Math Programming

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special thanks to codeforces

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1 Introduction

A field of programming that is constantly applied, so powerful as to make even a Codeforces Div 1 E problem trivial, math. There are several theorems in math programming that are important towards

2 Euler's Totient theorem

Let $p1, p2, ...pn$ be distinct prime factors of the integer n. $\phi(n) = n \cdot (1 - \frac{1}{p1}) \cdot (1 - \frac{1}{p2}) ... \cdot (1 - \frac{1}{pn})$ The reason we make this restriction is that on many programming competitions, time is restricted to one second, in which programming languages such as C++ can perform about this many computations.

3 Precision

(Google Code Jam Round 1A 2008) In this problem, you have to find the last three digits before the decimal point for the number $(3 + \sqrt{5})^n$.

For example, when $n = 5$, $(3 + \sqrt{5})^n = 3935.73982\ldots$ The answer is 935.

For $n = 2$, $(3 + \sqrt{5})^n = 27.4164079...$ The answer is 027. The large testcase has bounds of $2 \leq n \leq 2000000000$.

The difficulty comes from the fact that $\sqrt{5}$ is irrational and for n close to 2000000000 you would need a lot of precision and a lot of time if you wanted to use the naive solution. The key in solving the problem is a mathematical concept called conjugation. In our problem, we simply note that $(3-\sqrt{5})$ is a nice conjugate for $(3+\sqrt{5})$. Let us define

$$A := 3+\sqrt{5}, B := 3-\sqrt{5}, and X_n := A^n + B^n.$$

We first note that X_n is an integer. Experienced contestants may notice there is a linear recurrence on the X_i's. Indeed, this is not hard to find – the conjugation enters the picture again.

Notice that

$$A+B = 6, and AB = 4.$$

So A and B are the two roots of the quadratic equation $x^2-6x+4=0$. i.e.,

$$A^2 = 6A-4, and B^2 = 6B-4.$$

Looking at (1) and (6) together, we happily get

$$X^n + 2 = 6X^n + 1 - 4X^n.$$

Such recurrence can always be written in matrix form. It is somewhat redundant, but it is useful: Note we use fast matrix exponentiation. You can

$$\begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix} = B \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix} = B^n \begin{pmatrix} X_1 \\ X_0 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & -4 \\ 1 & 0 \end{pmatrix}$$

Figure 1: Matrix

learn this yourself online. Another solution □

Okay, that was a very instructive problem, but it illustrates that when possible, precision is the way way to do a problem.

4 A Few Problems

1. (Pretty Song) Let's define the simple prettiness of a word as the ratio of the number of vowels in the word to the number of all letters in the word.

Let's define the prettiness of a word as the sum of simple prettiness of all the substrings of the word. Find this value.

2. (Vasya and Polynomial) Vasya is studying in the last class of school and soon he will take exams. He decided to study polynomials. Polynomial is a function $P(x) = a_0 + a_1x^1 + ... + a_nx^n$. Numbers a_i are called coefficients of a polynomial, non-negative integer n is called a degree of a polynomial.

   Vasya has made a bet with his friends that he can solve any problem with polynomials. They suggested him the problem: "Determine how many polynomials $P(x)$ exist with integer non-negative coefficients so that , and , where and b are given positive integers"?

   Vasya does not like losing bets, but he has no idea how to solve this task, so please help him to solve the problem.

3. (New York Hotel) Think of New York as a rectangular grid consisting of N vertical avenues numerated from 1 to N and M horizontal streets numerated 1 to M. C friends are staying at C hotels located at some street-avenue crossings. They are going to celebrate birthday of one of them in the one of H restaurants also located at some street-avenue crossings. They also want that the maximum distance covered by one of them while traveling to the restaurant to be minimum possible. Help friends choose optimal restaurant for a celebration.

   Suppose that the distance between neighboring crossings are all the same equal to one kilometer.

5 More Material (Thanks to topcoder for the below)

Occasionally problems ask us to find the intersection of rectangles. There are a number of ways to represent a rectangle. For the standard Cartesian plane, a common method is to store the coordinates of the bottom-left and top-right corners.

Suppose we have two rectangles R1 and R2. Let (x_1, y_1) be the location of the bottom-left corner of $R1$ and (x_2, y_2) be the location of its top-right corner. Similarly, let (x_3, y_3) and (x_4, y_4) be the respective corner locations for R2. The intersection of R1 and R2 will be a rectangle R3 whose bottom-left corner is at $(max(x_1, x_3), max(y_1, y_3))$ and top-right corner at $(min(x_2, x_4), min(y_2, y_4))$. If $max(x_1, x_3)$ ¿ $min(x_2, x4)$ or $max(y1, y3) > min(y2, y4)$ then $R3$ does not exist, ie R1 and R2 do not intersect. This method can be extended to intersection in more than 2 dimensions as seen in CuboidJoin (SRM 191, Div 2 Hard).

6 More Problems

1. A table composed of 1000×1000 cells, each having a certain quantity of apples, is given. You start from the upper-left corner. At each step you can go down or right one cell. Find the maximum number of apples you can collect.

2. You are given two arbitrary strings of letters, of length 10^3 each. Devise an algorithm to find the length of the longest common sub-sequence of both of these strings. For example, 1123 is a subsequence of both 15120034 and 110230 because the digits $1, 1, 2, 3$ appear in order in both strings.

3. (Codeforces) There is a square matrix of size 1000 by 1000, consisting of non-negative integer numbers. You should find a path through it such that the path starts in the upper left cell of the matrix; each following cell is to the right or down from the current cell; and the path ends in the bottom right cell. Moreover, if we multiply together all the numbers along the way, the result should be the least "round". In other words, it should end in the least possible number of zeros.

4. (Project Euler) The number of divisors of 120 is 16. In fact 120 is the smallest number having 16 divisors. Find the smallest number with 2^{500500} divisors. Give your answer modulo 500500507.

7 Extra Practice/Problems

1. Codeforces, Project Euler

2. Problems in Elementary Number Theory (PEN)