Thomas Jefferson Physics Olympiad

A high school physics contest

This exam covers concepts related to chaos and deviations from equilibria and explores current insights into this topic and applications.

TJ Physics Team <tjhsstphysicsteam@gmail.com>

Deviations from Equilibria

If you have any questions or concerns, please email TJ Physics Team.

Contents

Rules

Teams will have a total of four days to complete the TJPhO. All teams are required to submit their response by uploading a PDF to a Google Form. Each submitted page should also have the problem number. All other formatting decisions are delegated to the teams themselves, with no one style favored over another. Teams may either submit a handwritten or typed response. If teams choose to submit a handwritten response, all handwriting must be legible, and any writing that cannot be read by the graders will not receive credit. If teams choose to submit a typed response, we suggest that teams use LaTeX for typesetting. The easiest way we've found to typeset LaTeX is with Overleaf.

The winning teams will be decided by the number of points. If two teams have the same number of points, the higher-ranking one will be the one who submitted first.

Even if you are not able to fully solve a problem, please write up what you can. We want to see how you approached the problem, and partial credit may be awarded.

Collaboration

Students participating in the competition may only correspond with other members of their team. No other correspondence is allowed, including mentors, teachers, professors, and other students. Teams may use any computational resources they might find helpful, such as Wolfram Alpha/Mathematica, Matlab, Excel, or programming languages (C++, Java, Python, etc). Teams are not allowed to use online/print resources with the exception of language documentation for any problems that may require computation. Teams are under no circumstances allowed to post content on online forums asking questions related to the exam. We welcome teams to email us if there are any questions or concerns.

Submission

Teams must submit their solutions through the provided Google Form by the deadline 11:59 PM EST on May 30th. Late submissions will not be considered. Solutions should be written in English and submitted as PDF documents with the .pdf extension. Only one person per team, the person who registered through the Google Form, should send the final submission.

Awards

The top five teams will receive 3 one-year subscriptions to WolframAlpha Notebook Edition

Sponsors

We are incredibly grateful to our sponsors, Dr. Adam Smith and Dr Abdel Hady Ebrahim, and Thomas Jefferson High School for Science and Technology for their support. We would also like thank our sponsor WolframAlpha for making this Olympiad possible.

Introduction

One fascinating aspect of our world is that many natural phenomena follow cyclical rules and regulations. For instance, a physics student will likely be familiar with simple harmonic motion: the movement of a simple pendulum, a simple harmonic oscillator, can be defined by a differential equation, and its period can be calculated. What happens, though, when these cycles are disturbed? Additionally, in what situations do these cycles appear?

Having been studied for centuries, equilibria appear in many disciplines, including but not limited to oscillations, hydrostatics, topography, astrophysics, and thermodynamics. Furthermore, there are many modern topics built off of equilibria, such as chaos theory and statistical physics. Current popular and intriguing questions in physics that pertain to equilibria include the notable Three Body Problem and Double Pendulum Problem.

The TJ Physics Team is proud to announce that the fifth annual TJ Physics Olympiad (TJPhO) focuses on this very topic! This year's contest spans a wide range of subjects, from the well-known pendulums to the lesser-known hydrostatic equilibria. We hope that our examination can introduce you to new physics topics in this fascinating world of deviations from equilibria, as well as provide a fun learning experience.

Happy exploring!

TJ Physics Team Officers

1 Problems

Hydrostatic Equilibrium (10 pts.)

1 Hydrostatics is the study of fluids at rest. The concept of hydrostatic equilibrium and its application in fluids and astrophysics will be explored in this problem.

a) (1 point) The Hydrostatic Equation, $\frac{dp}{dz} = -\rho g$ states that the pressure in a fluid at hydrostatic equilibrium varies constantly in respect to change in height. Given a uniform box of air in the Earth's atmosphere with density ρ, base area A, and height dz^*, derive the Hydrostatic Equation.

b) (4 points) Hydrostatic equilibrium also applies to stars. Given a star of mass M, radius r, and density function $\rho(r)$, derive a Hydrostatic Equation similar to the one in (a) regarding how the pressure in a star varies in respect to the distance to its center.

c) (3 points) The earth's density is not constant, but varies approximately according to $\rho(r) = C\left(1 - \frac{r}{R}\right)$, where C is a constant, R is the radius of the Earth, and r is the distance from the Earth's center. Using your equation from (b), find pressure as a function of the distance from the Earth's center.

d) (2 points) If the pressure of Earth's inner core is increased by an amount ΔP, find Earth's new gravitational field in terms of g.

Tunnel Through a Planet (7 pts.)

2 One popular thought experiment is, what if a tunnel was dug through the middle of the Earth? For this problem, consider a tunnel that was dug through the middle of a fictional Planet Epsilon. The density of the Planet Epsilon is constant at 3 kg/m^3. Approximate the radius of the Planet Epsilon as 12000 km.

a) (3 points) Assuming negligible friction, if a person modeled as a point with mass 50 kg fell into the tunnel from the surface of Planet Epsilon, what would be their period of oscillation?

b) (4 points) A person of mass m is placed, stationary, in this tunnel a distance of x away from the center of Planet Epsilon. The person is given a slight push toward the center. Find the person's position as a function of time.

Pendulum with Drag Force (8 pts.)

3 Let there be a pendulum set up with a massless string of length 1 meter and a mass of 1 kg hanging.

a) (2 point) The pendulum is displaced by an angle of $90°$. What is the maximum speed of the pendulum? Ignore air resistance.

b) (3 points)Now let us take into account air resistance. Let the force of air resistance be given by $F = \beta v^2$ with $\beta = 0.01$ kg/m. What is the maximum speed of the pendulum? Feel free to use computer software.

c) (3 points) Eventually, the pendulum will come to a near rest. At what time will the pendulum have lost 99% of its energy?

Hookean Springs (10 pts.)

4 Consider a spring with a linear restoring force of the form $F = -kx$, where k is a constant and x is the displacement from the equilibrium point. Suppose a block of mass m is attached to the end of the spring. The block is pulled a distance A from its equilibrium point and released.

a) (2 point) Using Newton's Laws, set up a differential equation that relates the position x to the amount of time t that has passed after the block is released.

b) (2 point) Assume that the solution for x has the form $x(t) = A\sin(\omega t)$, for some constant ω. Using this assumption, determine the value of ω and the period of oscillation of the block.

Now we consider the same problem as in part a, but we now take a slightly different approach. This new method will be useful later on.

c) (2 point) Let E be the total energy of the system. When the block is stretched at the maximum displacement A, we have $E = \frac{1}{2}kA^2$. Consider the block at a general position x. Use conservation of energy to set up a differential equation using the variables from part a.

d) (4 points) Using your equation from the previous subpart, find the period of oscillations of the block (note that to compute the integral through the differential equation, you may have to use a trigonometric substitution).

Nonlinear Spring Forces (10 pts.)

5 We now look into springs that do not necessarily obey Hooke's Law. Consider a spring with a nonlinear restoration force of the form $F = -kx^n$. Again, suppose a block of mass m is attached to the end of the spring and is pulled a distance A from its equilibrium point.

a) (2 points) When the block is released, the block will exhibit harmonic motion. Using dimensional analysis, determine how T varies with A in terms of n.

b) (2 points) When the block is pulled a distance A from its equilibrium point, the period of oscillations is T. Suppose the block is pulled an extra distance ΔA, with $\Delta A \ll A$. As a result, the period of oscillations changes by ΔT. Find ΔT in terms of ΔA, A, T, and n.

c) (3 points) Using the same procedure as in problem 4 parts c and d, use conservation of energy to set up an integral expression for the period of the subsequent oscillation. Do not evaluate the integral.

d) (3 points) Considering the special case of $n = 2$, use a calculator to find the period of the oscillation. Write your answer in the form $T = cf(m, k, A)$, where f is a function solely dependent on m, k, and A, and c is a constant expressed to three significant figures.

Pendulums (11 pts.)

6 Consider a bob of mass m attached to one end of a string of length L. The other end of the string is fixed to the ceiling. The bob is pulled from its equilibrium position by an angle θ and released.

a) (1 point) Suppose $\theta \ll 1$. Find the period of oscillations of the subsequent motion.

Now assume that θ is not negligible, so we cannot use small angle approximations.

b) (3 points) Using conservation of energy, determine an integral expression for the period of oscillations.

The earth is a rotating reference frame. As a result, a pendulum may experience deflection forces, most notably from the Coriolis force. Suppose the earth rotates at angular speed Ω.

c) (1 point) Suppose the pendulum is placed at the equator of the earth. Find the period of oscillations of the subsequent motion. You may use small angle approximations.

Now assume that the pendulum is moved to the North Pole. The pendulum is initially at rest at the origin $(0, 0)$. Now the pendulum bob is pulled from its equilibrium position by a small angle θ, with $\theta \ll 1$.

d) (2 point) At a general coordinate position (x, y), determine the Coriolis forces in the x and y directions. Be sure to clearly indicate the direction by including a negative sign when appropriate.

e) (2 point) The tension in the string of the pendulum also provides a restoring force. Find approximate expressions for the restoring tension forces in the x and y directions.

f) (2 points) Combining your results from the previous two subparts, determine a system of two differential equations representing the motion of the pendulum.

Classifying Equilibria (10 pts.)

7 If an object in a system is in equilibrium, that equilibrium can be classified into three main categories: stable, unstable, and neutral. If an object is in stable equilibrium, any small perturbations to the system result in forces that restore the object to its equilibrium point. If an object is in unstable equilibrium, any small perturbations to the system result in forces that cause the object to move even farther away from its equilibrium point. If an object is in neutral equilibrium, any small perturbations to the system result in the object staying stationary in its newly displaced position.

An object is in equilibrium if the net force acting on that object is 0 (assuming no torques). For each of the three given forces F as functions of positions x given below, determine all equilibrium points, and classify them as either stable, unstable, or neutral.

a) (1 point) $F(x) = x(x-1)(x-2)(x-3)(x-4)$

b) (1 point) $F(x) = \sin(x^2)$

c) (2 points) $F(x) = \sin(x^3)$

Equilibrium points, as well as their classifications, can also be found through potential energy functions of the system. For each of the three given potential energy functions (potential energy U as functions of position x) below, determine all equilibrium points, and classify them as either stable, unstable, or neutral.

d) (1 point) $U(x) = x^3 - x^2 - x$

e) (2 points) $U(x) = e^x \sin(x)$

f) (3 points) $U(x) = C((k/x)^{12} - (k/x)^6)$, where C and k are constants. This potential energy function is known as the Lennard-Jones model, which gives the potential energy between two atoms in a diatomic molecule.

Double Pendulum Pandemonium! (30 pts.)

8 You've probably seen a single pendulum, but when we add a second, things start to get wacky. Consider a double pendulum hanging from a ceiling.

a) (1 points) Draw an FBD and parameterize each mass with respect to the angle from each pivot.

b) (4 points) Give any closed-form differential equation for the angular position that could, hypothetically, be solved.

c) (2 point) Use Euler's integration to solve for $t = 10$ seconds given $m_1 = 1.5$ kg, $m_2 = 2$ kg, $\theta_1 = 30°$, and $\theta_2 = 45°$. What is the bound for the expected error?

d) (2 points) Google Runge-Kutta 4/5. Runge Kutta 4/5 is a 4th order numerical integration. For this problem alone you may reference Python or other language documentation and the Wikipedia reference on the topic. Implement the method and solve. What is the bound for the expected error?

e) (5 points) Create a simulation of a double pendulum in the framework of your choice. Submit an animation or website of it working.

f) (1 points) Count the number of times the pendulum flips. Plot this number using colors on a grid of varying θ from 0 to $\frac{\pi}{2}$. Do you notice any interesting patterns?

Now consider an inverted double pendulum.

g) (15 points) Repeat all the steps above for this type of pendulum.

2 Further Resources

For those interested in further resources, here is an enumeration of articles and web pages regarding current equilibria applications and news, as well as resources for learning about equilibria. Click on the hyperlinks for more information.

- Equilibrium Statistics Models
- Current Flow While in Equilibrium
- MIT OpenCourseWare Double Pendulum Lecture
- Society for Chaos Theory Resources
- Dwarf Planets and Hydrostatic Equilibrium
- Thermochemical Equilibrium
- The Three Body Problem
- Three Body System: Experimental Electronic Realization
- Chaos Theory Overview