Heavy-Light Decomposition

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1 Introduction

Brute-force algorithms on trees are often fast when the tree is nearly complete, or has small depth. Unfortunately, brute force is far too slow when the tree becomes more and more unbalanced and approximates a linked list.

Certain problems can be solved very nicely in a list with segment trees and similar data structures. Heavy-light decomposition provides us with a way to exploit the fact that long "chains," which are present in unbalanced trees, might be slow in brute force but can be sped up with other data structures.

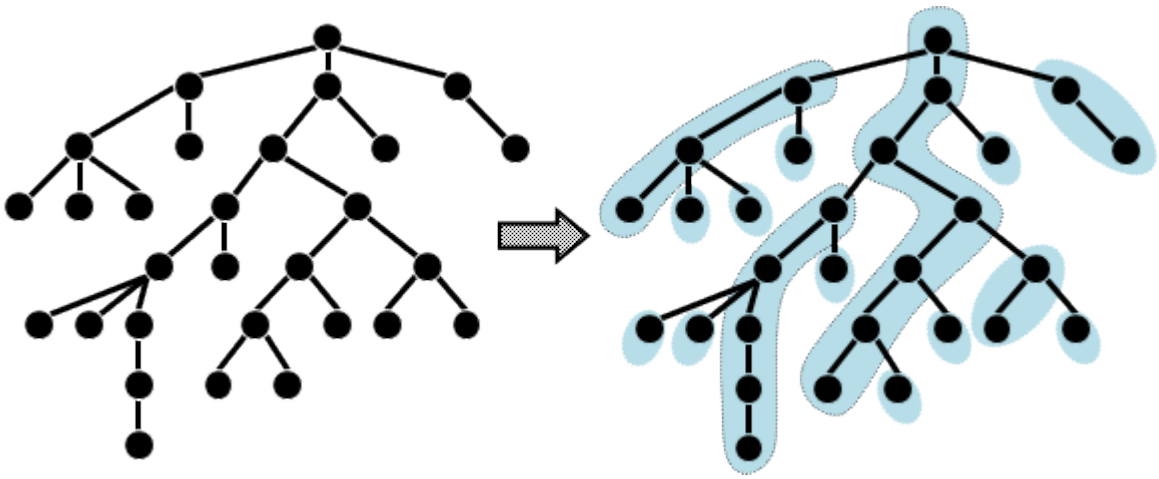
Heavy-light decomposition provides us a way to dynamically find the lowest common ancestor as well as a way to prove time complexities about tree data structures, like link-cut trees.

2 Heavy-Light

Heavy-light decomposition is a coloring of all the edges in a binary tree either heavy or light. For each vertex v, let $s(v)$ denote the number of nodes in the subtree with v as its head. Then, if u, w are the children of v, $s(v) = s(u) + s(w) + 1$.

We see that the smaller child u or w must have less than half the subtree size as v, and so that child exhibits qualities similar to those of a completely balanced tree. We color the edge connecting v to its lesser child **light**. We color the other edge **heavy**.

We see that from any node, the number of light edges needed to reach the head of the tree is at most $\log n$, as each light edge doubles the subtree size. Then, the number of "heavy chains" along the upwards path from any node is also of the order $\log n$.



http://www.csie.ntnu.edu.tw/ u91029/Heavy-LightDecomposition1.png

3 Problems

1. (USACO 2011 December Gold)

   Farmer John has N barren pastures ($2 \leq N \leq 100,000$) connected by $N-1$ bidirectional roads, such that there is exactly one path between any two pastures. Bessie, a cow who loves her grazing time, often complains about how there is no grass on the roads between pastures. Farmer John loves Bessie very much, and today he is finally going to plant grass on the roads. He will do so using a procedure consisting of M steps ($1 \leq M \leq 100,000$).

   At each step one of two things will happen:

   - FJ will choose two pastures, and plant a patch of grass along each road in between the two pastures, or,
   - Bessie will ask about how many patches of grass on a particular road, and Farmer John must answer her question.

   Farmer John is a very poor counter – help him answer Bessie's questions!

   Note: Technically heavy-light decomposition is banned on the IOI syllabus, and therefore on the USACO syllabus. However, that doesn't mean you will never see HL on USACO anymore – strongly connected components (SCC) is technically banned on IOI but was on a USACO monthly this year.

2. (SPOJ QTREE) You are given a tree (an acyclic undirected connected graph) with N nodes and edges numbered $1, 2, 3, \cdots, N-1$.

   We will ask you to perfrom some instructions of the following form:

   - CHANGE i ti : change the cost of the i-th edge to ti, or
   - QUERY a b : ask for the maximum edge cost on the path from node a to node b

3. (SPOJ GOT) There's a tree, with each vertex assigned a number. For each query (a, b, c), you are asked whether there is a vertex on the path from a to b, which is assigned number c?