# Binary Indexed Trees

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## 1 Introduction

A Binary Index Tree (BIT), also known as a Fenwick Tree, is used for range sums (usually). Namely, a BIT can do element updates and prefix sums ($a[1] + a[2] + ... + a[i]$; we one-index BITs for implementation-specific reasons) in $O(\log n)$. This is a tradeoff between a $O(n)$ update/$O(1)$ query prefix-sum solution and the $O(1)$ update/$O(n)$ query naive solution.

BITs are very useful, especially for their simple implementation.

## 2 BITs

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 6 | -2 | 3 | -10 | 2 | 2 | 0 | 12 | 4 | 1 | -1 | 6 | 5 | 2 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |

Figure 1: A sample array.

BITs rely on the idea that an integer can be decomposed into powers of two. Given an index $i$, we can find these powers of two by writing $i$ in binary. Then, we keep turning off the lowest bit until we reach zero. Say we want to find the prefix sum $a[1] + a[2] + ... + a[14]$:

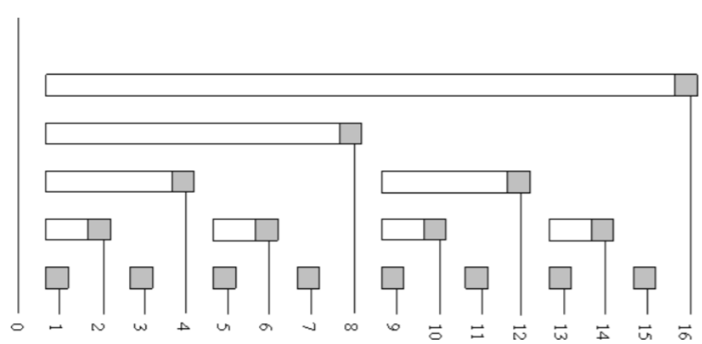
$$14 \to 1110 \to 1100 \to 1000 \to 0$$

How do we find the prefix sum with this?

Say we just went from $1110 \to 1100$. We just jumped from index $14 \to 12$. We can add the elements with indices 13 and 14 to a running sum, then recur on 12:

$$\texttt{1110}\ (14) \xrightarrow[\texttt{add}\ a[13]+a[14]]{} \texttt{1100}\ (12) \xrightarrow[\texttt{add}\ a[9]+...+a[12]]{} \texttt{1000}\ (8) \xrightarrow[\texttt{add}\ a[1]+...+a[8]]{} 0$$

Notice that every "step" (1110, 1100, and 1000), there's a unique range of indices denoted. That is, `1110` uniquely denotes indices `1101` and `1110`, or all numbers between the `1110` and $1 +$ (`1110` with the bottom bit removed). So we can map every number to a range of indices, and store the sum beforehand; `1110` stores $a[13] + a[14]$. See the illustration below.



## 2.1 Query

We discussed query above. But how do we find the lowest bit?

Taking advantage of the two's complement system ($-1 = 1...1111_2, -2 = 1...1110_2$ and so on), we can do this very easily. Say we're using $14 = 1110_2$. $-14 = 0010_2$ (with a bunch of ones in front). If we bitwise AND these two together, we get only the lowest bit set. This holds true in a general sense: let $i = (a1b)_2$, where $a$ and $b$ are parts of the binary number, and the one represents the lowest bit set. Then the negative is as follows: $-i = \tilde{}(a1b)_2 + 1 = \tilde{}a0\tilde{}b + 1$. But $b$ must consist of only zeros, since it's after the lowest set bit. Thus, we get $-i = (\tilde{}a1b)_2$. Bitwise AND-ing with $i$, we clearly see that only the lowest bit is set.

A C++ implementation is shown below.

```
int query(int i) {
    int ans = 0;
    for(; i>0; i-=(i & -i))
        ans += a[i];
    return ans;
}
```

Clearly, to do range queries, we can subtract in the same way we do with regular prefix sums:

```
int range(int i, int j) {
    return query(j) - (i>1?query(i-1):0);
}
```

## 2.2 Update

To update (add a value $v$) at a given index $i$, we want to add the value to all segments "above" $i$. Here I mean "above" in the sense of the diagram above – all segments that contain $i$.

Let's take 9. The sequence for segments "above" 9 is:

$$\texttt{9}\ (1001) \rightarrow \texttt{10}\ (1010) \rightarrow \texttt{12}\ (1100) \rightarrow 16(10000).$$

Notice that we're simply adding the lowest bit every time (why?). Then for each index we visit, we add $v$ to the value at this segment. Thus, the implementation is quite similar to query.

```
int update(int i, int v) {
    for(; i<=N; i += (i & -i))
        a[i] += v;
}
```

Note that for both update and query, we're only going through each bit once. Thus, the complexity is $O(\log n)$.

### 2.2.1 Range Updates

Range updates are a bit more involved, but can also be done in $O(\log n)$. The idea is to keep two BITs.

Let's say we want to find a given prefix sum to index $i$ (to find the range sum we can still subtract the prefix sums). To do this, we find all ranges that begin before $i$. Then, the answer is:

$$\sum_{ranges} max(i, r) * v - (l-1) * v$$

where $r$ is the right endpoint of a given range, $l$ is the left, and $v$ is the value. To calculate this, we can use one BIT to calculate the $i * v$ parts and the other to calculate the $r * v$ and $(l-1) * v$ parts.

Specifically, here's how we'd update:

1. BIT1.update(l, v)
2. BIT1.update(r+1, -v)
3. BIT2.update(l, (l-1)*v)
4. BIT2.update(r+1, -r*v)

To query $a[1] + ... + a[w]$: BIT1.query(w)*w - BIT2.query(w). BIT1 handles cases where $w$ falls within a range update; BIT2 handles the endpoints of the range updates.

## 3 Problems

1. You're given $n$ ($1 \leq n \leq 10^5$) horizontal line segments, each with inclusive endpoints $(x_1, y)$ and $(x_2, y)$ where $-10^9 \leq x_1 \leq x_2 \leq 10^9$. Each line segment has a value $v$ ($-10^9 \leq v \leq 10^9$).

   Answer each of $q$ ($1 \leq q \leq 10^5$) queries. Each query is of the form $x', a, b$, and asks you to sum the values of the $a$-th to the $b$-th (sorted by increasing $y$) line segments at the vertical line $x = x'$.

2. (Brian Dean, 2012) FJ has set up a cow race with N ($1 \leq N \leq 100,000$) cows running L laps around a circular track of length C ($1 \leq L, C \leq 25,000$). The cows all start at the same point on the track and run at different speeds, with the race ending when the fastest cow has run the total distance of $L * C$. FJ notices several occurrences of one cow overtaking another. Count the total number of crossing events during the entire race.

3. (Brian Dean, 2011) Farmer John has lined up his N ($1 \leq N \leq 100,000$) cows each with height $H_i$ ($1 \leq H_i \leq 1,000,000,000$) to take a picture of a contiguous subsequence of the cows, such that the median height is at least a certain threshold X ($1 \leq X \leq 1,000,000,000$). Count the number of possible subsequences.

4. (SPOJ BRCKTS) Given a bracket expression of length N ($1 \leq N \leq 30,000$), process M operations. There are two types of operations, a replacement, which changes the i-th bracket into its opposite, and a check, which determines whether a bracket expression is correct.