Modern Control Theory Notes

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Abstract

This is my personal note for *Control Theory and Practice, Advanced Course* (EL2520) in 2021 Spring. It is just a review outline for the exam (2021 June) so there might be a lot of mistakes.

1 Basics

1.1 Norm

In this course, vector norm (Eq. 1) and signal norm (Eq. 2) are both Euclidian norms, while system gain is infinity norm.

$$|z|^2 = z^T z \tag{1}$$

$$||z(t)||_\infty = \sqrt{\int_{-\infty}^{\infty} |z(t)|^2 \mathrm{d}t} \tag{2}$$

1.2 Gain

amplification of system S

$$\frac{||y||_2}{||u||_2} = \frac{||Su||_2}{||u||_2}$$

energy gain of system S

$$||S|| = \sup_{0<||u||_2<\infty} \frac{||Su||_2}{||u||_2}$$

(supremum of amplification). For scalar linear systems,

$$||S|| = ||G||_\infty$$

(L-∞ norm of transfer function). For $y = Ax$, it is

$$||A|| = \sqrt{\sigma_{max}(A)}$$

(sqrt of max singular value).

1.3 Stability

input-output stability
If $||S|| < \infty$, system S is input-output stable.
Small Gain Theorem
$||S_1||, ||S_2||$ both input-output stable; $||S_1|| \cdot ||S_2|| < 1$. Then the closed-loop system is i-o stable. (conservative if phase info ignored!)

For LTI systems, $||S_1 S_2|| < 1$ is sufficient to substitute the second condition.

2 SISO Systems

2.1 Transfer Functions

In a closed-loop system, we have six transfer functions: $\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{G_c}, SG, SF_y, SF_r$ for a controller with two DoFs (feedback & feedforward).
internal stability
equivalent to: For all inputs (including r, w_u, w, n) and outputs, the closed-loop system is input-output stable.

For SISO systems, internal stable iff the Gang of Four (S, SG, SF_y, F_r) stable.

sensitivity functions
In order to extend to MIMO cases, we shape sensitivity functions instead of margins.
S : disturbance attenuation; sensitivity to model uncertainty.

$|S(\mathrm{i}w)| < 1$ means the disturbance is attenuated. Otherwise it is amplified.
$\tilde{G} = G(1 + \Delta G) \Longrightarrow \tilde{G_c} = G_c(1 + S\Delta G)$

T : noise attenuation; robustness to model uncertainty

2.2 Robustness

robust stability
With system uncertainty ΔG, will the closed-loop system be stable?
Consider the system in "$\Delta G - T$" form. According to Small Gain Theorem, 1 ΔG stable, 2 T internally stable, 3 $||T\Delta G||_\infty < 1 \Longrightarrow$ system I-O stable.

One thing before checking robust stability: make sure the controller makes the nominal system G stable!

model sets
We often define a model set to analyze system robustness.
1 weight $\times$ relative uncertainty

$$G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)), ||\Delta_I||_\infty \leq 1$$

How to determine weights
Again, nominal stability! From SGT, then $||W_I T||_\infty \leq 1$.
Note that SGT requires $\Delta_I(s)$ to be stable. Thus, in this case G_p and G must have **the same number of RHP poles**.
Another approach if nominal model known: $|W_I(i\omega)| \geq |\frac{G_p(\mathrm{i}\omega)-G(\mathrm{i}\omega)}{G(\mathrm{i}\omega)}|$

② weight $\times$ inverse relative uncertainty

$$G_p(s) = G(s)(1 + W_{iI}(s)\Delta_{iI}(s))^{-1}, ||\Delta_{iI}||_\infty \leq 1$$

This approach is intended for uncertain number of RHP poles, since a controller is possible to change the stability. Δ_{iI} can still be stable. From SGT, $||W_{iI}S||_\infty \leq 1$.

robust performance
Desired performance of nominal model:

$$|W_p S| \leq 1, \forall\omega$$

With model uncertainty, robust performance:

$$|W_p S| + |W_I T| \leq 1, \forall\omega$$

2.3 Design Limitations

Design weights for sensitivity functions: W_S, W_T

$$W_S = \frac{1}{M_S} + \frac{\omega_{BS}}{s}$$
$$W_T = \frac{1}{M_T} + \frac{s}{\omega_{BT}}$$

Two parameters we need to consider: peak value M and bandwidth ω. Not all designs are feasible because of some trade-offs.
① $S + T = 1 \implies |W_s(\mathrm{i}\omega)| > 2$ and $|W_T(\mathrm{i}\omega)| > 2$ cannot hold at the same frequency
② if pole excess ≤ 2, $\exists\omega$, s.t. $|S(\mathrm{i}\omega)| > 1$
③ waterbed effect: cannot push down W^{-1} at all frequencies
④ interpolation constraints: consider RHP zeros (z), RHP poles (p), time delay (θ) of the system. First, we have $S(z) = 1, T(z) = 0, S(p) = 0, T(p) = 1$. Thus, if we tolerate arbitrarily large peaks ($M_S = \infty$)

$$\omega_{BS} \leq z, \omega_{BT} \geq p, \omega_{BS} \leq \frac{2}{\theta}$$

or if we tolerate $M_S = 2$ (more reasonable)

$$\omega_{BS} \leq \frac{z}{2}, \omega_{BT} \geq 2p, \omega_{BS} \leq \frac{1}{\theta}$$

If there is a pair of RHP zero and pole that are close, it results in a large peak

$$||S||_\infty \geq |\frac{z+p}{z-p}|$$

3 MIMO Systems

3.1 Transfer Matrices

Solve transfer matrices:
① output → against signal flow → input
② blocks: left to right
③ exit a loop: add $(I+L)^{-1}$
L: from exit agianst the signal flow of the loop
④ add parallell paths

A formula for comparing MIMO transfer matrices: push thru

$$A(I+BA)^{-1} = (I+AB)^{-1}A$$

Specially, when $B = I$, $A(I+A)^{-1} = (I+A)^{-1}A$

3.2 Poles and Zeros

pole
eigenvalues of A (minimal state-space realization); roots of pole polynomial
pole polynomial
$\lambda(s) = |sI - A|$; least common denominator of all minors of G(s)
zero
$G(z)$ looses rank.
zero polynomial
greatest common divisor of maximal normed minors of G(s)

minor: determinant of small square matrics by deleting some rows and columns of G

"normed" means the denominator is the pole polynomial

If the control signal u is known, it is easy to solve state-space model to get poles and zeros of closed-loop system.

direction

$$G(p)u_p = \infty \cdot y_p, G(z)u_z = 0 \cdot y_z$$

Equivalently

$$y_p^H G(p) = \infty \cdot u_p^H, y_z^H G(z) = 0 \cdot u_z^H$$

u and v are input and output directions, respectively, and H means conjugate transpose. u_z is in the null space of $G(z)$ while y_z is in the left null space of $G(z)$.
Zeros and poles at the same position but with different directions cannot be cancelled in MIMO cases!

3.3 Gain

The gain of a MIMO system depends on both direction and frequency.

$$\underline{\sigma}(G(i\omega)) \leq \frac{|Y(\mathrm{i}\omega)|}{|U(\mathrm{i}\omega)|} \leq \bar{\sigma}(G(\mathrm{i}\omega)) = |G(\mathrm{i}\omega)|$$

$$||G||_\infty = \sup_\omega |G(\mathrm{i}\omega)|$$

Singular Value Decomposition

$$A = U\Sigma V^H, Av_i = \sigma_i u_i$$

v_i: eigenvector of $A^H A$, u_i: eigenvector of AA^H, $\sigma_i = \sqrt{\lambda_i(A^H A)}$
useful: $\lambda(kA) = k\lambda(A)$

3.4 Internal Stability

For MIMO systems, stability of six transfer matrices: $S, SG, S_u, S_u F_y, F_r$ are required. (recall: for SISO, only the Gang of Four)
Some principles should be noticed:
RHP pole-zero cancellation between F_y and G will cause instability.
RHP poles of G must be retained as zeros of S;
RHP zeros of G must be retained as zeros of T.

$$S(p)y_p = 0 \cdot y_p, \quad y_z^H T(z) = 0 \cdot y_z^H$$

y_p: output pole direction, y_z: output zero direction

3.5 Decentralized Control and Decoupling

Relative Gain Array (RGA)

$$\Lambda(s) = \begin{bmatrix} \lambda_{11} & 1-\lambda_{11} \\ 1-\lambda_{11} & \lambda_{11} \end{bmatrix}$$

$$\lambda_{11} = \frac{1}{1 - \frac{g_{12}g_{21}}{g_{11}g_{22}}}$$

Sum of any row or column of RGA equals one. (for $N \times N$ matrices this also holds)

Should we use decentralized control?
First, compute $\Lambda(0)$ and determine input-output pairs according to signs of elements. Then check $\Lambda(\mathrm{i}w_c) \in (0.5, 3)$ to ensure that the interaction is weak.
decoupler
Try to make all elements are zero on the off-diagonal, e.g., $W = G^{-1}$

3.6 Robustness

For MIMO systems, it matters that we model the uncertainty at the input or output side. But $W(s)$ is still a scalar.
output uncertainty

$$G_p(s) = (I + W_O(s)\Delta_O(s))G(s), ||\Delta_O||_\infty \leq 1$$

robust stability condition:

$$||TW_O||_\infty < 1$$

input uncertainty

$$G_p(s) = G(s)(I + W_I(s)\Delta_I(s)), ||\Delta_I||_\infty \leq 1$$

robust stability condition:

$$||T_u W_I||_\infty < 1$$

$$T_u = F_y G(I + F_y G)^{-1}$$

***inverse input uncertainty**

$$G_p(s) = G(s)(I + W_{iI}(s)\Delta_{iI}(s))^{-1}, ||\Delta_{iI}||_\infty \leq 1$$

robust stability condition:

$$||S_u W_{iI}||_\infty < 1$$

$$S_u = I - T_u = (I + F_y G)^{-1}$$

3.7 Design Limitations

We need to design W_S and W_T so that

$$\bar{\sigma}(S(\mathrm{i}\omega)) \leq |W_S^{-1}(\mathrm{i}\omega)|, \forall \omega$$

$$\bar{\sigma}(T(\mathrm{i}\omega)) \leq |W_T^{-1}(\mathrm{i}\omega)|, \forall \omega$$

some basic limitations

① $S + T = I, S_u + T_u = I \Longrightarrow \bar{\sigma}(S), \bar{\sigma}(T)$ cannot < 0.5 at the same time; one peak implies another.

② waterbed effect: trade-offs between frequencies and directions, from Bode Sensitivity Integral

③ interpolation constraints: $\bar{\sigma}(S(z)) > 1$ so $W_S(z) < 1$. Similarly, $|W_T(p)| < 1$.

disturbance attenuation

$$z = S(s)g_d(s)d$$

For $\forall \omega$, given that $|d| < 1$, we hope $|z| < 1$, which indicates that

It is a good idea to scale the system.

$$||Sg_d||_\infty < 1$$

disturbance direction:

$$y_d(\mathrm{i}\omega) = \frac{g_d(\mathrm{i}\omega)}{|g_d(\mathrm{i}\omega)|}$$

① Check if there is any constraint on control signal u. Will it be sufficient?

② Before we design a controller, we should think about the system itself if there are RHP zeros.

When $y_z \perp y_d$, no limitation from this RHP zero; when $y_z \parallel y_d$, worst case!

From Maximum Modulus Theorem,

$$|y_z^H g_d(z)| < 1$$

It must be satisfied.

③ The requirement on S is in direction y_d:

$$\bar{\sigma}(S(\mathrm{i}\omega)y_d(\mathrm{i}\omega)) < \frac{1}{|g_d(\mathrm{i}\omega)|}, \forall \omega$$

3.8 Controller Design

Aim: design a controller to attenuate disturbance, noise and reduce control signal.
Approach 1: classical loop shaping
shape open-loop system: translate bounds on $\bar{\sigma}(S)$ and $\bar{\sigma}(T)$ into bounds on singular values of $L = GF_y$

$$\underline{\sigma}(L) >> 1 \Rightarrow \bar{\sigma}(S) \approx \frac{1}{\underline{\sigma}(L)}$$

$$\bar{\sigma}(L) << 1 \Rightarrow \bar{\sigma}(T) \approx \bar{\sigma}(L)$$

As there is no phase margin defined for MIMO systems, stability problems might exist. E.g., from the restrictions on W_S^{-1} and W_T^{-1}, will the slope of bode plot $\sigma_i(L) - \omega$ around crossover frequency be too large?

If we need more accurate bounds, there are some useful formulas:

$$\bar{\sigma}(A^{-1}) = \frac{1}{\underline{\sigma}(A)}$$

$$|\sigma_i(L) - 1| \leq \sigma_i(I + L) \leq \sigma_i(L) + 1 \text{ (from Fan's Theorem)}$$

$$\bar{\sigma}(A)\underline{\sigma}(B) \leq \bar{\sigma}(AB) \leq \bar{\sigma}(A)\bar{\sigma}(B)$$

***robust loop shaping**
In this course, we use Glover-McFarlane loop shaping to stabilize a loop-shaped system. More details can be found in lecture notes but it is not exam-related.

$$G(s) = (M(s) + \Delta_M(s))^{-1}(N(s) + \Delta_N(s))$$

$$\|\Delta_M(s)\Delta_N(s)\|_\infty \leq \epsilon$$

Approach 2: formulate and solve optimization problems
See next chapter.

4 Optimal Control

In this chapter, we review both a classical optimal control method (LQG) and modern ones (H_2, H_∞, MPC).

4.1 H_∞ Control

$$z = Sw + Tn$$

$$u = -FyS(w + n) = G_{wu}(w + n)$$

$$F_y = \underset{F_y}{\operatorname{argmin}} \left\| \begin{matrix} W_S S \\ W_T T \\ W_u G_{wu} \end{matrix} \right\|_\infty$$

extended system: $z_e = y = G_0 u + w$, G_0: open loop

① Formulate W_S, W_T, W_u: transform constraints into standard form

② Form the extended system G_{ec}: $(y, z_e) \rightarrow (u, u_e)$. Design extended signals

z_{ei}, u_e. E.g., $z_{e1} = W_S z = W_S(Gu + w)$, $u_e = w$. We hope the original problem is equivalent to a signal minimization problem.

$$\sup_w \frac{\|z_e\|_2}{\|w\|_2} = \|G_{ec}\|_\infty$$

③ Formulate a sub-optimal problem: $\|G_{ec}\|_\infty = \gamma$. Select a reasonable γ. (If you fails in the next step, return here.)
④ Derive controller based on state-space model of **open loop system** G_0. First scale the system so that $D^T M = 0, D^T D = I$. Then solve the Riccati Equation

$$A^T P + PA + M^T M + P(\gamma^{-2} N N^T - BB^T)P = 0$$

If such a γ exists, then $u(t) = -L_\infty \hat{x}(t), L_\infty = B^T P$. Use Laplace transform to get $F_y(s)$.
⑤ Moreover, ensure $A - BB^T P$ is stable.

4.2 LQG Control

LQG: linear system, quadratic cost, Gaussian noise

$$J = \lim_{T\to\infty} \frac{1}{T} \int_0^T [z^T Q_1 z + u^T Q_2 u] \mathrm{d}t$$

We also hope to reduce some signals z and control signal u but this time we consider euclidean norm. The problem can be divided into two sub-problems if separation principle holds.

Some terminologies: controllable, observable, stabilizable ($\exists L$ so that $A - BL$ stable), detectable ($\exists K$ so that $A - KC$ stable)

See lecture notes.

Step 1. solve LQ regulator
optimal linear state feedback

$$\dot{x} = Ax + Bu$$

$$z = Mx, u = -Lx$$

solve Riccati equation

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0, S \geq 0$$

$$L = Q_2^{-1} B^T S$$

$$u(t) = -Lx(t)$$

Step 2. solve Kalman filter
optimal observer

$$\dot{x} = Ax + Bu + Nv_1$$

$$y = Cx + v_2$$

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - c\hat{x})$$

solve Riccati equation

$$PA^T + AP - PC^T R_2^{-1} CP + NR_1 N^T = 0, P \geq 0$$

$$K_f = PC^T R_2^{-1}$$

closed-loop matrices
$x : A - BL, \hat{x} - x : A - KC$

Step 3. combine ($y \to u$)
controller (u, x), observer(u, x, y) $\to$ controller (u, y)
Eliminate state variables, so we get a controller based on observation.

4.3 H_2 Control

H_2 norm of SISO transfer function:

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\mathrm{i}\omega)|^2 \mathrm{d}\omega$$

Instead of H_∞ norm, here we consider H_2 norm. In fact, minimizing H_2 norm of a transfer function is equivalent to a LQ problem if we know $z = Mx + Du$ and $\min \|z(t)\|_2 \Leftrightarrow \min \|G\|_2^2$. H_2 considers all singular values while H_∞ cares about the greatest one.

Solving H_2 is just like solving H_∞. But $G(s)$ must be **strictly** proper and stable! Thus, we cannot always formulate weights as we wish.

4.4 Model Predictive Control (MPC)

MPC is a finite-horizon LQR problem. Thus, first we need to discretize the state-space model.

may be useful when dealing with MIMO:

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

Then we eliminate state variables $x_k, ..., x_{k+N}$ which results in a cost function $J(u)$. Generally, the optimal value of the last input $u^*_{k+N} = 0$, and the terms that are not related to u can be ignored. Finally it can be a constrained QP problem with respect to $u_k, ..., u_{k+N-1}$.

***anti-set windup**
Another approach dealing with hard constraints is so-called anti-set windup. We use a modified observer

$$\dot{\hat{x}} = A\hat{x} + Bu_p + K(y - C\hat{x})$$

where u_p is the actual input (may be saturated), and we get feedback from $u - u_p$.

$$\dot{\hat{x}} = (A - BL - KC)\hat{x} + Ky + B(u_p - u)$$

Then we can track input

$$U(s) = -F_y(s)Y(s) + W(s)(U_p(s) - U(s))$$