Computational Geometry

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"Geometry is just plane fun."

1 Introduction

Computational geometry problems show up on USACO contests fairly often. In this lecture, we will cover some techniques for manipulating points and lines, as well as some useful formulas.

2 Vectors

Rather than store points and lines directly, we will use vectors as a more convenient format to work with. In the three-dimensional case, a vector will take the form

$$\mathbf{v} = \vec{v} = \langle v_x, v_y, v_z \rangle = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

Vectors are used to represent directions; $\vec{v}$ represents a displacement of v_x, v_y, and v_z units in the positive x, y, and z directions.

2.1 Vector Basics

- The *magnitude* of a vector is given by $\sqrt{v_x^2 + v_y^2 + v_z^2}$.
- Vectors are added (or subtracted) component-wise, so that $\mathbf{u} + \mathbf{v} = \langle u_x + v_x, u_y + v_y, u_z + v_z \rangle$.
- Similarly, vectors can be multiplied component-wise by scalars: $k\mathbf{v} = \langle kv_x, kv_y, kv_z \rangle$
- For any $\mathbf{v}$, $-\mathbf{v}$ produces a vector of the same magnitude but in the opposite direction.

2.2 The Dot Product

The dot product takes two vectors and produces a scalar:

$$\mathbf{u} \cdot \mathbf{v} = \langle u_x \cdot v_x, u_y \cdot v_y, u_z \cdot v_z \rangle = uv\cos\theta$$

where the angle θ is the smaller angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Note that this means that perpendicular vectors will have a dot product of zero.

2.3 The Cross Product

The cross product takes two vectors and produces a third vector perpendicular to both of them. Written $\mathbf{u} \times \mathbf{v}$, this new vector has a magnitude of $uv\sin\theta$, where θ is the smaller angle between $\mathbf{u}$ and $\mathbf{v}$. The easiest way to remember how to find the cross product is to write $\mathbf{u}$ and $\mathbf{v}$ in a matrix and take the determinant:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \langle (u_yv_z - u_zv_y), (u_zv_x - u_xv_z), (u_xv_y - u_yv_x) \rangle$$

Careful! The cross product is not commutative. In fact, it is anticommutative: $\mathbf{u} \times \mathbf{v} = \textbf{-v} \times \mathbf{u}$

3 Geometric Formulas

3.1 Area of a Triangle

If we know the three vertices of a triangle $\triangle ABC$, we can find its area using the cross product. We pick a vertex (for example, A) and create a vector from that vertex to each of the other vertices (let $\mathbf{u} = \mathbf{b} - \mathbf{a}$ and $\mathbf{v} = \mathbf{c} - \mathbf{a}$). Then, the area of the triangle is given by half the magnitude of the cross product: $K = \frac{1}{2} |\mathbf{u} \times \mathbf{v}|$.

You can also use Heron's formula to find the area of a triangle with side lengths a, b, and c. First, we find the semiperimeter $s = \frac{a+b+c}{2}$. The area is then $K = \sqrt{s(s-a)(s-b)(s-c)}$.

3.2 Check if two lines are parallel

Checking whether two lines are parallel is very straightforward. Simply take two vectors, one lying in each line, and find their cross product. If the magnitude of the resulting vector is zero, the two lines are parallel.

Note that when you're coding this on a computer, you would check whether the magnitude is *almost* zero; that is, whether it is smaller than some threshold value ϵ.

3.3 Distance from a point to a line

Suppose we want to find the perpendicular distance from a point P to a line AB. We use position vectors to represent $\mathbf{P}$, $\mathbf{A}$, $\mathbf{B}$. The distance is then given by

$$d(P, AB) = \frac{|(\mathbf{P} - \mathbf{A}) \times (\mathbf{B} - \mathbf{A})|}{|\mathbf{B} - \mathbf{A}|}$$

This identity can easily be proven using a little trigonometry.

3.4 Distance from a point to a line segment

If the triangle $\triangle PAB$ is acute, we can simply use the distance formula given above. Otherwise, take the minimum of $d(P, A)$ and $d(P, B)$.

3.5 Check if a point is on a line or line segment

A point is on a line if the distance from the point to the line or line segment is 0.

3.6 Check if points on the same side of line

Caveat: this only works for two dimensions. If we want to check if points C and D are on the same side of line AB, we find the z components of $(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})$ and $(\mathbf{B} - \mathbf{A}) \times (\mathbf{D} - \mathbf{A})$. If the z components have the same sign, then C and D are on the same side of the line AB.

An easy way to check if two values have the same sign is to see whether their product is positive.

3.7 Check if a point is inside a polygon

We begin with the triangle, bounded by three lines. We know that the average of the vertices of the triangle is inside the triangle. So we check if this average point and the point in question are on the same side of each of the three sides of the triangle. If they are, the point we are concerned with is inside the triangle.

This technique will work for any polygon as long as you pick a point inside the boundary. However, taking the average of the vertices only guarantees a point inside if the polygon is convex. In the case of concave polygons, you will have to find the point in some other way.

Alternatively, draw any ray from the given point. The point is inside the polygon if and only if the ray intersects the polygon an odd number of times. This works for both convex and concave polygons. Note that the ray is not allowed to pass through any vertex of the polygon. If it does, simply pick a new ray.

3.8 Check if points are coplanar

To determine if a collection of points is coplanar, take any three points, A, B, and C. These three points define for us a plane that all the other points in the set must lie on. If for all other points D in the collection, $(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A}) \cdot (\mathbf{D} - \mathbf{A}) = 0$, then all of the points reside in the same plane.

3.9 Check if a pair of lines intersects

In two dimensions, two lines will intersect if they are not parallel. In three dimensions, two lines will intersect if they are not parallel and they are coplanar. You can check if two lines are coplanar by taking two distinct points from each line and checking if the four points are coplanar.

3.10 Line segment intersection

In two dimensions, two line segments AB and CD intersect if and only if A and B are on opposite sides of line CD **and** C and D are on opposite sides of line AB.

3.11 Check if a 2-dimensional polygon is convex

A convex polygon will have interior angles all less than π. We can check this by traversing the vertices in clock-wise order. For every triplet of vertices A, B, C, we calculate the cross product $(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A})$. If the z components of all of the cross products we find are positive, the polygon is convex.

4 Equality Checks

When implementing computational geometry programs, never directly compare two floats or doubles for equality. Instead, define A and B to be equal if $|A - B| < \epsilon$, where ϵ should be something like 10^{-9}.

5 Problems

1. Given a set of N line segments, all horizontal or vertical, find all intersection points among the segments.
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3. Each of Farmer John's cows grazes in a rectangular patch of his rectangular field. Given all the patches, find the total ungrazed area left in FJ's field. Note that grazing areas may overlap.
4. (IOI 98) Find the *perimeter* of the ungrazed land left in FJ's field.
5. (USACO OPEN10) Given N ($1 \leq \text{N} \leq 50000$) points in the plane, find the number of triangles containing the origin that can be formed from these points. No three of the points given are collinear.
6. Given N ($1 \leq \text{N} \leq 100,000$) lines in the plane, find the x-coordinate of the leftmost intersection.
7. (USACO DEC08) Given N ($1 \leq \text{N} \leq 250$) posts in the plane, what is the greatest number of posts Farmer John can include in a convex fence?
8. Given an N by N grid of 0's and 1's, compute the shortest distance to a 1 for every gridpoint. This can be done in $O(N^2)$ time.
9. Given a polygon representing the floor plan of a museum, determine the least number of stationary security guards needed to have every point in the building within sight range of at least one guard.

Geometric Formulas section borrowed from Johnathan Wang.