Network Flow

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10 March 2017

1 Introduction

Let's start by imagining a network of pipes. Water can flow through the pipes, each of which has a capacity, and flows in one direction. The pipes can also join together and split apart from each other. Water flows through the network of pipes starting from a source, and leading to a sink. We can represent this as a connected, weighted, and directed graph, a **flow network**, in which the edges represent pipes and the vertices represent splitting points. The weight of the edges represents the capacity of the pipe (e.g. the thickness).

There are also two special vertices: the source s and the sink t. A **flow** is sent from the source to the sink. More formally, for a graph $G(V, E)$, where $c(u, v)$ represents the capacity of the edge from u to v, the flow $f(u, v)$ from s to t must satisfy the following, where $|f|$ is the total flow from the source to the sink.

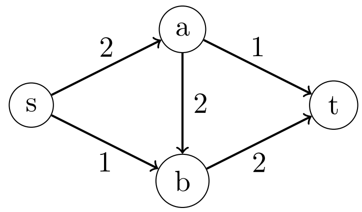
$$\begin{aligned}
f(u,v) &\le c(u,v) && \forall\ (u,v) \in E \\
f(u,v) &= -f(v,u) && \forall\ (u,v) \in E \\
\sum_{v \in V} f(u,v) &= 0 && \forall\ u \in V \setminus \{s,t\} \\
\sum_{v \in V} f(s,v) &= \sum_{v \in V} f(v,t) = |f|,
\end{aligned}$$

In a flow, we may not necessarily use up all of the capacity in an edge. After a certain amount of flow has been sent across an edge, the capacity that is left is known the **residual capacity**, which is defined as $c_f(u, v) = c(u, v) - f(u, v)$.

2 Maximum Flow

What is the maximum amount of flow $|f|$ that we can push from the source to the sink? In terms of pipes, what is the maximum volume of water over time that can pass through this pipe network? This is called the **maximum flow** problem.

First we'll try a greedy approach. Find a path from the source to the sink and send the max flow across it, which is the minimum capacity of any edge along this path. Then subtract this capacity from the residual capacity of every edge along the path, and repeat. This method is guaranteed to terminate, since we remove an edge each iteration. To see why this method does not work, look at the following graph:



The max flow from s to t is 3, but the greedy approach gives 2.

3 Ford-Fulkerson