PSTAT 120B

HW 5

05/19/21

By TJ Sipin

9.1)
$$f(y) = \begin{cases} \left(\frac{1}{\theta}\right) e^{-\frac{y}{\theta}}, & y > 0 \\ 0, & \text{o.w.} \end{cases}$$

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{(Y_1 + Y_2)}{2}, \quad \hat{\theta}_3 = \frac{(Y_1 + 2Y_2)}{3}$$

$$\hat{\theta}_5 = \overline{Y}$$

- $$\text{eff}(\hat{\theta}_1, \hat{\theta}_5) = \frac{\text{Var}(\hat{\theta}_5)}{\text{Var}(\hat{\theta}_1)}$$

$$\text{Var}(\hat{\theta}_1) = \text{Var}(Y_1) = \theta^2 \qquad \text{Var}(Y_i) = \theta^2$$

$$\begin{aligned} \text{Var}(\hat{\theta}_5) &= \text{Var}(\overline{Y}) \\ &= \text{Var}\left(\frac{Y_1 + Y_2 + Y_3}{3}\right) \\ &= \frac{1}{9}\left(\text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3)\right) \\ &= \frac{1}{9}(3\theta^2) = \frac{1}{3}\theta^2 \end{aligned}$$

$$\Rightarrow \text{eff}(\hat{\theta}_1, \hat{\theta}_5) = \frac{\frac{1}{3}\cancel{\theta^2}}{\cancel{\theta^2}} = \boxed{\frac{1}{3}}$$

$\text{eff}(\hat{\theta}_2, \hat{\theta}_5)$:

$$\text{Var}(\hat{\theta}_2) = \text{Var}\left(\frac{Y_1 + Y_2}{2}\right)$$

$$= \frac{1}{4} \cdot 2\theta^2 = \frac{1}{2}\theta^2$$

$$\Rightarrow \text{eff}(\hat{\theta}_2, \hat{\theta}_5) = \frac{1}{3}\cancel{\theta^2} \cdot \frac{1}{\frac{1}{2}\cancel{\theta^2}} = \boxed{\frac{2}{3}}$$

$\text{eff}(\hat{\theta}_3, \hat{\theta}_5)$:

$$\text{Var}(\hat{\theta}_3) = \text{Var}\left(\frac{Y_1 + 2Y_2}{3}\right) \Leftarrow Y_i \text{ independent}$$

$$= \frac{1}{9}\left(\theta^2 + 4\theta^2\right)$$

$$= \frac{5}{9}\theta^2$$

$$\Rightarrow \text{eff}(\hat{\theta}_3, \hat{\theta}_5) = \frac{1}{\cancel{3}}\cancel{\theta^2} \cdot \frac{\cancel{9}\,3}{5\cancel{\theta^2}} = \boxed{\frac{3}{5}}$$

9.17 $\overbrace{X_1, X_2, \ldots, X_n}^{\mu_1,\ \sigma_1^2}$ and $\overbrace{Y_1, Y_2, \ldots, Y_n}^{\mu_2,\ \sigma_2^2}$

Independent random samples

<u>WWTS</u>: $\bar{X} - \bar{Y}$ is a consistent estimator for $\mu_1 - \mu_2$

Note: $E(\bar{X}) = \mu_1$, $E(\bar{Y}) = \mu_2$

$V(\bar{X}) = \frac{\sigma_1^2}{n}$, $V(\bar{Y}) = \frac{\sigma_2^2}{n}$

- Must show $\hat{\theta} = \bar{X} - \bar{Y}$ is unbiased

$$E(\hat{\theta}) = E(\bar{X} - \bar{Y}) = \mu_1 - \mu_2$$
$$= E(\bar{X}) - E(\bar{Y})$$
$$= \mu_1 - \mu_2 \quad \checkmark$$

- Now must show

$$\lim_{n \to \infty} V(\bar{X} - \bar{Y}) = 0$$

Since $\bar{X}$, $\bar{Y}$ independent $\Rightarrow$

$$V(\bar{X} - \bar{Y}) = V(\bar{X}) - V(\bar{Y})$$
$$= \frac{\sigma_1^2}{n} - \frac{\sigma_2^2}{n}$$
$$\Rightarrow \lim_{n \to \infty} \left(\frac{\sigma_1^2}{n} - \frac{\sigma_2^2}{n}\right) = 0. \quad \checkmark \qquad \square$$

<u>9.20)</u> $Y \sim \text{bino}(n, p)$

WWTS: $\hat{\theta} = \frac{Y}{n}$ is consistent estimator of p.

or: $E\left(\frac{Y}{n}\right) = p$ & $\lim_{n \to \infty} V\left(\frac{Y}{n}\right) = 0$

$$E\left(\frac{Y}{n}\right) = \frac{E(Y)}{n} = \frac{1}{\cancel{n}} \cdot \cancel{n}p = p \quad \checkmark$$

$$V\left(\frac{Y}{n}\right) = \frac{1}{n^2} V(Y) = \frac{1}{\cancel{n^2}\, n} \cancel{n}p(1-p)$$

$$\lim_{n \to \infty} \frac{1}{n} p(1-p) = 0 \quad \checkmark$$

$\Rightarrow Y/n$ consistent estimator of p.

9.34) pdf: $f(y) = \begin{cases} \left(\frac{2y}{\theta}\right) e^{-y^2/\theta}, & y > 0 \\ 0, & \text{o.w.} \end{cases}$

$Y^2 \sim \exp$ $\quad E(Y^2) = \theta$

$Var(Y^2) = \theta^2$

WWTS: $W_n = \frac{1}{n} \sum_{i=1}^{n} Y_i^2$ is a consistent estimator for θ

- $E(W_n) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i)^2$

$$= \frac{1}{\cancel{n}} \cdot \cancel{n}\, \theta \quad ✓$$

- Now: $\lim_{n \to \infty} Var\left(\frac{1}{n} \sum_{i=1}^{n} Y_i^2\right) = 0?$

$$Var\left(\frac{1}{n} \sum_{i=1}^{n} Y_i^2\right) \overset{iid}{=} \frac{1}{n^{\cancel{2}}} \cdot \cancel{n} \cdot Var(Y_i^2)$$

$$= \frac{1}{n} \cdot \theta^2$$

$$\lim_{n \to \infty} \frac{1}{n} \theta^2 = 0 \quad ✓.$$

$\Rightarrow W_n$ is a consistent estimator for θ

9.40) $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim}$ Rayleigh (θ)

$f(y_i) = \begin{cases} \left(\frac{2y_i}{\theta}\right) e^{-y_i^2/\theta}, & y > 0 \\ 0, & \text{o.w.} \end{cases}$

WWTS: $\sum_{i=1}^{n} Y_i^2$ is sufficient for θ

$$L(y_1, y_2, \dots, y_n \mid \theta) = f(y_1, y_2, \dots, y_n \mid \theta)$$
$$= f(y_1 \mid \theta) \times f(y_2 \mid \theta) \times \dots \times f(y_n \mid \theta)$$
$$= \frac{2y_1}{\theta} e^{-y_1^2/\theta} \times \dots \times \frac{2y_n}{\theta} e^{-y_n^2/\theta}$$
$$= \left(\frac{2}{\theta}\right)^n (y_1 \times \dots \times y_n)\left(e^{-\frac{y_1^2}{\theta} - \dots - \frac{y_n^2}{\theta}}\right)$$
$$= \left(\frac{2}{\theta}\right)^n \left(\prod_{i=1}^n y_i\right)\left(e^{-\frac{1}{\theta}\sum y_i^2}\right)$$

Let $g(\sum y_i^2, \theta) = \left(\frac{2}{\theta}\right)^n e^{-\frac{1}{\theta}\sum y_i^2}$, and

$$h(y_1, \dots, y_n) = \prod_{i=1}^n y_i = y_1 y_2 \cdots y_n$$

$\Rightarrow \sum_{i=1}^n y_i^2$ is sufficient for θ.

9.58 Want: Use $\sum_{i=1}^n Y_i^2$ to find an MVUE of θ.

Let $w = Y_i^2$

$$\Rightarrow f_w(w) = f(\sqrt{w}) \frac{d\sqrt{w}}{dw}$$
$$= \left(\frac{\cancel{2}}{\theta}\right)\left(\cancel{\sqrt{w}}\, e^{-w/\theta}\right) \cdot \left(\frac{1}{\cancel{2}\cancel{\sqrt{w}}}\right)$$
$$= \frac{e^{-w/\theta}}{\theta}, \quad w > 0$$

$\Rightarrow Y_i^2 \sim \exp(\theta)$

$\Rightarrow \quad E(Y_i^2) = \theta \quad \Rightarrow \quad E(\sum_{i=1}^{n} Y_i^2) = n\theta$

$\Rightarrow \quad \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$ is an unbiased estimator for θ

$\Rightarrow$ Since $\hat{\theta}$ is a function of the sufficient statistic $\sum_{i=1}^{n} Y_i^2$, the estimator $\frac{1}{n}\sum_{i=1}^{n} Y_i^2 = \hat{\theta}$ is an MVUE for θ.