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HIGH-PRECISION THREE-DIMENSIONAL PHOTOGRAMMETRIC CALIBRATION AND OBJECT SPACE RECONSTRUCTION USING A MODIFIED DLT-APPROACH

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Abstract—Two modified DLT algorithms are presented that improve the accuracy of three-dimensional object space reconstruction by almost an order of magnitude when compared with conventional methods. The improvement in the linear modified DLT (MDLT) algorithm is achieved by satisfying certain orthogonality conditions in the form of a non-linear constraint, thereby effectively eliminating a redundant DLT parameter. In the non-linear MDLT algorithm, the improvement and computational stability results from the appropriate elimination of implicit variables from one side of the approximating relations and the corresponding reformulation of the objective function to be minimized. The highest reconstruction accuracy of 0.733 mm rms mean error was obtained with the non-linear MDLT algorithm. This corresponds to a spatial resolution of about one part in 2860 or 0.035% overall accuracy. The accuracy obtainable with the linear MDLT was found to be slightly less and about 0.041% (0.833 mm rms mean error).

INTRODUCTION

The biomechanical analysis of human movement necessitates the recording, by some means, of the motion to be analyzed. A method commonly employed is the spatial reconstruction of a motion from recorded two-dimensional optical images. In this method, certain parameters of the optical train have to be determined first by means of a calibration procedure, in order to make possible the reconstruction from image recordings of unknown point coordinates. Frequently, the conventional DLT approach (Marzan and Karara, 1975) is used for this calibration procedure. Wood and Marshall (1986) have recently reported on the comparatively poor accuracy that can be achieved with this method. In this paper we shall describe a modified DLT approach (the MDLT method) which produces results that improve the precision of the reconstruction by an order of magnitude.

THE MODIFIED DLT-METHOD

The basic photogrammetric relations

$$\begin{bmatrix} -\mu c \\ \eta-\eta_0 \\ \zeta-\zeta_0 \end{bmatrix} = \lambda A \begin{bmatrix} X-X_0 \\ Y-Y_0 \\ Z-Z_0 \end{bmatrix} \tag{1}$$

provide the connection between the spatial coordinates (X, Y, Z) of a point P located in object space and its theoretical image coordinates (η, ζ) as appearing on the comparator plane. In relation (1), λ denotes the scale factor for the size reduction of real objects to their comparator image; A, with elements $a_{ij} = a_{ij}(\phi_1, \phi_2, \phi_3)$, denotes the orientation matrix of the spatial coordinate system $O'X'Y'Z'$ relative to the comparator coordinate system $\Omega'\xi'\eta'\zeta'$; (X_0, Y_0, Z_0) are the spatial coordinates of the perspective center of a hypothetical camera (optical train); μc denotes the principal distance of that hypothetical camera; and (η_0, ζ_0) marks off the principal point at which the optical axis intersects the comparator plane.

In practice, the theoretical (ideal) image coordinates cannot be observed because of the presence of systematic errors which are due to linear lens (and possibly film) deformation, non-orthogonality of image axes (video, opto-electronic devices) and other linear comparator errors on the one hand, and non-linear and asymmetrical lens distortion on the other hand. Random errors resulting from the digitization process will be considered negligible in the present context.

Let the observed comparator coordinates of a point P be denoted by (η^*, ζ^*) and let the non-linear distortions $(\Delta\eta, \Delta\zeta)$ be defined (Marzan and Karara, 1975; Miller *et al.*, 1980) by

$$\Delta\eta = \eta'(K_1\varepsilon^2 + K_2\varepsilon^4 + K_3\varepsilon^6) + M_1(3\eta'^2 + \zeta'^2) + 2M_2\eta'\zeta' \tag{2}$$

$$\Delta\zeta = \zeta'(K_1\varepsilon^2 + K_2\varepsilon^4 + K_3\varepsilon^6) + M_2(3\zeta'^2 + \eta'^2) + 2M_1\eta'\zeta' \tag{3}$$

where

$$\eta' = \eta^* - \eta_0 \tag{4}$$

$$\zeta' = \zeta^* - \zeta_0 \tag{5}$$

$$\varepsilon^2 = \eta'^2 + \zeta'^2 \tag{6}$$

K_1, K_2, K_3 denote the coefficients of symmetrical lens distortion, and M_1, M_2 those of asymmetrical lens distortion.

All of the linear and non-linear image distortions can be incorporated in relation (1) by expressing (Miller *et al.*, 1980) $(\eta - \eta_0)$ and $(\zeta - \zeta_0)$ in the following way

$$\eta - \eta_0 = \lambda_\eta(\eta^* + \Delta\eta - \eta_0) \tag{7}$$

$$\zeta - \zeta_0 = \lambda_\zeta(\zeta^* + \Delta\zeta - \zeta_0) \tag{8}$$

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where λ_η and λ_ζ are the scale factors of the respective axes.

Equations (7) and (8) may be substituted into (1) from which λ can be eliminated. The resulting equations are given as

$$\eta^* + \Delta\eta - \eta_0 = -\frac{\mu c}{D\lambda_\eta}[a_{21}(X - X_0) + a_{22}(Y - Y_0) + a_{23}(Z - Z_0)] \quad (9)$$

$$\zeta^* + \Delta\zeta - \zeta_0 = -\frac{\mu c}{D\lambda_\zeta}[a_{31}(X - X_0) + a_{32}(Y - Y_0) + a_{33}(Z - Z_0)] \quad (10)$$

where

$$D = a_{11}(X - X_0) + a_{12}(Y - Y_0) + a_{13}(Z - Z_0). \quad (11)$$

By rearranging terms and neglecting, at first, the terms $\Delta\eta$, $\Delta\zeta$ representing the non-linear distortions, equations (9) and (10) can be written as

$$\eta^* = (L_1 X + L_2 Y + L_3 Z + L_4)/G \quad (12)$$

$$\zeta^* = (L_5 X + L_6 Y + L_7 Z + L_8)/G \quad (13)$$

where

$$G = L_9 X + L_{10} Y + L_{11} Z + 1. \quad (14)$$

It is obvious from equations (12–14) that, once they have been determined, the 11 DLT-parameters $L_1, \ldots, L_{11}$ permit the computation of the unknown spatial coordinates (X, Y, Z) of a point P whose image coordinates (η_i^*, ζ_i^*), $i = 1, \ldots, \kappa$, $\kappa \geqslant 2$, have been recorded by at least two cameras. In order to determine $L_1, \ldots, L_{11}$, a number v of control points $\bar{P}_j$, $j = 1, \ldots, v$, with known spatial coordinates $(\bar{X}_j, \bar{Y}_j, \bar{Z}_j)$ may be chosen and recorded by each of the cameras (optical trains) to be calibrated. By rearranging (12–14), a matrix equation results in which $(L_1, \ldots, L_{11})$ appears as the vector of unknowns. Linear least-squares techniques may then be employed to compute this vector from the overdetermined system.

In deriving (12–14) from (9–11), it is frequently overlooked that the latter equations contain only ten unknown parameters

$$\{\eta_0, \zeta_0, -\mu c/\lambda_\eta, -\eta c/\lambda_\zeta, \phi_1, \phi_2, \phi_3, X_0, Y_0, Z_0\},$$

while the DLT parameters constitute a set of 11 constants. Thus, one of the DLT parameters must be redundant. This is indeed the case. In a somewhat lengthy derivation it can be shown that if the orthogonality condition in the transformation matrix A appearing in (1) is to be satisfied, i.e. if

$$\sum_{j=1}^{3} a_{ij} a_{kj} = \delta_{jk} \qquad i, k = 1, 2, 3 \quad (15)$$

then the non-linear relation (see Appendix)

$$L_n - h(L_1, \ldots, L_{n-1}, L_{n+1}, \ldots, L_{11}) = 0 \quad (16)$$

must be included as a non-linear constraint in the computation of the 11 DLT parameters. If this is not done, the conditions (15) will, in general, be violated and the transformation between the spatial and the comparator coordinate system will be a non-orthogonal one, resulting in sizeable reconstruction errors. Results demonstrating the improvement achieved in reconstruction accuracy by incorporating the non-linear constraint (16) and using a special, numerically stable orthogonal transformation algorithm instead of the normal equations, will be presented below.

Next, we shall consider the inclusion into equations (9) and (10) of the non-linear distortions (2) and (3). Adding $\eta^* - \eta_0 = \eta'$ to both sides of equation (2) and $\zeta^* - \zeta_0 = \zeta'$ to both sides of equation (3), we obtain relations, whose right hand sides equal to those of (2) and (3) increased by η' and ζ' respectively, and whose left hand sides $\tilde{\eta} = \eta^* - \eta_0 + \Delta\eta$ and $\tilde{\zeta} = \zeta^* - \zeta_0 + \Delta\zeta$ are identical with those of (9) and (10) respectively. Thus, relationships similar to (12) and (13) may be derived also for $\tilde{\eta}$ and $\tilde{\zeta}$. However, since η_0 and ζ_0 no longer appear on the right hand sides of these relations, the corresponding parameters will be denoted by $J_1, \ldots, J_{11}$ in order to distinguish them from $L_1, \ldots, L_{11}$. The resulting relations are

$$\tilde{\eta} = (J_1 X + J_2 Y + J_3 Z + J_4)/H \quad (17)$$

$$\tilde{\zeta} = (J_5 X + J_6 Y + J_7 Z + J_8)/H \quad (18)$$

where

$$H = J_9 X + J_{10} Y + J_{11} Z + 1. \quad (19)$$

As mentioned above, the left hand sides of these equations, i.e. $\tilde{\eta}$ and $\tilde{\zeta}$ are respectively equal to the right hand sides of (2) and (3) increased by η' and ζ' respectively. In analogy to the linear case, it can be shown that, in order to satisfy the orthogonality conditions (15), three of the parameters $J_1, \ldots, J_{11}$ are redundant and have to be eliminated. Using the same control point procedure as for the linear case, the complete set of non-linear equations (17) and (18) is then solved for the unknown parameter set $\{\eta_0, \zeta_0, J_3, J_4, J_6, \ldots, J_{11}, K_1, K_2, K_3, M_1, M_2\}$ for each of the κ cameras. The interior and exterior camera parameters, except η_0 and ζ_0, can be computed from the above set of J-parameters.

It should be noted that in the present approach η_0 and ζ_0 appear in the expressions for $\tilde{\eta}$ and $\tilde{\zeta}$ in (17) and (18) on the left hand sides only and are no longer implicitly included in the J-parameters on the right hand sides. This is in contrast to the method of Marzan and Karara (1975), where η_0 and ζ_0 appear explicitly on the left hand sides of their equations (5) but also implicitly in the parameters $L_1, \ldots, L_{11}$ on the right hand sides. This may be the reason for the failure of the method, as reported by Miller *et al.* (1980), when applied to the non-linear case where more than the 11 DLT parameters must be estimated. Test results obtained by using the present approach will be presented under Results.

METHODS

The computerized versions of the above algorithms implementing the modified DLT (MDLT) method for linear and non-linear calibration are part of a more general Fortran 77 computer program which performs automatic calibration, data smoothing, removal of outliers, and complete three-dimensional or two-dimensional object space and/or configurational coordinate reconstruction from recorded optical images. Special high precision routines were developed for the linear and non-linear calibration procedures. These routines are numerically stable and employ orthogonal transformations (Wait, 1979) in place of the normal equations, and the Gauss–Newton–Marquart method in conjunction with Householder transformations for the non-linear case.

In order to test the accuracy of the results obtained by using the present MDLT algorithms and compare them with the results obtained by employing the conventional DLT method of Marzan and Karara (1975), a three-dimensional rectangular 0.425 m (depth) by 2.04 m (length) by 2 m (height) calibration frame was constructed using 40 mm angle-iron. Precision engineering ensured an accuracy of better than ± 1 mm in each of the three spatial coordinates $(\bar{X}_j, \bar{Y}_j, \bar{Z}_j)$ of the 30 control markers $\bar{P}_j$ attached to that frame. The markers were made of 40 mm square plates with a four-quadrant black-white pattern painted on them. This ensured a precise definition of the marker centers. The locations of the markers on the rectangular frame were as follows: markers 1–16 were fixed to the front of the structure in four approximately equally spaced columns of four markers each, while the remaining 14 markers were attached to the back of the structure in a similar fashion. The last back column on the far right carried only two markers. The numbering of the markers was done from top to bottom, left to right and front to back. Hence marker 1 was located at the left upper front corner followed vertically downwards by markers 2–4. The second front column began on top with marker 5, followed vertically by markers 6–8, etc. Similarly, marker 17 was located at the left upper back corner with the rest of the markers positioned as in the front of the structure. The marker distribution was such that the horizontal and vertical distances between them were not equal and such that the whole observation volume normally occupied by a person executing a walking stride was covered by them.

The right-handed Cartesian object space coordinate system had its origin in the middle of the lower back edge of the frame. The X-axis pointed out of the frame from the back to the front, the Y-axis was directed along the lower back edge from left to right, and the Z-axis pointed vertically upwards.

The frame structure was filmed at about 25 fps by a Locam II (Model 51) 16 mm high-speed camera with the camera positioned first at an angle of about 36° relative to the X-axis and then at an angle of about $-23°$. In the first position, the height (Z_0) and the horizontal distance of the camera perspective center from the spatial Z-axis were measured approximately and found to be 1.05 m and 4.29 m respectively. For the second position, the respective values were 1.30 m and 4.45 m. The two camera positions were chosen such as to ensure that all 30 markers are visible to the camera. A positive of the 16 mm film was then produced and projected by a Vanguard motion analyzer onto a digitizing table. For each camera position, one frame was chosen for the digitization (accuracy ± 0.1 mm) of the 30 control points. Each point was digitized four times and the mean was taken in order to reduce the influence of random errors.

To permit a direct comparison with the data reported by Wood and Marshall (1986), the following control point data files were created: a 30 control point file (30CP), an 11 control point file (11CP, using markers 1, 4, 6, 11, 13, 16, 18, 21, 26, 28, 29), a 7 control point file (7CP, using markers 1, 7, 9, 16, 20, 28, 29) and a Christmas-tree-like 11 control point configuration (11CPX, using markers 4, 6, 8, 9, 11, 12, 15, 21, 24, 28, 30).

In accordance with common practice (Miller *et al.*, 1980; Wood and Marshall, 1986), the 30 control points were redigitized as those points whose spatial coordinates are to be reconstructed. The calibration parameters required for these reconstructions were obtained by utilizing the various control point files and three different calibration algorithms. The first algorithm is the one proposed by Marzan and Karara and implemented in their computer program (Marzan and Karara, 1975). The second one implements the linear procedure of the present MDLT approach and is activated if the computer program MORECO is run with the option NLDIST = 0 (no correction for non-linear distortions but inclusion of the constraint (equation 16) preserving the orthogonality conditions (equation 15)). The third algorithm implements the correction of all linear and non-linear distortions in the present MDLT method. It is activated by selecting the run option NLDIST = 1 in program MORECO. These various run options were executed on an IBM 3083 digital computer and, for purposes of comparison, also on an IBM-XT personal computer with 640 KB memory. No significant differences could be observed between the results obtained on both machines.

RESULTS

The results summarized in Table 1 are categorized into those obtained by applying to the four different control point files (a) the conventional 11-DLT method of Marzan and Karara, (b) the linear MDLT algorithm and (c) the non-linear MDLT algorithm. For all of the above combinations and spatial coordinates the root mean square errors were calculated from

$$\mathrm{rms} = \left[\sum_{i=1}^{N} (q_i - \bar{q}_i)^2 / N \right]^{1/2}, \tag{20}$$

Table 1. RMS errors in the spatial coordinates (values in mm). Absolute values of the maximum deviations are given in brackets below the rms values

Algorithm used	Coordinate	Control point distribution 30CP	11CP	7CP	11CPX
	X	5.3	5.5	6.2	9.1
		(9.3)	(8.6)	(13.0)	(23.3)
Conventional	Y	4.2	4.1	4.4	8.7
11 DLT		(9.1)	(8.9)	(11.8)	(19.5)
(Marzan and Karara)	Z	4.8	4.9	5.2	8.8
		(10.7)	(7.8)	(11.0)	(17.1)
	Mean	4.767	4.833	5.267	8.867
	X	0.9	0.8	1.0	1.1
		(1.6)	(1.4)	(1.9)	(2.4)
Linear	Y	0.7	0.9	1.8	0.8
MDLT		(1.3)	(2.3)	(3.9)	(1.7)
	Z	0.9	0.9	1.2	1.0
		(1.5)	(1.8)	(2.5)	(2.0)
	Mean	0.833	0.867	1.330	0.967
	X	0.7	3.0		10.1
		(1.4)	(6.9)	—	(36.1)
Non-linear	Y	0.7	1.1		4.8
MDLT		(1.1)	(2.5)	—	(15.3)
	Z	0.8	1.1		3.5
		(1.2)	(2.7)	—	(11.4)
	Mean	0.733	1.733	—	6.133

where q_i is the reconstructed value of the quantity in question and $\bar{q}_i$ its real value.

For completeness, the set of computed internal and external camera parameter values, including the coefficients of non-linear distortions, are presented in Table 2 for the 30 control point configuration, for both camera positions and for the linear and non-linear MDLT computations. Some of the external parameter values (ϕ_1, ϕ_2, ϕ_3, $X_0 \approx d\cos\phi_3$, $Y_0 \approx d\sin\phi_3$, Z_0) can be compared directly with those that could be obtained approximately by direct measurement. The horizontal camera distances d for positions 1 and 2 were 4.29 m and 4.45 m respectively.

DISCUSSION

From Table 1 it can be inferred that both the linear and the non-linear MDLT algorithms produce results that markedly improve the accuracy of the object space reconstruction when compared with the conventional 11DLT method. In some cases, the improvement is of the order of one magnitude. The means of the rms errors for the conventional 11DLT approach were found to be about 26% lower than the corresponding mean values reported by Wood and Marshall (1986), but 67% higher than those described in the paper of Miller *et al.* (1980). These differences can probably be attributed to the different sizes of the observation volumes used by the various authors (Wood and Marshall used a 1.3 × 3.5 × 2.3 m space, in the present investigation 0.425 × 2.04 × 2.0 m were used, and Miller *et al.* recorded within a space of about 0.25 × 0.3 × 0.3 m encompassing their spatial linkage device).

The best reconstruction was obtained by using the non-linear MDLT algorithm on the 30CP data file. Not only is the mean rms error of 0.733 mm the lowest of all observed rms error means, but the mean value of 1.23 mm of the maximum deviations is also the lowest maximum deviation mean encountered. However, the precision that can be achieved with the non-linear MDLT algorithm depends strongly on the number and distribution of the control points. For the 11CP data file the rms error mean has already increased to 1.733 mm, while for the 11CPX configuration an unacceptably large mean value of 6.133 mm with a maximum deviation of 36.1 mm was found. In this connection it was interesting to note that at the control points of the 11CPX configuration the deviations were very small (approximately 0.5 mm). The large deviations occurred at the markers 1, 13, 16, 17, 20 and 29, all of which are furthest removed from the 11CPX control points. It is therefore apparent that for this configuration the non-linear algorithm yields a polynomial approximation which almost collocates at the control points but produces very poor extrapolation results outside the control point region. The non-linear algorithm could not be tested for the 7CP distribution since there would be only 2 × 7 = 14 equations for the 15 unknown parameter values $\{\eta_0, \zeta_0, \ldots, K_1, K_2, K_3, M_1, M_2\}$. In fact, when execution of the respective routine was attempted, the process terminated and the error message 'Not enough control points' appeared.

On the other hand, the linear MDLT algorithm produces acceptable results for all coordinates and all four sets of control point distributions, although the precision of the reconstruction deteriorates with a reduction in the number of control points. The rms mean error of 0.833 mm for the 30CP distribution is only slightly larger than that for the 30CP non-linear MDLT algorithm. However, the maximum deviations

Table 2. Measured and computed internal and external camera parameter values (units of measurement are given below the respective quantities)

Camera position	Calibration algorithm	η_0 (mm)	ζ_0 (mm)	B_1 (m)	B_2 (m)	ϕ_1 (°)	ϕ_2 (°)	ϕ_3 (°)	X_0 (m)	Y_0 (m)	Z_0 (m)	K_1 (m^{-2})	K_2 (m^{-4})	K_3 (m^{-6})	M_1 (m^{-1})	M_2 (m^{-1})
Position 1 (right position)	Measured values	—	—	—	—	0	0	36	3.48	2.51	1.05	—	—	—	—	—
	Linear MDLT	1.6	−14.1	0.873	0.875	−1.249	1.358	35.810	3.413	2.278	1.053	—	—	—	—	—
	Non-linear MDLT	1.5	−13.4	0.871	0.874	−1.278	1.398	35.816	3.419	2.282	1.053	−0.206	2.543	−0.200	0	−0.0009
Position 2 (left position)	Measured values	—	—	—	—	0	0	−23	4.08	−1.77	1.30	—	—	—	—	—
	Linear MDLT	−9.1	−11.9	0.846	0.849	−1.524	−1.409	−23.480	3.930	−1.737	1.285	—	—	—	—	—
	Non-linear MDLT	−9.0	−11.4	0.847	0.850	−1.507	−1.369	−23.462	3.947	−1.743	1.285	−0.190	2.488	−0.200	0.0012	−0.0013

of 1.6 mm for the X, 1.3 mm for the Y, and 1.5 mm for the Z-coordinate are significantly larger than the corresponding maximum deviations that occur in the non-linear 30CP calibration.

A brief look at Table 2 indicates good agreement between the camera parameters as computed by the two algorithms. A precise determination of the values of ϕ_1, ϕ_2, ϕ_3, X_0, Y_0 and Z_0 by direct measurement was, of course, not possible so that the measured values agree only approximately with the computed ones. It is noteworthy that the coefficients K_1, K_2, K_3 of symmetrical lens distortion and M_1, M_2 of asymmetrical lens distortion have about the same values for both camera positions. This is not surprising since the same system of lenses was used in both cases. The slight differences in the coefficients for the two positions are probably due to rotation of the lenses in the focusing process.

CONCLUSION

Considering that the real marker coordinates in object space could be determined with an accuracy of only ± 1 mm, the reconstruction precision achieved by the use of the present linear and, for the 30CP configuration, also by that of the non-linear MDLT algorithm can be considered satisfactory for purposes of motion analysis. Related to object space dimensions, these algorithms permit reconstructions with an average accuracy of about 0.035% (one part in 2860) which is in the range of the spatial resolution achieved by modern optoelectronic recording equipment. Such high reconstruction accuracy is essential if second derivatives are to be computed from the noise-corrupted position data, as is the case in modern gait analysis and general human motion research.

To summarize then, the present investigation has shown that high-precision three-dimensional object space reconstruction is possible by means of linear or non-linear modified DLT algorithms if the following restrictions are taken into account.

(1) The non-linear MDLT algorithm should be used only if a sufficiently large number (≥ 30) of control points is available all of which are evenly distributed throughout the observation volume. In this case, this algorithm yields the highest possible reconstruction precision.

(2) The linear MDLT algorithm should be used with at least 15 control points, evenly distributed within the observation volume. If less than 15 control points are used and/or the markers are not evenly distributed, the rms mean error can still be expected to be reasonably small but sizeable maximum deviations in some coordinates may occur. Less than seven control points should never be used.

Generally speaking, the linear MDLT algorithm is less sensitive to modifications in the control point configuration and computationally less expensive than the non-linear MDLT algorithm. However, for extremely accurate reconstructions, the application of the non-linear MDLT algorithm is indicated.

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$$\begin{bmatrix} \bar{X}_1 & \bar{Y}_1 & \bar{Z}_1 & 1 & 0 & 0 & 0 & 0 & -\bar{\eta}_1^*\bar{X}_1 & -\bar{\eta}_1^*\bar{Y}_1 & -\bar{\eta}_1^*\bar{Z}_1 \\ \bar{X}_2 & \bar{Y}_2 & \bar{Z}_2 & 1 & 0 & 0 & 0 & 0 & -\bar{\eta}_2^*\bar{X}_2 & -\bar{\eta}_2^*\bar{Y}_2 & -\bar{\eta}_2^*\bar{Z}_2 \\ \vdots & & \vdots & \vdots & & & & \vdots & \vdots & \vdots & \vdots \\ \bar{X}_v & \bar{Y}_v & \bar{Z}_v & 1 & 0 & 0 & 0 & 0 & -\bar{\eta}_v^*\bar{X}_v & -\bar{\eta}_v^*\bar{Y}_v & -\bar{\eta}_v^*\bar{Z}_v \\ 0 & 0 & 0 & 0 & \bar{X}_1 & \bar{Y}_1 & \bar{Z}_1 & 1 & -\bar{\zeta}_1^*\bar{X}_1 & -\bar{\zeta}_1^*\bar{Y}_1 & -\bar{\zeta}_1^*\bar{Z}_1 \\ 0 & 0 & 0 & 0 & \bar{X}_2 & \bar{Y}_2 & \bar{Z}_2 & 1 & -\bar{\zeta}_2^*\bar{X}_2 & -\bar{\zeta}_2^*\bar{Y}_2 & -\bar{\zeta}_2^*\bar{Z}_2 \\ \vdots & & \vdots & \vdots & & & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \bar{X}_v & \bar{Y}_v & \bar{Z}_v & 1 & -\bar{\zeta}_v^*\bar{X}_v & -\bar{\zeta}_v^*\bar{Y}_v & -\bar{\zeta}_v^*\bar{Z}_v \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_7 \\ L_8 \\ L_9 \\ L_{10} \\ L_{11} \end{bmatrix} = \begin{bmatrix} \bar{\eta}_1^* \\ \bar{\eta}_2^* \\ \vdots \\ \bar{\eta}_v^* \\ \bar{\zeta}_1^* \\ \bar{\zeta}_2^* \\ \vdots \\ \bar{\zeta}_v^* \end{bmatrix}$$

APPENDIX. ALGORITHM FOR THE SOLUTION OF THE MDLT-METHOD

The 11 DLT parameters $L_1, \ldots, L_{11}$ may be expressed in terms of the 16 parameters

$$(a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, X_0, Y_0, Z_0, \eta_0, \zeta_0, \mu c/\lambda_\eta, \mu c/\lambda_\zeta) \quad \text{(A1)}$$

as presented on p. 423 of Marzan and Karara (1975). It should, however, be noted that in the present context [see equation (1) of the main text] the negative ξ-axis and not the negative ζ-axis, as is the case in the treatment of Marzan and Karara (1975), points in the direction of the optical axis.

Of the 16 parameters in the set (A1) only 10 are independent, because of the orthogonality conditions (15) presented in the main text. Thus, one of the 11 DLT parameters must be redundant.

Expressions for the internal camera parameters η_0 and ζ_0 in terms of the DLT-parameters $L_1, \ldots, L_{11}$ are given by Marzan and Karara (1975) on p. 429 as equation (22), with x_0 and y_0 replacing η_0 and ζ_0 respectively.

For $i = 2$ and $k = 3$ it follows from equation (15) in the main text that

$$\eta_0\zeta_0 = E^2(L_1L_5 + L_2L_6 + L_3L_7) \quad \text{(A2)}$$

where

$$E = a_{11}X_0 + a_{12}Y_0 + a_{13}Z_0 \quad \text{(A3)}$$

in which the afore-mentioned expressions of Marzan and Karara (1975, pages 423 and 429) for η_0, ζ_0, etc. are to be substituted. However, an actual substitution would result in extremely bulky expressions, so that it turns out to be more practical to appropriately define the respective expressions and substitute the corresponding variables in a routine implementing the present algorithm.

At any rate, the final expression then follows directly from equation (A2) and has the general form

$$f(L_1, L_2, \ldots, L_{11}) = 0 \quad \text{(A4)}$$

which indicates that there exists a (non-linear) relationship between the 11 DLT parameters. For some of the 11 parameters it turns out that equation (A4) may be written in the form of equation (16) of the main text.

Having established the relationship (A4), the algorithm continues as follows. Assume that a number $v (v \geq 6)$ of control points $(\bar{X}_j, \bar{Y}_j, \bar{Z}_j)$, $j = 1, \ldots, v$, has been observed and their images recorded. Then linear least-squares techniques may be employed to compute initial estimates $\{L_1, L_2, \ldots, L_{11}\}$ from the overdetermined system of linear equations

where $(\bar{\eta}_j^*, \bar{\zeta}_j^*)$, $j = 1, \ldots, v$, are the observed comparator coordinates of the control points $\bar{P}_j$, $j = 1, \ldots, v$.

The solution vector $\{L_1, \ldots, L_{11}\}$ provides the starting values for the next step. Let it be assumed that $L_n = L_1$ in equation (16) of the main text. Then, for $j = 1, \ldots, v$, equations (12–14) of the main text can be written as

$$\bar{\eta}_j^* G_j - h(L_2, \ldots, L_{11})\bar{X}_j - L_2\bar{Y}_j - L_3\bar{Z}_j - L_4 = 0 \quad \text{(A5)}$$

$$\bar{\zeta}_j^* G_j - L_5\bar{X}_j - L_6\bar{Y}_j - L_7\bar{Z}_j - L_8 = 0 \quad \text{(A6)}$$

in which now only the 10 DLT parameters $L_2, \ldots, L_{11}$ appear. The function $h(L_2, \ldots, L_{11})$ is identical with the respective function appearing in (16) and is a result of the substitution into (A2) and (A3) of the respective expressions, as described above. In order to solve the non-linear system (A5), (A6) for $L_2, \ldots, L_{11}$, appropriate algorithms for the solution of systems on non-linear equations must be used. The value of L_1 may then be computed from (16). In this way, the 10 independent DLT-parameters have been determined and the 11th is computed from these ten.

In a similar fashion, the orthogonality conditions (15) are enforced also for the case of object space reconstructions involving the elimination of non-linear distortions.