HW-9
Math 117

Siddharth Deshpande

12/18/2020

3.1

- Because we are told that f and g are continuous and $f = g$ a.e. on $[a, b]$, we also know that $f - g$ is continuous and $f - g = 0$ a.e. on $[a, b]$.
- Let $(f - g)^{-1}(\mathbb{R} \setminus \{0\}) = A$. Then, we know that, the set A is the set of points where $f - g \neq 0$ and thus, where $f \neq g$.
- Therefore, we know by hypothesis that $\mu(A) = 0$. Furthermore, because $\mathbb{R} \setminus \{0\}$ is an open set, and the pre-image of an open set is an open set, we know that A is an open set as well.
- Thus, A is an open set of measure 0 and therefore, it must be that $A = \emptyset$ and therefore, $\emptyset = \{x \mid f(x) \neq g(x)\}$ and therefore, $f = g$ everywhere on $[a, b]$.
- Now, if we replace $[a, b]$ with a set E which is measurable, we want to show that this property still holds (or provide a counter example).
- Now, assume that $f = g$ almost everywhere on E. Then, similar to last time, we know that the function $f - g$ is also continuous on E and, $f - g = 0$ almost everywhere on E.
- Now, Let $A = (f - g)^{-1}(\mathbb{R} \setminus 0)$. This is the same as $A = \{X \mid f(x) \neq g(x)\}$. Furthermore, by hypothesis, we know that $\mu(A) = 0$. Finally, because $\mathbb{R} \setminus \{0\}$ is an open set, we know that the pre-image of an open set is an open set and therefore, $(f - g)^{-1}(\mathbb{R} \setminus \{0\} = A$ is also an open set.
- The only open set of measure 0 is going to be the empty set and therefore, $f = g$ everywhere on E.

3.3

- Let f be a continuous function on a measurable set E. Furthermore, assume that f is not continuous at a finite number of points. Let these such points be called the set A. Therefore, f is not continuous on $A \subseteq E$. Furthermore, because A is finite, we know that $m(A) = 0$ and thus, f is continuous almost everywhere on E.
- Now, define $F = E \setminus A$ which is the set of points where f is continuous. Thus, the set:

$$\{x \mid f(x) > c\} = \{x \in F \mid f(x) > c\} \bigcup \{x \in A \mid f(x) > c\}$$

for any number c can be shown to be measurable. Because the two sets are equivalent, we have that, showing the left side is measurable will suffice.
- We know that $m(A) = 0$ and $E \setminus A = F$ therefore, $m(E \setminus A) = m(F) = m(E) - m(A) = m(E) - 0 = m(E)$.
- Thus, we have that, because E is measurable, we have that $m(F) = m(E)$ and thus, F is also measurable.
- Therefore, we have that the set $\{x \mid f(x) > c\}$ is also measurable because the two sets on the RHS of the equation are measurable, their union is also measurable and thus, the LHS must also be measurable.
- Therefore, f is measurable on E.

3.9

- For this problem, we can call on the definition of Cauchy. Because the set E_0 is the set of points where $f_n(x)$ converges, we must have that $f_n(x)$ is cauchy. Therefore,

$$E_0 \supseteq \{x \mid \exists N \ni n, m \geq N, |f_n(x) - f_m(x)| < \frac{1}{k}\}$$

for some k. But, because E_0 is the set of points where $\{f_n(x)\}$ converges, we must have that, this is true for all k and thus,

$$E_0 = \bigcap_{k=1}^{\infty} \{x \mid \exists N \ni n, m \geq N, |f_n(x) - f_m(x)| < \frac{1}{k}\}$$

thus now, if we can prove that:

$$\{x \mid \exists N \ni n, m \geq N, |f_n(x) - f_m(x)| < \frac{1}{k}\}$$

is measurable then we are done.

- Because this has to apply for **all** $n, m \geq N$, we can re-write this set as:

$$\bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \{x \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}$$

for **some** N. Because this only has to be true for some number N and not all, we now have that, the original set can be written as:

$$\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \{x \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}$$

and now, we can completely re-write E_0 as:

$$E_0 = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \{x \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}$$

Furthermore, we know that the set $\{x \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}$ is measurable as, the f_n and f_m are measurable and therefore, $f_n - f_m$ is measurable and thus, $|f_n - f_m|$ is measurable, we know that the set

$$\{x \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}$$

must be measurable as well as $\frac{1}{k} \in \mathbb{R}$. Because E_0 is simply the countable union and intersection of such measurable sets, we know that E_0 itself must be measurable.

3.12

- We know by the simple approximation lemma that, for f bounded and measurable on E. We know that there are functions ϕ_k and ψ_k such that:

$$\phi_k \leq f \leq \psi_k$$

and

$$0 \leq \psi_k - \phi_k \leq \frac{1}{k}$$

so now, let the functione $\{\phi_k\}$ and $\{\psi_k\}$ be the sequence of functions for all k. Thus, we have that:

$$\lim_{k \to \infty} \psi_k - \phi_k \leq \lim_{k \to \infty} \frac{1}{k} = 0$$

which tells us that, $\lim_{k \to \infty} \psi_k = \lim_{k \to \infty} \phi_k$. But, because we have that:

$$\psi_k \leq f \leq \phi_k$$

for all k, we must have that this holds even as we go to ∞ and therefore,

$$\lim_{k \to \infty} \phi_k = f = \lim_{k \to \infty} \psi_k$$

Now, we need to show that such functions can exist where $\{\phi_k\}$ is an increasing sequence and $\{\psi_k\}$ is decreasing sequence.

- We know that $f \leq \psi_k$ for all k but, we also know that $\lim_{k \to \infty} \psi_k = f$ and therefore, we must have that the sequence ψ_k decreases to f.

- Using a similar argument we know that ϕ_k must converge increasingly to f and therefore, we have shown that these two functions exist.

3.14

- Let the sets $I_n = \{x \in E \mid |f(x)| > n\}$. We know that this collection of sets is countable as, $f(x) < \infty$ almost everywhere an that these sets are measurable.
- Therefore, now, we know that the collection of sets $\{I_n\}$ is a descending collection of sets and therefore, we can find a set I_k such that, $m^*(I_k) < \epsilon$ for some $\epsilon > 0$. Therefore, we now have that:

$$F = E \setminus I_k = \{x \in E \mid |f(x)| < k\}$$

because E is a measurable set and the I_n are measurable sets, we know that F is a measurable set and therefore, $m^*(E \setminus F) = m^*(E) - m^*(F) = m^*(I_k) < \epsilon$.
- Furthermore, for this set F, we have that, $|f(x)| \leq n$ and therefore, it is also bounded.

3.20

- We first need to remember that the characteristic function of a set A is given by:

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

- For

$$\chi_{A\cap B}$$

we have that,

$$\chi_{A\cap B} = \begin{cases} 1, & x \in A \cap B \\ 0, & x \notin A \cap B \end{cases}$$

- Because this characteristic function is 1 if and only if $x \in A$ and $x \in B$ which is the same as saying, if $\chi_A(x) = 1 = \chi_B(x)$. Furthermore, the characteristic function is 0 if and only if $x \notin A \cap B$ which means, $x \notin A$ or $x \notin B$ and thus, if $\chi_A(x) = 0$ or $\chi_B(x) = 0$. Therefore, we have that:

$$\chi_{A\cap B} = \chi_A \cdot \chi_B$$

- For the second property, we have that:

$$\chi_{A\cap B} = \begin{cases} 1, & x \in A \cup B \\ 0, & x \notin A \cup B \end{cases}$$

- Now, assume $x \notin A \cup B \Rightarrow \chi_{A\cup B} = 1$ therefore, we have that, $x \in A$ and/or $x \in B$. WLOG assume that $x \in A$ and $x \notin B$, then:

$$\chi_A + \chi_B - \chi_A \cdot \chi_B = 1 + 0 - 0 = 1$$

now assume that $x \in A$ and $x \in B$, then:

$$\chi_A + \chi_B - \chi_A \cdot \chi_B = 1 + 1 - 1 = 1$$

Now, assume that $x \notin A \cup B \Rightarrow \chi_{A\cup B} = 0$, we have that, $x \notin A$ and $x \notin B$, therefore, we have that:

$$\chi_A + \chi_B - \chi_A \cdot \chi_B = 0 + 0 - 0 = 0$$

therefore, we have that:

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$$

- Finally, for the last property,

$$\chi_{A^c} = \begin{cases} 1, & x \in A^c \Rightarrow x \notin A \\ 0, & x \notin A^c \Rightarrow x \in A \end{cases}$$

- Therefore, if $x \in A^c \Rightarrow \chi_{A^c} = 1$ we have that, $x \notin A \Rightarrow \chi_A = 0 \Rightarrow 1 - \chi_A = 1$.

- Now, if $x \notin A^c \Rightarrow \chi_A = 0$ we have that $x \in A \Rightarrow \chi_A = 1 \Rightarrow 1 - \chi_A = 1 - 1 = 0$.

- Therefore, we have that:

$$\chi_{A^c} = 1 - \chi_A$$

4.10

- By the properties of the characteristic function,

$$\chi_A = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

- Therefore, we have that, for any function f on E and $A \subseteq E$, we have that:

$$f \cdot \chi_A = \begin{cases} f, & x \in A \\ 0, & x \notin A \end{cases}$$

- Therefore, we have that:

$$\int_E f\chi_A = \int_{E \setminus A} f\chi_A + \int_A f\chi_A$$

- We know from the property above that, $f\chi_A = f$ everywhere on A and therefore,

$$\int_A f\chi_A = \int_A f$$

- Thus, we have that

$$\int_E f\chi_A = \int_A f$$

4.13

- Assume that, a sequence of functions is not uniformly bounded, that is, there exists no single constant M such that $|f_n| < M$ for all n.
- Consider the sequence of functions:

$$f_n = nx^{n-1}$$

on the set $E = (0, 1)$. We know this set has measure $m(E) = 1 - 0 = 1$ and, we also know that, the sequence of functions do not have a uniform bound.
- Thus, we have that,

$$\lim_{n\to\infty} \int_0^1 nx^{n-1}\, dx = x^n \Big|_0^1 = 1$$

but, we also have that,

$$\int_0^1 \lim_{n\to\infty} nx^{n-1} = 0$$

- Furthermore, we know that the function is defined on this part as, we are excluding 0 and 1 and therefore, we wont run into undefined epxressions such as 1^∞ or 0^∞.
- From this we can see that:

$$\lim_{n\to\infty} \int_E f_n = 1 \neq 0 = \int_E \lim_{n\to\infty} f_n$$

4.16

- Because f is non-negative, we know that $f(x) \geq 0$ for all x. Therefore, let

$$A = \{x \in E \mid f(x) = 0\}$$

and let

$$B = \{x \in E \mid f(x) > 0\}$$

- Therefore, we have that

$$\int_E f = \int_A f + \int_B f$$

but, because $f = 0$ on A, we have that:

$$\int_E f = \int_A f + \int_B f = \int_B f$$

Furthermore, we have that, $f(x) > 0$ for all B. Thus, if we want $\inf_E f = 0$, then we must have that $\int_B f = 0$ which will only happen if:

$$m(B) = 0$$

- As, if this was not the case and there is an open subset of B (meaning the subset can be made of intervals), then we will have that, $f(x) > 0$ on these intervals and thus, we will have that

$$\int_E f = \int_B f > 0$$

which would be a contradiction. Therefore, we must have that, $\int_B f = 0 \Rightarrow m(B) = 0$.

- Therefore, because $m(B) = 0$, we have that, $f(x) > 0$ almost nowhere and thus, $f(x) = 0$ almost everywhere.