On the existence, uniqueness and regularity of Harmonic map heat flow

Park JunYoung

June 2023

Abstract

This is a survey paper of some of the classical theories of harmonic map heat flow (henceforth HMHF). We first present the basic definitions and the derivation of the PDE. Then, we will mainly discuss problems of existence, uniqueness and regularity of the heat flow.

Contents

1 Introduction to Harmonic map heat flow

The concept of Harmonic map heat flow

It is well known that for a given domain $\Omega \subset R^n$, harmonic functions $u : \Omega \to R$ with fixed boundary condition are critical points Dirichlet energy $E[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 dx$ on $\{u \in W^{1,2}(\Omega) \mid u = g \text{ in trace sense}\}$. Similarly, given two fixed points p, q in a smooth Riemannian manifold (N, γ), a geodesic connecting the two points can be interpreted as a critical point of $I[f] = \int_I |f'(t)|^2 dt$ defined on $V = \{f : I = [0,1] \to M' \mid f(0) = p, f(1) = q\}$. Note that both objects are critical points of the same type of functional, namely the square integral of the gradient.

Eells and Sampson [1] generalized the standard Dirichlet energy functional to maps between smooth Riemannian manifolds. Given smooth Riemannian manifolds $(M, g), (N, \gamma)$, let $C^\infty(M; N)$ be set of smooth maps from M to N. For $u \in C^\infty(M; N),\ x \in M$, choose local coordinates near x and $u(x)$, U and W respectively.

Definition 1.1 *For $u \in C^\infty(M, N)$, we define the Dirichlet energy density at $x \in M$ as*

$$e(u)(x) = \frac{1}{2}\gamma_{ab}(u(x))\frac{\partial u^a}{\partial x^i}(x)\frac{\partial u^b}{\partial x^j}(x)g^{ij}(x)$$

Also, we define Dirichlet energy as

$$E(u) = \int_M e(u)(x)dM(x)$$

Above formula is independent of choice of local coordinates, which can be seen by change of coordinate.

We would like to generalize harmonic functions or geodesics to more general class of maps.

Definition 1.2 *$u \in C^\infty(M; N)$ is a harmonic map iff it is a critical point of E i.e $\frac{d}{dt}E(u_t)|_{t=0} = 0$ for all smooth variation of u.*

We now consider an intrinsic definition of Dirichlet Energy, which also implies coordinate independence. Given $u \in C^\infty(M, N)$, $Du \in \Gamma(u^*TN \otimes T^*M)$. The Dirichlet energy density is the square norm of Du w.r.t induced bundle metric $\langle \frac{\partial}{\partial y^a}(u(x)) \otimes dx^i(x), \frac{\partial}{\partial y^b}(u(x)) \otimes dx^j(x)\rangle = \gamma_{ab}(u(x))g^{ij}(x)$ on $u^*TN \otimes T^*M$.

$$e(u)(x) = \frac{1}{2}\langle \nabla u(x), \nabla u(x)\rangle_{u^*TN\otimes T^*M} = \frac{1}{2}\|\nabla u(x)\|^2_{u^*TN\otimes T^*M}$$

Hence

$$E(u) = \int_M \frac{1}{2}\|\nabla u(x)\|^2_{u^*TN\otimes T^*M}(x)dM(x)$$

The examples in the beginning imply that harmonic maps are likely to have various desirable properties. Hence, for given Riemannian manifolds, the existence of harmonic maps are of great interest, especially minimizers of E. One approach would be to construct minimizing sequence and take the weak limit of it. However, the weak limit of the sequence might not be in the initial homotopy class, meaning above method can fail. Thus, Eells and Sampson used heat flow method to prove that under curvature restriction on the target manifold, every smooth map is homotopic to a harmonic map. The idea is to view tension field as the negative L^2 gradient of Dirichlet energy functional. Thus for given smooth map u_0, by deforming it in the direction of tension field at every given time, the Dirichlet energy will decrease and if the function under deformation converges to say u_∞, we can expect that $\tau(u_\infty) = 0$ i.e. u_∞ is a harmonic map. This is analogous to the relationship between Laplace equation and heat equation, hence the name Harmonic map heat flow.

Formulation of the equation

Now we will derive the harmonic map heat flow equation. We follow the derivation in the book of Jost [2].

For given $\psi \in \Gamma(u^*TN)$, let $u(x,t) = exp_{u(x,t)}\psi(x)$, where exp is the Riemannian exponential map on N. Then u is a smooth variation of f with its variational field ψ. Also, note that

$$\langle \nabla u(x,t), \nabla u(x,t)\rangle_{u^*TN\otimes T^*M} = \langle \nabla_x u(x,t), \nabla_x u(x,t)\rangle_{u^*TN\otimes(T^*M\oplus T^*R)}$$

where D_x denote space derivative. Then since $u^*TN\otimes(T^*M\oplus T^*R)$ also has induced metric / connection

$$\frac{d}{dt}\frac{1}{2}\langle \nabla_x u(x,t), \nabla_x u(x,t)\rangle_{u^*TN\otimes(T^*M\oplus T^*R)}|_{t=0} = \langle \nabla_{\frac{\partial}{\partial t}}(\nabla_x u)|_{t=0}(x), \nabla_x u(x,0)\rangle$$

Now

$$\nabla_{\frac{\partial}{\partial t}}(\nabla_x u)|_{t=0}(x) = \nabla_{\frac{\partial}{\partial t}}(\frac{\partial u^a}{\partial x^i}\frac{\partial}{\partial y^a}\otimes dx^i) = (\nabla_{\frac{\partial}{\partial t}}\frac{\partial u^a}{\partial x^i}\frac{\partial}{\partial y^a})\otimes dx^i \ (\nabla_{\frac{\partial}{\partial t}}dx^i = 0)$$

$$= (\frac{\partial^2 u^a}{\partial t\partial x^i}\frac{\partial}{\partial y^a} + \frac{\partial u^a}{\partial x^i}\nabla_{\frac{\partial u}{\partial t}}\frac{\partial}{\partial y^a})\otimes dx^i$$

Since $\nabla_{\frac{\partial}{\partial y^b}}\frac{\partial}{\partial y^a} = \nabla_{\frac{\partial}{\partial y^a}}\frac{\partial}{\partial y^b}$,

$$(\frac{\partial^2 u^a}{\partial t\partial x^i}\frac{\partial}{\partial y^a} + \frac{\partial u^a}{\partial x^i}\nabla_{\frac{\partial u}{\partial t}}\frac{\partial}{\partial y^a})\otimes dx^i = (\frac{\partial^2 u^b}{\partial t\partial x^i}\frac{\partial}{\partial y^b} + \frac{\partial u^a}{\partial x^i}\frac{\partial u^b}{\partial t}\nabla_{\frac{\partial}{\partial y^a}}\frac{\partial}{\partial y^b})\otimes dx^i$$

$$= (\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial u^b}{\partial t}\frac{\partial}{\partial y^b})\otimes dx^i = \nabla(\frac{\partial u^b}{\partial t}\frac{\partial}{\partial y^b})$$

Since $\frac{\partial u^b}{\partial t}\frac{\partial}{\partial y^b}$ is the variational field of u, we conclude that

$$\frac{d}{dt}\frac{1}{2}\langle \nabla_x u(x,t), \nabla_x u(x,t)\rangle_{u^*TN\otimes(T^*M\oplus T^*R)}|_{t=0} = \langle \nabla(\psi^a\frac{\partial}{\partial y^a}), \nabla u(x)\rangle_{u^*TN\otimes T^*M}$$

where the covariant derivative on the r.h.s is the induced connection on $u^*TN\otimes T^*M$. Therefore

$$\frac{d}{dt}E(u)|_{t=0} = \int_M \langle \nabla(\psi^a\frac{\partial}{\partial y^a}), \nabla u(x)\rangle_{u^*TN\otimes T^*M}(x)dM(x)$$

By using Stokes theorem, one can find a formal L^2 adjoint operator of ∇, denoted by ∇^*

$$\nabla^* F = -trace\nabla F \in \Gamma(u^*TN)$$

for $F \in \Gamma(u^*TN\otimes T^*M)$ Then

$$\int_M \langle \nabla(\psi^a\frac{\partial}{\partial y^a}), \nabla u(x)\rangle_{u^*TN\otimes T^*M}dM = \int_M \langle \psi^a\frac{\partial}{\partial y^a}, \nabla^*\nabla u(x)\rangle_{u^*TN}dM$$

Therefore, $\nabla^* \nabla u(x)$ is the L^2 gradient of E. Therefore, u is a solution of the heat flow iff its variation w.r.t time is the negative gradient of E, which is $\tau(u) = -\nabla^* Du(x)$. Thus, we have the following definition.

Definition 1.3 *$u : M \times [0, T_{max}) \to N$ is a solution of Harmonic map heat flow with initial condition $u_0 \in C^1(M; N)$ iff $u \in C^\infty(M \times (0, T_{max}); N) \cap C^1(M \times [0, T_{max}); N)$ and*

$$(a).\ \lim_{t \to 0} u(x,t) = u_0(x) \text{ in } C^1(M, N)$$

$$(b).\ \frac{\partial u}{\partial t} = \tau(u) \text{ for any } (x,t) \in M \times (0, T_{max})$$

In terms of local coordinate, above equation becomes

$$\frac{\partial u^c}{\partial t} = \Delta_M u^c + \Gamma^c_{ab}(u(x,t)) \frac{\partial u^a}{\partial x^i} \frac{\partial u^b}{\partial x^j} g^{ij}(x) = 0 \quad c = 1, 2, ..., dim(N)$$

Thus, our primary interest would be

(a) Does the heat flow have a unique solution $u(x,t)$?

(b) Is $T_{max} = \infty$? In other words, is there a global solution?

(c) If so, will $u(\cdot, t)$ converge(subconverge) to some harmonic map as $t \to \infty$?

2 With curvature assumption

The first answer was given by Eells and Sampson [1] and Hartman [3].

Theorem 2.1 (Eells and Sampson 1964) *Let (M, g), (N, γ) be a closed Riemannian manifold. Assume N has nonpositive curvature. Then for every smooth u_0 there exists a unique global smooth solution of*

$$\frac{\partial u}{\partial t} = \tau(u)$$

with initial condition u_0. Moreover, it subconverges in $C^\infty(M, N)$ to a harmonic map as $t \to \infty$.

Remark 2.1 *The original paper by Eells and Sampson deals the case when N is just complete. In that case, to control the nonlinear term with $e(u)$, they imposed an embedding condition. This condition is superfluous when N is compact, and later Philip Hartman proved that this condition can be removed. Also, subconvergence to harmonic map was later improved to full convergence by Hartman.*

The proof can be divided into four steps.

1. Find an alternative system of P.D.E so that we don't need to work with local coordinates of N. This is done by using Nash's embedding theorem.

2. Show that the PDE system from step 1 has unique short time smooth solution for every smooth initial data.

3. Establish a uniform bounds the energy density by using Bochner's formula for HMHF. This immediately gives us uniform bound on all higher order derivatives of u.

4. Use the estimates from step 3 to show all time existence of solution and sub-convergence to a harmonic map.

We will follow the exposition of [4] and [5]. We first need the following lemma.

Lemma 2.1 *Let $\phi : (N,\gamma) \to (R^L, g_0)$ be a Riemannian embedding, which exists by Nash's embedding theorem. Then $u \in C^\infty(M \times [0,T]; N)$ is a solution of HMHF iff for $W(x,t) = \phi(u(x,t))$,*

$$\Delta W - \frac{\partial W}{\partial t}$$

is perpendicular to $\phi(N) \subset R^L$, i.e for each $x \in M$, $\tau(W) - \frac{\partial W}{\partial t} \subset T_{W(x)}R^L$ is normal to $T_{W(x)}N$. Here, Δ is the Laplace operator on M.

Proof: Let $(U,\{x^i\})$, $(V,\{y^a\})$ be local coordinate of $x \in M$, $u(x) \in N$ respectively. Then, for each $W = (W^1, ..., W^L)$

$$\frac{\partial}{\partial t} W^\alpha = \frac{\partial \phi^\alpha}{\partial y^a} \frac{\partial u^a}{\partial t}$$

and

$$\Delta W^\alpha = g^{ij}[\frac{\partial^2 W^\alpha}{\partial x^i \partial x^j} - \Gamma^{Mk}_{ij} \frac{\partial W^\alpha}{\partial x^k}] = \frac{\partial \phi^\alpha}{\partial y^a}[g^{ij}(\frac{\partial^2 u^a}{\partial x^i \partial x^j} - \Gamma^{Mk}_{ij} \frac{\partial u^a}{\partial x^k})] + \frac{\partial^2 \phi^\alpha}{\partial y^a \partial y^b} \frac{\partial u^a}{\partial x^j} \frac{\partial u^b}{\partial x^j} g^{ij}$$

Adding and subtracting $\Gamma^{Na}_{bc} \frac{\partial \phi^\alpha}{\partial y^a} \frac{\partial u^b}{\partial x^i} \frac{\partial u^c}{\partial x^j} g^{ij}$, we have

$$\nabla \phi^\alpha(\tau(u)) + (\frac{\partial^2 \phi^\alpha}{\partial y^b \partial y^c} - \Gamma^{Na}_{bc} \frac{\partial \phi^\alpha}{\partial y^a}) \frac{\partial u^b}{\partial x^i} \frac{\partial u^c}{\partial x^j} g^{ij}$$

By collecting all $1 \le \alpha \le L$, we have

$$\Delta W - \frac{\partial W}{\partial t} = D\phi(\tau(u) - \frac{\partial u}{\partial t}) + h^\alpha_{bc} \frac{\partial u^b}{\partial x^i} \frac{\partial u^c}{\partial x^j} g^{ij} e_\alpha$$

where e_α is standard orthonormal basis of R^L. Clearly the 1st term in the RHS is tangent to $\phi(N)$. We claim that each $h^\alpha_{bc}e_\alpha$ is normal to N. Since ϕ is an isometric embedding, the covariant derivative on N is projection of covariant derivative in R^L. This immediately implies that

$$h^\alpha_{bc}e_\alpha = [\nabla^{u^*TR^L}_{\frac{\partial}{\partial y^b}} D\phi(\frac{\partial}{\partial y^c})]^\perp$$

Therefore, the 2nd term is finite sum of normal vectors, hence normal. This completes the proof of the lemma.

Notice that the normal term is the 2nd fundamental form of N, A_N. Therefore, if $u \in C^\infty(M; N)$ solves HMHF, then for $W = \phi(u)$

$$\frac{\partial W}{\partial t} - \Delta W = -trace(W^* A_N)$$

We now show the converse, namely if $W \in C^\infty(M; R^L)$ solves above system of PDE with initial data $W(\cdot, 0) \in C^\infty(M; N)$, then $W \in C^\infty(M; N)$ for all $t > 0$ and u such that $W = \phi(u)$ solves HMHF. The 2nd assertion is immediate as long as we show $W \in C^\infty(M; N)$.

For this, let $N \subset U$ be a tubular neighborhood of N in R^L and let $\pi : U \to N$ be a projection map. Let $I = \{t | W(M, s) \subset N \text{ for all } 0 \leq s \leq t\}$. By given assumption, $0 \subset I$. Since W is continuous and N is closed, I is closed as well. We show that I is open. For each $t \in I$, since W is smooth, we can find $\epsilon > 0$ so that any $|s - t| < \epsilon$, the image lies within U. Define $\rho : M \times (t - \epsilon, t + \epsilon) \to R^L$, $\rho(x, t) = |W(x, t) - \pi(W(x, t))|^2$. We compute

$$\frac{\partial}{\partial t}\rho(x, t) = 2(W - \pi(W)) \cdot (\Delta W - D\pi(\Delta W) - trace(W^* A_N))$$

Note that $D\pi(\Delta W) + trace(W^* A_N) = \Delta(\pi(W))$. Therefore

$$\frac{\partial}{\partial t}\rho(x, t) = 2(W - \pi(W)) \cdot (\Delta(W - \pi(W)))$$

Take the integral over M and integrate by parts, then we have

$$\frac{1}{2}\frac{d}{dt}\int_M \rho dM(x) = -\int_M |D(W - \pi(W))|^2 dM(x) \leq 0$$

Therefore, if $\rho(x, t) = 0$, then $\rho(x, s) = 0$ for all $t \leq s < t + \epsilon$. This implies that I is open.

Thus, I is both open and closed and nonempty. Thus image of W is in N as long as it is defined, thus proving the converse. This allows us to work with HMHF as a semilinear parabolic system on M.

We will use the semilinear parabolic system to show short time existence and uniqueness of Cauchy problem with smooth initial data. To do so, we briefly go over theory of heat equation on closed M.

Lemma 2.2 *Let (M, g) be a closed Riemannian manifold. Then for every $f \in C^0(M)$, there exists a unique $u \in C^\infty(M \times [0, \infty)) \cap C^0(M \times [0, \infty))$ which solves*

$$\frac{\partial u}{\partial t} - \Delta u = 0 \text{ in } M \times R_+$$

and

$$u(x, 0) = f(x)$$

Above lemma can be proved by constructing a 'masterkey' called heat kernel.

Definition 2.1 *Let (M, g) be a closed Riemannian manifold. A heat kernel H is a heat kernel iff $H \in C^\infty(M \times M \times (0, \infty))$ satisfies*

1. $\partial_t H - \Delta_y H(x, y, t) = 0$ *in* $M \times (0, \infty)$ *for each fixed* x
2. $\lim_{t \to 0} \int_M H(x, y, t) f(y) dM(y) = f(x)$ *for each* $f \in C^0(M)$

There are two different ways to construct heat kernel. One is to use eigenfunction of Δ_M. By Hodge theorem, we can find smooth L^2 orthonormal eigenbasis of the Laplace Beltrami operator. Then we can mimic the construction of heat kernel on S^1 via Fourier series. Then at least formally, we expect that

$$H(x, y, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$$

where λ_i is the ith eigenvalue with $\lambda_0 = 0$, and ϕ_i are the corresponding L^2 orthonormal eigenbasis. This sum is justified by the sufficient growth of λ_i which can be obtained by using several techneiques such as Moser's iteration. Details can be found in P.Li's book [6].

Another method is to find a parametrix, or approximate fundamental solution, and slowly correct it by using Volterra series. Since manifolds are locally Euclidean, we choose parametrix to be

$$H_k(x, y, t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2(x, y)}{4t}\right) \eta_k(x, y, t)$$

for $k > \frac{n}{2} + 2$, where $\eta_k(x, y, t) = (v_0(x, y) + v_1(x, y)t + ... + v_k(x, y)t^k)\phi_{2\epsilon}(r(x, y)) \in C^\infty(M \times M \times [0, \infty))$, $\phi_{2\epsilon} \in C_c^\infty(R)$, $\phi_{2\epsilon}(x) = 1$ when $x \leq \epsilon$ and $\phi_{2\epsilon}(x) = 0$ when $x \geq 2\epsilon$. Here we choose $\epsilon < \frac{inj(M)}{100}$. The v_i are chosen so that

$$\partial_t H_k - \Delta_y H_k = O(t^{k - \frac{n}{2}})$$

Then the actual heat kernel is

$$H(x, y, t) = H_k(x, y, t) + [\sum_{k=1}^{\infty} (-1)^k \{(\partial_t - \Delta_y) H_k\}^* k] * H_k(x, y, t) = H_k + \phi_k * H_k$$

where $A*B(x,y,t) = \int_0^t \int_M A(x,z,s)B(z,y,t-s)dzds$ and $*k$ is k times iterating $*$, namely $A^{*k} = A^{*(k-1)} * A$. Note that by induction, since $k > \frac{n}{2} + 2$, the series converge in $C^2(M \times M \times [0,T])$ and

$$|D_x^r D_y^s \partial_t^l \phi_k| \leq C(T) t^{k - \frac{n}{2} - \frac{|r|+|s|}{2} - l}$$

for all $t \in [0,T]$ for each $T > 0$. With this, we have sufficient regularity to perform $\partial_t - \Delta_y$. Details on this construction can be found in Rosenbloom [7].

For each $t > 0$, denote $e^{t\Delta} f$ as the solution of the heat equation with initial data f at time t. One of the key properties of heat operator is that it instantaneously increases regularity, i.e even though f is not regular, $e^{t\Delta} f$ is smooth for $t > 0$. In fact, by Bernstein's method combined with Holder space interpolation, one can show that for any $f \in C^a(M)$ for $a \geq 0$, $\|e^{t\Delta} f\|_{C^b(M)} \leq C(a,b,M) t^{\frac{a-b}{2}} \|f\|_{C^a(M)}$ for $b \geq a$.

Now, we come back to HMHF. We first look for weak solution.

Definition 2.2 *$u : M \times [0, T_{max}) \to R^L$ is a weak solution with smooth initial condition u_0 iff $u \in C^0([0, T_{max}); C^1(M; N))$ and*

$$u(t) = \int_0^t e^{(t-s)\Delta} F(u)(s) ds + e^{t\Delta} u_0$$

where

$$F(u) = -trace(u^* A_N) = -\pi_{ab}^c(u) \frac{\partial u^a}{\partial x^i} \frac{\partial u^b}{\partial x^j} g^{ij}$$

where $\pi_{ab}^c : U \to R$, $N \subset U$ is the 2nd derivative of projection map defined in the tubular neighborhood of N w.r.t standard coordinate in R^L.

First, we show uniqueness of weak solution.

Lemma 2.3 *For $i = 1, 2$, let $u_i : M \times [0, T_i) \to R^L$ be weak solutions of the heat flow with initial condition u_0. Then*

$$u_1(x,t) = u_2(x,t)$$

whenever $(x,t) \in M \times [0, \min(T_1, T_2))$

Proof: Let $I = \{t < \min(T_1, T_2) \mid u_1(s) = u_2(s) \text{ for all } 0 \leq s \leq t\}$. Then by assumption, I is nonempty and closed. We show it is open. Let $t_0 \in I$. Then since N is closed and u_1, u_2 are continuous, we can find $\epsilon > 0$ so that $t_0 + \epsilon < \min(T_1, T_2)$. We may further assume that for $|t - t_0| < \epsilon$, u_1 and u_2 both lie in the tubular neighborhood of N say U and are close enough so that the segment connecting $u_1(x,t)$ and $u_2(x,t)$ also lie in U. We can also find some $0 < C = C(u_1, u_2, t_0, \epsilon) < \infty$ such that

$$\|u_i(\cdot, s)\|_{C^1(M;N)} \leq C$$

for all $0 \leq s \leq t_0 + \epsilon$. The assumption on t_0 togethor with definition of weak solution yields

$$|u_1(x,s) - u_2(x,s)| \leq \int_{t_0}^{s} |e^{(s-r)\Delta}(F_1(r) - F_2(r))|dr$$

for $t_0 \leq s < t_0 + \epsilon$. Then

$$\sup_{t_0 \leq r \leq s} \|u_1(r) - u_2(r)\|_{C^1(M;R^L)} \leq C(s-t_0)^{1/2} \sup_{t_0 \leq r \leq s} \|F_1(r) - F_2(r)\|_{C^0(M;R^L)}$$

Note that

$$\begin{aligned}|F_1^c(y,r) - F_2^c(y,r)| = | \int_0^1 \frac{d}{dv}[\pi^c_{ab}(vu_1(y,r) + (1-v)u_2(y,r)) \\ \times (v\frac{\partial u_1^a}{\partial x^i} + (1-v)\frac{\partial u_2^a}{\partial x^i})(v\frac{\partial u_1^a}{\partial x^i} + (1-v)\frac{\partial u_2^a}{\partial x^i})]g^{ij}(y)dv| \\ \leq C(|u_1^d - u_2^d|(|Du_1|^2 + |Du_2|^2) + |Du_1 - Du_2|(|Du_1| + |Du_2|)) \\ \leq C\|u_1(\cdot,r) - u_2(\cdot,r)\|_{C^1(M;N)}\end{aligned}$$

Notice that since N is closed, derivatives of π are all uniformly bounded. Thus we have

$$\sup_{t_0 \leq r \leq s} \|u_1(r) - u_2(r)\|_{C^1(M;R^L)} \leq C(s-t_0)^{1/2} \sup_{t_0 \leq r \leq s} \|u_1(r) - u_2(r)\|_{C^1(M;R^L)}$$

Therefore, if we choose $C(s-t_0)^{1/2} < 1$, then $\sup_{t_0 \leq r \leq s} \|u_1(r) - u_2(r)\|_{C^1(M;R^L)} = 0$ i.e $u_1(r) = u_2(r)$ in $C^1(M;N)$ for all $t_0 \leq r \leq s$. This implies that I is open. Therefore $I = [0, \min(T_1, T_2))$, thus proving uniquenss of weak solution.

Remark 2.2 *It is known that if u is a classical solution of HMHF, then it is also weak solution in above sense. Therefore lemma 2.3 also implies uniqueness of classcial solution as well.*

We will now prove short time existence of weak solution. The key is to interpret the weak solution as a fixed point of

$$\Phi(u)(t) = \int_0^t e^{(t-s)\Delta} F(u)(s)ds + e^{t\Delta}u_0$$

We will find a suitable Banach space and prove that T is a well defined contraction mapping, thus proving existence of weak solution.

Lemma 2.4 *Let (M,g) be a closed manifold. Then for any smooth u_0, there exists a weak solution u as in definition 2.2. Moreover, if $T_{max} < \infty$, T_{max} is characterized by*

$$\limsup_{t \to T_{max}} \|Du(\cdot,t)\|_{C^0(M;R^L)} \to \infty$$

Proof: Define $X_{\gamma,T} = \{u \in C^0([0,T]; C^1(M; R^L) \mid \|u(t) - u_0\|_{C^1(M;R^L)} \leq \gamma\}$ for $\gamma, T > 0$ to be determined. Then $X_{\gamma,T}$ with $C^0([0,T]; C^1(M; R^L)$ norm is a Banach space. We find $\gamma > 0$ and T so that (a) Φ is well defined map on $X_{\gamma,T}$ to itself and (b) Φ is a contraction map.

(a) Since N is closed, there exists $\gamma = \gamma(N) > 0$ so that any point in R^L with $dist(x, N) \leq \gamma$ is in the tubular neighborhood of N, U. Then for any $u \in X_{\gamma,T}$, $\Phi(u)(t)$ is defined. Now we show that for small T, $\Phi(u) \in X_{\gamma,T}$. Since u_0 is smooth, $e^{t\Delta}u_0 \to u_0$ in $C^1(M; R^L)$ as $t \to 0$. Therefore, we can find $T_1 = T_1(u_0, M) > 0$ so that

$$\|e^{t\Delta}u_0 - u_0\|_{C^1} \leq \frac{\gamma}{2}$$

Also, by regularity of heat operator

$$\|\int_0^t e^{(t-s)\Delta}F(u)(s)ds\|_{C^1} \leq Ct^{1/2}\|u\|^2_{C^0([0,T];C^1)} \leq Ct^{1/2}(\|u_0\|_{C^1} + \gamma)^2$$

Therefore, we can find $T_2 = T_2(u_0, \gamma, M)$ so that the RHS is bounded by $\frac{\gamma}{2}$. This implies that Φ is well defined map on $X_{\gamma,T}$ to itself as long as $0 < T \leq \min(T_1, T_2)$.

(b) We now wish to make Φ a contraction map on $X_{\gamma,T}$. By following the proof of lemma 2.3, we obtain

$$\|\Phi(u) - \Phi(v)\|_{C^0([0,T];C^1)} \leq C(\gamma, u_0, M)T^{1/2}\|u - v\|_{C^0([0,T];C^1)}$$

Therefore, by choosing $C(\gamma, u_0, M)T^{1/2} < 1$, Φ is indeed a contraction map.

Thus, by contraction mapping principle, Φ has unique fixed point in $X_{\gamma,T}$, thus proving short time existence.

Now we give the characterization of $T_{max} < \infty$. Assume that

$$\limsup_{t \to T_{max}} \|Du(\cdot, t)\|_{C^0(M)} = K < \infty$$

Then since $[0, T_{max} - \epsilon]$ is compact for any $\epsilon > 0$, we may assume that $\sup_{0 \leq r < T_{max}} \|F(u)(r)\|_{C^0} \leq K$. Then the regularity of heat operator implies that for $0 \leq t, s < T_{max}$

$$\|u(t) - u(s)\|_{C^1} \leq C(t - s)^{1/2}K + \|e^{t\Delta}u_0 - e^{s\Delta}u_0\|_{C^1}$$

Therefore one can extend u continuously to some $u(T_{max}) \in C^1(M; N)$. By using short time existence with initial value $u(T_{max})$ and concatenating with original solution, we can extend our weak solution beyond T_{max}, contradicting the maximality of T_{max}. This shows that the solution can blow up in finite time only when the energy density blows up to infinity.

Now, we show that our weak solution is in fact smooth. This can be done by bootstrapping argument. For any $t < T_{max}$, $F(u)$ is uniformly bounded by the finite time blow up criteria. Therefore

$$\| \int_0^t e^{(t-s)\Delta} F(u)(s) ds \|_{C^{1+\alpha}} \leq C t^{\frac{1-\alpha}{2}} \|F(u)\|_{C^0(M\times[0,t])} < \infty$$

Therefore, from the equation, we can see that $u(t) \in C^{1+\alpha}$. This implies that $F(u)(t) \in C^{\alpha}$. Then again we can use the regularity of heat operator to increase the regularity of u even further. Such a cyclic argument is called bootstrapping argument. By increasing the regularity of u, we increase the regularity of the source term as well, which further increases the regularity of u. We use bootstrapping argument and the integral equation to increase the time regularity likewise, thus obtaining the fact that u is in fact a smooth solution to HMHF.

We now obtain a uniform bound on energy density $e(u)(t) = \frac{1}{2}\|Du\|^2_{C^0(M)}(t)$.

Lemma 2.5 (Uniform bound on energy density) *Let $u : M \times [0, T_{max}) \to N$ be a smooth solution of heat flow with smooth initial data u_0. If (N, γ) has nonpositive sectional curvature, then*

$$e(u) \leq C \max(\sup_M e(u_0), E_0)$$

Proof: The key is the following Bochner type formula

$$\begin{aligned} Le(u) = &-\|\nabla Du\|^2 - \langle Du(Ric^M(e_\alpha), Du(e_\alpha)\rangle \\ &+ R^N(u)(Du(e_\alpha), Du(e_\beta), Du(e_\beta), Du(e_\alpha)) \end{aligned}$$

Where e_α denote local orthonormal frame at x in TM. One way to prove above formula is since we know that the energy density is independent of local coordinates, we use geodesic normal coordinates centered at x and $u(x)$ respectively, which will reduce the computation significantly. Since M is compact, there exists $R = R(M) > 0$ such that

$$-\langle Df_t(Ric^M(e_\alpha), Df_t(e_\alpha)\rangle \leq R\langle Df_t(e_\alpha), Df_t(e_\alpha)\rangle = Re(f_t)(x)$$

Combined with the nonpositive curvature assumption, we have

$$Le(u)(x) \leq Re(u)(x)$$

Therefore $e(u)(x)$ is a nonnegative smooth subsolution of parabolic equation

$$(\partial_t - \Delta)u = Cu$$

Write $v(x,t) = e(u)(x)\exp(-Ct)$ Then $(\partial_t - \Delta)v \leq 0$ If $0 \leq t \leq 1$, since v is continuous up to $t = 0$, by the weak maximum principle

$$e(u)(x) \leq \sup_{x\in M} e(u_0)(x) \exp(Ct) \leq C \sup_{x\in M} e(u_0)(x)$$

When $t > 1$, we can use parabolic Moser iteration on each $P_{\epsilon'}((x,t)) = \{(y,s) \in M \times R^+ | r(x,y) < \epsilon', t - \epsilon'^2 \leq s \leq t\}$ for $\epsilon' = \min(1, \epsilon)$. Then since $\int_M e(u)(x,t)dM(x)$ is nonincreasing w.r.t t,

$$e(u)(x,t) \leq C \exp(Ct) \int_{P_{\epsilon'}((x,t))} e(u)(y,s) \exp(-Cs) dM(y) ds$$

$$\leq C \exp(Ct) \int_{t-1}^{t} \exp(-Cs) ds \int_M e(f_0)(y) dM(y) \leq C \int_M e(f_0)(y) dM(y)$$

thus proving the claim.

We now use the uniform bound on energy density to prove full time existence and convergence to harmonic map. $T_{max} = \infty$ is immediate from the characterization of finite time blow up. Thus, we will show convergence to some harmonic map.

For each $t > 1$, we can write

$$u(t) = \int_{t}^{t-1} e^{(t-s)\Delta} F(u)(s) ds + e^{\Delta} u(t-1)$$

Since N is closed, u is uniformly bounded in C^0. the local energy density bound implies that $F(u)(t)$ is also uniformly bounded. Therefore, the regularity of heat operator

$$\|u(t)\|_{C^k(M;N)} \leq C = C(k, N, M, u_0)$$

for all $k \geq 0$. Then, we may extract subsequence of t so that $t_n \to \infty$ and $u(t_n) \to u(\infty)$ in any C^k. Also notice that since

$$\frac{d}{dt} E(u(t)) = -\int_M |\partial_t u|^2 dM(x)$$

we may again extract subsequence so that

$$\partial_t u \to 0 \text{ in } L^2$$

The PDE implies that uniform C^k bound on $u(t)$ implies uniform C^k bound on $\partial_t u$ as well. Therefore, by interpolating with the uniform bounds on C^k norm, we finally prove that $\partial_t \to 0$ is any C^k as well. Then the PDE passes through the limit and the time derivative vanishes. This proves that $u(\infty)$ is a smooth harmonic map, completing the proof of Eells and Sampson's theorem.

2.1 Improvement by Hartman

The theorem by Eells and Sampson says that u_t subconverges to a harmonic map in C^2. It turns out that u_t strongly converges smoothly to a harmonic map, a result by Hartman [3]. We will also discuss the structure of collection of harmonic maps homotopic to a given $u_0 \in C^\infty(M; N)$.

Theorem 2.2 (Full convergence to harmonic map) *Under the same assumptions in Eells and Sampson's theorem, $u(\cdot,t) \to u_\infty$ smoothly as $t \to \infty$.*

Remark 2.3 *When N is not compact, one needs to further assume that the global unique solution u has precompact image in N. The original paper proves that under the curvature assumption, the global solution has precompact image iff homotopy class of u_0 contains some harmonic map. This assertion fails without the curvature assumption, as can be seen from the example given in [3].*

Proof: Above theorem and other results in this section all rely on the following lemma.

Lemma 2.6 *Let M, N be closed Riemannian manifolds, with N having non-positive curvature. Let $F : M \times [0,a] \to N$ be a smooth map. Define $f : M \times [0,a] \times [0,\infty) \to N$ be defined as solution of HMHF at time t for initial data $x \to F(x,u)$. Define $Q(x,u,t) = \|\frac{\partial f}{\partial u}\|_N^2$. Then for each fixed $0 \le u \le a$, Q is a nonnegative subsolution of standard heat equation on M, i.e*

$$\frac{\partial Q}{\partial t} - \Delta Q \le 0$$

Proof: By direct computation using local coordinates, we obtain

$$\frac{\partial Q}{\partial t} - \Delta Q = 2trace(R^N(Df(\cdot), \frac{\partial f}{\partial u}, \frac{\partial f}{\partial u}.Df(\cdot)) - 2|D(\frac{\partial f}{\partial u})|^2$$

The curvature assumption implies that the RHS is nonpositive, thus completing the proof of the lemma.

To prove full convergence, let $\{t_n\}_{n\in N}$ so that $t_N \to \infty$ and $u_{t_n} \to u_\infty$ in each $C^k(M;N)$. Let $\delta = inj(N) > 0$. Then for sufficiently large n

$$\sup_M dist(u_{t_n}(x), u_\infty(x)) < \delta$$

Define $F(x,u) : M \times [0,1] \to N$ so that for each $x \in M$, $F(x,\cdot)$ is the minimizing geodesic connecting $u_{t_n}(x)$ and $u_\infty(x)$. By uniqueness of solution of HMHF, we see that

$$f(x,0,t) = u_{t_n+t}(x),\ f(x,1,t) = u_\infty(x)$$

By applying the maximum principle on Q in lemma 2.6 for our f, we obtain

$$dist(u_{t_n+t}(x), u_\infty(x)) \le \int_0^1 |\frac{\partial f}{\partial u}| du \le \max_{M\times[0,1]} Q^{1/2}(x,u,t)$$

$$\le \max_{M\times[0,1]} Q^{1/2}(x,u,0) = \sup_M dist(u_{t_n}(x), u_\infty(x))$$

where the last equality is due to the fact that $f(x,u,0)$ is a minimizing geodesic. Therefore, for any $\epsilon > 0$, if we choose t_n so that $\sup_M dist(u_{t_n}(x), u_\infty(x)) < \min(\epsilon,\delta)$, then for all $t > t_n$, $\sup_M dist(u_t(x), u_\infty(x)) < \min(\epsilon,\delta)$ as well. This proves strong convergence in $C^0(M;N)$. Also, recall that we have uniform bound on all derivatives of u for $t \geq 1$. Therefore, by interpolation, we also obtain strong convergence in any $C^k(M;N)$, thus proving theorem 2.2.

As a result of Eells and Sampson with Hartman, in the case when N is closed with nonpositive curvature, all the three questions we had in the introduction are answered with an affirmative. Recall that the purpose of HMHF is to find a minimizer of Dirichlet energy functional. Therefore, two natural questions arise. (d) Is u_∞ obtained by heat flow is actually a minimizer withen the homotopy class of u_0?

(e) If (d) is the case, Is u_∞ a unique minimizer?

Under the nonpositive curvature assumption, the answer to (d) is yes, which we will now see. The answer to (e) requires further assumptions concerning the image of u_∞ and strict negativity of curvature.

Lemma 2.7 (Hartman, 1967) *For given $u_0 \in C^\infty(M;N)$, let $H(u_0)$ denote the collection of all harmonic maps homotopic to u_0. Then $H(u_0)$ is path connected. In addition, if any $u_{\infty 0}, u_{\infty,1} \in H(u_0)$ such that*

$$\sup_M dist(u_{\infty 0}(x), u_{\infty 1}(x)) < \delta$$

there exists a homotopy $U_\infty : M \times [0,1] \to N$ with endpoints $u_{\infty 0}$ and $u_{\infty 1}$ so that for each x, $U_\infty(x,s)$ is unique length minimizing geodesic with total length independent of x and each $U(x,s)$ is a harmonic map.

Remark 2.4 *If we take N to be standard Euclidean space, the 2nd part of above lemma easily follows since every harmonic map on closed M to Euclidean space is a constant map.*

Proof: Let $u_{\infty 0}, u_{\infty,1} \in H(u_0)$. Then since both are smooth maps, we can find a smooth homotopy $H : M \times [0,1] \to N$ with endpoints $u_{\infty 0}, u_{\infty,1}$. Then we consider $U(x,s,t)$ to be solution of HMHF with initial data H. Then, for each $s \in [0,1]$, $U(x,s,t) \to U(x,s,\infty) = U_\infty(x,s)$ smoothly which is a harmonic map. Also, since $U(x,s,0)$ are also homotopic to u_0, so is $U_\infty(x,s)$. This means for each $s \in [0,1]$, $U_\infty(x,s) \in H(u_0)$.

We now show continuity of U_∞ in s. This directly follows from lemma 2.6. For any $t > 0$ and $s, s' \in [0,1]$

$$\sup_M dist(U(x,s,t), U(x,s',t)) \leq C|s - s'|$$

where C is independent of t. Therefore, by letting $t \to \infty$, we obtain

$$\sup_M dist(U_\infty(x,s), U_\infty(x,s')) \leq C|s - s'|$$

Therefore U_∞ is a path in $H(u_0)$. This proves the first half of lemma 2.7.

For the 2nd half, note that we can take the initial H as a geodesic homotopy, meaning for each $x \in M$, $H(x,s)$ is the unique minimizing geodesic connecting $u_{\infty,0}(x)$ and $u_{\infty,1}(x)$. This additional fact together with lemma 2.6 and the strong maximum principle implies that

$$|\frac{\partial u}{\partial s}|(x,s,t) = \phi(s)$$

i.e independent of both x, t. This implies that for all $x \in M, t > 0$, $u(x,s,t)$ is still the unique minimizing geodesic connecting $u_{\infty,0}(x)$ and $u_{\infty,1}(x)$, meaning $U(x,s,t) = H(x,s)$ for all $t > 0$. Therefore $U_\infty = H$ is the desired geodesic homotopy.

As a corollary of lemma 2.7, for any two $u_{\infty,0}, u_{\infty,1} \in H(u_0)$, one can find a piecewise smooth homotopy U_∞ such that for each s subinterval, U_∞ is the geodesic homotopy in the 2nd half of lemma 2.7. Indeed, by first take the path H in $H(u_0)$ given by the 1st half of lemma 2.7. Then, using continuity, we subdivide the interval into $0 = s_0 < s_1 < .. < s_n = 1$ so that for each i

$$\sup_M dist(H(x,s_i), H(x,s_{i+1})) < \delta$$

Then by the 2nd half of lemma, we can find a geodesic homotopy for each $H(x,s_i), H(x,s_{i+1})$. Collecting these together yields our piecewise smooth geodesic homotopy.

We finally answer (d).

Theorem 2.3 (Hartman, 1967) *Let M, N be closed Riemannian manifolds, with N having nonpositive curvature. Then for each $u_0 \in C^\infty(M;N)$, $H(u_0)$ is nonempty and the Dirichlet energy functional E is constant on $H(u_0)$. As a result, the u_∞ obtained from heat flow method is indeed a minimizer of E.*

Proof: We only need to show that E is constant on $H(u_0)$. For any two $u_{\infty,0}, u_{\infty,1} \in H(u_0)$ let U_∞ be the piecewise smooth geodesic homotopy as above. We claim that $\phi(s) = E(U_\infty(\cdot, s))$ is constant. This claim follows once we show that ϕ is constant on each smooth subinterval. Thus, we may assume U_∞ is a smooth geodesic homotopy. Then since for each s, $U_\infty(x,s)$ is harmonic, the 1st variational formula implies that $\phi'(s) = 0$. Therefore $\phi(s)$ is constant, thus proving theorem 2.3.

In conclusion, when N is closed with nonpositive curvature, the heat flow method for harmonic maps successfully finds a minimizer of the Dirichlet energy functional.

3 Without curvature assumption, dimension two case

We now ask a natural question, What would the long time behavior of the solution be if there were no curvature assumption. When $dim(M) = 2$, Struwe [8] showed that a unique global weak solution exists, and it is smooth on $M \times (0, \infty) - \{(x_k, t_k)\}|k = 1, 2, .., L\}$ i.e is smooth away from finitely many points in spacetime. Moreover, he showed that as time progresses, there are jumps in the Dirichlet energy, and these jumps are caused by formation of 'bubbles'. More specifically, there are energy concentrations near these singular points, which after proper rescaling, subconverges to a nontrivial harmonic map defined on a 2-sphere, which is called a bubble. We also remark that this global solution is not neccesarily smooth everywhere, as we will see from an example by Chang-Ding-Ye [9].

We introduce some preliminary concepts. Since we are looking for distributional solution, it is natural to define Sobolev space of maps between manifolds. Note that we are assuming that (M, g) and (N, γ) are both closed. Thus, by Nash's imbedding theorem, we can isometrically embed (N, γ) in a Euclidean space R^L.

Definition 3.1 *Sobolev space of maps $W^{k,p}(M; N)$ consists of $f \in W^{k,p}(M; R^L)$ such that $f(x) \in N$ a.e [dM].*

Note that the Sobolev space defined above depends on the imbedding. Since we are working with Dirichlet energy, or the L^2 norm of derivative, we are mostly interested in $W^{1,2}(M; N)$.

Recall that set of smooth functions is dense in $W^{1,2}(\Omega)$ where $\Omega \in R^d$ is a domain with smooth boundary. Schoen and Uhlenbeck [10] showed that this is also the case when $dim(M) = 2$.

Lemma 3.1 *If $(M, g), (N, \gamma)$ are closed Riemannian manifolds with $dim(M) = 2$, then $C^\infty(M; N)$ is dense in $W^{1,2}(M; N)$*

Proof: Since (M, g) is closed, by using partition of unity, we can write any $u \in W^{1,2}(M; N)$ as finite sum of $u_i \in W^{1,2}(M; N)$ which is supported in some coordinate patch. Therefore, WLOG we assume after pullback to coordinate patch, $u \in W^{1,2}(B_1(0); N) \subset W^{1,2}(B_1(0); R^L)$ with $supp(u) \in B_{\frac{1}{2}}(0)$ where $B_r(0)$ is a disk of radius r centered at 0 in R^2. Let $u^\epsilon(x)$ be standard molification of u for sufficiently small ϵ. Since we defined Sobolev space of maps as a subset of standard Sobolev space, such molification is well defined, but note that these are maps into R^L not N. For any $x \in M$, the following Poincare type inequality holds

$$\frac{1}{\epsilon^2} \int_{B_\epsilon(x)} |u(y) - u^\epsilon(x)|^2 dM(y) \leq C \int_{B_\epsilon(x)} |\nabla u(y)|^2 dM(y)$$

where C is independent of ϵ. Since above inequality has scaling invariance, E.T.S above for some fixed $\epsilon > 0$. So let $\epsilon = 1$. Assume the contrary. Then we can construct $u_n \in W^{1,2}(B_1(x); R^L)$ such that

$$\int_{B_1(x)} |\nabla u_n(y)|^2 dM(y) \to 0$$

and

$$\int_{B_1(x)} |u_n(y) - u_n^1(x)|^2 dM(y) = 1$$

By compactness, there is some $u \in W^{1,2}(B_1(x); R^L)$ such that

$$\int_{B_1(x)} |\nabla u(y)|^2 dM(y) = 0$$

with

$$\int_{B_1(x)} |u(y) - u^1(x)|^2 dM(y) = 1$$

The first equality means u is a.e constant on $B_\epsilon(x)$. Then the 2nd equality can't hold since the integrand is 0 a.e, thus proving the Poincare type estimate.

Since $|Du|^2 \in L^1(M)$, by choosing sufficiently small $\epsilon > 0$

$$dist(N, u^\epsilon) \leq \frac{1}{\epsilon^2} \int_{B_\epsilon(x)} |u(y) - u^\epsilon(x)|^2 dM(y) \leq \delta$$

where $\delta = \frac{1}{100} dist(N, \partial U)$ with U being the tubular neighborhood of N in R^L. Thus if we let π be the projection of U on N, $u_\epsilon(x) = \pi(u^\epsilon(x)) \in C^\infty(B_1(0); N)$ is well defined. Since $\|u^\epsilon - u\|_{W^{1,2}(B_1(0);R^L)} \to 0$ and π is smooth,

$$\|u_\epsilon - u\|_{W^{1,2}(B_1(0);N)} \to 0$$

as $\epsilon \to 0$. This completes the proof of the lemma.

Remark 3.1 *Unlike in the case of Sobolev space of functions, class of smooth maps in general are not dense in Sobolev spaces of maps [5]. If $kp > dim(M)$, by Sobolev embedding, $W^{k,p}(M; N) \subset C^0(M; N)$, thus by using Whitney's approximation theorem, we also obtain density. In case when $kp = dim(M)$, the argument in the proof of above lemma can be used to obtain density. However, in general if $kp < dim(M)$, set of smooth maps might not be dense in Sobolev space . For example, $\phi(x) = \frac{x}{|x|} \in W^{1,2}(B^3, S^2)$ can't be approximated by smooth maps. In particular, if $dim(M) = 2$, then $C^\infty(M; N)$ is dense in $W^{1,2}(M; N)$. This is one reason why $dim(M) = 2$ is a desirable situation.*

We define distributional solution of HMHF.

Definition 3.2 *$u : M \times [0,\infty) \to N$ is a weak solution of harmonic map heat flow with initial data $u_0 \in W^{1,2}(M;N)$ iff $u \in L^2_{loc}([0,\infty); W^{2,2}(M;N))$ with $\partial_t u \in L^2_{loc}([0,\infty); L^2(M; R^L)$ and*

$$1. \int_0^T \int_M (\partial_t u - \Delta u)\phi(x,t) dM(x)dt = \int_0^T \int_M -trace(u^* A_N)\phi dM(x)dt$$

for all $\phi \in C_c^\infty(M \times [0,\infty); R^L)$, and

$$2.\ u(x,0) = u_0(x) \text{ in } W^{1,2}(M;N)$$

Note that after redefining on a measure 0 set in $[0,\infty)$,
$u \in C^0([0,\infty); W^{1,2}(M;N))$, hence the initial condition makes sense.

Now, we state the theorem of Struwe [8].

Theorem 3.1 (Struwe, 1985) *Let (M,g), (N,γ) be closed Riemmanian manifolds. Also, assume that $dim(M) = 2$. Then for any $u_0 \in W^{1,2}(M;N)$, there is a unique global weak solution $u(x,t)$ which is regular away from finitely many points in $M \times (0,\infty)$. These singular points (x,T) can be charecterized as*

$$\limsup_{t \to T-} E_R(u(\cdot,t),x) > \epsilon_1$$

where $E_R(u(\cdot,T),x) = \int_{B_R(x)} |Du(\cdot,t)|^2 dx$ and $\epsilon_1 > 0$ is some fixed constant only dependent on M,N. Finally, for such (x,T), there is $R_m \to 0$, $t_m < T, t_m \to T$, $x_m \to x$ and a nontrivial harmonic map $\overline{u_0} : R^2 \to N$ such that

$$u(R_m \cdot + x, t_m) \to \overline{u_0}(\cdot)$$

locally in $W^{2,2}(M;N)$. $\overline{u_0}$ has finite energy and can be extended to a smooth harmonic map from 2-sphere.

The strategy is to construct an approximating sequence of smooth solution of the heat flow. We first approximate the initial data $u_0 \in W^{1,2}(M;N)$ with $(u_0)_m \in C^\infty(M;N)$ in $W^{1,2}(M;N)$. Then by Eells and Sampson's theorem, we know that short time regular solution exists for each u_m. We will show that by possibly dropping some finite terms, u_m exists for at least for some $T > 0$ independent of m and that

$$\|u\|_{L^2([0,T];W^{2,2}(M;N))} + \|\partial_t u\|_{L^2([0,T];L^2(M;R^L))} \leq C$$

where C is independent of m. Then, we can extract a weakly convergent subsequence to some u. Aubin-Lions lemma allows the nonlinear term to pass through the limit as well, meaning u is the desired weak solution for short time. We will then show that even though the flow hits singular points, we will be able to extend it as a global weak solution pass such points. The key tool in the analysis is estimate of local energy.

Several ingredients

We will first show that when a smooth solution of heat flow with smooth initial data u does not have 'large' energy concentration, then it has bounded Sobolev space norm, with bound independent of u. This allows us to approximate a weak solution via sequence of smooth solutions with smooth initial data.

Definition 3.3 *Let u be a smooth solution of the heat flow. For given $R > 0$, $x \in M$, $t \in [0,T]$, define*

$$E_R(u(\cdot,t);x) = \int_{B_R(x)} |\nabla u(y,t)|^2 dM(y)$$

$$\epsilon(R;u,T) = \sup_{(x,t)\in M^T} E_R(u(\cdot,t);x)$$

$$E_0 = \int_M |\nabla u_0(y)|^2 dM(y)$$

Lemma 3.2 *Let u be a smooth solution of the heat flow with smooth initial data. Then*

$$\int_{M^T} |\partial_t u|^2 dM(x)dt \leq C$$

for some $C = C(N, E_0) > 0$.

Proof: Multiply $\partial_t u$ and integrate over M^T. Then since $\partial_t u$ is tangent to N, the nonlinear term vanishes, hence

$$\int_{M^T} |\partial_t u|^2 dM(x)dt \leq \int_{M^T} |\partial_t u|^2 dM(x)dt + E(u(T)) = E(u_0)$$

Lemma 3.3 *There exists $c, R_0 > 0$ dependent only on M, N such that for any smooth u, $0 \leq a \leq b \leq T$*

$$\int_a^b \int_M |\nabla u|^4 dM(x)dt \leq c\epsilon(R;u,T)(\int_a^b \int_M |\nabla^2 u|^2 dM(x)dt$$

$$+\frac{b-a}{R^2}\int_a^b \int_M |\nabla u|^2 dM(x)dt)$$

This lemma states that $t \to \int_0^t \int_M |\nabla u|^4 dM(x)ds$ is absolutely continuous whenever $u \in V(M^T;N)$.

Proof: Note that since M is closed, there exists $K.R_0 > 0$ depending only on M such that for any $0 < R \leq R_0$, we can find a cover of M by $B_{\frac{R}{2}}(x_i)$ so that the maximal number of intersections of $B_R(x_i)$ is bounded by K. This is called uniformly locally finite covering. This is true because since M is closed, one can isometrically embed M in some R^L. Since M is closed, there is $\delta = \delta(M) > 0$

so that any point $p \in R^L$ with $d(p, M) \leq \delta$ can be projected to M.

Then, for simplicity, for every $x \in M \subset R^L$, by rigid motion in R^L, we can assume that $x = 0$ and $T_x M \subset T_x R^L$ is $R^m \times \{0\}$. Then one can locally express M as a graph $(x, \phi(x))$ for $|x| << 1$, for some $\phi \in C^\infty(R^m)$, where $\phi(0) = D\phi(0) = 0$. This means that ϕ is locally written as a quadratic form with error term of order $|x|^3$. This implies that locally, Euclidean distance and geodesic distance between two points in M are comparable, hence there exists an open set $0 \in U \subset M$ and $r' = r'(x, M) > 0$ so that for any $x \in U$, if $0 < r < r'$, then

$$B^M_{r/2}(x) \subset B_r(x) \cap M \subset B^M_{2r}(x)$$

Since M is closed, above local property holds for any $x \in M$ and some $R_0 > 0$.

Then, for every $0 < R \leq R_0$, one can use tubular neighborhood to 'project' uniformly locally finite covering of R^L to M to obtain the desired covering.

Now we prove for any $0 < R \leq R_0$,

$$\int_a^b \int_{B_R} |\nabla u|^4 dM(x)dt \leq c\epsilon(R; u, T)(\int_a^b \int_{B_R(x_0)} |\nabla^2 u|^2 dM(x)dt$$
$$+\frac{1}{R^2}\int_a^b \int_{B_R(x_0)} |\nabla u|^2 dM(x)dt)$$

For each $a \leq t \leq b$

$$\int_{B_R(x_0)} |\nabla u(x,t)|^4 dM(x)dt \leq c\int_{B_R(x_0)} |Du - \overline{\nabla u}|^4 dM(x) + c\int_{B_R(x_0)} |\overline{\nabla u}|^4 dM(x)$$

where

$$\overline{\nabla u}(t) = \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} \nabla u(x,t)dM(x)$$

By Poincare's inequality and Holder inequality

$$\int_{B_R(x_0)} (|\nabla u - \overline{\nabla u}|^2)^2 dM(x) \leq c\int_{B_R(x_0)} |\nabla u - \overline{\nabla u}|^2 dM(x)\int_{B_R(x_0)} |\nabla^2 u|^2 dM(x)$$

By scaling invariance, note that above c is independent of R. Integrate w.r.t t from $a \leq t \leq b$.

$$\int_a^b \int_{B_R} |\nabla u|^4 dM(x)dt \leq c \sup_{t\in[a,b]} \int_{B_R(x_0)} |\nabla u - \overline{\nabla u}|^2 dM(x)\int_a^b \int_{B_R(x_0)} |\nabla^2 u|^2 dM(x)dt$$
$$+c\frac{1}{|B_R(x_0)|^3}\int_a^b |\int_{B_R(x_0)} \nabla u dM(x)|^4 dt$$
$$\leq c \sup_{t\in[a,b]} \int_{B_R(x_0)} |\nabla u|^2 dM(x)\int_a^b \int_{B_R(x_0)} |\nabla^2 u|^2 dM(x)dt$$

$$+c\frac{b-a}{|B_R(x_0)|}\int_a^b(\int_{B_R(x_0)}|\nabla u|^2dM(x))^2dt$$

We used Holder inequality twice in the last inequality. Since M is closed, by Bishop's volume comparison theorem, there is some $C=C(M)>0$ so that for $0<R\leq 1$, $|B_R(x_0)|\geq CR^2$ for any $x_0\in M$. Hence, we have

$$\int_a^b\int_{B_R}|\nabla u|^4dM(x)dt\leq c\epsilon(R;u,T)(\int_a^b\int_{B_R(x_0)}|\nabla^2u|^2dM(x)dt$$

$$+\frac{b-a}{R^2}\int_a^b\int_{B_R(x_0)}|\nabla u|^2dM(x)dt)$$

for all $0<R\leq R_0$.

To prove the lemma, cover M with the uniformly locally finite covering by balls $\{B_R(x_i)\}_{i=1,..,N}$. Then

$$\int_a^b\int_M|\nabla u|^4dM(x)dt\leq\sum_{i=1}^N\int_a^b\int_{B_R(x_i)}|\nabla u|^4dM(x)dt$$

$$\leq c\epsilon(R;u,T)\sum_{i=1}^N(\int_a^b\int_{B_R(x_0)}|\nabla^2u|^2dM(x)dt+\frac{b-a}{R^2}\int_a^b\int_{B_R(x_0)}|\nabla u|^2dM(x)dt)$$

$$\leq cK\epsilon(R;u,T)(\int_a^b\int_M|\nabla^2u|^2dM(x)dt+\frac{b-a}{R^2}\int_a^b\int_M|\nabla u|^2dM(x)dt)$$

With this lemma, we show that if a smooth solution u with smooth initial data has small $\epsilon(R;u,T)$ is small, then $\|u\|_{L^2([0,T];W^{2,2}(M;N))}$ is uniformly bounded.

Lemma 3.4 *There exists $\epsilon_1>0$ dependent only on M,N such that all smooth solution of the heat flow with smooth initial data with the following property*

There exists some $0<R\leq 1$ such that $\epsilon(R;u,T)\leq\epsilon_1$

satisfies

$$\|u\|^2_{L^2([0,T];W^{2,2}(M;N))}\leq C(1+\frac{T}{R^2})$$

for some $c=c(M,N,E_0)$.

Proof: Since N is closed, there exists $C=C(N)>0$ so that for any $x\in N\subset R^L$, $|x|\leq C$. Then since $0<R\leq 1$

$$\int_{M^T}|u|^2dM(x)dt\leq C|M|T\leq C(M,N)(1+\frac{T}{R^2})$$

Since u is smooth solution of heat flow with smooth initial data,

$$\int_M|\nabla u(x,t)|^2dM(x)$$

is a nonincreasing function and is continuous at $t = 0$. Thus

$$\int_M |\nabla u(x,t)|^2 dM(x) \leq E_0$$

Take the inner product between HMHF equation and $-\Delta u$ and integrate over $M^T = M \times [0,T]$. Then

$$\frac{1}{2}\int_{M^T} \frac{d}{dt}|\nabla u|^2 dM(x)dt + \int_{M^T} |\Delta u|^2 dM(x)dt = \int_{M^T} -trace(u^* A_N)\Delta u dM(x)dt$$

$$\leq C\int_{M^T} |\Delta u||\nabla u|^2 dM(x)dt \leq \frac{1}{2}\int_{M^T} |\Delta u|^2 dM(x)dt + C\int_{M^T} |\nabla u|^4 dM(x)dt$$

by Holder and Young's inequality. Now, by integrating by parts, because of the metric tensor coefficients, we have

$$\int_{M^T} |\nabla^2 u|^2 dM(x)dt \leq \int_{M^T} |\Delta u|^2 dM(x)dt + C\int_{M^T} |\nabla u|^2 dM(x)dt$$

where C only depends on M, N. Also, by lemma 3.3

$$\int_{M^T} |\nabla u|^4 dM(x)dt \leq C\epsilon(R; u, T)(\int_{M^T} |\nabla^2 u|^2 dM(x)dt + \frac{T}{R^2}E_0)$$

$$\leq C\epsilon_1(\int_{M^T} |\nabla^2 u|^2 dM(x)dt + \frac{T}{R^2}E_0)$$

where C only depends on M, N. Thus, by combining these two, we have

$$(\frac{1}{2} - C\epsilon_1)\int_{M^T} |\Delta u|^2 dM(x)dt \leq -\frac{1}{2}\int_{M^T} \frac{d}{dt}|\nabla u|^2 dM(x)dt + CTE_0 + C\epsilon_1\frac{T}{R^2}E_0)$$

$$\leq 2E_0 + (C + C\epsilon_1)\frac{T}{R^2}E_0$$

Let $C\epsilon_1 \leq \frac{1}{4}$. Then we have

$$\int_{M^T} |\Delta u|^2 dM(x)dt \leq CE_0(1 + \frac{T}{R^2})$$

By using

$$\int_{M^T} |\nabla^2 u|^2 dM(x)dt \leq \int_{M^T} |\Delta u|^2 dM(x)dt + C\int_{M^T} |\nabla u|^2 dM(x)dt$$

this leads to

$$\int_{M^T} |\nabla^2 u|^2 dM(x) \leq CE_0(1 + \frac{T}{R^2})$$

Therefore, by combining all 3 results, we have

$$\|u\|^2_{L^2([0,T];W^{2,2}(M;N))} \leq C(1 + \frac{T}{R^2})$$

for some $c = c(M, N, E_0)$.

Thus, we have the following lemma.

Lemma 3.5 *There exists $\epsilon_1 = \epsilon_1(N, M)$ so that if u be a smooth solution of heat flow with smooth initial data such that for some $0 < R \leq 1$, $T > 0$, $\epsilon(R; u, T) \leq \epsilon_1$ then*

$$\|u\|^2_{L^2([0,T];W^{2,2}(M;N))} + \|\partial_t u\|^2_{L^2([0,T];L^2(M;N))} \leq C(1 + \frac{T}{R^2})$$

for $C = C(M, N, E_0)$

This theorem shows us that if we could control $\epsilon(R; u_n, T)$ where u_n is the approximating sequence of smooth solutions with smooth initial data, then we would have weak convergence in Sobolev space.

Lemma 3.6 (local energy estimate) *Let u be a regular solution of the heat flow with smooth initial data. Fix $x_0 \in M$. Then for $R < \frac{1}{2} inj(M)$, $T > t_0$*

$$E_R(u(\cdot, T); x_0) \leq E_{2R}(u(\cdot, t_0); x_0) + C\frac{T - t_0}{R^2} E_0$$

Proof: Let $\eta \in C^\infty(M)$ such that $\eta = 1$ on $B_R(x_0)$, $supp\eta \subset B_{2R}(x_0)$, $|\nabla\eta| \leq \frac{C}{R}$. By taking the inner product between $\partial_t u - \Delta u$ with $\partial_t u\eta^2$ and integrating on M, because u is a solution of the heat flow and $\partial_t u\eta^2$ is tangential to $N \subset R^L$

$$\int_M |\partial_t u|^2\eta^2 dM(x) - \int_M \Delta u \partial_t u \eta^2 dM(x) = 0$$

By integration by parts

$$\int_M |\partial_t u|^2\eta^2 dM(x) + \frac{1}{2}\frac{d}{dt}(\int_M |\nabla u|^2\eta^2) dM(x) + \int_M \nabla u \partial_t u 2\eta\nabla\eta dM(x) = 0$$

Then

$$-\int_M \nabla u \partial_t u 2\eta\nabla\eta dM(x) \leq \int_M |\partial_t u|^2\eta^2 dM(x) + C\int_M |\nabla u|^2|\nabla\eta|^2 dM(x)$$

$$\leq \int_M |\partial_t u|^2\eta^2 dM(x) + \frac{C}{R^2}E_0$$

where the last inequality is due to the fact that the Dirichlet energy is nonincreasing w.r.t time. Thus, we have

$$\frac{1}{2}\frac{d}{dt}(\int_M |\nabla u|^2\eta^2) dM(x) \leq \frac{C}{R^2}E_0$$

Integrate from t_0 to T and we have the result.

We now prove an ϵ regularity criterion which is key to determining the size of singular set. For any $z_0 = (x_0, t_0) \in M \times (0, T]$, let $P_{r_0}(z_0) = \{(x, t) \mid r(x, x_0) < r_0,\ t_0 - r^2 < t \leq t_0\}$. Since M is closed, we assume $0 < r_0 \leq \overline{r} = \overline{r}(M) \leq 1$ so that $B_{r_0}(x)$ lie in some compact geodesically convex normal coordinate neighborhood.

Theorem 3.2 (ϵ-regularity theorem) *There exists some $\epsilon_0 = \epsilon_0(N) > 0$ such that for any solution of the heat flow $u \in C^\infty(\overline{P_{r_0}(z_0)}; N)$ with*

$$\sup_{[t_0 - r_0^2, t_0]} E_{r_0}(u(\cdot, t); x_0) \leq \epsilon_0$$

Then

$$\sup_{z \in P_{\frac{r_0}{2}}(z_0)} |\nabla u| \leq C r_0^{-1}$$

for some $C = C(N)$.

Proof: First, if u is constant, then above trivially holds, so we assume $\nabla u \neq 0$. By the assumption on r_0, we can assume that $B_{r_0}(x_0)$ lies in some compact normal coordinate neighborhood centered at x_0. Hence, we can assume that u is defined on $P_{r_0} = \{(x,t) | |x| < r_0, x \in R^n, t_0 - r_0^2 < t \leq t_0\}$. We consider $r_0^2(1-\rho)^2 \sup_{P_{\rho r_0}} |\nabla u|^2 = \phi(\rho)$ for $\rho \in [0,1]$. By smoothness of u, ϕ is a continuous function w.r.t ρ, thus obtains a maximum in $[0,1]$. Clearly, it must obtain maximum in $(0,1)$, or else we would have $|\nabla u| = 0$ on P_{r_0} which is not the case. Let that $\rho = \sigma \in (0,1)$. Also, for that σ, there is $z_1 = (x_1, t_1) \in P_{\sigma r_0}$ such that $e_0 = |\nabla u|^2(z_1) = \sup_{P_{\sigma r_0}} |\nabla u|^2$. Now, we split into 2 cases.

(a) If $r_0^2 e_0 (1-\sigma)^2 \leq 4$, then

$$\phi(\frac{1}{2}) = \frac{1}{4} r_0^2 \sup_{P_{\frac{r_0}{2}}} |\nabla u|^2 \leq \phi(\sigma) \leq 4$$

Thus by going back to $P_{\frac{r_0}{2}}(z_0)$, we are done.

(b) If $r_1^2 = r_0^2 e_0 (1-\sigma)^2 > 4$, we define v on $P = \{(x,t) | |x| < r_1 \text{ in } R^n, -r_1^2 < t \leq 0\}$ as

$$v(x,t) = u(x_1 + e_0^{-\frac{1}{2}} x, t_1 + e_0^{-1} t)$$

where we consider u to be defined on P_{r_0} via the normal coordinate system. Note that this is well defined since

$$|x_1 + e_0^{-\frac{1}{2}} x| \leq |x_1| + r_1 e_0^{-\frac{1}{2}} \leq r_0(\sigma + \frac{1-\sigma}{2}) < r_0$$

and

$$t_0 - (t_1 + e_0^{-1} t) \leq (\sigma^2 + \frac{(1-\sigma)^2}{4}) r_0^2 < r_0^2$$

Note that $|\nabla v|^2(0) = 1$. For simplicity, let us write $\phi(x,t) = (x_1 + e_0^{-\frac{1}{2}} x, t_1 + e_0^{-1} t) \in P_{r_0}$ for $(x,t) \in P$. Now, by the Bochner type formula for $e(u) = |Du|^2$, and the fact that M, N are closed, the following holds

$$(\partial_t - \Delta_M) e(u) \leq C(e(u) + e(u)^2)$$

Thus, $e(v) = |\nabla v|^2 = e_0^{-1}e(u)$ on P satisfies the uniformly parabolic PDE

$$(\partial_t - \Delta)e(v)$$

$$= \partial_t e(v) - \frac{1}{\sqrt{|g|(\phi(x,t))}}\frac{\partial}{\partial x^i}(g^{ij}(\phi(x,t))\sqrt{|g|(\phi(x,t))}\frac{\partial}{\partial x^j}e(v)) \leq C(e(v)e_0^{-1} + e(v)^2)$$

Since $r_0^2 e_0(1-\sigma)^2 > 4$ with $r_0 \leq 1$, $e_0 > 1$, hence the R.H.S can be majorized as

$$C(e(v)e_0^{-1} + e(v)^2) \leq (C + e(v))e(v)$$

Note that the differential operator w.r.t x in above pde is just the Laplace-Beltrami operator expressed in coordinate form, hence it is indeed uniformly parabolic operator. Since

$$|\nabla v|^2 \leq e_0^{-1} \sup_{P_{r_0(\frac{1}{2}+\frac{\sigma}{2})}} |\nabla u|^2 = (\frac{1}{2} - \frac{\sigma}{2})^2 \sup_{P_{r_0(\frac{1}{2}+\frac{\sigma}{2})}} |\nabla u|^2 e_0^{-1}(\frac{1}{2} - \frac{\sigma}{2})^{-2}$$

$$\leq 4$$

we can write

$$\partial_t e(v) - Le(v) \leq Ce(v) \text{ in } P$$

Note that since $r_1 > 1$, above also holds in $\tilde{P} = \{(x,t)||x| \leq 1 \text{ in } R^n, -1 \leq t \leq 0\}$. Thus, by applying Moser's iteration, and by change of variable

$$1 = |\nabla v|^2(0) \leq C\int_{\tilde{P}} |\nabla v|^2 dxdt \leq C\int_{\phi(\tilde{P})} e_0|\nabla u|^2 dxdt \leq C\epsilon_0$$

Thus, by choosing $C\epsilon_0 < 1$, above is impossible, i.e $e_0(1-\sigma)^2 r_0^2 \leq 4$. Since C is a constant from the iteration, and the ellipticity of L and C in the PDE are dependent only on M, N, we conclude that C, hence ϵ_0 is only dependent on M, N. This proves the theorem.

As in the proof of Eells and Sampson's theorem, by using fundamental solutions, Schauder theory and parabolic bootstrapping argument, we can obtain a uniform bound on higher order derivatives of u near z_0, which means that the weak solution will also be regular near z_0.

We let $\epsilon_2 = \min(\epsilon_0, \epsilon_1) = \epsilon_2(M, N) > 0$.

Existence and uniqueness of weak solution

We first show that weak solution of heat flow is unique. For any $T < \infty$, Let $u, v \in L^2([0,T]; W^{2,2}(M;N))$ with $\partial_t u, \partial_t v \in L^2([0,T]; L^2(M;N))$ be two weak solution of harmonic map heat flow with initial data $u_0 \in W^{1,2}(M;N)$. Let $w = u - v$. As mentioned above, we assume that u, v are refined on measure

0 set on $[0,T]$ so that they are both contained in $C^0([0,T];W^{1,2}(M;N))$. By testing w, we have

$$|\int_0^T\int_M \partial_t w w dM(x)dt - \int_0^T\int_M \Delta w w dM(x)dt|$$

$$= |\int_M w(x,T)^2 dM(x) - 0 + \int_0^T\int_M |\nabla w|^2 dM(x)dt|$$

$$\leq \frac{1}{2}\int_0^T\int_M |\nabla w|^2 dM(x)dt + C\int_0^T\int_M |w|^2(|\nabla u|^2+|\nabla v|^2)dM(x)dt$$

where we used Young's inequality. By Holder inequality

$$\int_0^T\int_M |w|^2(|\nabla u|^2+|\nabla v|^2)dM(x)dt \leq (\int_0^T\int_M |w|^4 dM(x)dt)^{\frac{1}{2}}$$

$$\times(\int_0^T\int_M (|\nabla u|^4+|\nabla v|^4)dM(x)dt)^{\frac{1}{2}}$$

For the 2nd integral, we will use lemma 3.3. Since $u,v \in C^0([0,T];W^{1,2}(M;N)))$, we may assume that

$$\epsilon(R;u,T), \epsilon(R;v,T) \leq C = C(u,v,T)$$

Then for each $\epsilon > 0$, there is some $0 < S = S(M,N,u,v,\epsilon) \leq 1$ so that

$$\int_0^S\int_M (|\nabla u|^4+|\nabla v|^4)dM(x)dt \leq \epsilon$$

In case of the first integral, we proceed similar to the proof of lemma 3.3. Note that since $w(\cdot,t) \in W^{1,2}(M;N)$ for a.e t, $|w|^2 \in W^{1,1}(M;N)$. Thus, by Sobolev and Holder inequality

$$\int_M |w|^4 dM(x) \leq C(\int_M |w|^2 dM(x) + \int_M |\nabla w|^2 dM(x))^2$$

for a.e. $0 \leq t \leq S$. Thus by integrating over t, and using that $S \leq 1$, we have

$$\int_0^S\int_M |w|^4 dM(x)dt \leq C(\sup_{0\leq s\leq S}\int_M |v(x,s)|^2 dM(x) + \int_0^S\int_M |\nabla w|^2 dM(x)dt)^2$$

By combining these estimates, note that if we take $\epsilon > 0$ to be small enough, we will have some $0 < S \leq 1$ so that

$$\sup_{0\leq t\leq S}\int_M |w(x,t)|^2 dM(x) + \int_0^t\int_M |\nabla w|^2 dM(x)dt \leq 0$$

implying that $w = 0$ for $0 \leq t \leq S$. By iterating this process, we have $u = v$, thus proving uniqueness.

Now, we will construct a weak solution which is regular away from finitely many points. Let $\{u_{m0}\} \subset C^\infty(M;N)$ such that $u_{m0} \to u_0$ in $W^{1,2}(M;N)$. Then since $|\nabla u_0|^2 \in L^1(M)$, there is some $0 < R_0 \le 1$, m_0 such that $\sup_{x\in M} \int_{B_R(x)} |\nabla u_0|^2 dx \le \frac{\epsilon_2}{4}$ and $\int_M |\nabla u_{m0} - \nabla u_0|^2 dx \le \frac{\epsilon_2}{4}$ for all $m \ge m_0$. Then

$$\int_{B_R(x)} |\nabla u_m|^2 dM(x) \le \frac{\epsilon_2}{2}$$

$$\int_M |\nabla u_{m0}|^2 dM(x) \le 2E_0$$

for all $m \ge m_0$. Therefore, by lemma 3.5, there exists $T = T(\epsilon_2, R)$ such that

$$\epsilon(R; u_m, T) \le \epsilon_2$$

where u_m is the short time smooth solution of the heat flow with initial data u_{m0}. By lemma 3.5, for $m \ge m_0$

$$\|u_m\|_{L^2([0,T];W^{2,2}(M;N)} + \|\partial_t u_m\|_{L^2([0,T];L^2(M;TN)} \le C = C(N, M, E_0)$$

This implies that $u_m \to u$ in $L^2([0,T]; W^{2,2}(M;N))$, $\partial_t u_m \to \partial_t u$ in $L^2(M^T;N)$ weakly. Also, by Aubin-Lions lemma, $u_m \to u$ strongly in $L^2([0,T]; W^{1,2}(M;N)))$. This implies that the integral equation passes through the limit, thus obtaining

$$\int_0^T \int_M (\partial_t u - \Delta u)\phi(x,t) dM(x)dt = \int_0^T \int_M -trace(u^* A_N)\phi dM(x)dt$$

To see that $u(0) = u_0$ in $W^{1,2}(M;N)$, we may instead show that $u(0) = u_0$ in $L^2(M;N)$.

$$\frac{d}{dt}\|u_n(t) - u(t)\|^2_{L^2(M;N)} = 2\langle \partial_t u_n(t) - \partial_t u(t), u_n(t) - u(t)\rangle_{L^2(M;N)}$$

Therefore, by integrating over $[0,t]$, we have

$$\begin{aligned}\|u_n(t) - u(t)\|^2_{L^2(M;N)} &- \|u_n(0) - u(0)\|^2_{L^2(M;N)} \\ &= 2\langle \partial_t u_n - \partial_t u, u_n - u\rangle_{L^2([0,t];L^2(M;N))}\end{aligned}$$

Since $\partial_t u_n \to \partial_t u$ weakly in $L^2([0,t]; L^2(M;N))$ and $u_n \to u$ strongly in $L^2([0,t]; L^2(M;N))$, the RHS converges to 0 for any t as $n \to \infty$. In particular, if we choose $t > 0$ so that $\|u_n(t) - u(t)\|^2_{L^2(M;N)} \to 0$, then we have

$$\|u_0 - u(0)\|^2_{L^2(M;N)} = \lim_{n\to\infty} \|u_n(0) - u(0)\|^2_{L^2(M;N)} = 0$$

Therefore, $u(0) = u_0$ in $L^2(M;N)$, hence in $W^{1,2}(M;N)$ as well. This implies that u is the weak solution of HMHF with initial data u_0 up to some $T > 0$.

Before extending our short time solution, we first discuss the regularity of u. For every $\delta > 0$, for each $(x,t) \in M \times [\delta, T]$, we can find some small $r > 0$ independent of (x,t) so that

$$P_r((x,t)) \subset M \times (0,T]$$

and

$$\sup_{(t-r^2,t]} E_r(u_m(\cdot,s);x) \leq \epsilon(r;u_m,T) \leq \epsilon_0$$

Then, by ϵ-regularity theorem

$$|\nabla u_m|(x,t) \leq \sup_{P_{r/2}((x,t))} |\nabla u_m| \leq Cr^{-1} = K(\delta)$$

hence

$$\sup_{(x,t)\in M\times[\delta,T]} |\nabla u_m| \leq K(\delta)$$

for all $m \geq m_0$. Therefore, as we did in proof of Eells and Sampson,

$$\|u_m\|_{C^{1+\alpha,\frac{\alpha}{2}}(M\times[\delta',T])} \leq K = K(\delta')$$

for all $\epsilon > 0$, $0 < \delta < \delta' < T$. for $\delta > 0$ as well. Just like in the proof of Eells and Sampson, by using heat kernel and Schauder theory, above implies that

$$\|u_m\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(M\times[\delta'',T])} \leq K = K(\delta'')$$

for $0 < \delta' < \delta'' < T$ Note that the constant K differs as we proceed, but still remains independent of m. Thus we can find a subsequence convergent in $C^2(M \times [\delta'',T];N)$ to some $v \in C^{2+\alpha,1+\frac{\alpha}{2}}(M \times [\delta'',T];N)$. However, since we know that $u_m \to u$ in $L^2([0,T];W^{2,2}(M;N))$, we have

$$u = v \text{ in } M \times [\delta'',T]$$

hence $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(M \times [\delta'',T];N)$. Since $\delta > 0$ is arbitrary, we conclude that

$$u \in C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(M \times (0,T];N)$$

thus smooth.

Thus u will evolve as a smooth solution for some time. Now, We claim that first time singularity as a smooth solution occurs at (x_0,T_0) when

$$\limsup_{t\to T_0-} E_R(u(\cdot,t);x_0) > \epsilon_1$$

for all $R > 0$. Assume not, i.e there is some $r_0 = r_0(x_0) > 0$ such that

$$\limsup_{t\to T_0-} E_{r_0}(u(\cdot,t);x_0) \leq \epsilon_1$$

Then, there exists $\eta > 0$, independent of t such that

$$E_{r_0}(u(\cdot,t);x_0) \le 2\epsilon_1$$

for all $t \in [T_0 - 2\eta, T_0)$. Then by choosing $r_1 = \min(\sqrt{\eta}, r_0)$, we have

$$\sup_{t\in[t_0-r_1^2,t_0]} E_{r_1}(u(\cdot,t);x_0) \le 2\epsilon_1 \le \epsilon_0$$

for all $t \in [T_0 - \eta, T_0)$. Therefore, by the ϵ-regularity theorem, we have

$$\sup_{B_{\frac{r_1}{2}}(x_0)} |\nabla u(\cdot,t_0)| \le C_0 = C_0(N)$$

for all $t \in [T_0 - \eta, T_0)$. However, this contradicts the finite maximal time characterization in Eells and Sampson's theorem, thus proving the claim.

Now, we will show that there are finitely many such singular point when $t = T_0$. Let S be collection of singular points at T_0. Let $F = \{x_i\} \subset S$ be any finite subset of S. Then there is some $R > 0$ such that

$$B_R(x_i) \cap B_R(x_j) = \emptyset \ (i \ne j)$$

Then for any $t \in (0, T_0)$ close to T_0

$$C|F|\epsilon_1 \le \sum_{i=1}^{|F|} E_R(u(\cdot,t);x_i) \le E(t) \le E_0$$

hence

$$|F| \le \frac{E_0}{C\epsilon_1} < \infty$$

Since F was arbitrary, we conclude that $|S| < \infty$.

Now, we will extend our weak solution beyond T_0. Since the energy functional is nonincreasing in $[0, T_0)$ we have

$$\|u(\cdot,t)\|_{W^{1,2}(M;N)} \le C$$

with C independent of t. Then, by possibly extracting a subsequence $t_k \to T_0$, there is some $u(x,T_0) \in W^{1,2}(M;N)$ such that $u(\cdot,t_k)$ converges in $W^{1,2}(M;N)$. Since

$$\int_M |\nabla u(x,T_0)|^2 dM(x) \le E_0 < \infty$$

by repeating above construction for new initial condition $u(x,T_0) \in W^{1,2}(M;N)$, we can find another weak solution $u_2(x, T_0+t) \in L^2_{loc}([0,T'); W^{2,2}(M;N))$ with initial condition $u(x,T_0)$, which is smooth in $M \times (0, T')$. We extend our weak solution by concatenating u and u_2, i.e

$$\tilde{u}(x,t) = \begin{cases} u_1(x,t) = u(x,t) \text{ when } t < T_0 \\ u_2(x,t) \text{ when } T_0 \le t < T_1 \end{cases} \tag{1}$$

Letting u now be the extended version, clearly u satisfies HMHF in distributional sense, since we can split the time interval into $[0, T_0]$ and $[T_0, T_1)$ and u_1, u_2 are both weak solutions of HMHF in each time subinterval.

We also note that there are finitely many singular points at $t = T_0$. This implies that away from these points, the ϵ regularity criterion is still valid up to time T_0. Thus the extended u is smooth even at $t = T_0$ away from singular points. Therefore, we conclude that this extended weak solution $u : M \times [0, T_1)$ is smooth except for the singular points at $t = T_0$.

We will again face the same situation at $t = T_1$. Repeat above procedure, then one can keep extending the weak solution to a global weak solution. Thus, we have a global weak solution of the heat flow with initial data $u_0 \in W^{1,2}(M; N)$.

Formation of bubbles

We now study the behavior of the weak solution near singular points. We first give a more detailed estimate of

$$\int_M |\nabla u(x, T_0)|^2 dM(x)$$

Let $S = \{x_1, x_2, ..., x_k\}$ be set of singular points at $t = T_0$. Then choose $r_0 > 0$ so that $r(x_i, x_j) > 2r_0$ for $i \neq j$. Then for any $0 < r < r_0$

$$\int_{M - \cup_{i=1}^k B_r(x_i)} |\nabla u(x, T_0)|^2 dM(x)$$

$$= \lim_{n\to\infty} (\int_M |\nabla u(x, t_n)|^2 dM(x) - \sum_{i=1}^k \int_{B_r(x_i)} |\nabla u(x, t_n)|^2 dM(x))$$

$$\leq \lim_{n\to\infty} \int_M |\nabla u(x, t_n)|^2 dM(x) - |S|\epsilon_1$$

Thus, by letting $r \to 0$, because the Dirichlet energy is nonincreasing, we have

$$\int_M |Du(x, T_0)|^2 dM(x) \leq \lim_{t\to T_0-} \int_M |Du(x, t)|^2 dM(x) - |S|\epsilon_1$$

This implies that there are finitely many points in spacetime where there is singularity. This is because when there are $T_1 < ... < T_k$ moments when singularity occur, then for $t > T_k$

$$0 \leq E(t) \leq E_0 - s\epsilon_1$$

where s denote the total number of singular points in spacetime up to $t = T_k$. Therefore, $s \leq E_0/\epsilon_1 < \infty$.

Notice that as we go beyond singular points, the Dirichlet energy suddenly decreases at least ϵ_1 times the number of singularity. We will now show that this 'sudden loss of energy' is due to formation of bubbles in the flow.

Theorem 3.3 (Bubbles at singularity) *Let $u : M \times [0, \infty) \to N$ be the global weak solution of the heat flow constructed above. Let (x_0, t_0) be the singular point in the flow, i.e there is $R_0 > 0$ such that*

$$\limsup_{t \to t_0 -} E_R(u(\cdot, t); x_0) > \epsilon_1$$

for all $0 < R \leq R_0$. Then there exist $x_m \to x_0$, $t_m < t_0$ with $t_m \to t_0$, $R_m \leq R_0$ with $R_m \to 0$ and a regular harmonic map $\overline{u_0} : R^2 \to N$ such that

$$\tilde{u}_m(x) = u_m(x_0 + R_m x, t_m + R_m^2 \tau_m) \to \overline{u_0}$$

weakly in $W^{2,2}_{loc}(R^2; N)$, where $\tau_m < 0$ will be determined later. Also, $\overline{u_0}$ has finite energy, thus can be extended to a smooth harmonic map from S^2 to N.

WLOG, we can choose R_0 to be small enough so that $R_0 \leq \frac{1}{2} inj(M)$. Thus for each $t_0 - \delta < t < t_0$, we can understand $u(x, t)$ restricted to $r(x, x_0) \leq R_0$ as a smooth function defined on $B_{r_0}(0) \subset R^2 = T_{x_0}M$. Thus for every $K \subset\subset R^2$, we can find some $R > 0$ so that $K \subset B_R(0)$. Since $R_m \to 0$ as $m \to \infty$, we can assume that for all $x \in R^2$ such that $|x| \leq R$, $|R_m x| \leq R_0$. Thus for sufficiently large m, u_m is well defined smooth function on $K \subset R^2$, and we can consider the convergence of u_m in $W^{2,2}(K; N)$. The theorem says that this is actually the case.

Since the flow has finitely many singular points in the flow, S, the set of all singular points is discrete. Hence, we can choose $\rho \leq \min(\frac{R_0}{2}, \frac{1}{2} inj(M))$ so that if $\{x_i\}_{i=0,1,2,..,k}$ denote singular points at $t = t_0$, then $r(x_0, x_i) \leq 5\rho$ and for all $t_1 = t_0 - \rho^2 \leq t < t_0$, $u(\cdot, t) \in C^\infty(M; N)$. This also means that for any $0 < \delta < \rho$

$$u \in C^\infty(\overline{(B_\rho(x_0)} - B_\delta(x_0)) \times [t_1, t_0])$$

We first show that for any $0 < \delta < \rho$, there exists $R = R(\delta) > 0$ and $t_1 < t(\delta) < t_0$ such that for any $t \in [t(\delta), t_0)$, $x \in \overline{B_\rho(x_0)} - B_\delta(x_0)$, $0 < r \leq R$

$$E_r(u(\cdot, t); x) < \epsilon_1$$

Assume not. Then for some $\delta > 0$, we can find $R_n \to 0$, $t_n \to t_0$ and $x_n \in \overline{(B_\rho(x_0)} - B_\delta(x_0)$ so that

$$E_{R_n}(u(\cdot, t_n); x_n)) \geq \epsilon_1$$

Since $\overline{(B_\rho(x_0)} - B_\delta(x_0)$ is compact, we may assume that $x_n \to x \in \overline{(B_\rho(x_0)} - B_\delta(x_0)$. For any $R > 0$, since $R_n \to 0$ and $x_n \to x$, there is some n_0 so that for $n > n_0$, $B_{R_n}(x_n) \subset B_R(x)$, thus

$$E_R(u(\cdot, t_n); x) \geq E_{R_n}(u(\cdot, t_n); x_n) \geq \epsilon_1$$

Since $t_n \to t_0$, this implies that for any $R > 0$

$$\limsup_{t\to t_0} E_R(u(\cdot,t);x) \geq \epsilon_1$$

which means x is a singular point, hence a contradiction.

Now, for each $0 < R \leq \rho$, let

$$\phi_R(t) = \sup_{t_1\leq s\leq t\ x\in\overline{B_\rho(x_0)}} E_R(u(\cdot,s);x)$$

$\phi_R(t)$ is continuous on $[t_1, t_0)$ for all fixed $R > 0$ by the uniform continuity of $|\nabla u(y,s)|^2$ on $M \times [t_1, t_0 - \epsilon]$ for any fixed $\epsilon > 0$. Also $\phi_R(t)$ is a nondecreasing function of t by definition. Finally, by the characterization of singularity, for any $0 < R \leq \rho$

$$\lim_{t\to t_0-} \phi_R(t) \geq \limsup_{t\to t_0-} E_R(u(\cdot,t);x_0) > \epsilon_1$$

Now, for each integer m, choose $\delta_m = \frac{\rho_0}{5m}$. Then we can choose $R_m = R(\delta_m) > 0$ and $t_m = t(\delta_m)$ such that for all $x \in \overline{B_\rho(x_0)} - B_{\delta_m}(x_0)$ and $t \in [t_m; t_0)$

$$E_{R_m}(u(\cdot,t);x) < \epsilon_1$$

Since we can freely increase $t_m < t_0$, choose $t_m \to t_0$. By possibly shrinking R_m, we can additionally assume that

$$\phi_{R_m}(t_m) < \epsilon_1$$

Since we can freely shrink R_m, choose R_m so that $R_m \to 0$ as $m \to \infty$. By intermediate value theorem, we can find $\tau_m \in (t_m, t_0)$ so that

$$\phi_{R_m}(t_m) = \epsilon_1$$

Again, by regularity of u, there exists $x_m \in \overline{B_\rho(x_0)}$ so that

$$\sup_{t_1\leq s\leq\tau_m\ x\in\overline{B_\rho(x_0)}} E_{R_m}(u(\cdot,s);x) = \phi_{R_m}(\tau_m) = \epsilon_1 = E_{R_m}(u(\cdot,\tau_m);x_m)$$

By the assumption on R_m, $x_m \in B_{\delta_m}(x_0)$. We redefine $t_m = \tau_m$, then we chose x_m, t_m, R_m so that

$$x_m \to x_0,\ t_m < t_0,\ t_m \to t_0,\ R_m \to 0$$

and

$$\sup_{t_1\leq s\leq t_m\ x\in\overline{B_\rho(x_0)}} E_{R_m}(u(\cdot,s);x) = \phi_{R_m}(t_m) = \epsilon_1 = E_{R_m}(u(\cdot,t_m);x_m)$$

By lemma 3.5 and lemma 3.4, there is some constant $C > 0$ independent of m so that for all $t \in [t_m - CR_m^2, t_m]$

$$E_{2R_m}(u(\cdot,t);x_m) \geq \frac{\epsilon_1}{2}$$

and

$$\int_{t_m - CR_m^2}^{t_m} \int_M |\nabla^2 u|^2 dM(x) dt \le cE_0$$

Also, on $\overline{B_\rho(x_0)} \subset M$, we are using geodesic normal coordinate centered at x_0. This implies that on $B_\rho(0) \subset R^2 = T_{x_0}M$, there are 2 metric tensors, one pulled back from M, and the other g_0, the standard metric on R^2. By compactness of $\overline{B_\rho(x_0)}$, we can find some $0 < C' \le 1$, independent of u, m so that

$$C'g(y) \le g_0(y) \le C'^{-1} g(y)$$

and

$$C'dM(y) \le dy \le C'^{-1} dM(y)$$

for all $y \in \overline{B_\rho(0)}$.

Now, we define $D_m = \{x \in R^2 | 2C'^{-1} R_m x + x_m \in B_\rho(x_0)\}$, where $B_\rho(x_0)$ indicates the image of $B_\rho(x_0) \subset M$ under geodesic normal coordinate centered at x_0, and let $u_m(x,t) = u(x_m + R_m x, t_m + R_m^2 t)$ for all $x \in D_m$ and $-C \le t \le 0$. Then

$$E_{2C'^{-1}}(u_m(\cdot,t);0) = \int_{B_{2C'^{-1}}(0)} R_m^2 |\nabla u(x_m + R_m x, t_m + R_m^2 t)|^2 dx$$

$$= \int_{B_{2R_m C'^{-1}}(x_m)} |\nabla u(x, t_m + R_m^2 t)|^2 dx \ge C' \int_{B_{2R_m}(x_m)} |\nabla u(x, t_m + R_m^2 t)|^2 dM(x)$$

$$= C' E_{2R_m}(u(\cdot, t_m + R_m^2 t); x_m) \ge C' \frac{\epsilon_1}{2} > 0$$

for all $-C \le t \le 0$. Also since for all $x \in D_m$, $-C \le t \le 0$

$$E_{C'}(u_m(\cdot,t);x) = \int_{B_{C'R_m}(x_m + R_m x)} |\nabla u(y, t_m + R_m^2 t)|^2 dy$$

$$\le C'^{-1} \int_{B_{R_m}(x_m + R_m x)} |\nabla u(y, t_m + R_m^2 t)|^2 dM(y)$$

$$= C'^{-1} E_{R_m}(u(\cdot, t_m + R_m^2 t); x_m) \le C'^{-1} \epsilon_1$$

$$\sup_{x \in D_m,\ -C \le t \le 0} E'_C(u_m(\cdot,t);x) \le C'^{-1} \epsilon_1$$

By similar calculations, we also have

$$\int_{-C}^{0} \int_{D_m} |\nabla^2 u_m|^2 dx dt \le K$$

and

$$\int_{-C}^{0} \int_{D_m} |\partial_t u_m|^2 dx dt \to 0$$

as $m \to \infty$. Then, for each m, we can choose $\tau_m \in [-C, 0]$ so that

$$(1). \int_{D_m} |\nabla^2 u_m(x, \tau_m)|^2 dx \le K'$$

$$(2). \int_{D_m} |\partial_t u_m(x, \tau_m)|^2 dx \to 0$$

$$(3). \int_{D_m} |\nabla u_m(x, \tau_m)|^2 dx \le K'$$

$$(4). \int_{B_{2C'^{-1}}(0)} |\nabla u_m(x, \tau_m)|^2 dx \ge c > 0$$

for fixed constants K' and c. (1), (3), together with closedness of M, N implies that $\tilde{u_m}(x) = u_m(x, \tau_m)$ defined on D_m have uniformly bounded $W^{2,2}(M; N)$ norm. Also, since D_m expands to entire R^2 as $m \to \infty$, we conclude that $\tilde{u_m} \to \overline{u_0}$ weakly in $W^{2,2}_{loc}(R^2; N)$, and thus by compactness, converge in $W^{1,2}_{loc}(R^2; N)$.

Now, if we write the heat flow PDE in local coordinate form and send $m \to \infty$, by (2), ∂_t term vanishes. Also, the coefficients from the Laplace-Beltrami operators have the form $a^{ij}(x_m + R_m x)$ for each $x \in R^2$. Thus, as $m \to \infty$, these are converge to $a^{ij}(x_0)$, meaning the Laplace term converges to Laplace-Beltrami operator fixed at x_0, and since we are using geodesic normal coordinate centered at x_0, we conclude that the 2nd order term converges to $\Delta \overline{u_0}$, where Δ is standard Laplacian on R^2. These 2 facts tells us that $\overline{u_0} \in W^{2,2}(R^2; N)$ is a weak harmonic map from (R^2, g_0) to N. By (3), $\overline{u_0}$ has finite energy. Thus $\overline{u_0} \in W^{2,2}(R^2; N)$ is a weakly harmonic map from R^2 to N with finite energy.

Now, we show that it is smooth. $\overline{u_0} \in W^{2,2}(R^2; N) \subset W^{1,p}(R^2; N)$ for all $1 < p < \infty$. Since

$$\Delta \overline{u_0} = trace(\overline{u_0}^* A_N) = F(\overline{u_0})$$

$F(\overline{u_0}) \in L^p(R^2; N)$ for all sufficiently large p. By Calderon-Zygmund theory, this implies that

$$\overline{u_0} \in W^{2,p}(R^2; N)$$

for all large p. Thus $\overline{u_0} \in C^{1,\alpha}(R^2; N)$ for all $0 < \alpha < 1$, and from here, by Schauder theory, $\overline{u_0} \in C^\infty(R^2; N)$. Thus $\overline{u_0}$ is smooth harmonic map with finite energy. Thus, by the work of Sacks and Uhlenbeck, $\overline{u_0}$ can be extended to harmonic map from S^2 to N. By (4), $\overline{u_0}$ is a nontrivial harmonic map, implying the extension is a bubble.

Remark 3.2 *Theorem 3.3 on bubble formation also holds for* $t_0 = \infty$. *Since* $E(u(\cdot, t)) \le E_0 < \infty$ *for all* $t > 0$ *and*

$$\int_0^\infty \int_M |\partial_t u|^2 dM(x) dt \le E_0 < \infty$$

we can always find a sequence $t_n \to \infty$ so that

$$\|\partial_t u(t_n)\|_{L^2(M)} \to 0$$

and

$$u(t_n) \to u_\infty$$

weakly in $W^{1,2}(M;N)$. Then for each $x_0 \in M$, we ask if there is some $R > 0$ so that

$$\limsup_{t\to\infty} E_R(u(\cdot,t);x_0) \le \epsilon_1 \cdots (*)$$

Again, since $E_0 < \infty$, there are at most finitely many such points in M.

If $*$ holds for x_0, then we can a uniform control on the derivatives of $u(\cdot,t_n)$ near x_0, thus the convergence is in fact strong. This strong convergence away from finitely many points implies that u_∞ is smooth and is harmonic away from the singular points. Thus, by extending u_∞ smoothly to entire M, u_∞ is a smooth harmonic map from M to N.

On the other hand, if $*$ fails, then we can repeat the proof of theorem 3.3 to show that bubbles form centered at x_0. Therefore, the formation of bubbles are responsible for the failure of strong convergence in $W^{1,2}(M;N)$. Following works have shown that the formation of such bubbles (possibly more than one for each blow up point) is the only obstruction of strong convergence [5].

4 Finite time blow-up

By the work of Struwe, we now know that when M is a surface, then a unique global weak solution exists for any regular initial data. Then, the next question would be is this global weak solution actually a global smooth solution. The following work of Chang-Ding-Ye shows us that this is not the case.

They considered the evolution problem of harmonic map from D^2 to S^2, both with standard metric, with Dirichlet boundary condition. Chang-Ding-Ye [9] proved that under certain conditions on the initial data, the solution of the heat flow $u(\cdot,t)$ forms singularity in finite time.

Theorem 4.1 (Chang-Ding-Ye, 1992) *Let $u(x,t) : D^2 \times [0,T_{\max}) \to S^2$ be a smooth unique solution of*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2 u \\ u(0,x) = u_0(x) \\ u(t,\cdot)|_{\partial D^2} = u_0(\cdot)|_{\partial D^2} \end{cases} \tag{2}$$

where $u_0 \in C^1(D^2)$. We additionally assume that

$$u_0(x) = \left(\frac{x}{|x|}\sin h_0(|x|), \cos h_0(|x|)\right)$$

where $h_0 \in C^1([0,1])$ *with* $h_0(0) = 0$*. Then for certain initial conditions,* $T_{\max} < \infty$ *and*

$$\limsup_{t \to T_{\max}-} |\nabla u(0,t)|^2 = \infty$$

The idea is to construct a family of subsolutions which exhibits finite time blow up, and use maximum principle. We remark that R.Hamilton proved short time existence for HMHF with Dirichlet boundary condition [11].

We first show that the solution of HMHF also possesses the symmetry the initial data has.

Lemma 4.1 *If* $u(x,t)$ *is the solution of the heat flow from* D^2 *to* S^2 *with Dirichlet boundary condition and initial condition*

$$u_0(x) = \left(\frac{x}{|x|}\sin h_0(|x|), \cos h_0(|x|)\right)$$

where $h_0 \in C^1([0,1])$ *with* $h_0(0) = 0$*, then*

$$u(x,t) = \left(\frac{x}{|x|}\sin h(t,|x|), \cos h(t,|x|)\right)$$

with h *satisfying*

$$\begin{cases} \frac{\partial h}{\partial t} = h_{rr} + \frac{u_r}{r} - \frac{\sin h \cos h}{r^2} = \tau(h) \\ h(0,r) = h_0(r) \\ h(t,0) = h_0(0) = 0 \ , \ h(t,1) = h_0(1) \end{cases} \tag{3}$$

Proof: For any $\theta_0 \in [0, 2\pi]$ consider rotation along z axis by θ_0 angle R_{θ_0}. Note that by the particular structure of u_0, both $u(r, \theta + \theta_0, t)$ and $R_{\theta_0}u(r,\theta,t)$ are solution to the initial-boundary value problem with initial condition

$$u_{0,\theta_0}(r,\theta) = u_0(r, \theta + \theta_0) = R_{\theta_0}u(r,\theta)$$

The uniqueness of HMHF implies that

$$u(r, \theta + \theta_0, t) = R_{\theta_0}u(r,\theta,t)$$

Then, by letting $\theta = 0$, $\theta_0 = \theta$ we have

$$u(r,\theta,t) = R_\theta u(r,0,t)$$

We now determine $u(r,0,t)$. We can write

$$u(r,0,t) = (\sin h(r,t)\cos\nu(r,t), \sin h(r,t)\sin\nu(r,t), \cos h(r,t))$$

Therefore, if we can show that $\nu = 0$, then lemma 4.1 is proved. By plugging in our new expression of $u(r,\theta,t)$ into the HMHF equation, we can obtain a system of PDE h and ν satisfy. Then, we can use maximum principle to show that $\nu = 0$. Details can be found in [12]

Proof of theorem 4.1: By above lemma, the blow up of u is equivalent to the blow up of h, namely the maximal time T such that

$$h \in C^0([0,T) \times [0,1]) \cap C^\infty((0,T) \times [0,1])$$

We pose several conditions on h_0. We first assume that

$$h_0(1) = b > \pi$$

Now, we will construct a subsolutions of (2). Notice that all solutions of

$$\tau(\phi) = 0$$

with $\phi(0) = 0, \phi(r) > 0$ for $r > 0$ is given by

$$\phi(r,\mu) = \arccos\left(\frac{\mu^2 - r^2}{\mu^2 + r^2}\right)$$

for each $\mu > 0$. For any $0 < \epsilon < 1$, set $a = 1 + \epsilon$ and define

$$\theta(r) = \phi(r^a, \mu)$$

Choose $\mu = \mu(\epsilon)$ so large that

$$\cos\theta(r) > \frac{1}{1+\epsilon}$$

$$\theta(1) \leq b - \pi$$

These two conditions can both be satisfied because for any $0 \leq r \leq 1$

$$\lim_{\mu\to\infty} \phi(r,\mu) = 0$$

Next, we consider $\lambda(t)$ which satisfies the following initial value problem

$$\lambda'(t) = -\delta\lambda^\epsilon(t) \quad \lambda(0) = 1$$

where $\delta > 0$. We will now choose $\delta > 0$ so that

$$f(r,t) = \phi(r,\lambda(t)) + \theta(r)$$

is a subsolution of the original problem. Note that

$$\frac{\partial f}{\partial t} - \tau(f_t) \leq \frac{2\delta\lambda^\epsilon r}{\lambda^2 + r^2} - \frac{2\mu\epsilon r^{\epsilon-1}}{\mu^2 + 1}$$

Set $s = \frac{r}{\lambda} \in R^+$, then above is equivalent to

$$\delta\frac{s^{2-\epsilon}}{1+s^2} \leq \frac{2\mu\epsilon}{2\mu^2 + 2}$$

Note that $\frac{s^{2-\epsilon}}{1+s^2}$, viewed as a function of $s \in R^+$ has a maximum on R^+, say $M(\epsilon)$. Then if

$$\delta \leq \frac{2\mu\epsilon}{2\mu^2+2}\frac{1}{M(\epsilon)}$$

then f is a subsolution.

Now, we additionally assume $h_0(0,r) \leq f(0,r)$ for all $r \in [0,1]$. Also, since

$$h(t,0) = 0 = f(t,0) \quad h(t,1) = b > \pi \geq f(t,1)$$

$h \leq f$ on the parabolic boundary of $[0,T) \times [0,1]$. Now, we need a comparison principle.

Lemma 4.2 *Let $f, g \in C^0([0,1] \times [0,T_{max})) \bigcap C^\infty((0,1) \times (0,T_{max}))$ be super / subsolutions of*

$$\frac{\partial h}{\partial t} = h_{rr} + \frac{u_r}{r} - \frac{\sin h \cos h}{r^2}$$

respectively. If $f(0,t) = g(0,t) = 0$, $f \geq g$ on the parabolic boundary of $\Omega = [0,1] \times [0,T_{max})$ then

$$f \geq g$$

on Ω.

Proof: We argue by contradiction. Assume at $(p,T) \in (0,1) \times (0,T_{max})$, $f < g$. Let $h = f - g$. Then by subtracting the differential inequalities, we have

$$\frac{\partial h}{\partial t} \geq h_{rr} + \frac{h_r}{r} - \frac{\sin f \cos f - \sin g \cos g}{r^2}$$

$$\sin f \cos f - \sin g \cos g = (\int_0^1 \cos^2(sf + (1-s)g) - \sin^2(sf + (1-s)g) ds)h$$
$$= p(f,g)h$$

Choose any $T < T_{max}$. Then since $f, g \in C^0([0,1] \times [0,T])$ and $f(0,t) = g(0,t) = 0$, there exists some $\delta = \delta(T, f, g) > 0$ such that

$$f, g \in [-\frac{\pi}{8}, \frac{\pi}{8}]$$

for all $(x,t) \in [0,\delta] \times [0,T]$. This implies that

$$p > 0$$

on $[0,\delta] \times [0,T]$. Also, since p is continuous on $[\delta,1] \times [0,T]$, we conclude that there exists $K > 0$ such that

$$p > -K$$

on $[0,1] \times [0,T]$. We define

$$\hat{h} = e^{Kt}h$$

Then $\hat{h}$ satisfies

$$\frac{\partial \hat{h}}{\partial t} - \hat{h}_{rr} - \frac{\hat{h}_r}{r} + (p+K)\hat{h} \geq 0$$

We will show that $\hat{h} \geq 0$ on $[0,1] \times [0,T]$, contradicting the assumption.

For $\epsilon > 0$, let

$$\phi^\epsilon(x,t) = \hat{h} + \epsilon e^t$$

Then

$$\frac{\partial \phi^\epsilon}{\partial t} - \phi^\epsilon_{rr} - \frac{\phi^\epsilon}{r} + (p+K)\phi^\epsilon \geq \epsilon e^t > 0$$

Also, by the assumptions on the boundary value, $\phi^\epsilon \geq 0$ on the boundary. Assume ϕ^ϵ obtains strictly negative minimum value in $(0,1) \times (0,T]$. Then this is a contradiction because at that point, above inequality cannot hold. Thus

$$\hat{h} \geq -\epsilon e^T$$

Since $\epsilon > 0$ is arbitrary, we have $\hat{h} \geq 0$ i.e $h \geq 0$ on $[0,1] \times [0,T]$, which is a contradiction. Thus

$$f \geq g$$

on Ω, completing the proof.

By applying the comparision principle, we have

$$h \leq f \quad \text{in } [0,T) \times [0,1]$$

Since $f(t,0) = h(t,0)$, this means

$$|f_r(t,0)|^2 \leq |h_r(t,0)|^2$$

for all $t \in [0,T)$. Note that

$$f_r(t,0) = \frac{2}{\lambda(t)}$$

and by solving the ODE

$$\lambda(t) = [1 - (1-\epsilon)\delta t]^{\frac{1}{1-\epsilon}}$$

Let $T_0 = \frac{1}{(1-\epsilon)\delta} < \infty$, then

$$\lim_{t \to T_0} |f_r(t,0)|^2 = \infty$$

This means

$$\lim_{t \to T_0} |h_r(t,0)|^2 = \infty$$

as well, meaning $T \leq T_0 < \infty$, i.e h, hence u formulates energy concentration at the origin and thus blows up in finite time.

Remark 4.1 *By a more thorough analysis, one can show that the 2nd condition on h_0 is superfluous, hence the HMHF always blows up in finite time whenever $h_0(1) > \pi$. Chang also proved that if conversely $h_0(1) \leq \pi$, then the HMHF has smooth global solution. He also showed that if $h_0(1) = \pi$, then the smooth global solution does not converge strongly to a harmonic map [12].*

5 Higher dimensional case

We now consider when $M = R^m$ for $m > 2$. By using a monotonicity formula adapted for Harmonic map heat flow, Struwe was able to prove a ϵ regularity theorem. This theorem asserts that when the weighted energy is small enough, one obtains an upper bound for energy density, which in turn, gives uniform bound on the derivatives of smooth solution u.

As a direct application of the monotonicity formula and ϵ regularity theorem, one can establish a partial regularity result which estimates the size of singular point set. We will also show that if the initial data $u_0 \in W^{1,2}(M; N)$ has 'small energy', then there is a unique global smooth solution with initial data u_0 which converges to a constant function as $t \to \infty$.

As usual, we assume $N \subset R^L$, by Nash's embedding theorem. We first fix some notations.

Definition 5.1 *For $z_0 = (x_0, t_0) \in R^m \times R$, and $r > 0$, let*

$$P_r(z_0) = \{(x,t) | |x - x_0| < r, |t_0 - t| < r^2\}$$

$$Q_r(z_0) = \{(x,t) | |x - x_0| < r, t_0 - r^2 < t \leq t_0\}$$

$$S_r(z_0) = R^m \times \{t = t_0 - r^2\}$$

$$T_r(z_0) = R^m \times \{t_0 - 4r^2 < t < t_0 - r^2\}$$

$$G_{z_0}(x,t) = (4\pi(t_0 - t))^{-\frac{m}{2}} \exp(-\frac{|x - x_0|^2}{4(t_0 - t)}) \text{ when } t < t_0$$

Monotonicity formula

Let $u : R^m \times [0,T] \to N \subset R^L$ be smooth solution of HMHF with bounded $u, \nabla u$ and $E(u(t)) = \frac{1}{2} \int_{R^m} |\nabla u|^2(x,t) dx$. Note that this combined with parabolic theory implies that all higher derivatives of u are uniformly bounded. For convenience, we will extend u globally in time by letting $\partial_t u - \Delta u = 0$ for $t > T$. Then by the maximum principle, the uniform bounds on $u, \nabla u$ are preserved after the extension. For any $z_0 = (x_0, t_0) \in R^m \times R_+$, define

$$\Phi(z_0, R; u) = \frac{1}{2} R^2 \int_{S_R(z_0)} |\nabla u|^2 G_{z_0} dx$$

and

$$\Psi(z_0, R; u) = \int_{T_R(z_0)} |\nabla u|^2 G_{z_0} dxdt$$

for $0 < R \leq R_0 = \sqrt{t_0}$.

Notice that both HMHF and standard heat equation have rescaling invariance by $u_R(x,t) = u(Rx, R^2 t)$ for $R > 0$ and translation invariance $u_{z_0}(x,t) = u(x+x_0, t+t_0)$. We first show that above weighted energies are invariant under rescaling and translation. By change of variable $x' = x - x_0, t' = t - t_0$

$$\Phi(z_0, R; u) = \frac{1}{2}R^2 \int_{S_R(z_0)} |\nabla u|^2 G_{z_0} dx = \frac{1}{2}R^2 \int_{S_R(0)} |\nabla u_{z_0}|^2 G_0 dx' = \Phi(0, R; u_{z_0})$$

Again by change of variable $x'' = Rx', t'' = R^2 t'$

$$\frac{1}{2}R^2 \int_{S_R(0)} |\nabla u_{z_0}|^2 G_0 dx' = \frac{1}{2} \int_{S_1(0)} |\nabla (u_{z_0})_R|^2 G_0 dx'' = \Phi(0, 1; (u_{z_0})_R)$$

Therefore, Φ has translation and rescaling invariance. For Ψ, note that by change of variable $t = t_0 - r^2$

$$\Psi(z_0, R; u) = \int_{T_R(z_0)} |\nabla u|^2 G_{z_0} dxdt = 4 \int_R^{2R} \Phi(z_0, r; u) \frac{dr}{r}$$

Therefore

$$4 \int_R^{2R} \Phi(z_0, r; u) \frac{dr}{r} = 4 \int_R^{2R} \Phi(0, r; u_{z_0}) \frac{dr}{r} = \Psi(0, R; u_{z_0})$$

and by change of variable $r = Rr'$

$$4 \int_R^{2R} \Phi(0, r; u_{z_0}) \frac{dr}{r} = 4 \int_1^{2} \Phi(0, r'; (u_{z_0})_R) \frac{dr'}{r'} = \Psi(0, 1; (u_{z_0})_R)$$

Therefore, Ψ also has translation and rescaling invariance.

We now prove that both Φ and Ψ are nondecreasing w.r.t R.

Lemma 5.1 (Monotonicity formula) *$\Phi(z_0, R; u)$ and $\Psi(z_0, R; u)$ are non-decreasing w.r.t R, as long as the weighted energies are defined.*

Proof: By translation invariance, we may assume $z_0 = 0$ and let $u : R^m \times [-t_0, \infty) \to R^L$. For $0 < R \leq \sqrt{t_0}$, $\Phi(0, R; u) = \Phi(0, 1; u_R)$. Smoothness and boundedness of u and exponential decay implies that $\Phi(0, 1; u_R)$ is differentiable w.r.t R. By rescaling invariance, for any $0 < R_1 < \sqrt{t_0}$

$$\frac{d}{dR}\Phi(0, R; u)|_{R=R_1} = \frac{d}{dR}\Phi(0, R/R_1; u_{R_1})|_{R=R_1} = \frac{1}{R_1}\frac{d}{dR}\Phi(0, R; u_{R_1})|_{R=1}$$

Therefore, if we show that

$$\frac{d}{dR}\Phi(0,R;u)|_{R=1} \geq 0$$

for any u, then $\frac{d}{dR}\Phi(0,R;u)| \geq 0$ for all possible R.

$$\frac{d}{dR}\Phi(0,R;u)|_{R=1} = \frac{d}{dR}\Phi(0,1;u_R)|_{R=1} = \frac{1}{2}\int_{S_1(0)} \frac{d}{dR}|\nabla u_R|^2 G dx$$

$$= \frac{1}{2}\int_{R^m} 2\nabla(u_R)\cdot\nabla(\frac{d}{dR}u_R)G(x,-1)dx|_{R=1}$$

$$= \int_{R^m} \nabla u\cdot\nabla(\nabla u\cdot x - 2\frac{\partial}{\partial t}u)G(x,-1)dx$$

$$= -\int_{R^m} \Delta u\cdot(\nabla u\cdot x - 2\frac{\partial}{\partial t}u)G - \int_{R^m}\nabla u(\nabla u\cdot x - 2\frac{\partial}{\partial t}u)\nabla G dx$$

Now, recall that u either solves HMHF or standard heat equation in R^m. If u solves $HMHF$, then $\nabla u\cdot x - 2\frac{\partial}{\partial t}u$ is tangent to N. Then because $(\partial_t - \Delta)u$ is normal to N, we can replace Δu with $\partial_t u$. If u solves standard heat equation, then above replacement is trivial. Since $\nabla G(x,-1) = \frac{x}{2}G(x,-1)$ our integral becomes

$$\int_{R^m} 2G|\nabla u\cdot\frac{x}{2} - \frac{\partial}{\partial t}u|^2 dx \geq 0$$

Thus $\frac{d}{dR}\Phi(0,R;u)|_{R=1} \geq 0$, establishing the monotonicity for Φ.

For Ψ, for $0 < R_0 \leq R_1 \leq \frac{\sqrt{t_0}}{2}$

$$\Psi(z_0,R_0;u) = 4\int_{R_0}^{2R_0}\frac{1}{r}\Phi(z_0,r;u)dr = 4\int_{R_1}^{2R_1}\frac{1}{r}\Phi(z_0,r\frac{R_0}{R_1};u)dr$$

$$\leq 4\int_{R_1}^{2R_1}\frac{1}{r}\Phi(z_0,r;u)dr = \Psi(z_0,R_1;u)$$

with the last inequality due to $\frac{R_0}{R_1} \leq 1$ and monotonicity of Φ.

Now, we prove ϵ-regularity theorem, which allows us to gain uniform bound on $|Du|$ given that the weighted energy is sufficiently small.

Theorem 5.1 (ϵ-regularity theorem) *There is some $\epsilon_0 = \epsilon_0(N,m) > 0$, $c = c(N,m) > 0$, $\delta = \delta(N,m,E_0,\inf(R,1)) > 0$ such that whenever a smooth solution $u : R^m \times [-4R_0^2, 0] \to N$ with $E(u(t)) \leq E_0$ satisfies*

$$\Psi(0,R;u) < \epsilon_0 \textit{ for some } 0 < R \leq R_0$$

then

$$\sup_{Q_{\delta R}} |\nabla u|^2 \leq c(\delta R)^{-2}$$

Proof: We again extend u to positive time by letting it solve standard heat equation. If u is constant, then the conclusion holds trivially, hence we assume u is not constant. Define

$$f(\sigma) = (2\delta R - \sigma)^2 \sup_{\overline{Q_{\sigma R}}} |\nabla u|^2$$

for $\sigma \in [0, 2\delta R]$, where $\delta \in (0, \frac{1}{4})$ will be determined. By smoothness of u, f is continuous and $f(0) = f(2\delta R) = 0$. Since u is nonconstant, there is $\sigma_0 \in (0, 2\delta R)$ so that $\max_{\sigma \in [0,2\delta R]} f(\sigma) = f(\sigma_0)$. We define $e_0 = \sup_{\overline{Q_{\sigma_0}}} |\nabla u|^2$. Then $f(\sigma) \leq (2R\delta - \sigma_0)^2 e_0$ for $0 \leq \sigma \leq 2\delta R$.

If $(2\delta R - \sigma_0)^2 e_0 \leq 4$, by letting $\sigma = R\delta$

$$f(\delta R) = R^2 \delta^2 \sup_{\overline{P_{\delta R}}} |\nabla u|^4 \leq 4$$

thus the conclusion holds.

If $(2\delta R - \sigma_0)^2 e_0 > 4$, let $z_0 = (x_0, t_0) \in \overline{Q_{\sigma_0}}$ so that $e(u)(z_0) = e_0$. We define on $Q_1(0)$

$$v(x,t) = u(x_0 + \frac{x}{\sqrt{e_0}}, t_0 + \frac{t}{e_0})$$

Note that since $(2\delta R - \sigma_0)^2 e_0 > 4$, above function is well defined. By the translation and rescaling invariance of HMHF, v also solves HMHF in $Q_1(0)$. Also, $e(v)(x,t) = \frac{1}{e_0} e(u)(x_0 + \frac{x}{\sqrt{e_0}}, t_0 + \frac{t}{e_0})$. Recall that $e(u)$ satisfies the following Bochner type formula

$$(\partial_t - \Delta) e(u) = -\|\nabla Du\|^2 - \langle Du(Ric^M(e_\alpha), Du(e_\alpha)\rangle$$
$$+ R^N(u)(Du(e_\alpha), Du(e_\beta), Du(e_\beta), Du(e_\alpha))$$

Since $M = R^m$, the Ricci curvature term vanishes, and since N is closed, the Riemann curvature term is bounded by $Ce(u)^2$ for some C depending only on N. Hence, $e(u)$ satisfies

$$(\partial_t - \Delta) e(u) \leq C(N) e(u)^2$$

Thus

$$(\partial_t - \Delta) e(v) \leq C(N) e(v)^2$$

holds in $Q_1(0)$ as well. Just like in $dim(M) = 2$ case,

$$e(v)(0,0) = 1$$

and

$$\sup_P e(v) \leq 4$$

Thus we have

$$(\partial_t - \Delta)e(v) \le 4C(N)e(v) = Ce(v)$$

By parabolic Moser iteration, and change of variable

$$e(v)(0,0) = 1 \le C \int_{Q_1(0)} e(v)dxdt \le Ce_0^{\frac{m}{2}} \int_{Q_{1/\sqrt{e_0}}(z_0)} e(u)dxdt$$

$$\le Ce_0^{\frac{m}{2}} \int_{P_{1/\sqrt{e_0}}(z_0)} |\nabla u|^2 dxdt$$

Our objective is to make $Ce_0^{\frac{m}{2}} \int_{P_{1/\sqrt{e_0}}(z_0)} e(u)dxdt < 1$ so that we can rule out $(2\delta R - \sigma_0)^2 e_0 > 4$ case.

Put $\rho_0 = 1/\sqrt{e_0}$. Then first note that $\rho_0 < R\delta - \frac{\sigma_0}{2}$, thus $\rho_0 + \sigma_0 < 2R\delta$. We will show that for the extended u, we can find $\delta = \delta(N, m, E_0, inf\{R, 1\})$ so that whenever $r + \sigma < 2\delta R$ with $z_0 \in P_\sigma(0)$

$$r^m \int_{P_r(z_0)} |\nabla u|^2 dxdt \le c\epsilon_0 + \frac{1}{2}$$

where c is a generic constant. If we take $c\epsilon_0 < \frac{1}{4}$, above inequality applied to $\sigma = \sigma_0, r = \rho_0$ implies that

$$Ce_0^{\frac{m}{2}} \int_{P_{1/\sqrt{e_0}}(z_0)} e(u)dxdt < \frac{3}{4}$$

which is the desired result.

The idea is to estimate above quantity by

$$\int_{T_r(t_0+2r^2)} |\nabla u|^2 G_{(x_0,t_0+2r^2)} dxdt$$

By monotonicity formula and the fact that $t_0 + 2r^2$ is small compared to R, above can be majorized to

$$C[\int_{T_R(0)} |\nabla u|^2 G_{(x_0,t_0+2r^2)} dxdt + \int_{T_{\frac{R}{2}}(0)} |\nabla u|^2 G_{(x_0,t_0+2r^2)} dxdt]$$

Direct computation shows us that withen $T_R \cup T_{R/2}$,

$$G_{(x_0,t_0+2r^2)(x,t)} \le CG(x,t) + CR^{-2} \exp\left((2-m)\ln R - c\delta^{-2}\right)$$

Let $\epsilon > 0$ be given. Then if $R > 1$, take $\exp(-c\delta^{-2}) < \epsilon$ and if $R \le 1$, take $\exp\left((2-m)\ln R - c\delta^{-2}\right) < \epsilon$. Since $m > 2$, in any case, we have

$$\exp\left((2-m)\ln R - c\delta^{-2}\right) < \epsilon$$

Then by applying this condition to above inequality, and replacing $G_{(x_0,t_0+2r^2)}$ in the integrals with the RHS, we obtain

$$C[\int_{T_R(0)} |\nabla u|^2 G_{(x_0,t_0+2r^2)} dxdt + \int_{T_{\frac{R}{2}}(0)} |\nabla u|^2 G_{(x_0,t_0+2r^2)} dxdt]$$

$$\leq C[\Psi(0,R;u) + \Psi(0,R/2;u)] + C\epsilon E_0 \leq 2C\Psi(0,R;u) + C\epsilon E_0$$

where the last inequality is due to monotonicity formula. Therefore, by letting $C\epsilon E_0 < \frac{1}{2}$ and choosing δ accordingly, we have

$$r^m \int_{P_r(z_0)} |\nabla u|^2 dxdt \leq c\epsilon_0 + \frac{1}{2}$$

which completes the proof.

Several consequences

Since a local bound on $|\nabla u|$ implies local smoothness by parabolic regularity theory, the ϵ regularity theorem can be used as a criteria to rule out singular points. As a result, we can estimate the size of singular point set.

Theorem 5.2 (Partial regularity) *Suppose $u : R^m \times R_+ \to N \subset R^L$ is a limit of a sequence of regular solutions of u_k with uniformly finite energy*

$$E(u_k(t)) \leq E_0 < \infty$$

in the sense that $E(u(t)) \leq E_0$ a.e and $\nabla u_k \to \nabla u$ weakly in $L^2(Q)$ for any compact $Q \subset R^m \times R_+$. If we let S be points where u is not smooth, then S has locally finite m dimensional Hausdorff measure w.r.t parabolic metric. Also, there exists $t_0 = t_0(m, N, E_0)$ such that after t_0, u is fully smooth and converges to a constant map in C^1_{loc} as $t \to \infty$.

Before going into the proof, we first define parabolic Hausdorff measure of dimension $\alpha \geq 0$.

Definition 5.2 *Let $S \subset R^m \times R_+$. For each $\delta > 0$, we define*

$$H^\alpha_\delta(S) = \inf\{\sum_{i=1}^{\infty} r_i^m \mid S \subset \bigcup_{i=1}^{\infty} P_{r_i}(z_i), r_i \leq \delta\}$$

Then, the α dimensional parabolic Hausdorff measure is

$$H^\alpha(S) = \lim_{\delta \to 0} H^\alpha_\delta(S)$$

Above definition makes sense because $H^\alpha_\delta(S)$ is nondecreasing as $\delta \to 0$, so the limit exists (possibly ∞). Notice that if S has finite m dimensional Hausdorff measure, then S has $m+1$ Lebesgue measure 0. Therefore theorem 5.2 implies that u is smooth a.e w.r.t Lebesgue measure on $R^m \times R_+$. Proof of theorem 5.2: We define

$$\Sigma = \bigcap_{R>0} \{z_0 \mid \liminf_{k\to\infty} \int_{T_R(z_0)} |\nabla u_k|^2 G_{z_0} dxdt \geq \epsilon_0\}$$

The ϵ regularity theorem asserts that if $z_0 \notin \Sigma$, then we can find some small parabolic cylinder centered at z_0 independent of k so that $|\nabla u_k|$ is uniformly bounded. Then by Schauder theory, after shrinking the cylinder, we obtain a uniform local $C^{2+\alpha}$ bound for u_k. By extracting subsequence, we can assume that $u_k \to v$ in $C^{2+\alpha}(Q)$ where Q is the small cylinder centered at z_0. Since $\nabla u_k \to \nabla u$ weakly in L^2, we see that $u = v \in C^{2+\alpha}(Q)$. Since the PDE passes through the limit under this convergence, we see that u also solves HMHF in the cylinder. Then u is smooth and its derivatives are bounded as well. This implies that S is contained in Σ.

Therefore, we may estimate size of Σ instead. We take any compact subset of Σ, say $\tilde{\Sigma}$. For any $z_0 \in \tilde{\Sigma}$, any $0 < R \leq \frac{\sqrt{t_0}}{2}$, we can choose infinite k so that

$$\int_{T_R(z_0)} |\nabla u_k|^2 G_{z_0} dxdt \geq \epsilon_0$$

For any $\epsilon > 0$, we can find $C(\epsilon)$ so that for any $(x,t) \in T_R(z_0)$, if $|x - x_0| \geq C(\epsilon)R$, then $G_{z_0}(x,t) \leq cG_{z_0+(0,R^2)}(x,t)$. Since $G_{z_0}(x,t) \leq CR^{-m}$ for all $T_R(z_0)$, we have

$$\epsilon_0 \leq \int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0} dxdt \leq cR^{-m} \int_{P_{C(\epsilon)R}(z_0)} |\nabla u_k|^2 dxdt$$
$$+\epsilon \int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0+(0,R^2)} dxdt$$

We now estimate the integral on far right. Since for $t_0 - 4R^2 \leq t \leq t_0 - R^2$, $G_{z_0+(0,R^2)}(x,t) \leq C\frac{1}{(t_0+R^2-t)^{m/2}} \leq C\frac{1}{t_0^{m/2}}$,

$$\epsilon \int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0+(0,R^2)} dxdt \leq C\epsilon \frac{1}{t_0^{m/2-1}} E_0$$

Since $\tilde{\Sigma}$ is compact, all t_0 are bounded away from 0, hence by taking $\epsilon > 0$ small enough, we can make the far right integral less than $\frac{1}{2}\epsilon_0$.

Therefore, if we choose $R > 0$ small enough so that $P_{C(\epsilon)R}(z_0) \subset R^m \times R_+$, then we have a cover of $\tilde{\Sigma}$ by $\{P_{C(\epsilon)R}(z_0)\}_{z_0 \in \tilde{\Sigma}}$. Since $C(\epsilon)R$ are uniformly bounded, by Vitali covering lemma for parabolic cylinders, we can extract countable subfamily of mutually disjoint cylinders such that 5 times enlarged cylinders cover $\tilde{\Sigma}$. Note that our estimate gives

$$(C(\epsilon)R)^m \leq C\epsilon_0^{-1} \int_{P_{C(\epsilon)R}(z_0)} |\nabla u_k|^2 dxdt$$

The mutual disjointness allows us to convert the sum of the integrals as integral over $\bigcup P_{C(\epsilon)R}(z_0)$, giving

$$\sum R_i^m \leq C\epsilon_0^{-1} \sum \int_{P_{R_i}(z_i)} |\nabla u_k|^2 dxdt = C\epsilon_0^{-1} \int_{\cup P_{R_i}(z_i)} |\nabla u_k|^2 dxdt \leq C(E_0)\epsilon_0^{-1}$$

Therefore

$$H^m(\tilde{\Sigma}) \leq \sum 5^m R_i^m \leq C(E_0, m, \epsilon_0) < \infty$$

Therefore, Σ has locally finite m dimensional Hausdorff measure, hence so does S.

Now, we show that if t_0 is large enough, u is fully regular. This can be seen if $\Sigma \cap \{z|t > t_0\} = \emptyset$. Choose $R > 0$ so that $4R^2 \leq t_0$. Then

$$\int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0} dxdt \leq CR^{2-m} E_0 \leq Ct_0^{2-m/2} E_0$$

Since $2-m < 0$, choosing t_0 large enough makes the RHS smaller than ϵ_0. Thus $z_0 \notin \Sigma$. Thus u is smooth after t_0. Also, the ϵ regularity theorem states that $|\nabla u| \leq C(t-t_0)^{-1/2} \to 0$ as $t \to \infty$. This shows that u converges to a constant map as $t \to \infty$, thus completing the proof.

Another application of the ϵ regularity theorem is that for initial data $u_0 \in W^{1,2}(R^m; N)$ with sufficiently small energy, the Cauchy problem for HMHF has global smooth solution converging to a constant map as $t \to \infty$.

Theorem 5.3 (Struwe, 1989) *There exists $\epsilon_1 > 0$ depending on C', N, m such that for any $u_0 \in W^{1,2}(R^m; N) \cap W^{1,\infty}(R^m; N)$ with $\|\nabla u_0\|_{L^\infty(M)} \leq C'$, $E_0 < \epsilon_1$, there exists a unique $u \in C^\infty(R^m \times R^+; N) \cap C^0(R^m \times [0, \infty); N)$ such that it is the solution of the heat flow with initial data u_0, which converges to a constant map as $t \to \infty$.*

Similar to $dim(M) = 2$ case, the strategy is to construct an approximating sequence $u_{m0} \in C^\infty(R^m; N)$ by using mollifier so that $u_{m0} \to u_0$ in $W^{1,2}(R^m; N) \cap C^0(R^m; N)$. However, unlike $dim(M) = 2$ case, set of smooth functions in general is not dense inside the Sobolev space. This is why we require the initial energy bound.

By direct calculation, $E(u_{m0}) \leq cE_0$ and $\|\nabla u_{m0}\|_{L^\infty} \leq cC'$ with $c = c(N)$. Note that although R^m is not compact, by using the same method using Euclidean heat kernel as in Eells and Sampson, one can show that for each u_{n0}, smooth local solutions converging to u_{n0} in $C^1(R^m; N)$ as $t \to 0$ exists. Denote them by u_n, then we claim that they are in fact global smooth solutions. This can be achieved by obtaining a uniform bound on $|\nabla u_n|$ using the ϵ-regularity theorem. To do so, we first show that for sufficiently small $\epsilon_1 > 0$

$$\sup_{R>0} R^2 \int_{R^m} |\nabla u_{n0}|^2 G_{(x_0, R^2)}(x, 0) dx < \frac{\epsilon_0}{2}$$

for all $x_0 \in R^m$. For $0 < R < e^{-m} < 1$,

$$G_{(x_0,R^2)}(x,0) \leq \begin{cases} CR^{-m} & |x-x_0| \leq 2R|\ln R| \\ CR^{-m+|\ln R|} & |x-x_0| > 2R|\ln R| \end{cases} \tag{4}$$

Thus by spliting R^m into $B_{2R|\ln R|}(x_0)$ and the rest, and applying above estimates, we have

$$\int_{R^m} |\nabla u_{n0}|^2 G_{(x_0,R^2)} dx \leq C|\ln R|^m \|\nabla u_0\|_{L^\infty} + CR^{-m+|\ln R|}E_0$$

$$\leq C|\ln R|^m \|\nabla u_0\|_{L^\infty} + CE_0$$

where the last inequality is by $0 < R < e^{-m}$. Since

$$C|\ln R|^m \|\nabla u_0\|_{L^\infty} + CE_0 \leq CC'|\ln R|^m + C\epsilon_1$$

By choosing $R_1 = R_1(C', N, m) \leq e^{-m}$ so that

$$CC'|\ln R|^m \leq \frac{\epsilon_0}{4}R^{-2}$$

for all $0 < R \leq R_1$ and $\epsilon_1 < \frac{\epsilon_0}{4C}$, we have

$$C|\ln R|^m + C\epsilon_1 \leq \frac{\epsilon_0}{4}R^{-2} + \frac{\epsilon_0}{4} = \frac{\epsilon_0}{2}R^{-2}$$

for all $0 < R \leq R_1$. If $R > R_1$, then

$$\int_{R^m} |\nabla u_{n0}|^2 G_{(x_0,R^2)}(x,0)dx \leq \epsilon_1 CR^{-m} = \epsilon_1 CR_1^{-m+2}R^{-2}$$

Hence, if we choose $\epsilon_1 CR_1^{-m+2} < \frac{\epsilon_0}{2}$, then we again have

$$\int_{R^m} |\nabla u_{n0}|^2 G_{(x_0,R^2)}(x,0)dx \leq \frac{\epsilon_0}{2}R^{-2}$$

Note that we imposed 2 conditions on ϵ_1 and these conditions only rely on N, m, C', thus so is $\epsilon_1 > 0$.

Since for all n

$$R^2 \int_{R^m} |\nabla u_{n0}|^2 G_{(x_0,R^2)} dx \leq \frac{\epsilon_0}{2}$$

for each u_n, $z_0 = (x_0, t_0) \in R^m \times [0, t_0]$,

$$\Phi(z_0, \sqrt{t_0}; u_n) = \frac{t_0}{2}\int_{R^m} |\nabla u_{n0}|^2 G_{(x_0,t_0)} dx \leq \frac{\epsilon_0}{4}$$

thus

$$\Psi(z_0, \sqrt{t_0}; u_n) = 4\int_{\sqrt{t_0}/2}^{\sqrt{t_0}} \frac{\Psi(z_0, r; u_n)}{r} dr \leq \frac{\epsilon_0}{4}4 = \epsilon_0$$

By ϵ-regularity theorem, we have

$$|\nabla u_n(z_0)|^2 \leq c\delta^{-2}t_0^{-1}$$

Moreover, since $|\nabla u_{n0}| \leq CC'$, one can show that $|\nabla u_{n0}| \leq C$ uniformly for $0 \leq t \leq t'$. Combining the two estimates, we see that u_n has uniformly bounded $|\nabla u_n|$ for all $t \geq 0$, which implies that u_n is a smooth global solution.

$u_n(\cdot,t)$ is uniformly bounded in $C^1(R^m;N)$ for all $t \geq 0$. Moreover, by the heat kernel estimates and schauder theory, uniform bound on higher deriatives in and compact subset of $R^m \times R^+$ is also obtained. Thus, $u_n \to u$ in $C^\infty_{loc}(R^m \times R^+;N)$. Also , since $u_{n0} \to u_0$ in $W^{1,2}(R^m;N) \cap C^0(R^m;N)$, the integral representation of u_n implies that $u \in C^0(R^m \times [0,\infty);N)$, and $u(\cdot,t) \to u_0$ in $C^0(R^m;N)$. Therefore $u \in C^\infty(R^m \times R^+;N) \cap C^0(R^m \times [0,\infty);N)$, and solves the equation with initial condition u_0. Moreover, since

$$|\nabla u_n(x,t)|^2 \leq c\delta^{-2}t^{-1}$$

so does u, meaning $Du \to 0$ uniformly as $t \to \infty$. This means $u \to u_\infty = constant$.

We now prove uniqueness of $u \in C^\infty(R^m \times R^+;N) \cap C^0(R^m \times [0,\infty);N)$ of the heat flow with initial condition $u_0 \in W^{1,2}(R^m;N) \cap W^{1,\infty}(R^m;N)$. Let u,v be two such solutions. As before, we will use the heat kernel representation of solution. If we let $\Gamma(x,t) = (4\pi t)^{-\frac{m}{2}}\exp(-\frac{|x-y|^2}{4t})$ be heat kernel on R^m, by the regularity of u,v we can represent the solutions as

$$u(x,t) = \int_0^t \int_{R^m} \Gamma(x-y,t-s)A_N(u)(\nabla u,\nabla u)(y,s)dyds + \int_{R^m} \Gamma(x-y,t)u_0(y)dy$$

and same for v. We define $\phi(t) = \sup_{0\leq s\leq t} \|\nabla u(\cdot,s)\|_{L^\infty}$, where ∇u denote the weak derivative of $u(\cdot,t)$. Since $u_0 \in W^{1,\infty}(R^m;N)$ and $u(\cdot,t) \in C^\infty$ for $t > 0$, ϕ is well defined. Then for some fixed $\frac{1}{2} < \alpha < 1$

$$\phi(t) \leq \int_0^t \int_{R^m} |\nabla_x\Gamma(x-y,t-s)A_N(u)(\nabla u,\nabla u)(y,s)|dyds$$

$$+\int_{R^m} \Gamma(x-y,t)|\nabla u_0(y)|dy \leq Ct^{1-\alpha}\phi(t) + C'$$

Therefore, for some fixed $T > 0$ independent of u

$$\phi(t) \leq \frac{C'}{1-Ct^{1-\alpha}} \leq \frac{C'}{2}$$

Also, for $t > T$, by the proof of existence, the ϵ-regularity gives us

$$\phi(t) \leq \min(\frac{C'}{2}, c\delta T^{-\frac{1}{2}})$$

Same holds for v as well. Thus If we let $U(t) = \sup_{0\leq s\leq t}(|\nabla u|^2 + |\nabla v|^2 + |\nabla u| + |\nabla v|)$, then $U(t) \leq C$ for all $t \geq 0$. Then, by using the same method as in proof of Eells and Sampson's theorem, if we let $\Psi(t) = \sup_{0\leq s\leq t} \|u - v\|_{C^1(R^m;R^L)}(s)$, then

$$\Psi(t) \leq C\Psi(t)t^{1-\alpha}$$

for some fixed $\frac{1}{2} < \alpha < 1$. Thus, if $Ct^{1-\alpha} \leq \frac{1}{2}$, then $\Psi(t) = 0$ i.e $u = v$ in $R^m \times [0, t]$. By iterating this, we have $u = v$, hence uniqueness.

References

[1] J. Eells and J. H. Sampson, "Harmonic mappings of riemannian manifolds," *American Journal of Mathematics*, vol. 86, no. 1, pp. 109–160, 1964.

[2] J. Jost, *Chapter 8 Harmonic Maps between Riemannian Manifolds*, pp. 419–494. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011.

[3] P. Hartman, "On homotopic harmonic maps," *Canadian Journal of Mathematics*, vol. 19, p. 673–687, 1967.

[4] M. E. Taylor, *Nonlinear Parabolic Equations*, pp. 313–411. New York, NY: Springer New York, 2011.

[5] F.-H. Lin and C. Wang, *The analysis of harmonic maps and their heat flows*. World Scientific, 2008. Includes bibliographical references and index.

[6] P. Li, *Geometric Analysis*. Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2012.

[7] S. Rosenberg, *The Laplacian on a Riemannian Manifold: An Introduction to Analysis on Manifolds*. EBSCO ebook academic collection, Cambridge University Press, 1997.

[8] M. Struwe, *On the Evolution of Harmonic Mappings of Riemannian Surfaces*. Universität Bonn. SFB 72. Approximation und Optimierung, 1984.

[9] K.-C. Chang, W. Y. Ding, and R. Ye, "Finite-time blow-up of the heat flow of harmonic maps from surfaces," *Journal of Differential Geometry*, vol. 36, no. 2, pp. 507 – 515, 1992.

[10] R. Schoen and K. Uhlenbeck, "Boundary regularity and the Dirichlet problem for harmonic maps," *Journal of Differential Geometry*, vol. 18, no. 2, pp. 253 – 268, 1983.

[11] R. Hamilton, *Harmonic Maps of Manifolds with Boundary*. Lecture Notes in Mathematics, Springer Berlin Heidelberg, 2006.

[12] K. Chang and W.-Y. Ding, *A Result on the Global Existence for Heat Flows of Harmonic Maps from D2into S2*, pp. 37–47. 07 2011.