(K_{j+1} - K_j)} + B_{bid}(t_0, T_i)\right) \tag{3.5}$$

By combining the two conditions, we get the final "soft" Sharkjaw arbitrage free condition that is for $BEUI_{mod}(K_j)$ to be such that:

$$Lower\_Shark\_tooth_j < BEUI_{mod}(K_j) < Upper\_Shark\_tooth_j \;\; \forall j = 2; ...; n_{iK} - 1 \; \forall i = 1; ...; n_T$$

with

$$Lower\_Shark\_tooth_j = \max\left(\frac{P_{ask}(K_j, T_i) - P_{bid}(K_{j+1}, T_i)}{(K_{j+1} - K_j)} + B_{ask}(t_0, T_i); \frac{P_{bid}(K_j, T_i) - P_{ask}(K_{j+1}, T_i)}{(K_{j+1} - K_j)} + B_{bid}(t_0, T_i)\right)$$

$$Upper\_Shark\_tooth_j = \min\left(\frac{C_{ask}(K_{j-1}, T_i) - C_{bid}(K_j, T_i)}{(K_j - K_{j-1})}; \frac{C_{bid}(K_{j-1}, T_i) - C_{ask}(K_j, T_i)}{(K_j - K_{j-1})}\right)$$

In practice, the arbitrage may be expected to appear for far out the money or far in the money quoted strikes. However that's regions of the market where the bid-ask spreads are expected to be the largest by several dozens of bps whereas these same bid-ask spreads are much more tightened near the money. Moreover the quoted strikes tend to be more concentrated around the money, so the Shark Jaw test may not be so easy to pass for strike near the money.

4 The Calibration Model Scores with the Shark Jaw Test

4.1 The Absolute Calibration Arbitrability Score

Given all the preceding developments, for any given maturity $T_j \in Cal_T$ and the corresponding set of quoted strike $\{K_j\}_{j=1;...;n_{iK}}$, a first calibration score, the Absolute Calibration Score for an arbitrage tolerance threshold of x basis points of the underlying spot is defined like:

$$ACA(x) = \frac{\sum_{i=1}^{n_T} \sum_{j=2}^{n_{iK}} \frac{\nu_j}{\sqrt{T_i}} \left\{ \mathbf{1}_{\{CT(K_{j-1}, K_j, T_i) + x > 0\}} \times \mathbf{1}_{\{PT(K_{j-1}, K_j, T_i) + x > 0\}} \right\}}{\sum_{i=1}^{n_T} \sum_{j=2}^{n_{iK}} \frac{\nu_j}{\sqrt{T_i}}} \times 10$$

with

$$CT(K_{j-1}, K_j, T_i) = C(K_{j-1}, T_i) - C(K_j, T_i) - (K_j - K_{j-1}) BEUI(K_j, T_i)$$

$$PT(K_{j-1}, K_j, T_i) = P(K_j, T_i) - P(K_{j-1}, T_i) - (K_j - K_{j-1}) BEDI(K_{j-1}, T_i)$$

where $w_A(i, j)$ is a function weighting each possible arbitrage. Two kinds of weighting were tested:

- $w_A(i, j) = \frac{\nu_j}{\sqrt{T_i}}$ with ν_j being the classic Black Scholes Vega evaluated for the strike level K_j , the maturity T_i and the corresponding ATM implied volatility. Although the Vega ν_j is present in the formula, it is not here to take into account the vega risk exposure in itself but to get a proxy for the liquidity. Indeed the risk exposure is taken into account by the overall threshold tolerance level x measured in bps.

- $w_A(i,j) = 1$

In practice, the two weighting functions delivers pretty much the same results in all cases. So we keep the unitary weighting to make the things simpler.

This measure gives a score comprised between 0 and 10 of the arbitrability of the stochastic volatility model used to fit the market of vanilla options. Examples of this measures for a small set of stochastic volatility models is given and compared to the classic MAE (Mean Absolute Error) and RMSE (Root Mean Squared Error) usually given and commented when the fitting quality of such model is appreciated. In particular, in these examples, we will see that the best model in terms of MAE or RMSE may not be at all the best one in terms of ACA measure. This proves the usefulness of the ACA measure which tracks down the existence of arbitrage more than the pure average level of calibration error. Said otherwise, what is the most important is not to be wrong but to be arbitrage free.

What the ACA doesn't tell you is where a stochastic volatility model is arbitrable in the market volatility surface. That's the purpose of the Arbitrogram presented in the next subsection

4.2 The Arbitrogram

For any one stochastic volatility model under scrutiny, the Arbitrogram is a map of market points $\{k_j; T_i\}_{i=1;\ldots;n_T; j=1;\ldots;n_{iK}}$ (with $k_j = \frac{\ln(K_j)}{\sigma_{ATM}(T_i)\sqrt{T_i}}$ being the cumulated volatility normalized log-strike) for which the value is either 1 if the model is arbitrage free or 0 otherwise.

For example, for a simple Black-Scholes model calibrated on an example of Market Implied Volatility Surface generated by a Double Heston stochastic volatility model with Bates-like jumps (See the section 5.1), the corresponding Arbitrogram is:

Arbitrogrammetry of a Black Scholes model with a x=0 bps tolerance threshold

k	Maturity 1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,92	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,85	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,77	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,69	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,62	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,54	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,46	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,38	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,31	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,23	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,15	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,08	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,00	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,08	NOK	NOK	NOK	NOK	ok	ok	ok	ok	ok
0,15	NOK	NOK	NOK	NOK	ok	ok	ok	ok	ok
0,23	NOK	NOK	NOK	ok	ok	ok	NOK	NOK	NOK
0,31	NOK	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK
0,38	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
0,46	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
0,54	NOK	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,62	NOK	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,69	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,77	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,85	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,92	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
1,00	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK

The corresponding ACA score is 1.36/10 which is pretty terrible. In particular, for $k = 0$, the Black Scholes model is arbitrable whatever the maturity is. The rare arbitrage free points are located on the upper tail of the distribution. As the formula 2.4 tolds us, the Black Scholes is all the more arbitrage free as the volatility skew is near from 0, and precisely this skew tends to be weaker in absolute value in the upper tail of the market distribution.

Now, if the arbitrage tolerance threshold is raised to 5 bps from 0, the ACA calibration score of the Black Scholes records a significant improvment passing from 1.36 / 10 to 5.31 /10.

Arbitrogrammetry of a Black Scholes model
with a x=5 bps tolerance threshold

	Maturity								
k	1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,92	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,85	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,77	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,69	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,62	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,54	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,46	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,38	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,31	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,23	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,15	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,08	ok	ok	ok	ok	ok	ok	ok	NOK	NOK
0,00	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,08	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,15	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,23	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,31	ok	ok	ok	ok	ok	ok	ok	NOK	NOK
0,38	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
0,46	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,54	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,62	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,69	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,77	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,85	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,92	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
1,00	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK

The Arbitrogram helps to see that the market points that are still arbitrable are no longer in the short term part of the Volatility Surface but more on the wings of the volatility smile for longer maturity. More detailed example exist in the section 5. Before dwelving into this last section, an ultimate Scoring measure, the Exotic Calibration Arbitrability (ECA) score is presented which measures the added value of a stochastic volatility model over a simple Black Scholes in its ability to be arbitrage free.

4.3 The Exotic Calibration Arbitrability Score

For any one stochastic volatility model under scrutiny, the Exotic Calibration Arbitrability (ECA) score is based on the calculus of the ACA score and measures the over performance of this model over the Black Scholes in terms of the ACA score. Like the ACA, the ECA score depends on a given tolerance threshold level x expressed in basis points of the underlying spot and is given by the following formula:

$$ECA\left(x\right)=\frac{\max\left(ACA\left(x\right)-ACA_{BS}\left(x\right);0\right)}{\max\left(10-ACA_{BS}\left(x\right);0\right)}$$

The idea behind the ECA score is it is useless to use a stochastic volatility if it generates more arbitrages than the simplest Black Scholes model in which the volatility is constant over time.

To illustrate the usefulness of the ACA and ECA scoring developed here, the next section compares the calibration performance of several models on the ground of simulated market data set.

5 Examples and Comparison of Calibration Scoring

The idea of this section is to show the added value of the ACA and ECA scores over the classic statistical measures like the Mean Absolute Error or the Root Mean Squared Error extensively used in papers dedicated to stochastic volatility models. As the purpose of this section is to give examples of the methods more than to identify stylized facts from the real market data, the market data used here to assess the calibration performance of each model and methods is generated by simulation with a Double Heston stochastic volatility model with a Bates-like jump component.

5.1 A simulated market data

5.1.1 Using a model to simulate Market data

The market data is generated by pricing options with an underlying which the SDE is assumed to be:

$$dX_t=X_t\left(\sqrt{V_{1t}}dW_{1t}^X+\sqrt{V_{2t}}dW_{2t}^X+\left(J-1\right)dN_t\right)$$

with

$$X_{t_0} = X_0 = 1.0; \quad V_{1t_0} = v_1; \quad ; V_{2t_0} = v_2$$
$$dV_{1t} = \kappa_1 \left(\theta_1 - V_{1t}\right) dt + \sigma_1 \sqrt{V_{1t}} dB_{1t}^v$$
$$dV_{2t} = \kappa_2 \left(\theta_2 - V_{2t}\right) dt + \sigma_2 \sqrt{V_{2t}} dB_{2t}^v$$
$$\left\langle dW_{1t}^X dB_{1t}^v \right\rangle = \rho_1 dt$$
$$\left\langle dW_{2t}^X dB_{2t}^v \right\rangle = \rho_2 dt$$
$$\left\langle dW_{1t}^X dW_{2t}^X \right\rangle = \left\langle dW_{1t}^X dB_{2t}^v \right\rangle = \left\langle dW_{2t}^X dB_{1t}^v \right\rangle = \left\langle dB_{1t}^X dB_{2t}^v \right\rangle = 0$$
$$J \to \mathcal{LN}\left(\mu; \delta\right)$$

This model is a Double Heston with a Bates like jump component (DHB) which the jump distribution is a log-normal one while the intensity λ of the Poisson process N_t is constant. The good new with this model is:

- it allows a quite rich dynamic allowing a not so unrealistic simulation of vanilla options market quotes with well chosen values for the parameter set $p = \{v_1; \kappa_1; \theta_1; \sigma_1; \rho_1; v_2; \kappa_2; \theta_2; \sigma_2; \rho_2; \mu; \delta; \lambda\}$;
- it easy to price vanilla options with it as its characteristic function is known in closed formula given the Affine nature of the model (See [6] for more details on this class of Affine models) ;
- it allows to retrieve some classic models as special cases: the Christoffersen ([5]) Double Heston (DH) without jumps (with $\lambda = 0$), the Bates (B) model ($v_2 = \kappa_2 = \theta_2 = \sigma_2 = \rho_2 = 0$), the Heston (H) model (with $v_2 = \kappa_2 = \theta_2 = \sigma_2 = \rho_2 = \lambda = 0$), and of course the Black Scholes (BS) model (with all model coefficients being null except v_1) to which is added a Black Scholes model with a time-dependant volatility (BS(T)).

Using the DHB model for a set of n_K log strikes $\{k_j\}_{j=1;\ldots;n_K}$ for each maturity $T_i \in Cal_T$ such that the simulated quoted maturities are {1W, 1M, 3M, 6M, 1Y, 2Y, 3Y, 4Y, 5Y}, a set of $n_K \times n_T$ vanilla options are priced with the following values for the coefficients of the DHB model:

Coefficient	value
v_1	0.085
κ_1	25.37
θ_1	0.0227
σ_1	1.05
ρ_1	-0.99

Coefficient	value
v_2	0.0244
κ_2	8.8
θ_2	0.0273
σ_2	0.6827
ρ_2	-0.8417
μ	0.002
δ	0.15
λ	1.25

So as to illustrate the method all the above models have been fit to simulated market data, our analysis will be focused on the method and the scores and not on dragging fundamental conclusions on the real behavior of the market.

5.1.2 The common cost function

To set all the above models ((DH), (H), (B), (BS) and (BS(T)) on a same foot of equality, the cost function to minimize in the model parameter vector p during the calibration process is based on the Mean Squared Error such that:

$$C(p) = \left(\sum_{j=1}^{n} w_j \left[C_{mkt}\left(K_j, T\right) - C_{mod}\left(K_j, T, p\right) \right]^2 + Pen\left(p\right) \right)$$

with:

- w_j equalling the implied volatility for the current pair (strike, maturity): the advantage of this choice is to offer a good balance between the necessity to match quotations with low levels of strike and the necessity to not overweight these strikes compared to the other strike levels. An usual counterpart seen in is to take the inverse of the vega Black Scholes. Compared to the weighting by the implied volatility, this choice of the inverse Vega has the drawback to overweight long maturity compared to the shortest one and to overweight the strikes in the money call for which the volatility skew is weaker.

- $Pen\,(p)$ a penalty function which equals 0 when the one or both of the Feller conditions ($2\kappa_1\theta_1 \geq \sigma_1^2$ and $2\kappa_2\theta_2 \geq \sigma_2^2$) specific to the Heston, Bates and Double Heston model are respected and equals $+\infty$ otherwise. The penalty function is also used to rule out negative value of the volatility and intensity of the jump component.

The optimal values of p is then defined like:

$$p = \arg\min_p \sum_{j=1}^{n} w_j \left[C_{mkt}\left(K_j, T\right) - C_{mod}\left(K_j, T\right)\right]^2$$

To solve this problem for each tested model, the Nelder-Mead algorithm has been used (see [15]). While using this method, the penalty function $Pen\,(p)$ equals 1×10^{200} when the conditions are not respected

5.1.3 The results

First the values of the calibrated coefficients for the different models are given by the following table:

	Black Scholes	Heston	Bates	Double Heston
$v_1{}^{\dagger}$	0.0761	0,0998	0,1032	0.0933
κ_1		5,4044	22,9408	3.7900
$\theta_1{}^{\dagger}$		0,0660	0,0692	0.0515
σ_1		0,7010	0,4847	0.3608
ρ_1		-0,4153	-0,9995	-0.7148
$v_2{}^{\dagger}$				1.5361e-4
κ_2				6.3340
$\theta_2{}^{\dagger}$				0.0135
σ_2				0,0062
ρ_2				-0,9666
μ			-0,1214	
δ			0,1321	
λ			5,17E-01	

† : this is a variance.

Note that in addition to these four models, a time varying Black model (BLACK SIGMA(T)) is among the list of the studied models. Its SDE is:

$$dX_t = X_t \sigma\left(T\right) dW_t^X$$

$$X_{t_0} = X_0$$

with $\sigma\left(T\right)$ being a deterministic function of time (For more details on this model, see [18]).

The three stochastic volatility models built upon CIR processes all respect the Feller condition. The Bates model is in sharp contrast with the two others as it exhibits much higher speed of mean reversion and spot/vol correlation. The Double Heston exhibits a two regimes behavior of volatility with one beginning at a level near from 30% and mean reverting to 22% while the second one begins at a much lower level (1.2% !) and mean reverting to 11.6% two times faster while also having a strong anti-correlation with the spot. Given these stark constrasts between these three models, one may wonder which one will perform the best.

First let's see which one of the tested models performs the best when using the classic Mean Absolute Error. The following table allows us to retrieve the conclusions obtained in many other previous papers related to the comparative performance of stochastic volatility models:

1. The Jump-Diffusions always outperforms the pure diffusions: Indeed, the MAE of the Bates is two times lower than the Heston model or the Double Heston model ;

2. The Double Heston is only very slightly better than the Heston model ;

3. Of course, the different Black models are far away behind the other models and this is justifying the need for a Stochastic Volatility model.

Model	MAE in bps
BLACK	72
BLACK SIGMA(T)	42
BLACK "SMILED"	N/A
Bates	**15**
Heston	26
Double Heston	**25**

Note that even the Bates is not good for a day to day use in production: an mean absolute error of 15 bps is way too high. Now let's run the SharkJaw test for all these models and calculate the corresponding ACA and ECA scores. A priori, the same ordering is expected but, guess what ? The Table (5.1.3)below shows it is not the case:

		ACA SCORE		ECA SCORE	
Model	**MAE in bps**	0 bps	5bps	0 bps	5 bps
BLACK	72	1,36	5,31	N/A	N/A
BLACK SIGMA(T)	42	1,67	5,49	0,36	0,38
BLACK "SMILED"	N/A *	1,58	5,58	0,26	0,57
Bates	**15**	**2,34**	**4,43**	**1,13**	**0,00**
Heston	26	**4,65**	**8,61**	**3,81**	**7,03**
Double Heston	**25**	**0,49**	**7,01**	**0,00**	**3,63**

* Black "Smiled": each option is priced with its corresponding implied volatility

The ACA and ECA scores are given here with two different threshold tolerance levels in bps: 0 bps and 5 bps. In practice, the higher the level the better will be the scores. However the Call Triangle and the Put triangles being bounded functions of the underlyings, their prices have also a higher bound that are not so high and so obtaining 10 / 10 with a sufficiently high threshold tolerance level doesn't mean your model is performing well, it means your tolerance level is way too high. In practice, from a pure modelling point of view, the level to consider is 0 bps as theoretically speaking any existence of arbitrage is bad. From a trading point of view, what matters is not really this level of X bps but how to exploit the existence of call and/or put arbitrage but this matter is beyond the scope of this paper.

Now model-wise, what is striking here is how the Double Heston and the Bates underperform while the Heston overperforms the other models. In the details:

- when the threshold tolerance level is 0 bps: all models exhibit a massive presence of arbitrages. Worse than that, the arbitrogrammetry shows this presence is not necessarily on the tails of the distribution but rather everywhere and especially on the ATM backbone (see the Arbitrograms of the model below)

Arbitrogrammetry of Stochastic Volatility models with a 0 bps tolerance level

Black Scholes model

	Maturity								
k	1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,92	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,85	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,77	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,69	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,62	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,54	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,46	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,38	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,31	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,23	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,15	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,08	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,00	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,08	NOK	NOK	NOK	NOK	ok	ok	ok	ok	ok
0,15	NOK	NOK	NOK	NOK	ok	ok	ok	ok	ok
0,23	NOK	NOK	NOK	ok	ok	ok	NOK	NOK	NOK
0,31	NOK	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK
0,38	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
0,46	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
0,54	NOK	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,62	NOK	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,69	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,77	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,85	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,92	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
1,00	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK

Heston model

	Maturity								
k	1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	ok	NOK	ok	ok	ok	ok	ok	ok	ok
-0,92	ok	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,85	ok	NOK	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,77	ok	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,69	ok	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,62	ok	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,54	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,46	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,38	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,31	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,23	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,15	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,08	NOK	NOK	ok	ok	NOK	NOK	NOK	NOK	ok
0,00	NOK	NOK	ok	ok	ok	ok	ok	ok	ok
0,08	NOK	NOK	ok	ok	ok	ok	ok	ok	ok
0,15	NOK	NOK	ok	ok	ok	ok	ok	ok	ok
0,23	NOK	NOK	ok	ok	ok	ok	ok	ok	ok
0,31	NOK	NOK	ok	ok	ok	ok	ok	ok	NOK
0,38	NOK	NOK	ok	ok	ok	ok	ok	NOK	NOK
0,46	NOK	ok	ok	ok	ok	ok	ok	NOK	NOK
0,54	NOK	ok	ok	ok	ok	ok	NOK	NOK	NOK
0,62	NOK	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,69	NOK	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,77	NOK	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,85	NOK	ok	NOK	ok	NOK	NOK	NOK	NOK	NOK
0,92	ok	ok	NOK	ok	NOK	NOK	NOK	NOK	NOK
1,00	ok	NOK	NOK	ok	ok	ok	ok	ok	ok

Double Heston model

	Maturity								
k	1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	ok	NOK	NOK	ok	ok	ok	ok	ok	ok
-0,92	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,85	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,77	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,69	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,62	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,54	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,46	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,38	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,31	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,23	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,15	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,08	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,00	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,08	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,15	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,23	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,31	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,38	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,46	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,54	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,62	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,69	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,77	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,85	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,92	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
1,00	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK

Bates model

	Maturity								
k	1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	ok	ok	ok	ok	ok	ok	ok	ok	ok
-0,92	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,85	NOK	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,77	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,69	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,62	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,54	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,46	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,38	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,31	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,23	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,15	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,08	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,00	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,08	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,15	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,23	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,31	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,38	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,46	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,54	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,62	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,69	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,77	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,85	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
0,92	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK
1,00	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK	NOK

- a more detailed exam of the arbitrogrammetry of the Bates and Double Heston is needed to understand why they do perform so badly compared to the Heston one:
- The following two tables split the result of the SharkJaw test for the call triangles and the put triangles. It shows the Up and In Binaries are way globally too expensive which implies more negative values for the Call Triangles. The consequence of this global over-estimation is the global underestimation of the Down and In Binaries which leads to the total absence of arbitrages for the Put Triangle with the Bates model:

Call Triangle Arbitrages in bps (Bates model)

k	Maturity 1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,92	0,1	0,0	0,0	0,7	3,0	7,2	10,7	13,6	15,9
-0,85	0,0	0,0	0,0	0,8	3,3	7,9	11,8	15,0	17,6
-0,77	0,0	0,0	0,0	0,9	3,6	8,7	13,0	16,5	19,4
-0,69	0,0	0,0	0,0	1,0	4,0	9,6	14,2	18,0	21,2
-0,62	0,0	0,0	0,0	1,2	4,5	10,4	15,4	19,6	23,1
-0,54	0,0	0,0	0,0	1,4	4,9	11,3	16,7	21,2	25,0
-0,46	0,0	0,0	0,0	1,6	5,4	12,2	18,0	22,8	26,8
-0,38	0,0	0,0	0,0	1,8	5,9	13,1	19,3	24,4	28,7
-0,31	0,0	0,0	0,1	2,1	6,4	14,1	20,5	26,0	30,5
-0,23	0,0	0,0	0,3	2,4	7,0	15,0	21,7	27,5	32,2
-0,15	0,0	0,0	0,5	2,8	7,6	15,9	22,9	28,9	33,9
-0,08	0,0	0,0	0,7	3,2	8,2	16,8	24,0	30,2	35,3
0,00	0,0	0,0	1,0	3,6	8,7	17,6	25,0	31,3	36,6
0,08	0,0	0,0	1,3	4,0	9,3	18,3	25,9	32,3	37,6
0,15	0,0	0,0	1,6	4,5	9,9	19,0	26,7	33,1	38,3
0,23	0,0	0,0	1,9	4,9	10,4	19,6	27,2	33,6	38,8
0,31	0,0	0,0	2,3	5,4	10,9	20,0	27,6	33,9	38,9
0,38	0,0	0,1	2,7	5,9	11,3	20,4	27,8	33,9	38,6
0,46	0,0	0,3	3,1	6,3	11,7	20,5	27,8	33,6	37,9
0,54	0,0	0,5	3,5	6,8	12,0	20,6	27,5	32,9	36,7
0,62	0,0	0,7	3,9	7,1	12,2	20,4	26,9	31,8	35,1
0,69	0,0	0,9	4,3	7,4	12,2	20,1	26,1	30,4	32,9
0,77	0,0	1,1	4,7	7,6	12,2	19,5	25,0	28,6	30,3
0,85	0,0	1,3	4,9	7,8	12,0	18,8	23,6	26,4	27,1
0,92	0,0	1,5	5,1	7,8	11,8	17,9	22,0	23,8	23,4
1,00	0,0	1,7	5,3	7,7	11,3	16,8	20,0	20,8	19,2

Put Triangle Arbitrages in bps (Bates model)

k	1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	0	0	0	0	0	0	0	0	0
-0,92	0	0	0	0	0	0	0	0	0
-0,85	0	0	0	0	0	0	0	0	0
-0,77	0	0	0	0	0	0	0	0	0
-0,69	0	0	0	0	0	0	0	0	0
-0,62	0	0	0	0	0	0	0	0	0
-0,54	0	0	0	0	0	0	0	0	0
-0,46	0	0	0	0	0	0	0	0	0
-0,38	0	0	0	0	0	0	0	0	0
-0,31	0	0	0	0	0	0	0	0	0
-0,23	0	0	0	0	0	0	0	0	0
-0,15	0	0	0	0	0	0	0	0	0
-0,08	0	0	0	0	0	0	0	0	0
0,00	0	0	0	0	0	0	0	0	0
0,08	0	0	0	0	0	0	0	0	0
0,15	0	0	0	0	0	0	0	0	0
0,23	0	0	0	0	0	0	0	0	0
0,31	0	0	0	0	0	0	0	0	0
0,38	0	0	0	0	0	0	0	0	0
0,46	0	0	0	0	0	0	0	0	0
0,54	0	0	0	0	0	0	0	0	0
0,62	0	0	0	0	0	0	0	0	0
0,69	0	0	0	0	0	0	0	0	0
0,77	0	0	0	0	0	0	0	0	0
0,85	0	0	0	0	0	0	0	0	0
0,92	0	0	0	0	0	0	0	0	0
1,00	0	0	0	0	0	0	0	0	0

According to the values in bps of the arbitrages above is the fact the longer the maturity the higher they are: as the quoted strikes are expresses as equally spaced in cumulated colatility to the maturity, their values are more spaced with the maturity which means their maximum possible value is greater. this may also be related to a kind of positive calendar spread property that would apply to the call and put triangles but this needs to be mathematically proved.

- At the opposite, in the case of Double Heston model, the arbitrages are more equally split between the call and put triangles:

Call Triangle Arbitrages in bps (Double Heston model)

k	Maturity 1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
-0,92	0,3	0,0	0,0	0,9	3,8	7,9	11,0	13,5	15,7
-0,85	0,3	0,0	0,0	1,0	4,0	8,2	11,3	13,8	16,0
-0,77	0,2	0,0	0,0	1,0	4,2	8,5	11,6	14,1	16,2
-0,69	0,2	0,0	0,0	1,1	4,3	8,6	11,7	14,2	16,3
-0,62	0,1	0,0	0,0	1,1	4,5	8,8	11,8	14,3	16,3
-0,54	0,0	0,0	0,0	1,2	4,6	8,9	11,9	14,2	16,1
-0,46	0,0	0,0	0,0	1,3	4,7	8,9	11,8	13,9	15,7
-0,38	0,0	0,0	0,0	1,5	4,8	8,9	11,6	13,6	15,1
-0,31	0,0	0,0	0,0	1,6	4,9	8,7	11,2	13,0	14,4
-0,23	0,0	0,0	0,0	1,7	4,9	8,5	10,8	12,3	13,4
-0,15	0,0	0,0	0,0	1,9	5,0	8,2	10,2	11,5	12,3
-0,08	0,0	0,0	0,0	2,1	5,0	7,8	9,5	10,4	10,9
0,00	0,0	0,0	0,2	2,3	4,9	7,4	8,6	9,2	9,2
0,08	0,0	0,0	0,5	2,5	4,8	6,8	7,6	7,8	7,4
0,15	0,0	0,0	0,8	2,7	4,7	6,1	6,5	6,2	5,3
0,23	0,0	0,0	1,1	3,0	4,5	5,3	5,2	4,4	3,1
0,31	0,0	0,0	1,5	3,2	4,3	4,5	3,8	2,5	0,6
0,38	0,0	0,0	1,9	3,4	4,0	3,6	2,4	0,5	0,0
0,46	0,0	0,0	2,3	3,5	3,7	2,6	0,8	0,0	0,0
0,54	0,0	0,2	2,7	3,7	3,3	1,5	0,0	0,0	0,0
0,62	0,0	0,5	3,1	3,8	2,9	0,4	0,0	0,0	0,0
0,69	0,0	0,8	3,4	3,8	2,4	0,0	0,0	0,0	0,0
0,77	0,0	1,0	3,7	3,8	1,9	0,0	0,0	0,0	0,0
0,85	0,0	1,3	3,9	3,7	1,4	0,0	0,0	0,0	0,0
0,92	0,0	1,6	4,1	3,6	0,8	0,0	0,0	0,0	0,0
1,00	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0

Put Triangle Arbitrages in bps (Double Heston model)

k	1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	0,0	0,2	0,4	0,0	0,0	0,0	0,0	0,0	0,0
-0,92	0,0	0,3	0,5	0,0	0,0	0,0	0,0	0,0	0,0
-0,85	0,0	0,4	0,6	0,0	0,0	0,0	0,0	0,0	0,0
-0,77	0,0	0,5	0,6	0,0	0,0	0,0	0,0	0,0	0,0
-0,69	0,0	0,6	0,7	0,0	0,0	0,0	0,0	0,0	0,0
-0,62	0,0	0,7	0,7	0,0	0,0	0,0	0,0	0,0	0,0
-0,54	0,0	0,8	0,7	0,0	0,0	0,0	0,0	0,0	0,0
-0,46	0,0	0,9	0,7	0,0	0,0	0,0	0,0	0,0	0,0
-0,38	0,1	0,9	0,6	0,0	0,0	0,0	0,0	0,0	0,0
-0,31	0,1	1,0	0,5	0,0	0,0	0,0	0,0	0,0	0,0
-0,23	0,2	1,0	0,4	0,0	0,0	0,0	0,0	0,0	0,0
-0,15	0,2	1,0	0,2	0,0	0,0	0,0	0,0	0,0	0,0
-0,08	0,2	1,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,00	0,3	0,9	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,08	0,3	0,8	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,15	0,3	0,7	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,23	0,3	0,6	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,31	0,3	0,4	0,0	0,0	0,0	0,0	0,0	0,0	0,0
0,38	0,3	0,2	0,0	0,0	0,0	0,0	0,0	0,0	2,0
0,46	0,3	0,0	0,0	0,0	0,0	0,0	0,0	1,7	4,8
0,54	0,3	0,0	0,0	0,0	0,0	0,0	0,9	3,9	7,7
0,62	0,3	0,0	0,0	0,0	0,0	0,0	2,5	6,2	10,7
0,69	0,3	0,0	0,0	0,0	0,0	0,7	4,3	8,6	13,8
0,77	0,2	0,0	0,0	0,0	0,0	1,8	6,0	11,0	17,0
0,85	0,2	0,0	0,0	0,0	0,0	3,0	7,7	13,4	20,1
0,92	0,2	0,0	0,0	0,0	0,0	4,1	9,4	15,8	23,3
1,00	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0

- When the Tolerance Threshold level is raised from 0 bps to 5 bps, the situation gets better for all the models.

Arbitrogrammetry of Stochastic Volatility models with a 5 bps tolerance level

Black Scholes model

k	Maturity 1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,92	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,85	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,77	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,69	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,62	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,54	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,46	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
-0,38	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,31	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,23	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
-0,15	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,08	ok	ok	ok	ok	ok	ok	ok	NOK	NOK
0,00	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,08	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,15	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,23	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,31	ok	ok	ok	ok	ok	ok	ok	NOK	NOK
0,38	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
0,46	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,54	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,62	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,69	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,77	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,85	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,92	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
1,00	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK

Heston model

k	Maturity 1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	ok	ok	ok	ok	ok	ok	ok	ok	ok
-0,92	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
-0,85	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
-0,77	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
-0,69	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
-0,62	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
-0,54	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
-0,46	ok	ok	ok	ok	ok	ok	ok	NOK	NOK
-0,38	ok	ok	ok	ok	ok	ok	ok	NOK	NOK
-0,31	ok	ok	ok	ok	ok	ok	ok	ok	ok
-0,23	ok	ok	ok	ok	ok	ok	ok	ok	ok
-0,15	ok	ok	ok	ok	ok	ok	ok	ok	ok
-0,08	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,00	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,08	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,15	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,23	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,31	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,38	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,46	ok	ok	ok	ok	ok	ok	ok	ok	NOK
0,54	ok	ok	ok	ok	ok	ok	ok	ok	NOK
0,62	ok	ok	ok	ok	ok	ok	ok	NOK	NOK
0,69	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
0,77	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
0,85	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
0,92	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
1,00	ok	ok	ok	ok	ok	ok	ok	ok	ok

Double Heston model

k	Maturity 1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	ok	ok	ok	ok	ok	ok	ok	ok	ok
-0,92	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,85	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,77	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,69	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,62	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,54	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,46	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,38	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,31	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,23	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,15	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,08	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,00	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,08	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,15	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
0,23	ok	ok	ok	ok	ok	NOK	NOK	ok	ok
0,31	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,38	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,46	ok	ok	ok	ok	ok	ok	ok	ok	ok
0,54	ok	ok	ok	ok	ok	ok	ok	ok	NOK
0,62	ok	ok	ok	ok	ok	ok	ok	NOK	NOK
0,69	ok	ok	ok	ok	ok	ok	ok	NOK	NOK
0,77	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
0,85	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
0,92	ok	ok	ok	ok	ok	ok	NOK	NOK	NOK
1,00	ok	ok	ok	ok	ok	ok	ok	ok	ok

Bates model

k	Maturity 1W	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
-1,00	ok	ok	ok	ok	ok	ok	ok	ok	ok
-0,92	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,85	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,77	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,69	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,62	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,54	ok	ok	ok	ok	ok	NOK	NOK	NOK	NOK
-0,46	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,38	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,31	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,23	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,15	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
-0,08	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,00	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,08	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,15	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,23	ok	ok	ok	ok	NOK	NOK	NOK	NOK	NOK
0,31	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,38	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,46	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,54	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,62	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,69	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,77	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,85	ok	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK
0,92	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK
1,00	ok	ok	NOK	NOK	NOK	NOK	NOK	NOK	NOK

However there are stark contrasts there:

- Indeed the situation is so much better for the Black Scholes model that the ECA score get far worse for the Bates model although its ACA is getting better. This means that compared to the Black Scholes model, the Bates model fails in getting better results in terms of arbitrage in the call and put triangles: according to the table (5.1.3) whereas the ACA score of the Bates model goes to 4.43 from 2.34, its ECA score falls to zero. For a model that has the best fit to the market data in terms of MAE, this is showing the MAE is not of much help to settle whether the model is arbitrage free or not.
- For the Double Heston, at the opposite to the Bates, the calibration scoring is better for both the ACA and the ECA scores as they raises respectively from 0.49 to 7.01 and from 0 to 3.63. Although the Double Heston exhibits arbitrage almost everywhere in the Volatility Surface, the average arbitrage in CT and PT is 5.97 bps with a maximum equalling 16.3 bps. In contrast, the average arbitrage for the Bates equals 15.23 bps while the maximum equals 38.9 bps. Needless to say which model has the most pathologic calibration

5.2 Tests on real market data

Based on the market quotes of vanilla options written on the SP500 Index of the 23 may 2022, the SharkJaw test is run on the same stochastic volatility models than above to which are added the Bergomi Local Stochastic Volatility models one factor and two factors as they are defined in [8] and [12].

5.2.1 The LSV Model one and two factors

The SDE driving the dynamic of the underlying X_t in the Bergomi LSV 2 factors is:

$$dX_t = X_t a\left(t, X_t\right) \sqrt{\xi_t^t} dW_t$$

$$\sigma_t = a(t, X_t)\sqrt{\xi_t^t}$$
$$X_t = X_0$$
$$\xi_t^T = \xi_{t_0}^T \exp\left(2\nu x_t^T - 2\nu^2 h(t,T)\right)$$
$$x_t^T = \alpha_\theta \left((1-\theta) e^{-\kappa_1(T-t)} Y_{1t} + \theta e^{-\kappa_2(T-t)} Y_{2t}\right)$$
$$\alpha_\theta = \frac{1}{\sqrt{(1-\theta)^2 + \theta^2 + 2\rho_{1,2}\theta(1-\theta)}}$$
$$dY_{1t} = -\kappa_1 Y_{1t} dt + dB_{1t}$$
$$dY_{2t} = -\kappa_2 Y_{2t} dt + dB_{2t}$$

with

$$h(t,T) = (1-\theta)^2 e^{-2\kappa_1(T-t)} E\left(Y_{1t}^2\right) + \theta^2 e^{-2\kappa_2(T-t)} E\left(Y_{2t}^2\right) + 2\theta(1-\theta) e^{-(\kappa_1+\kappa_2)(T-t)} E(Y_{1t}Y_{2t})$$
$$E\left(Y_{1t}^2\right) = \frac{1-e^{-2\kappa_1(T-t)}}{2\kappa_1}$$
$$E\left(Y_{2t}^2\right) = \frac{1-e^{-2\kappa_2(T-t)}}{2\kappa_2}$$
$$E(Y_{1t}Y_{2t}) = \rho_{1,2}\frac{1-e^{-(\kappa_1+\kappa_2)(T-t)}}{(\kappa_1+\kappa_2)}$$
$$\langle dB_{1t}, dB_{2t}\rangle = \rho_{1,2} dt$$

$$\langle dW_t, dB_{1t}\rangle = \rho_1 dt$$
$$\langle dW_t, dB_{2t}\rangle = \rho_2 dt$$

To obtain the one factor Bergomi LSV model, we must set $\theta = 0$ so that $\alpha_\theta = 1$, $h(t,T) = e^{-2\kappa_1(T-t)} E\left(Y_{1t}^2\right)$ and $x_t^T = e^{-\kappa_1(T-t)} Y_{1t}$. These two models are not calibrated by the minimization of a cost function like in the subsection (5.1.2)but by relying on the particle method as it is described in [8]. Of course, the cost function defined in (5.1.2) is used to compare the performance of all the models in terms of pricing errors.

5.2.2 The results

First the values of the calibrated coefficients for the different pure stochastic volatility models are given by the following table:

	Black Scholes	Heston	Bates	Double Heston	Double Heston+Jump
$v_1{}^\dagger$	0,229241636	0,0899	0,1006	0,0396	0,0051
κ_1		5,7533	5,0972	2182,5304	2,7580
$\theta_1{}^\dagger$		0,0584	0,0549	0,0050	0,0060
σ_1		0,8185	0,7467	0,3122	0,1822
ρ_1		-0,9877	-0,9947	0,0381	-0,2720
$v_2{}^\dagger$				0,0685	0,0627
κ_2				1,3608	1,3505
$\theta_2{}^\dagger$				0,0532	0,0568
σ_2				0,3804	0,3915
ρ_2				-0,9950	-0,9947
μ			0,0066		-0,0243
δ			0,0035		0,0008
λ			0,0038		0,0015

† : this is a variance.

The values of the stochastic volatility parameters of the Bergomi LSV model are the following ones:

	LSV Bergomi 1F	LSV Bergomi 2F
ν	4.3	4.3
κ_1	3.8	3.8
ρ_1	-0.92	-0.92
θ		0.5
κ_2		1.0
ρ_2		-0.75

Whereas these two LSV models have their leverage function $a(t, X_t)$ been calibrated by using the Particle Method, the LSV Bergomi 1F have also been calibrated by solving of the corresponding 2D Fokker-Planck PDE respected by the underlying's density across the time and the two "spatial" dimensions X_t and σ_t (For more details, see [8]).

First let's see which one of the tested models performs the best when using the classic Mean Absolute Error.

Analyzing the MAE The following table allows us to retrieve the conclusions obtained in many other previous papers related to the comparative performance of stochastic volatility models:

Model	MAE in bps	Max Error in bps
BLACK	54	289
BLACK SIGMA(T)	52	246
Bates (Heston + Jump)	18	146
Heston	20	177
Double Heston	**16**	109
Double Heston + Jump	16	130
Dupire	**1**	**5**
LSV1F PDE 2D	11	42
LSV 1F 20 k Particles	3	21
LSV 2F 20 k Particles	2	68

Calibration errors on vanillas

Here all the pure stochastic volatility models deliver almost the same performance as their MAE ranges from 16 bps to 20 bps. They differ more by the maximum error observed: While combining the appreciation of these two metrics the Double Heston seems to be the best but given the fact one of its two CIR process exhibits a very strong speed of mean reversion, one may wonder if the Double Heston with log-normal jump is not the best in practice as its estimated parameters appears to be more within reason. One common feature of all these models is they all get a extremely strong negative correlation between the variance and the underlying spot. This is a well known stylised fact of the market. In the end, whatever we may think of these calibration results, we may wonder if the overall MAE is way to high to have an arbitrage free model. This depends on which strikes the biggest calibration errors are observed: if they appear for strikes far from the money, this is an issue we may live with. However as the table below shows it, that is not the case as a lot of errors higher than 20 bps appear near the money.

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%	0,0	1,5	7,5	18,0	26,2	46,6	55,3	59,7	61,3	61,4	58,6
70%	0,0	3,7	14,5	29,0	37,4	56,1	60,9	63,6	61,2	58,1	49,4
80%	0,0	9,4	24,4	38,3	42,7	54,4	55,7	57,3	51,5	44,7	26,8
90%	0,0	23,3	28,2	33,9	34,1	41,5	40,1	40,0	30,8	20,7	10,7
95%	0,2	28,1	21,5	23,1	22,4	27,4	25,6	25,1	16,0	4,4	33,6
100%	12,9	15,8	1,4	2,8	5,4	9,2	7,1	6,2	1,1	13,7	58,3
105%	0,0	3,5	15,7	15,2	11,2	6,8	10,4	12,8	18,9	31,5	83,5
110%	0,0	1,1	11,2	15,2	15,6	16,2	22,1	26,6	34,1	46,2	106,1
120%	0,0	0,5	1,3	1,4	0,2	8,1	20,6	29,9	42,9	55,3	130,3
130%	0,0	0,1	0,3	1,0	1,3	3,0	0,4	4,5	18,0	33,7	120,5
140%	0,0	0,0	0,1	0,2	0,2	1,6	4,7	8,8	6,7	3,0	85,5
150%	0,0	0,0	0,0	0,0	0,0	0,7	2,9	7,8	12,9	13,3	44,2
160%	0,0	0,0	0,0	0,0	0,0	0,3	1,6	5,1	10,1	15,2	11,9
170%	0,0	0,0	0,0	0,0	0,0	0,1	0,9	3,4	6,5	11,9	6,4
180%	0,0	0,0	0,0	0,0	0,0	0,0	0,5	2,3	3,9	8,2	13,8
190%	0,0	0,0	0,0	0,0	0,1	0,1	0,2	1,4	2,2	5,1	15,8
200%	0,0	0,0	0,0	0,0	0,1	0,1	0,0	0,7	1,1	2,9	15,6
250%	0,0	0,0	0,0	0,1	0,1	0,2	0,2	0,2	0,3	0,3	7,8

Figure 5.2.2.a: Error calibration of the Double Heston+ jumps model on vanilla options for the SP500 as of 23 may, 2022

At the opposite, the Dupire model delivers a much lower MAE of only 1 bps with a maximum error of 5 bps. At first glance, this may look good but the bad new is the distribution of errors over the maturity and the moneyness (Figure (5.2.2)) shows us this Dupire calibartion appears to be problematic with the biggest errors appearing near the money for almost every maturity whereas the pricing of the vanillas was done by Monte-Carlo with 200 000 paths in Sophis. As current pricing uses way less paths, this is casting a doubt on the overall quality of the pricing with the Dupire model whereas theoretically speaking this model must match perfectly the market data. This is all the more a concern as the quality of the Dupire model has a direct impact on the calibration of the LSV models that rely on it.

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%	0,0	0,8	1,1	1,3	1,4	1,0	0,7	0,4	0,4	0,4	0,3
70%	0,0	1,5	1,6	1,5	1,3	0,7	0,4	0,1	0,1	0,1	0,7
80%	0,0	2,0	1,5	1,2	0,7	0,3	0,3	0,7	0,8	0,9	1,2
90%	0,0	2,0	0,8	0,2	0,7	1,2	1,3	1,6	1,8	1,6	1,5
95%	0,2	1,3	0,1	1,0	1,7	2,1	2,2	2,3	2,4	2,0	1,7
100%	0,1	2,2	2,9	3,1	2,9	3,2	3,1	3,1	2,9	2,5	1,9
105%	0,1	4,8	4,4	4,4	4,0	3,7	3,8	3,7	3,4	2,9	2,1
110%	0,0	1,8	2,9	3,4	3,9	3,9	3,9	3,9	3,7	3,2	2,3
120%	0,0	0,0	0,3	0,9	1,4	2,5	3,1	3,5	3,6	3,2	2,5
130%	0,0	0,0	0,0	0,1	0,2	0,8	1,4	1,9	2,4	2,4	2,6
140%	0,0	0,0	0,0	0,0	0,1	0,2	0,6	1,0	1,3	1,6	2,2
150%	0,0	0,0	0,0	0,0	0,0	0,1	0,3	0,6	0,8	1,1	1,8
160%	0,0	0,0	0,0	0,0	0,0	0,0	0,2	0,4	0,6	0,8	1,3
170%	0,0	0,0	0,0	0,0	0,0	0,0	0,1	0,2	0,3	0,5	1,0
180%	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,1	0,2	0,4	0,8
190%	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,1	0,1	0,3	0,8
200%	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,1	0,1	0,2	0,7
250%	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,0	0,3

Figure : Error calibration of the Dupire model on vanilla options for the SP500 as of 23 may, 2022

And indeed, the calibration errors of the LSV are globallyt higher than the ones observed for the Dupire (See (5.2.2)). This demonstrates the quality of the Dupire calibration is a upper bound for the quality of calibration of the LSV models

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%	0,0	0,4	3,1	3,2	3,6	5,6	10,0	13,9	15,6	16,7	21,7
70%	0,0	1,1	5,3	5,6	6,4	8,1	13,5	17,2	19,4	20,4	24,2
80%	0,0	2,8	9,7	9,7	10,7	11,9	18,1	21,4	23,5	24,1	26,6
90%	0,0	4,8	14,8	12,6	13,4	12,5	19,7	23,0	25,6	25,8	28,3
95%	0,2	1,0	11,5	7,7	9,1	9,9	18,3	22,5	25,4	25,8	28,5
100%	11,3	12,0	2,2	5,7	3,4	2,2	13,7	19,4	22,8	23,9	28,3
105%	0,0	30,8	33,3	34,8	29,4	15,2	2,0	11,5	16,7	18,7	26,9
110%	0,0	8,6	25,5	42,7	45,5	37,4	17,4	3,3	5,1	8,8	22,6
120%	0,0	0,1	0,0	2,9	6,1	18,8	24,5	24,5	22,2	19,5	1,1
130%	0,0	0,1	0,9	1,2	1,4	0,6	3,2	9,1	15,8	22,1	23,4
140%	0,0	0,1	0,9	1,2	1,5	3,1	3,6	2,0	3,2	9,3	28,7
150%	0,0	0,1	0,8	1,0	1,2	2,7	4,3	5,6	2,9	1,0	20,6
160%	0,0	0,0	0,6	0,8	0,9	2,2	4,0	6,3	5,4	3,3	11,1
170%	0,0	0,0	0,5	0,6	0,7	1,7	3,6	6,1	6,3	5,4	3,9
180%	0,0	0,0	0,5	0,5	0,6	1,3	3,1	5,8	6,5	6,4	0,8
190%	0,0	0,0	0,4	0,5	0,5	1,0	2,7	5,5	6,4	7,0	3,4
200%	0,0	0,0	0,4	0,4	0,5	0,8	2,3	5,1	6,1	7,2	4,4
250%	0,0	0,0	0,3	0,3	0,3	0,4	0,9	3,0	3,7	4,9	6,3

Figure : Error calibration of the LSV 1F model calibrated by 2D PDE solving on vanilla options for the SP500 as of 23 may, 2022

If the same LSV model is calibrated by relying on the Particle Method of [9], the things get better although the errors are still getting high (> 10 bps) for some of the near the money strike levels.

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%	0,0	0,1	1,3	0,5	0,9	1,5	3,2	4,2	4,0	3,1	6,6
70%	0,0	0,2	2,2	1,1	1,5	0,8	2,8	3,3	3,0	1,4	4,5
80%	0,0	0,2	3,6	1,8	1,5	0,8	2,6	2,3	2,0	0,1	4,8
90%	0,0	0,7	3,7	0,9	1,1	5,5	0,8	0,9	0,9	0,3	6,1
95%	0,2	4,8	0,7	4,7	4,0	7,4	2,2	1,2	0,9	0,6	6,8
100%	11,3	9,4	2,1	7,7	7,6	9,5	3,2	1,1	1,0	1,7	7,7
105%	0,0	10,0	7,2	11,9	11,8	12,9	4,9	1,7	0,7	1,7	8,8
110%	0,0	3,4	5,6	11,2	11,8	14,4	7,4	3,7	0,3	0,6	8,9
120%	0,0	0,1	0,8	0,8	1,3	1,0	2,2	2,3	1,7	2,3	3,4
130%	0,0	0,3	0,6	1,2	1,9	4,0	3,9	2,7	1,4	0,1	3,2
140%	0,0	0,2	0,5	0,8	1,2	3,0	4,0	4,6	3,4	2,5	4,0
150%	0,0	0,2	0,4	0,6	0,8	1,9	2,9	4,2	3,8	3,0	1,7
160%	0,0	0,2	0,4	0,5	0,6	1,4	2,3	3,6	3,7	2,8	0,8
170%	0,0	0,2	0,3	0,4	0,6	1,1	1,9	3,2	3,4	2,9	2,8
180%	0,0	0,1	0,3	0,4	0,5	1,0	1,6	3,0	3,2	3,0	3,9
190%	0,0	0,1	0,3	0,3	0,4	0,9	1,4	2,9	3,2	3,2	4,2
200%	0,0	0,1	0,2	0,3	0,4	0,8	1,3	2,9	3,1	3,4	4,0
250%	0,0	0,1	0,2	0,2	0,3	0,5	0,8	2,1	2,3	2,7	4,4

Figure : Error calibration of the LSV 1F model calibrated by the Particle Method with 20 000 particles on vanilla options for the SP500 as of 23 may, 2022

Last, the recourse to a two factors Bergomi LSV model roughly delivers as the same overall quality of calibration (Figure (5.2.2)) as the errors appears to be almost the same than in the one factor case: The errors are smaller in the long term but are higher between 1 Y and 2 Y of maturity shows it (note using a two factor model has other virtues in terms of dynamics of the volatility but that is not the focus here).

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%	0,0	0,3	1,9	2,2	2,9	3,1	5,0	5,0	6,3	5,5	7,3
70%	0,0	0,2	2,6	2,5	3,3	1,9	4,0	3,2	4,9	3,6	4,7
80%	0,0	0,7	3,9	2,1	2,8	0,5	1,3	0,1	1,5	0,5	2,6
90%	0,0	0,1	3,5	0,4	0,1	5,8	4,2	5,1	2,9	3,5	1,1
95%	0,2	4,4	0,8	3,3	2,6	7,4	5,8	6,2	4,1	4,4	0,9
100%	11,3	9,0	1,2	5,7	5,9	9,0	6,6	6,3	4,8	5,1	1,2
105%	0,0	9,7	5,4	9,3	9,7	12,1	8,2	6,1	5,3	5,9	1,9
110%	0,0	3,2	4,4	9,3	9,8	13,9	10,1	6,9	5,8	6,7	2,2
120%	0,0	0,0	0,6	1,4	2,0	1,4	2,7	3,5	4,1	6,9	0,2
130%	0,0	0,1	0,4	1,0	1,6	3,1	4,0	2,7	0,8	2,0	2,4
140%	0,0	0,1	0,3	0,5	0,8	2,4	4,0	4,4	3,1	1,4	1,3
150%	0,0	0,1	0,2	0,4	0,6	1,7	2,8	3,9	3,0	2,0	1,0
160%	0,0	0,0	0,2	0,3	0,4	1,4	2,2	3,2	2,6	2,2	2,6
170%	0,0	0,0	0,1	0,2	0,4	1,2	2,0	2,7	2,5	2,3	3,4
180%	0,0	0,0	0,1	0,2	0,3	1,0	1,9	2,4	2,5	2,5	3,8
190%	0,0	0,0	0,1	0,2	0,2	0,8	1,7	2,2	2,5	2,6	3,6
200%	0,0	0,0	0,1	0,1	0,2	0,7	1,6	2,1	2,3	2,8	3,3
250%	0,0	0,0	0,0	0,0	0,1	0,3	0,8	1,3	1,4	1,8	3,3

Figure : Error calibration of the LSV 2F model calibrated by the Particle Method with 20 000 particles on vanilla options for the SP500 as of 23 may, 2022

Then the calibration scoring base upon the absence of arbitrage proves to be useful to distinguish the one factor and the two factors LSV models. More generally, the problem with a measure of pricing error is that it does not a give clear clue of whether the model is arbitrable for the corresponding couple of strike and maturity. Indeed, an error of 5 bps has not the same meaning if 1/ the corresponding strike is or is not far from the money and 2/ if the maturity is small or high.

Analysis of the calibration scoring Using the SharkJaw test for all the preceding models, the following ACA scores are obtained:

Model	ACA Score	SharkJaw Arbitrage Errors in bps Average (1)	SharkJaw Arbitrage Errors in bps Maximum	ECA SCORE 0 bps
BLACK	5,56	20,05	54,74	N/A
BLACK SIGMA(T)	5,40	19,33	54,06	0,00
Bates (Heston + Jump)	7,88	5,54	22,00	5,23
Heston	7,47	6,04	24,98	4,32
Double Heston	8,18	**3,11**	**12,47**	5,91
Double Heston + Jump	**8,33**	3,83	13,42	6,25
Dupire	**9,39**	11,40	32,51	8,52
LSV1F PDE 2D	9,04	8,46	35,67	7,27
LSV 1F 20 k Particles	9,44	12,06	30,67	8,41
LSV 2F 20 k Particles	8,18	**1,98**	**16,42**	5,45

Average over the existing arbitrages (no null value is counted for couple (K,T) for which there is no arbitrage)

Among the pure stochastic volatility models, the Double Heston and the Double Heston + jumps dominate the other ones. As expected, the Dupire model has the best scoring but it does not obtain the maximal notation (10/ 10) which means its calibration method and/or its pricing of european binaries by Monte-carlo with 200 000 pathes remains too inprecise to avoid exhibiting arbitrages according to the SharkJaw test. More interesting is the results of the LSV models: The one factor model has a scoring very near from the Dupire meaning their overall calibration performance is mainly penalized by the one of the Dupire model on which the LSV is built.

A priori, any well calibrated model must be self consistent and be arbitrage free in terms of market triangles so that market triangles evaluated with models prices for both the calls, puts and binaries must always be positive. To check this self consistency for all the tested models, the SharkJaw test has been run except the model prices have been used instead of the market prices for the calls and the puts. In the table (5.2.2) below, while all the pure Stochastic Volatility pass the test without revealing any self consistency issue, the LV and the LSV models fail to do so:

Model	Self-Consistency SharkJaw ACA Score	Self-Consistency SharkJaw Avg error in bps	Self-Consistency SharkJaw Max error in bps
BLACK	10,0	0	0
BLACK SIGMA(T)	10,0	0	0
Bates (Heston + Jump)	10,0	0	0
Heston	10,0	0	0
Double Heston	10,0	0	0
Double Heston + Jump	10,0	0	0
Dupire	9,39	11,0	32
LSV1F PDE 2D	9,34	11,4	33
LSV 1F 20 k Particles	**9,40**	**11,3**	33
LSV 2F 20 k Particles	9,39	2,1	19

Table Self consitency SharkJaw test

Again this shows how hard LV and LSV models are hard to master: while they are theoretically designed to be self consistent in terms of absence of arbitrage, due to the diverse numeric intervening during the calibration which generates multiple accumulating imprecisions, there are still works to avoid this kind of results. It also shows the arbitrages exhibited by the results of the SharkJaw test in the table (5.2.2) probably mainly come from the precision of the numeric methods embedded into the calibration process of these models. A detail view of the arbitrage error is given by the figure:

DUPIRE 1 Million pathes

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%								0,9	7,8	13,4	32,1
70%										1,6	26,2
80%											15,3
90%										1,7	12,4
95%											10,1
100%											7,4
105%											3,6
110%											
120%											
130%											
140%											
150%											
160%											
170%											
180%											
190%											
200%											
250%											

LSV Bergomi 1F calibrated by PDE2D 1 Million pathes

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%								0,6	7,9	13,8	33,2
70%										2,2	25,8
80%											15,6
90%										1,9	12,3
95%											10,4
100%											8,1
105%											5,6
110%											
120%											
130%											
140%											
150%											
160%											
170%											
180%											
190%											
200%											
250%											

LSV Bergomi 1F calibrated by Particle Method 1 Million pathes

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%								2,2	9,2	14,9	32,9
70%										1,9	24,8
80%											15,1
90%										1,0	12,0
95%											9,7
100%											7,2
105%											4,2
110%											
120%											
130%											
140%											
150%											
160%											
170%											
180%											
190%											
200%											
250%											

LSV Bergomi 2F calibrated by Particle Method 1 Million pathes

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%										4,5	18,9
70%											4,4
80%											
90%											4,4
95%											2,5
100%											
105%											
110%											
120%											
130%											
140%											
150%						0,5					
160%						0,9	1,1				
170%						0,6	1,3	1,5	1,5	0,6	
180%							1,1	1,4	1,8	1,7	
190%							0,9	1,1	1,6	1,9	
200%							0,6	0,9	1,3	1,7	0,9
250%							0,8	1,8	2,1	1,4	0,7

Table : maximum error in bps
by moneyness and maturity in the Self consistency Shark Jaw test for the LV-LSV models

While the LSV2F exhibits more arbitrage, they are less pronunced as except the error of 19 bps for the couple (60%, 20/12/2024), the other arbitrage doesn't exceed 5 bps. On the contrary, the arbitrages observed for the LSV 1F remains around or superior 10 bps for near the money strike at the last maturity.

Comin back to the results of the SharkJaw test in table (5.2.2), the scoring of the LSV 2 factors is interesting as it equals the one of the Double Heston (DH) and is even beaten by the Double Heston + Jumps (DHJ). Moreover while the average arbitrage is lower than the ones of the two Double Heston models, its maximum value is slightly higher.

A close scrutiny of the corresponding arbitrograms shows us the LSV 2F and the DHJ model both exhibit arbitrages when the moneyness is higher than 150 % when the maturity is above 6M for the LSV 2F and above 10 M for the DHJ. The difference is the corresponding area (K, T) where the LSV 2F is significantly bigger than the one for the DHJ. In contrast, the DHJ also exhibits more scattered arbitrages with some arbitrages in the lower tail for short term maturities between 1W and 3M

The Double Heston + Jumps model delivers the following arbitrogram (figure):

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%	1	1	1	1	1	1	1	1	1	1	1
70%	1	0	0	0	1	1	1	1	1	1	1
80%	1	0	1	1	1	1	1	1	1	1	1
90%	1	1	1	1	1	1	1	1	1	0	0
95%	1	1	1	1	1	1	1	0	0	0	0
100%	1	1	1	1	1	1	1	1	0	0	0
105%	1	1	1	1	1	1	1	1	1	1	0
110%	1	1	1	1	1	1	1	1	1	1	1
120%	1	1	1	1	1	1	1	1	1	1	1
130%	1	1	1	1	1	1	1	1	1	1	1
140%	1	1	1	1	1	1	1	1	1	1	1
150%	1	1	1	1	1	1	0	1	1	1	0
160%	1	1	1	1	1	1	0	0	0	1	0
170%	1	1	1	1	1	1	1	0	0	0	0
180%	1	1	1	1	1	1	1	0	0	0	1
190%	1	1	1	1	1	1	1	0	0	0	1
200%	1	1	1	1	1	1	1	0	0	0	1
250%	1	1	1	1	1	1	1	1	1	1	1

Figure: Arbitrogram of the Double Heston + Jumps as of 23 may, 2022

while the LSV 2 factors delivers the following one:

	24/05/2022	24/05/2022	24/05/2022	24/05/2022	24/05/2022	24/05/2022	24/05/2022	24/05/2022	24/05/2022	24/05/2022	24/05/2022
60%	1	1	1	1	1	1	1	1	1	0	0
70%	1	1	1	1	1	1	1	1	1	1	0
80%	1	1	1	1	1	1	1	1	1	1	1
90%	1	1	1	1	1	1	1	1	1	1	0
95%	1	1	1	1	1	1	1	1	1	1	0
100%	1	1	1	1	1	1	1	1	1	1	0
105%	1	1	1	1	1	1	1	1	1	1	1
110%	1	1	1	1	1	1	1	1	1	1	1
120%	1	1	1	1	1	1	1	1	1	1	1
130%	1	1	1	1	1	1	1	1	1	1	1
140%	1	1	1	1	1	1	1	1	1	1	1
150%	1	1	1	1	0	0	1	1	1	1	1
160%	1	1	1	1	1	0	0	0	1	1	1
170%	1	1	1	1	1	0	0	0	0	1	1
180%	1	1	1	1	1	0	0	0	0	0	1
190%	1	1	1	1	1	0	0	0	0	0	1
200%	1	1	1	1	1	1	0	0	0	0	0
250%	1	1	1	1	1	0	0	0	0	0	0

Figure: Arbitrogram of the Bergomi LSV 2 factors calibrated with the Particle Method with 20 k particles and 200 000 pathes as of 23 may, 2022

A close scrutiny of the arbitrages revealed by the SharkJaw test:

- let think the errors appear all the more frequently the maturity is long. This is misleading because the scale of strike is a scale of moneyness which doesn't vary with the maturity. Then in the short term, the only meaningful strikes are in the range 95%-105% where few quotations are available in this market data set. If this data set would have a range of strike scaled with the cumulated ATM volatility at maturity like it is in the tests run on simulated data in the section (5.1), more arbitrage will appear in the short term.
- If the Shark Jaw test is only observed, the Double Heston + Jumps may seem to perform as well as the LSV Bergomi 2F. However if the calibration errors on vanillas are also taken into account, the LSV 2F is way better as it has a MAE almost ten time smaller. This illustrates the fact the calibration on vanillas and the existence of arbitrages in Market triangles are different matters. This shows the usefulness of the SharkJaw to track down the existence of arbitrage in the model under scrutiny.
- Among all the LSV models, the Bergomi factors 2 factors is the only one to show small values of arbitrage that don't exceed 5 apart from one quoted point. Then, even if it shows a greater number of arbitrages than the different LSV 1 factor here, it seems to be the only one for which exhibited arbitrages may not be really exploitable by traders. All the other models exhibit arbitrages we may consider exploitable given their relative high value.

BLACK

	24/05/2022	17/06/2022	15/07/2022	19/09/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%											
70%		2,2	8,1	17,3	22,7	30,3	24,8	16,9	5,7		
80%		6,5	16,4	18,4	12,0						24,6
90%						7,1	18,0	24,7	28,4	32,1	41,3
95%					5,1	24,5	33,6	38,6	39,3	41,5	48,2
100%				8,7	15,9	31,0	39,6	44,6	45,0	45,6	52,0
105%					5,4	24,9	34,3	40,0	43,0	43,3	50,6
110%								4,5	20,3	27,5	54,7
120%											9,3
130%				1,2	3,1	2,9					
140%					2,4	13,2	21,1	20,9	18,7	11,6	
150%						6,5	15,8	22,2	27,3	30,0	18,9
160%						2,2	8,0	14,8	20,7	26,6	33,6
170%						0,6	3,5	8,4	13,1	18,8	34,3
180%							1,4	4,2	7,7	12,0	29,8
190%							0,5	1,8	4,3	7,2	24,6
200%								0,7	2,3	4,2	19,6
250%											15,0

BLACK SIGMA(T)

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%											
70%		2,2	7,9	15,3	18,3	18,6	13,4	8,0	1,6		
80%		5,6	8,3								6,4
90%						17,5	24,6	28,9	30,1	31,6	34,6
95%				9,5	14,1	31,1	37,9	41,4	40,5	41,1	43,1
100%				10,2	17,4	32,8	41,2	45,9	45,6	45,4	48,5
105%						22,3	33,4	39,8	43,1	43,2	48,7
110%								1,6	19,4	27,6	54,1
120%											13,5
130%			1,8	9,2	14,0	17,2	3,1				
140%				2,5	6,5	22,4	29,5	27,0	21,4	10,7	
150%					1,7	11,4	21,5	27,0	29,6	29,3	9,3
160%						4,5	11,5	18,3	22,4	25,9	24,4
170%						1,7	5,6	10,7	14,4	18,3	25,9
180%						0,6	2,5	5,7	8,6	11,7	22,5
190%							1,1	2,7	4,9	6,9	18,4
200%								1,3	2,7	4,0	14,5
250%											6,6

Maximum Arbitrage values in bps in the space (M%, Maturity) for Black and Black Sigma(T) models

DOUBLE HESTON + JUMPS

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%											
70%		2,2	4,9	2,8							
80%		2,9									
90%										2,9	11,9
95%								1,6	1,4	3,6	13,2
100%									0,8	2,3	13,4
105%											10,6
110%											
120%											
130%											
140%											
150%							0,7				11,4
160%							0,6	1,2	1,9		10,9
170%								1,1	2,3	2,5	4,6
180%								0,9	1,7	2,8	
190%								0,6	1,1	2,2	
200%								0,9	1,3	3,1	
250%											

HESTON

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%											
70%		1,4									
80%											9,1
90%								0,9	4,7	9,0	21,1
95%								4,5	7,2	11,1	23,6
100%								4,1	7,5	10,6	24,3
105%									4,0	6,8	21,4
110%											
120%			1,8	5,2							
130%				0,8	1,4	1,0					
140%						1,3	1,1			0,7	1,1
150%							1,7	1,8		2,3	22,1
160%							0,7	2,0	2,5		24,3
170%								1,1	2,7	2,4	17,0
180%								0,8	1,7	3,0	8,9
190%								0,6	1,0	2,2	3,7
200%								0,6	1,0	2,8	0,9
250%											

DOUBLE HESTON

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%											
70%		2,2	4,4	1,4							
80%		2,2									
90%									0,7	2,6	10,4
95%							0,5	2,4	1,6	3,3	11,9
100%								0,7	1,0	2,2	12,5
105%											10,1
110%											
120%											
130%											
140%											
150%							0,6				7,1
160%							0,7	1,2	1,5		7,7
170%								1,1	2,3	2,4	2,7
180%								0,9	1,7	2,8	
190%								0,6	1,1	2,2	
200%								0,8	1,2	3,1	
250%											

BATES

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%											
70%		1,3									
80%											3,4
90%									3,2	7,1	18,3
95%								3,4	5,6	9,2	20,9
100%								3,0	6,0	8,9	22,0
105%									2,8	5,4	19,5
110%											
120%			1,8								
130%				0,8	1,5						
140%						1,3	1,9				
150%							1,8	3,0			16,4
160%							0,8	2,1	3,7		18,1
170%								1,2	3,0	3,7	11,2
180%								0,9	1,9	3,3	4,1
190%								0,7	1,2	2,4	
200%								1,1	1,5	3,5	
250%											

Maximum Arbitrage values in bps in the space (M%, Maturity) for pure stochastic volatility models

DUPIRE

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%								1,1	8,2	13,9	32,5
70%										2,4	26,6
80%											15,7
90%										2,1	12,6
95%											10,3
100%											7,6
105%											3,8
110%											
120%											
130%											
140%											
150%											
160%											
170%											
180%											
190%											
200%											
250%											

LSV Bergomi 1F PDE 2D 1M Pathes

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%								3,9	11,7	17,5	35,7
70%										5,9	28,3
80%											17,3
90%										1,8	12,5
95%											10,1
100%											6,8
105%											1,3
110%											
120%											
130%											
140%											
150%											
160%											1,6
170%											2,2
180%											1,3
190%											0,5
200%											
250%							0,7	1,1	0,7		

LSV Bergomi 1F 20k Particles 1M Pathes

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%								1,2	8,3	13,2	30,7
70%											25,1
80%											16,4
90%										1,9	12,7
95%											10,6
100%											8,3
105%											4,4
110%											
120%											
130%											
140%											
150%											
160%											
170%											
180%											
190%											
200%											
250%											

LSV Bergomi 2F 20k Particles 1 M Pathes

	24/05/2022	17/06/2022	15/07/2022	19/08/2022	16/09/2022	16/12/2022	17/03/2023	16/06/2023	15/09/2023	15/12/2023	20/12/2024
60%										2,7	16,4
70%											2,3
80%											
90%											4,1
95%											2,8
100%											1,2
105%											
110%											
120%											
130%											
140%											
150%					0,6	1,2					
160%						1,2	1,7	1,0			
170%						0,8	1,5	2,1	1,6		
180%						0,7	1,3	1,7	1,8	1,6	
190%						0,5	1,0	1,3	1,6	1,7	
200%							0,8	1,1	1,4	1,6	1,2
250%						0,8	1,5	2,6	3,1	2,3	0,7

6 Conclusion:

The purpose of this paper was to present new arbitrability bounds for the implied volatility skew and convexity and to define a scoring of model calibration based upon the central concept of arbitrage in relation with real market quotes of vanillas whereas until now, only classic statistical measures like the MAE, the MSE or the RMSE were used by practitionneer.

The Sharkjaw arbitrability bounds for the implied volatility skew has been shown to supplant existing bounds either given the natural bounds 0 and 1 of a probability measure or given by the works from [7] and [21]. Given the ease to compute these bounds and the fact they get more and more narrower while the market gets more and more quotes, this makes the Sharkjaw bounds the ones to consider to track arbitrage in the market.

To proceed, an arbitrage test, the SharJaw test, has been developed: this test exploits the intricated nature of the vanilla options to check arbitrages while opposing model prices of European Binaries to market quotes of vanillas. Two calibration scoring, the Absolute Calibration Arbitrability (ACA) score and the Exotic Calibration Arbitrability (ECA) score, were developed. The first one, the ACA, proves measuring the fitting quality of a given model with a classical statistical measure like the MAE, MSE or the RMSE may be misleading as a model with the lowest value of such a measure may exhibit significant arbitrages.

The second score, the ECA, measures the capacity of a model to avoid the arbitrages better than a Black Scholes may do. It proves to be useful in addition to the ACA to challenge the capacity of a stochastic volatility model to provide better calibration results in terms of arbitrage than a simple Black Scholes.

In addition, a graphic tool, the arbitrogram has been developed and helps to understand where a model fails in the space (K, T) of strike and maturity. It shows arbitrages are far from being confined in the tails of the underlying's distribution.

In theory, by construction the family of Local Volatility and Local Stochastic Volatility models must provide a perfect match to the vanilla market quotes with no possible arbitrages in order that the ACA and ECA must equal 10 in all possible market cases. In practice, the calibration of a Dupire Local Volatility now may seem to be a well mastered matter. However in the case of the LSV models, the calibration methods may fail by significant margins in terms of bps. Indeed, the current mainstream methodologies based on the Particle Method designed by Guyon-Labordère [9] and [14] endure important difficulties to make their LSV model to keep a perfect match to the vanilla market quotes. For example, in [8], the LSV Bergomi 2 factors shows important deviation up to 20 or 30 bps from the market prices when the volatility of volatility becomes too high. The Sharkjaw tests run in the section (5) shows our intuition is right as their results suggest that the LSV models as well as the LV exhibit arbitrages whereas their overall performance in terms of calibration to the vanilla market is much better than the classic stochastic volatility models and near to be perfect.

In the end, the Shark Jaw test and the ACA calibration score are new useful tool to check the existence of arbitrage resulting from a calibration exercise. Coupled with classic statistical measures, it will help practitionneers to have a clearer view of the performance of their model calibration: the statistical measures give a quantitative measure of the calibration quality in terms of pricing error whereas the ACA score gives a qualitative appreciation of it (the model is arbitrage free or not).

Annex

The proof of proposition (2)

Proof. Let the quantities $I(j)$ and $I_x(j)$ be defined like

$$\begin{cases} I(j) = \int_{K_j}^{K_{j+1}} f(x)\,dx \\ I_x(j) = \int_{K_j}^{K_{j+1}} xf(x)\,dx \end{cases} \quad \forall j = 2; ...; n_{i,k} - 1$$

$$\begin{cases} I(0) = \int_0^{K_1} f(x)\,dx & I_x(0) = \int_0^{K_1} xf(x)\,dx \\ I(n_{i,k}) = \int_{K_{n_{i,k}}}^{+\infty} f(x)\,dx & I_x(n_{i,k}) = \int_{K_{n_{i,k}}}^{+\infty} xf(x)\,dx \end{cases}$$

Then the quantities $I(j)$are linked to the undiscounted values of the european down and in and up and in Binaries, $BDI(K_j, T)$ and $BUI(K_j, T)$, such that:

$$BDI(K_j, T) = \sum_{u=1}^{u<j} \int_{K_u}^{K_{u+1}} f(x)\,dx = \sum_{u=0}^{u<j} I(u) \tag{6.1}$$

$$BUI(K_j, T) = \sum_{u=j}^{u \leq n_{i,k}} \int_{K_u}^{K_{u+1}} f(x)\,dx = \sum_{u=j}^{u \leq n_{i,k}} I(u) \tag{6.2}$$

These undiscounted values $BDI(K_j, T)$ and $BUI(K_j, T)$ equal then respectively the probabilities $P(X_T < K_j)$ and $P(X_T > K_j)$. Then by assuming the market quotes of calls and puts to be arbitrage, the conditions (1.3) guarantees their value to be probability values (i.e. inferior to 1 and superior to 0). Indeed:

$$\begin{cases} P(K_{j+1}, T) - P(K_j, T) - (K_{j+1} - K_j) P(X_T < K_j) & \geq 0 \\ C(K_{j-1}, T) - C(K_j, T) - (K_j - K_{j-1}) P(X_T > K_j) & \geq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 1 \geq \frac{P(K_{j+1},T)-P(K_j,T)}{(K_{j+1}-K_j)} \geq P(X_T < K_j) \\ P(X_T < K_j) \geq 1 + \frac{C(K_{j-1},T)-C(K_j,T)}{(K_j-K_{j-1})} \end{cases}$$

and as the market quotes are assumed to be arbitrage free, we have $0 < \frac{P(K_{j+1},T)-P(K_j,T)}{(K_{j+1}-K_j)} < 1$ and $-1 < \frac{C(K_{j-1},T)-C(K_j,T)}{(K_j-K_{j-1})} < 0$ so that:

$$0 \leq 1 + \frac{C(K_{j-1}, T) - C(K_j, T)}{(K_j - K_{j-1})} \leq P(X_T < K_j) \leq \frac{P(K_{j+1}, T) - P(K_j, T)}{(K_{j+1} - K_j)} \leq 1$$

$$\Leftrightarrow 0 \leq P(X_T < K_j) \leq 1$$

Replacing (6.1) and (6.2) in (1.3) yields:

$$\begin{cases} P(K_{j+1}, T) - P(K_j, T) - (K_{j+1} - K_j) \sum_{u=0}^{u<j} I(u) & \geq 0 \\ C(K_{j-1}, T) - C(K_j, T) - (K_j - K_{j-1}) \sum_{u \geq j}^{n_{i,k}} I(u) & \geq 0 \end{cases}$$

Given the market quotes, a limit value, $I_{max}(j)$, of $I(j)$ may be computed. Given the first equation of (1.3), we must have:

$$C(K_{j-1}, T) - C(K_j, T) - (K_j - K_{j-1}) \left(I(j) + \sum_{u>j}^{n_{i,k}} I(u) \right) \geq 0$$

$$\Leftrightarrow I(j) \leq I_{max}(j)$$

$$\Leftrightarrow I_{max}(j) = \frac{C(K_{j-1}, T) - C(K_j, T)}{(K_j - K_{j-1})} - \sum_{u>j}^{n_{i,k}} I(k) \quad \forall j = 2; ...; n_{i,K} \tag{6.3}$$

Likewise the quantity $I(j)$ is linked to the quotations of the puts via the positivity constraint lying on the put triangle $PT(K_j, K_{j+1}, T)$. Indeed we have

$$PT(K_j, K_{j+1}, T) \geq 0 \;\; \forall j = 1; ...; n_{i,K} - 1$$

$$\Leftrightarrow P_{mkt}\left(K_{j+1}\right)-P_{mkt}\left(K_{j}\right)-\left(K_{j+1}-K_{j}\right)\left(\sum_{u=0}^{u<j}I\left(u\right)\right)\geq 0$$

$$\Leftrightarrow P_{mkt}\left(K_{j+1}\right)-P_{mkt}\left(K_{j}\right)-\left(K_{j+1}-K_{j}\right)\left(1-\sum_{u\geq j}^{n_{i,k}}I\left(u\right)\right)\geq 0$$

$$\Leftrightarrow P_{mkt}\left(K_{j+1}\right)-P_{mkt}\left(K_{j}\right)-\left(K_{j+1}-K_{j}\right)\left(1-I\left(j\right)-\sum_{u>j}^{n_{i,k}}I\left(u\right)\right)\geq 0$$

A minimum for $I\left(j\right)$ may then be identified:

$$I\left(j\right)\geq 1-\frac{P_{mkt}\left(K_{j+1}\right)-P_{mkt}\left(K_{j}\right)}{K_{j+1}-K_{j}}-\sum_{u>j}^{n_{i,k}}I\left(u\right)$$

$$\Leftrightarrow I\left(j\right)\geq I_{min}\left(j\right)\quad\forall j=1;...;n_{i,K}-1$$

$$\Leftrightarrow I_{min}\left(j\right)=1-\frac{P_{mkt}\left(K_{j+1}\right)-P_{mkt}\left(K_{j}\right)}{K_{j+1}-K_{j}}-\sum_{u>j}^{n_{i,k}}I\left(u\right)\quad\forall j=1;...;n_{i,K}-1\tag{6.4}$$

Combining (6.4) and (6.3) yields:

$$I_{min}\left(j\right)\leq I\left(j\right)\leq I_{max}\left(j\right)\tag{6.5}$$

with

$$1-\frac{P_{mkt}\left(K_{j+1}\right)-P_{mkt}\left(K_{j}\right)}{K_{j+1}-K_{j}}-\sum_{u>j}^{n_{i,k}}I\left(u\right)<I\left(j\right)<\frac{C\left(K_{j-1},T\right)-C\left(K_{j},T\right)}{\left(K_{j}-K_{j-1}\right)}-\sum_{u>j}^{n_{i,k}}I\left(u\right)\quad\forall j=1;...;n_{i,K}-1\tag{6.6}$$

Note that if the market quotes are arbitrage free, we have $\frac{P_{mkt}(K_{j+1})-P_{mkt}(K_{j})}{K_{j+1}-K_{j}}>0$ and $\frac{C(K_{j-1},T)-C(K_{j},T)}{(K_{j}-K_{j-1})}>0$ such that

$$0<I_{min}\left(j\right)\leq I\left(j\right)\leq I_{max}\left(j\right)<1\quad\forall j=1;...;n_{i,K}-1$$

That means $I\left(j\right)$ is a probability value for any j.

For $j=n_{i,K}$, no market quote for any strike K superior to $K_{n_{i,k}}$ exists in the market in order that the minimum may only given by the constraint that $I\left(n_{i,k}\right)$ is a probability which implies $0\leq I\left(n_{i,k}\right)\leq 1.0$, then (6.6) becomes:

$$0<I\left(n_{i,k}\right)<\frac{C\left(K_{n_{i,k}-1},T\right)-C\left(K_{n_{i,k}},T\right)}{\left(K_{n_{i,k}}-K_{n_{i,k}-1}\right)}$$

The rhs being a proxy for the delta, if the vanilla market quotes are arbitrage free, this rhs is inferior to one. Last, the case for $j=0$ may be almost ignored as $I\left(0\right)=1-\sum_{u>j}^{n_{i,k}}I\left(u\right)$. The only constraint is then to have:

$$0\leq 1-\sum_{u>j}^{n_{i,k}}I\left(u\right)\leq 1$$

Combining the equations (2.4) and (2.1) defining the quantity $I\left(j\right)$ and the cdf, we have:

$$\begin{cases}I\left(n_{i,K}\right)=P\left(X_{T}<K_{n_{i,k}}\right)=1-\left(\delta_{n_{i,K}}+\nu_{n_{i,K}}s\left(K_{n_{i,K}}\right)+1\right)\\ I\left(j\right)=P\left(X_{T}<K_{j+1}\right)-P\left(X_{T}<K_{j}\right)=\delta_{j+1}+\nu_{j+1}s\left(K_{j+1}\right)+1-\left(\delta_{j}+\nu_{j}s\left(K_{j}\right)+1\right)\quad\forall j=1;...;n_{i,K}-1\end{cases}$$

with

$$\delta_{j}=DeltaK^{BS}\left(K_{j};\sigma\left(K_{j}\right)\right)=\frac{\partial C_{BS}\left(K_{j};\sigma\left(K_{j}\right)\right)}{\partial K_{j}}$$

$$\nu_{j}=Vega^{BS}\left(K_{j};\sigma\left(K_{j}\right)\right)=\frac{\partial C_{BS}\left(K_{j};\sigma\left(K_{j}\right)\right)}{\partial\sigma\left(K_{j}\right)}$$

hence

$$\begin{cases} I\left(n_{i,K}\right) = -\delta_{n_{i,K}} - \nu_{n_{i,K}} s\left(K_{n_{i,K}}\right) \\ I\left(j\right) = \delta_{j+1} - \delta_j + \nu_{j+1} s\left(K_{j+1}\right) - \nu_j s\left(K_j\right) \quad \forall j = 1; ...; n_{i,K} - 1 \end{cases}$$

$$\begin{aligned} \sum_{u>j}^{n_{i,k}} I\left(u\right) &= \sum_{u>j}^{n_{i,k}} \delta_{u+1} - \delta_u + \nu_{u+1} s\left(K_{u+1}\right) - \nu_u s\left(K_u\right) \\ &= -\delta_{n_{i,K}} - \nu_{n_{i,K}} s\left(K_{n_{i,K}}\right) + \sum_{u>j}^{n_{i,k}-1}\left(\delta_{u+1} - \delta_u\right) + \sum_{u>j}^{n_{i,k}-1}\left(\nu_{u+1} s\left(K_{u+1}\right) - \nu_u s\left(K_u\right)\right) \end{aligned}$$

$$\begin{aligned} \sum_{u>j}^{n_{i,k}} I\left(u\right) &= -\delta_{j+1} - \nu_{j+1} s\left(K_{j+1}\right) \\ &= \sum_{u>j}^{n_{i,k}} -\delta_{j+1} + \sum_{u>j}^{n_{i,k}}\left(\nu_{u+1} s\left(K_{u+1}\right) - \nu_u s\left(K_u\right)\right) \end{aligned}$$

Replacing these values in (6.5) delivers

$$\begin{gathered} I_{min}\left(j\right) < \delta_{j+1} - \delta_j + \nu_{j+1} s\left(K_{j+1}\right) - \nu_j s\left(K_j\right) < I_{max}\left(j\right) \\ \Leftrightarrow I_{min}\left(j\right) - \left(\delta_{j+1} - \delta_j\right) < \nu_{j+1} s\left(K_{j+1}\right) - \nu_j s\left(K_j\right) < I_{max}\left(j\right) - \delta_{j+1} - \delta_j \\ \Leftrightarrow I_{min}\left(j\right) - \left(\delta_{j+1} - \delta_j\right) - \nu_{j+1} s\left(K_{j+1}\right) < -\nu_j s\left(K_j\right) < I_{max}\left(j\right) - \left(\delta_{j+1} - \delta_j\right) - \nu_{j+1} s\left(K_{j+1}\right) \\ \Leftrightarrow \frac{\left(\delta_{j+1} - \delta_j\right) + \nu_{j+1} s\left(K_{j+1}\right) - I_{max}\left(j\right)}{\nu_j} < s\left(K_j\right) < \frac{\left(\delta_{j+1} - \delta_j\right) + \nu_{j+1} s\left(K_{j+1}\right) - I_{min}\left(j\right)}{\nu_j} \end{gathered}$$

As

$$\begin{cases} I_{max}\left(j\right) & = \frac{C(K_{j-1},T) - C(K_j,T)}{(K_j - K_{j-1})} - \sum_{u>j}^{n_{i,k}} I\left(k\right) \\ I_{min}\left(j\right) & = 1 - \frac{P_{mkt}(K_{j+1}) - P_{mkt}(K_j)}{K_{j+1} - K_j} - \sum_{u>j}^{n_{i,k}} I\left(u\right) \end{cases}$$

with

$$\begin{cases} I_{max}\left(n_{iK}\right) = \frac{C(K_{j-1},T) - C(K_j,T)}{(K_j - K_{j-1})} & I_{max}\left(1\right) = 1.0 \\ I_{min}\left(n_{iK}\right) = 0 & I_{min}\left(1\right) = 1 - \frac{P_{mkt}(K_{j+1}) - P_{mkt}(K_j)}{K_{j+1} - K_j} - \sum_{u>j}^{n_{i,k}} I\left(u\right) \end{cases}$$

and because $\sum_{u>j}^{n_{i,k}} I\left(k\right) = P\left(X_{T_i} > K_{j+1}\right) = \left(-\delta_{j+1} - \nu_{j+1} s\left(K_{j+1}\right)\right)$, we have:

$$\begin{cases} I_{max}\left(j\right) & = \frac{C(K_{j-1},T) - C(K_j,T)}{(K_j - K_{j-1})} + \delta_{j+1} + \nu_{j+1} s\left(K_{j+1}\right) \\ I_{min}\left(j\right) & = 1 - \frac{P_{mkt}(K_{j+1}) - P_{mkt}(K_j)}{K_{j+1} - K_j} + \delta_{j+1} + \nu_{j+1} s\left(K_{j+1}\right) \end{cases}$$

in order that

$$\begin{aligned} & \frac{\left(\delta_{j+1} - \delta_j\right) + \nu_{j+1} s\left(K_{j+1}\right) - \left(\frac{C(K_{j-1},T) - C(K_j,T)}{(K_j - K_{j-1})} + \delta_{j+1} + \nu_{j+1} s\left(K_{j+1}\right)\right)}{\nu_j} \\ & < s\left(K_j\right) < \\ & \frac{\left(\delta_{j+1} - \delta_j\right) + \nu_{j+1} s\left(K_{j+1}\right) - \left(1 - \frac{P_{mkt}(K_{j+1}) - P_{mkt}(K_j)}{K_{j+1} - K_j} + \delta_{j+1} + \nu_{j+1} s\left(K_{j+1}\right)\right)}{\nu_j} \end{aligned}$$

$$\begin{gathered} \frac{-\delta_j - \left(\frac{C(K_{j-1},T) - C(K_j,T)}{(K_j - K_{j-1})}\right)}{\nu_j} < s\left(K_j\right) < \frac{-\delta_j - \left(1 - \frac{P_{mkt}(K_{j+1}) - P_{mkt}(K_j)}{K_{j+1} - K_j}\right)}{\nu_j} \\ \frac{-\delta_j - \frac{C(K_{j-1},T) - C(K_j,T)}{(K_j - K_{j-1})}}{\nu_j} < s\left(K_j\right) < \frac{-\delta_j - 1 + \frac{P_{mkt}(K_{j+1}) - P_{mkt}(K_j)}{K_{j+1} - K_j}}{\nu_j} \quad \forall j = 2; ...; n_{iK} - 1 \end{gathered}$$

Using the Box trade identity which implies:

$$\frac{P_{mkt}\left(K_{j+1}\right)-P_{mkt}\left(K_{j}\right)}{K_{j+1}-K_{j}}=\frac{C_{mkt}\left(K_{j+1}\right)-C_{mkt}\left(K_{j}\right)}{K_{j+1}-K_{j}}+1$$

we have

$$\frac{-\delta_{j}+\frac{C(K_{j},T)-C(K_{j-1},T)}{(K_{j}-K_{j-1})}}{\nu_{j}}<s\left(K_{j}\right)<\frac{-\delta_{j}+\frac{C_{mkt}(K_{j+1})-C_{mkt}(K_{j})}{K_{j+1}-K_{j}}}{\nu_{j}}\quad\forall j=2;...;n_{iK}-1$$

As no market quote exists for $j=0$, $j=n_{i,K}$, as these the lower bound for $j=1$ and the upper bound for $j=n_{i,K}$ may be obtained by taking the corresponding limits of the call price when its strike goes either to the infinite or to zero i.e:

$$\begin{cases}\lim_{K\to 0}C\left(K,T\right) & =X_{0}\\ \lim_{K\to+\infty}C\left(K,T\right) & =0\end{cases}$$

This implies

$$\frac{-\delta_{1}-1}{\nu_{1}}<s\left(K_{1}\right)<\frac{-\delta_{1}+\frac{C_{mkt}(K_{2})-C_{mkt}(K_{1})}{K_{2}-K_{1}}}{\nu_{1}}\quad\text{when } j=1$$

$$\frac{-\delta_{n_{iK}}-\frac{C\left(K_{n_{iK}-1},T\right)-C\left(K_{n_{iK}},T\right)}{\left(K_{n_{iK}}-K_{n_{iK}-1}\right)}}{\nu_{n_{iK}}}<s\left(K_{n_{iK}}\right)<\frac{-\delta_{n_{iK}}}{\nu_{j}}\quad\text{when }\forall j=n_{iK}$$

so that:

$$s_{min}\left(K_{j}\right)<s\left(K_{j}\right)<s_{max}\left(K_{j}\right)\quad\forall j=1;...;n_{iK}$$

$$\begin{cases}s_{min}\left(K_{j}\right) & =\frac{\frac{C(K_{j},T)-C(K_{j-1},T)}{(K_{j}-K_{j-1})}-\delta_{j}}{\nu_{j}}\quad\forall j=2;...;n_{iK}\\ s_{min}\left(K_{1}\right) & =\frac{\frac{C(K_{1},T)-X_{0}}{K_{1}}-\delta_{1}}{\nu_{1}}\quad\text{when } j=1\end{cases}$$

$$\begin{cases}s_{max}\left(K_{j}\right) & =\frac{\frac{C_{mkt}(K_{j+1})-C_{mkt}(K_{j})}{K_{j+1}-K_{j}}-\delta_{j}}{\nu_{j}}\quad\forall j=1;...;n_{iK}-1\\ s_{max}\left(K_{n_{i,k}}\right) & =\frac{-\delta_{j}}{\nu_{j}}\quad\text{when } j=n_{i,K}\end{cases}$$

Last, we need to show $s_{min}\left(K_{j}\right)\leq s_{max}\left(K_{j}\right)$ if and only if the market quotes $C\left(K_{j-1},T\right)$, $C\left(K_{j},T\right)$ and $C\left(K_{j+1},T\right)$ are arbitrage free in terms of butterfly. This is very easy to show as:

$$\begin{aligned}& s_{min}\left(K_{j}\right)\leq s_{max}\left(K_{j}\right)\\ \Leftrightarrow\ & \frac{\frac{C(K_{j},T)-C(K_{j-1},T)}{(K_{j}-K_{j-1})}-\delta_{j}}{\nu_{j}}\leq\frac{\frac{C_{mkt}(K_{j+1})-C_{mkt}(K_{j})}{K_{j+1}-K_{j}}-\delta_{j}}{\nu_{j}}\\ \Leftrightarrow\ & \frac{C\left(K_{j},T\right)-C\left(K_{j-1},T\right)}{\left(K_{j}-K_{j-1}\right)}\leq\frac{C_{mkt}\left(K_{j+1}\right)-C_{mkt}\left(K_{j}\right)}{\left(K_{j+1}-K_{j}\right)}\end{aligned}\tag{6.7}$$

This last condition being nothing less than the condition of positivity of a butterfly strategy built upon three consecutive strike K_{j-1}, K_{j} and K_{j+1}while the distance between the strike are not necessarily equal. Indeed, the corresponding butterfly is built by:

- buying one call of strike K_{j-1} ;
- selling $1+\frac{(K_{j}-K_{j-1})}{(K_{i+1}-K_{i})}$ calls of strike K_{j};
- and buying $\frac{(K_{j}-K_{j-1})}{(K_{j+1}-K_{j})}$ calls of strike K_{j+1}.

The value of this market butterfly being always positive or null, we must have:

$$C\left(K_{j-1}\right)-\left(1+\frac{\left(K_{j}-K_{j-1}\right)}{\left(K_{j+1}-K_{j}\right)}\right)C\left(K_{j}\right)+\frac{\left(K_{j}-K_{j-1}\right)}{\left(K_{j+1}-K_{j}\right)}C\left(K_{j+1}\right)\geq 0$$

$$\Leftrightarrow C\left(K_{j-1}\right)-C\left(K_{j}\right)+\left(K_{j}-K_{j-1}\right)\frac{C\left(K_{j+1}\right)-C\left(K_{j}\right)}{\left(K_{j+1}-K_{j}\right)}\geq 0$$

$$\Leftrightarrow\frac{C\left(K_{j+1}\right)-C\left(K_{j}\right)}{\left(K_{j+1}-K_{j}\right)}\geq\frac{C\left(K_{j}\right)-C\left(K_{j-1}\right)}{\left(K_{j}-K_{j-1}\right)}$$

This last condition being (6.7), this ends the proof.

Q.E.D. □

Proof of the Theorem (8)

For any given quoted maturity T_i, given a set $\mathbb{K}_i$ of quoted strikes, the following set of inequalities for the volatility skew $s(K) := \frac{\mathrm{d}\sigma(K,T)}{\mathrm{d}K}$ for all $K \in \mathbb{K}_i$ comes from the crossing of theorems (6) and (2) :

$$\begin{cases} \max\left(s_{min}(K_j);\mathcal{S}^{Fukawasa}_{down};\mathcal{S}^{prob}_{down}\right) < \frac{\mathrm{d}\sigma(K_j)}{\mathrm{d}K} < \min\left(s_{max}(K_j);\mathcal{S}^{prob}_{up}\right) & \text{if } K_j < X_0\exp\left(-\frac{\sigma^2\tau}{2}\right) \\ \max\left(s_{min}(K_j);\mathcal{S}^{Fukawasa}_{down};\mathcal{S}^{prob}_{down}\right) < \frac{\mathrm{d}\sigma(K_j)}{\mathrm{d}K} < \min\left(\mathcal{S}^{Fukawasa}_{up};s_{max}(K_j);\mathcal{S}^{prob}_{up}\right) & \text{if } X_0\exp\left(-\frac{\sigma^2\tau}{2}\right) < K_j < X_0\exp\left(\frac{\sigma^2\tau}{2}\right) \\ \max\left(s_{min}(K_j);\mathcal{S}^{prob}_{down}\right) < \frac{\mathrm{d}\sigma(K_j)}{\mathrm{d}K} < \min\left(\mathcal{S}^{Fukawasa}_{up};s_{max}(K_j);\mathcal{S}^{prob}_{up}\right) & \text{if } K_j > X_0\exp\left(\frac{\sigma^2\tau}{2}\right) \end{cases}$$

we already know from (2.15) that $\mathcal{S}^{prob}_{down}(j) < s_{min}(j) < s_{max}(j) < \mathcal{S}^{prob}_{up}(j)$. This means we only need to prove:

$$\begin{cases} \mathcal{S}^{Fukawasa}_{down}(j) < \mathcal{S}^{prob}_{down}(j) \\ \mathcal{S}^{prob}_{up}(j) < \mathcal{S}^{Fukawasa}_{up}(j) \end{cases}$$

The existence of the Fukawasa bounds depending on the level of the strike, this means we need to prove

$$\begin{cases} \frac{-1}{Kd_1\sqrt{\tau}} < \frac{-1-\delta}{\nu} & \forall K < K_{ZDS} \\ \mathcal{S}^{prob}_{up}(j) < \mathcal{S}^{Fukawasa}_{up}(j) & \forall K > K_{ZDS-P} \end{cases}$$

Last, we may prove now in a last theorem that the Sharkjaw bounds are always the narrowest ones. Let's prove that for $k < \frac{\sigma^2\tau}{2}$ (i.e $K < K_{ZDS}$), we have:

$$\frac{-1}{Kd_1\sqrt{\tau}} < \frac{-1-\delta_j}{\nu_j}$$

$$\Leftrightarrow \frac{-1}{Kd_1\sqrt{\tau}} < \frac{N(d_2)-1}{X_0\sqrt{\tau}n(d_1)}$$

$$\Leftrightarrow \frac{-1}{d_1} < \frac{N(d_2)-1}{n(d_2)}$$

As $d_1 > 0$ for $k < \frac{\sigma^2\tau}{2}$, we have:

$$-1 < \left(\frac{N(d_2)-1}{n(d_2)}\right)d_1$$

Now as $d_2 < d_1$, we have :

$$\left(\frac{N(d_2)-1}{n(d_2)}\right)d_2 < \left(\frac{N(d_2)-1}{n(d_2)}\right)d_1$$

and then we need to show $-1 < \left(\frac{N(d_2)-1}{n(d_2)}\right)d_2$ for $k < \frac{\sigma^2\tau}{2}$. This may be done for two cases: when $k < -\frac{\sigma^2\tau}{2}$ for which $d_2 > 0$ and when $-\frac{\sigma^2\tau}{2} < k < \frac{\sigma^2\tau}{2}$ (the case $k = \frac{\sigma^2\tau}{2}$ is trivial). When $k < -\frac{\sigma^2\tau}{2}$, $d_2 > 0$, and the result is immediate as for any $x > 0$, we know the normal cdf and densities are such that:

$$1 - N(x) \leq \frac{n(x)}{x} \quad \forall x > 0 \tag{6.8}$$

This means we have:

$$1 - N(d_2) \leq \frac{n(d_2)}{d_2}$$

$$\Leftrightarrow -1 < \frac{(N(d_2)-1)}{n(d_2)}d_2$$

When $-\frac{\sigma^2\tau}{2} < k < \frac{\sigma^2\tau}{2}$, the same property of the normal law may be used by passing from d_2 to $-d_2$ to apply the same result. This is done using the symetry properties of the normal law

$$-1 < \left(\frac{N(d_2)-1}{n(d_2)}\right)d_2$$

$$\frac{-1}{d_2} > \left(\frac{N(d_2)-1}{n(d_2)}\right)$$

As $\frac{N(d_2)-1}{n(d_2)} < \frac{N(d_2)}{n(d_2)}$, our conclusion requires to show

$$\frac{-1}{d_2} > \left(\frac{N(d_2)}{n(d_2)}\right)$$

using the classic symetry properties of the normal law, we have:

$$\frac{1}{-d_2} > \left(\frac{1-N(-d_2)}{n(-d_2)}\right)$$

Then this part of the proof ends by invoking the inequality 6.8. Last let's prove that for $k > -\frac{\sigma^2\tau}{2}$ (i.e. $K > K_{ZDS-P}$), we have:

$$\frac{-\delta_j}{\nu_j} < \frac{-1}{K\sqrt{\tau}d_2}$$

$$\Leftrightarrow \frac{N(d_2)}{X_0\sqrt{T}n(d_1)} < \frac{-1}{K\sqrt{\tau}d_2}$$

$$\Leftrightarrow \frac{N(d_2)}{K\sqrt{T}n(d_2)} < \frac{-1}{K\sqrt{\tau}d_2}$$

$$\Leftrightarrow \frac{N(d_2)}{n(d_2)} < \frac{-1}{d_2}$$

As $d_2 < 0$ for all $k > -\frac{\sigma^2\tau}{2}$, we have:

$$-d_2\frac{N(d_2)}{n(d_2)} < 1$$

$$\Leftrightarrow -d_2\frac{(1-N(-d_2))}{n(-d_2)} < 1$$

Then for any $x > 0$, we know the normal cdf and densities are such that:

$$1-N(x) \leq \frac{n(x)}{x}$$

which ends the proof.

Q.E.D.

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