Game Theory

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This lecture will present some interesting aspects of games from a computer science perspective. Much owed to Sreenath Are's December 2013 lecture on the same topic.

1 Definitions

Most games that will come up in contest problems are *combinatorial*, which we will define to include the following characteristics:

- Players alternate taking turns.
- Deterministic rules specify which moves can be made.
- There is perfect information: both players know everything about the state of the game and the rules.
- The game ends when one player cant make a move. This player either wins or loses.
- Every game ends.

In this lecture, we will focus on *impartial* games, combinatorial games in which available moves do not determine on the player.

2 Positions

Although a mathematical treatment of game theory would include other more specialized techniques for analyzing games, we will focus here on a very broad technique: Analyzing positions. The basic idea is to consider positions from which one player can force a win no matter how his/her opponent plays, called *winning positions*, and work backwards from there. Conversely, the other player is said to be in a *losing position*. This means that in a position P when it is player A's turn to move:

1. If player A can make a move that turns P into a losing position, then P is a winning position.
2. If no matter how player A moves, P becomes a winning position, then P is a losing position.

Thus, once we determine from the rules the terminal positions where a win or loss is defined, we can determine for each position P in an impartial game whether P is a winning or losing position. This same method also determines the winning strategy.

3 Nim

Nim is the best-known impartial game and also presents an excellent example of how we ought to analyze games. Nim consists of k heaps of stones. Players alternate turns taking any positive number of stones from one heap. The player to take the last stone wins. We will denote a Nim position with a_i stones in each heap as $(a_1, \cdots, a_k)$.

If there is one heap, the first player wins. If there are two heaps, we claim that only positions of the form (a, a) are losing positions. The terminal position $(0, 0)$ is a loss, and once in a position (a, a), it takes more than one move to reach a different (b, b). This gives us a clear winning strategy.

Unfortunately, this finding is rather unmotivated. All you can do is try playing the game and look for a characteristic that switches back and forth.

4 Some Examples

1. On a chalkboard are written all numbers of the form $2^a 3^b$ for nonnegative integers a, b. On a players turn, he/she chooses a number n on the board and erases all multiples of it. The person to erase the number 1 loses. Who has the winning strategy? What if we only start with numbers of the form 3^b and $2 \cdot 3^b$?

2. Determine winning and losing positions for a Nim game of $(a, b, 2)$.

3. Consider a game with 1 token on a 4×4 grid. Two players alternate moving the token either any number of squares to the right or to the leftmost row above. If the first player to run out of moves loses,

   (a) Determine winning and losing positions.

   (b) Determine winning and losing positions if, when moved right, the token can only be moved 1 or 2 spaces.

   (c) Determine positions if the grid has size $m \times 3n + 1$.

   (d) Determine positions for any grid.

5 Multiple-Heap Nim

We claim that the multiple-heap Nim position $(a_1, \cdots a_k)$ is a losing position iff $a_1 \oplus a_2 \oplus \cdots \oplus a_k = 0$, where $\oplus$ denotes the bitwise exclusive or. Call this xor expression the *sum* S of the position.

This is essentially a generalization of the two-heap case. Again, we begin with the terminal position, which has $S = 0$. Now, for a position $(a_1, \cdots, a_k)$, if $S \neq 0$, S must have a bit somewhere that is 1 that is also a 1 in the bitwise representations of at least one of $a_1, \cdots, a_k$. Since $n \oplus a_i < a_i$, we can take stones from heap i such that it has $n \oplus a_i$ stones left. The result is a position with bitwise xor $n \oplus a_i \oplus (n \oplus a_i) = 0$.

Any move from a position with bitwise xor 0 will flip one of the bits in the result, giving a position with nonzero bitwise xor.

6 Problems

1. Generalize our results in Nim to the case where the player who takes the last stone loses.

2. Two players play a game with a number of coins. They take turns flipping either one or two of the coins from heads to tails. The last player to make a move wins. Given the initial positions of the coins, determine the winner if both players play optimally.

3. Two players play a game with finitely many markers placed on the nonnegative half of the number line. Each turn, a player moves one of the markers to an unoccupied spot to the left (toward 0), without crossing over any other markers. The game ends when one player cannot make a move, and this player loses. Given the initial positions of the markers, determine the winner if both players play optimally.

4. Two players write the numbers from 1 to n on a blackboard. They then take turns erasing a number and all powers of it from the board. The person to erase the last number wins. Given n, determine the winner if both players play optimally.

5. Two players take turns marking a single square on an $n \times 2$ grid. Once a square has been marked, it is not allowed to mark that square or the three neighboring squares in the other column. Equivalently, it must be possible at any point to draw a curve from the top of the grid to the bottom that only passes through unmarked squares. The last player to mark a square wins. Given n and the already marked squares, determine the winner if both players play optimally.

6. Consider a sequence. Two players move alternately by selecting a number from either the left or the right end of the sequence. That number is then deleted from the board, and its value is added to the score of the player who selected it. A player wins if his sum is greater than his opponents. Find the best possible score of each player, assuming both play optimally.