Axiom 1 (Reflexiveness for Equality): $x = x$.

Axiom 2 (Equality Used): If $x = y$, then $\Phi_{[a \to x]} \Leftrightarrow \Phi_{[a \to y]}$.

Theorem 3 (Symmetry and Transitivity for Equality):
(1) If $x = y$, then $y = x$.
(2) If $x = y$ and $y = z$, then $x = z$.

Theorem 4 (Equality Used): If $x = y$, then:
(1) $\Phi_{[a \to x]} \Leftrightarrow \Phi_{[a \to y]}$
(2) $\smiley_{[a \to x]} = \smiley_{[a \to y]}$
(3) $\Phi \Leftrightarrow \Phi_{[x \to y]}$
(4) $\smiley = \smiley_{[x \to y]}$

Axiom 5 (Membership): $a \in \{\, x \mid \Phi \,\} \Leftrightarrow \Phi_{[x \to a]}$.
Warning: this axiom is broken.

Definition 6 (Inclusion): $a \subseteq b \Leftrightarrow (\, \forall z \mid z \in a \Rightarrow z \in b \,)$.

Theorem 7 (Reflexivity and Transitivity for Inclusion):
(1) $a \subseteq a$.
(2) If $a \subseteq b$ and $b \subseteq c$, then $a \subseteq c$.

Axiom 8 (Equality Gained): If $(\, \forall z \mid z \in a \Leftrightarrow z \in b \,)$, then $a = b$.

Theorem 9: $y = \{\, z \mid z \in y \,\}$.

Theorem 10 (Equality as Double Inclusion): $a = b$ if and only if $a \subseteq b$ and $b \subseteq a$.

Definition 11 (Empty Set): $\{\} = \{\, x \mid x \neq x \,\}$

Theorem 12 (Empty Set is Empty): $x \notin \{\}$.

Theorem 13 (Minimality of the Empty Set):
(1) $\{\} \subseteq a$.
(2) If $a \subseteq \{\}$, then $a = \{\}$.

Theorem 14 (Only the Empty Set is Empty): If $x \notin a$ for all x, then $a = \{\}$.

Theorem 15 (Nonemptiness): $a \neq \{\} \Leftrightarrow (\, \exists x \mid x \in a \,)$.

Definition 16 (Singleton): $\{x\} = \{\, a \mid a = x \,\}$.

Definition 17 (Doubleton): $\{x, y\} = \{\, a \mid (a = x) \vee (a = y) \,\}$.

Definition 18 (More Ways to List Elements):

(1) $\{x, y, z\} = \{\, a \mid (a = x) \vee (a = y) \vee (a = z) \,\}$
(2) $\{x, y, z, u\} = \{\, a \mid (a = x) \vee (a = y) \vee (a = z) \vee (a = u) \,\}$
(3) $\{x, y, z, u, v\} = \{\, a \mid (a = x) \vee (a = y) \vee (a = z) \vee (a = u) \vee (a = v) \,\}$
(4) $\{x, y, z, u, v, w\} = \{\, a \mid (a = x) \vee (a = y) \vee (a = z) \vee (a = u) \vee (a = v) \vee (a = w) \,\}$

Theorem 19 (Singleton and Inclusion): $x \in S$ if and only if $\{x\} \subseteq S$.

Theorem 20 : $\{x, y\} = \{y, x\}$

Theorem 21 : $\{x\} = \{y\}$ if and only if $x = y$.

Theorem 22 : $\{x, y\} = \{u\}$ if and only if $x = y = u$.

Theorem 23 : If $\{x, y\} = \{u, v\}$, then either $x = u$ and $y = v$, or $x = v$ and $y = u$.

Definition 24 (Being a Singleton): A set is said to be a *singleton* iff it has the form $\{x\}$ for some x. That is, we call s a singleton iff $(\, \exists x \mid s = \{x\} \,)$.

Theorem 25 (Existence and Uniqueness in Terms of Singleton):
A set s is a singleton if and only if it has a unique element. That is, s is a singleton if and only if

$$(\, \exists x \mid x \in s \wedge (\, \forall y \mid y \in s \Rightarrow y = x \,)\,)\,.$$

Definition 26 (Occupant): Let s be a singleton. The *occupant* of s is denoted by $\operatorname{occ}(s)$ and defined by

$$\operatorname{occ}(s) = \{\, z \mid (\, \forall x \mid x \in s \Rightarrow z \in x \,)\,\}\,.$$

Theorem 27 (Essence of Occupant): $\operatorname{occ}(\{g\}) = g$.

Definition 28 (Intersection and Union):

(1) $a \cap b = \{\, x \mid (x \in a) \wedge (x \in b) \,\}$
(2) $a \cup b = \{\, x \mid (x \in a) \vee (x \in b) \,\}$

Theorem 29 (Algebraic Properties of Intersection and Union):

(1)	$a \cap b = b \cap a$	(commutativity of $\cap$)
(2)	$a \cup b = b \cup a$	(commutativity of $\cup$)
(3)	$(a \cap b) \cap c = a \cap (b \cap c)$	(associativity of $\cap$)
(4)	$(a \cup b) \cup c = a \cup (b \cup c)$	(associativity of $\cup$)
(5)	$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$	($\cup$ distributes over $\cap$)
(6)	$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$	($\cap$ distributes over $\cup$)
(7)	$a \cap a = a$	(idempotence of $\cap$)
(8)	$a \cup a = a$	(idempotence of $\cup$)
(9)	$a \cup \{\} = a$	($\{\}$ is an identity for $\cup$)
(10)	$a \cap \{\} = \{\}$	($\{\}$ is an annihilator for $\cap$)

Theorem 30 (Maximality of Intersection):

The set $b \cap c$ is a subset of both b and c, and it contains anything that is a subset of both b and c. That is,

(1) $b \cap c \subseteq b$
(2) $b \cap c \subseteq c$
(3) If $a \subseteq b$ and $a \subseteq c$, then $a \subseteq b \cap c$.

Theorem 31 (Minimality of Union):

The set $b \cup c$ contains both b and c, and it is a subset of any set that contains both b and c. That is,

(1) $b \subseteq b \cup c$
(2) $c \subseteq b \cup c$
(3) If $b \subseteq a$ and $c \subseteq a$, then $b \cup c \subseteq a$.

Theorem 32 (Inclusion in Terms of $\cap$ and $\cup$):

The following are equivalent:

(1) $a \subseteq b$
(2) $a \cap b = a$
(3) $a \cup b = b$

Definition 33 (Relative Complement): $a \setminus b = \{\, x \mid (x \in a) \wedge (x \notin b) \,\}$

Theorem 34 (De Morgan's Law - Set Theory Version):

(1) $a \setminus (b \cap c) = (a \setminus b) \cup (a \setminus c)$
(2) $a \setminus (b \cup c) = (a \setminus b) \cap (a \setminus c)$

Theorem 35 (Iterated Relative Complement): $a \setminus (a \setminus b) = a \cap b$.

Definition 36 (Power Set): $\mathcal{P}(x) = \{\, y \mid y \subseteq x \,\}$.

Theorem 37: $\mathcal{P}(\{x\}) = \{\{\}, \{x\}\}$.

Definition 38 (Ordered Pair): $(x, y) = \{\{x\}, \{x, y\}\}$.

Theorem 39 (Essence of Pairs):

$$(x, y) = (u, v) \text{ if and only if } x = u \text{ and } y = v.$$

Definition 40: A set is called a *pair* if and only if it has the form (a, b) for some a and b. That is, we call x a pair if and only if

$$(\, \exists a, b \mid x = (a, b) \,)\, .$$

A set s is called a *set of pairs* or a *relation* if every element of s is a pair.

Definition 41 (Cartesian Product):

$$A \times B = \{\, x \mid (\, \exists a, b \mid x = (a, b) \,\wedge\, a \in A \,\wedge\, b \in B \,)\,\}\, .$$

Theorem 42 (Essence of Cartesian Product):

$$(y, z) \in A \times B \text{ if and only if } y \in A \text{ and } z \in B.$$

Theorem 43 (Cartesian Products and Emptiness):
(1) $A \times B$ is nonempty if and only if A and B are both nonempty.
(2) $A \times B$ is empty if and only if either A is empty or B is empty.

Theorem 44: If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.

Theorem 45:

$$(A \times C) \cap (B \times D) \subseteq (A \cap B) \times (C \cap D).$$

Theorem 46: If A is nonempty and $A \times C \subseteq B \times D$, then $C \subseteq D$.

Definition 47 (Domain and Range): Let S be a set of pairs. The *domain* and the *range* of S are denoted respectively by $\mathrm{dom}(S)$ and $\mathrm{ran}(S)$, and they are defined as follows:

$$\begin{aligned} \mathrm{dom}(S) &= \{\, x \mid (\, \exists y \mid (x, y) \in S \,)\,\} \\ \mathrm{ran}(S) &= \{\, y \mid (\, \exists x \mid (x, y) \in S \,)\,\} \end{aligned}$$

Definition 48 (Function): A set f is said to be a *function* iff f is a set of pairs and

$$(\, \forall x, y, z \mid (x, y), (x, z) \in f \;\Rightarrow\; y = z \,)\, .$$

Theorem 49 : If f is a function and $x \in \mathrm{dom}(f)$, then

$$\{\, y \mid (x, y) \in f \,\}$$

is a singleton.

Definition 50 (Function Evaluation): Let f be a function and let $x \in \mathrm{dom}(f)$. By Theorem 49, there is a unique y such that $(x, y) \in f$. We denote this unique y by "$f(x)$", and we call it *the value of f at x*. Formally, we are making the definition

$$f(x) = \mathrm{occ}(\{\, y \mid (x, y) \in f \,\}).$$

Theorem 51 (Essence of Evaluation): If f is a function then

(1) If $x \in \mathrm{dom}(f)$, then $(x, f(x)) \in f$.
(2) If $(x, y) \in f$, then $y = f(x)$.
(3) If $x \in \mathrm{dom}(f)$, then $(x, y) \in f \Leftrightarrow y = f(x)$.

Theorem 52 : If f is a function, then $y \in \mathrm{ran}(f)$ if and only if $y = f(x)$ for some $x \in \mathrm{dom}(f)$.

Theorem 53 (Equality of Functions): If f and g are functions, then $f = g$ if and only if $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $f(x) = g(x)$ for all $x \in \mathrm{dom}(f)$.

Definition 54 (Mapping):

$$f : X \to Y \quad \Leftrightarrow \quad \begin{array}{ll} (& f \text{ is a function} \quad \wedge \\ & \mathrm{dom}(f) = X \quad \wedge \\ & \mathrm{ran}(f) \subseteq Y \qquad). \end{array}$$

Theorem 55 : If f is a function, then $f : X \to Y$ if and only if $f \subseteq X \times Y$ and $X \subseteq \mathrm{dom}(f)$.

Theorem 56 (Piecewise Function Definition):
If f and g are functions and $\mathrm{dom}(f) \cap \mathrm{dom}(g) = \{\}$, then $f \cup g$ is a function.

Definition 57 (Injectivity, Surjectivity, and Bijectivity):
Assume that $f : X \to Y$.

(1) f is *injective* iff for all $x, y \in X$, we have $f(x) = f(y) \Rightarrow x = y$.
(2) f is *surjective onto* Y iff $\mathrm{ran}(f) = Y$.
(3) f is *bijective onto* Y iff f is both injective and surjective onto Y.

Theorem 58 : Assume that f is a function. Then f is injective if and only if

$$(\, \forall x, y, z \mid (x, y), (z, y) \in f \;\Rightarrow\; x = z \,)\,.$$

Theorem 59 : Assume that $f : X \to Y$. Then f is surjective onto Y if and only if

$$(\, \forall y \mid y \in Y \;\Rightarrow\; (\, \exists x \mid x \in X \;\wedge\; f(x) = y \,)\,)\,.$$

Definition 60 (Composite):

$$S \circ T = \{\, (x, z) \mid (\, \exists y \mid (x, y) \in T \;\wedge\; (y, z) \in S \,)\,\}\,.$$

Theorem 61 (Associativity of Composition): $S \circ (T \circ U) = (S \circ T) \circ U$

Theorem 62 (Domain and Range of Composite):

(1) If $\mathrm{ran}(T) \subseteq \mathrm{dom}(S)$, then $\mathrm{dom}(S \circ T) = \mathrm{dom}(T)$.

(2) $\mathrm{ran}(S \circ T) \subseteq \mathrm{ran}(S)$

Theorem 63 (Essence of Composition):

(1) If f and g are functions, then $g \circ f$ is a function and

$$(g \circ f)(x) = g(f(x))$$

for all $x \in \mathrm{dom}(g \circ f)$.

(2) If $f : X \to Y$ and $g : Y \to Z$, then $g \circ f : X \to Z$.

Definition 64 (Identity Function): The *identity function on* A is denoted by id_A and defined by

$$\mathrm{id}_A = \{\, (x, x) \mid x \in A \,\}\,.$$

Theorem 65 (Facts about id$_A$):

(1) $\mathrm{id}_A : A \to A$.

(2) $\mathrm{id}_A(x) = x$ for all $x \in A$.

(3) If $f : A \to B$, then $f \circ \mathrm{id}_A = f$.

(4) If $g : B \to A$, then $\mathrm{id}_A \circ g = g$.

Definition 66 (Reverse): The *reverse* of S is denoted by $S^{\dashv}$ and defined by

$$S^{\dashv} = \{\, (y, x) \mid (x, y) \in S \,\}\,.$$

Theorem 67 (Essence of Reversal): $(x, y) \in S \Leftrightarrow (y, x) \in S^{\dashv}$.

Theorem 68 (Double Reversal): If S is a set of pairs, then

$$S^{\dashv\dashv} = S.$$

Theorem 69 (Reversal, Domain, and Range):

(1) $\mathrm{dom}(S^{\dashv}) = \mathrm{ran}(S)$.

(2) $\mathrm{ran}(S^{\dashv}) = \mathrm{dom}(S)$.

Theorem 70 (Reversal and Composition): $(S \circ T)^{\dashv} = T^{\dashv} \circ S^{\dashv}$.

Definition 71 (Invertibility): If f is a function, then f is said to be *invertible* if and only if $f^{\dashv}$ is also a function. In this context, we will refer to $f^{\dashv}$ as *the inverse* of f.

Theorem 72 (Invertibility of Injections): If f is a function, then $f^{\dashv}$ is a function if and only if f is injective.

Theorem 73: If $f : X \to Y$ and f is injective, then $f^{\dashv} : \mathrm{ran}(f) \to X$.

Theorem 74 : If f is an invertible function, $x \in \mathrm{dom}(f)$, and $y \in \mathrm{ran}(f)$, then

$$f(x) = y \;\Leftrightarrow\; x = f^{\dashv}(y)$$

Theorem 75 : If $f : X \to Y$ and f is a bijection onto Y, then $f^{\dashv} : Y \to X$ and $f^{\dashv}$ is a bijection onto X.

Theorem 76 : If $f : X \to Y$ and f is injective, then $f^{\dashv} \circ f = \mathrm{id}_X$.

Theorem 77 (Inverse and Composition): If $f : X \to Y$ and f is a bijection onto Y, then
(1) $f \circ f^{\dashv} = \mathrm{id}_Y$.
(2) $f^{\dashv} \circ f = \mathrm{id}_X$.

Theorem 78 (Composition, Injectivity, and Surjectivity):
Assume that $f : X \to Y$ and $g : Y \to Z$.
(1) If f and g are injective, then $g \circ f$ is injective.
(2) If $g \circ f$ is injective, then f is injective.
(3) If f is surjective onto Y and g is surjective onto Z, then $g \circ f$ is surjective onto Z.
(4) If $g \circ f$ is surjective onto Z, then g is surjective onto Z.
(5) If f is bijective onto Y and g is bijective onto Z, then $g \circ f$ is bijective onto Z.

Theorem 79 (Cancellation of Composite): Assume that $f, g : X \to Y$.
(1) If $h : Y \to Z$, h is injective, and $h \circ f = h \circ g$, then $f = g$.
(2) If $h : Z \to X$, h is surjective onto X, and $f \circ h = g \circ h$, then $f = g$.

Theorem 80 (Left-Invertible Functions): Assume that $f : X \to Y$. If

$$(\, \exists g \mid g : Y \to X \;\wedge\; g \circ f = \mathrm{id}_X \,),$$

then f is injective.

Theorem 81 (Right-Invertible Functions): Assume that $f : X \to Y$. If

$$(\, \exists g \mid g : Y \to X \;\wedge\; f \circ g = \mathrm{id}_Y \,),$$

then f is surjective onto Y.

Axiom 82 (Axiom of Choice): If $f : X \to Y$ and f is surjective onto Y, then there exists a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

Theorem 83 (Bijectivity and Isomorphism): Assume that $f : X \to Y$. Then f is a bijection onto Y if and only if

$$(\, \exists g \mid g : Y \to X \;\wedge\; g \circ f = \mathrm{id}_X \;\wedge\; f \circ g = \mathrm{id}_Y \,).$$

Definition 84 (Image): The *image of A under S* is denoted by $S[A]$ and defined by

$$S[A] = \{\, y \mid (\, \exists x \mid x \in A \wedge (x, y) \in S \,) \,\}.$$

Theorem 85 (Image for Functions): Assume that $f : X \to Y$.

(1) If $A \subseteq X$, then

$$f[A] = \{\, f(a) \mid a \in A \,\}.$$

In other words,

$$y \in f[A] \;\Leftrightarrow\; (\, \exists a \mid y = f(a) \wedge a \in A \,).$$

(2)

$$f^{\dashv}[B] = \{\, x \mid x \in X \wedge f(x) \in B \,\}.$$

So if $x \in X$, then

$$x \in f^{\dashv}[B] \;\Leftrightarrow\; f(x) \in B.$$

Theorem 86 (Image and Composition): $(g \circ f)[A] = g[f[A]]$.

Theorem 87 : Assume that $f : X \to Y$, $A \subseteq X$, and $B \subseteq Y$.

(1) $f[f^{\dashv}[B]] \subseteq B$.
(2) $A \subseteq f^{\dashv}[f[A]]$.
(3) If $B \subseteq \mathrm{ran}(f)$, then $f[f^{\dashv}[B]] = B$.
(4) If f is injective, then $f^{\dashv}[f[A]] = A$.

Theorem 88 (Preimage Plays Nice): Assume that $f : X \to Y$, $A \subseteq Y$, and $B \subseteq Y$.

(1) $f^{\dashv}[A \cup B] = f^{\dashv}[A] \cup f^{\dashv}[B]$
(2) $f^{\dashv}[A \cap B] = f^{\dashv}[A] \cap f^{\dashv}[B]$
(3) $f^{\dashv}[A \setminus B] = f^{\dashv}[A] \setminus f^{\dashv}[B]$

Definition 89 (Successor): $\mathrm{next}(x) = x \cup \{x\}$.

Definition 90 (Certain Numbers):

$$\begin{array}{ll} 0 = \{\} & 5 = \operatorname{next}(4) \\ 1 = \operatorname{next}(0) & 6 = \operatorname{next}(5) \\ 2 = \operatorname{next}(1) & 7 = \operatorname{next}(6) \\ 3 = \operatorname{next}(2) & 8 = \operatorname{next}(7) \\ 4 = \operatorname{next}(3) & 9 = \operatorname{next}(8) \end{array}$$

Definition 91 (Inductive): A set J is said to be *inductive* if it contains 0 and it contains $\operatorname{next}(n)$ for every n that it contains. That is,

$$J \text{ is inductive } \Leftrightarrow \ (0 \in J \wedge (\forall n \mathrel{¦} n \in J \Rightarrow \operatorname{next}(n) \in J\,))$$

Definition 92: $\mathbb{N} = \{\, n \mathrel{¦} (\forall J \mathrel{¦} J \text{ is inductive } \Rightarrow \ n \in J\,)\,\}$.

Theorem 93 (Essence of $\mathbb{N}$):
(1) $\mathbb{N}$ is inductive.
(2) If J is inductive, then $\mathbb{N} \subseteq J$.

Theorem 94 (Mathematical Induction):
If

$$\Phi_{[x \to 0]}$$

and

$$\big(\forall n \mathrel{¦} n \in \mathbb{N} \ \Rightarrow \ (\Phi_{[x \to n]} \ \Rightarrow \ \Phi_{[x \to \operatorname{next}(n)]})\big)$$

both hold, then it follows that

$$\big(\forall n \mathrel{¦} n \in \mathbb{N} \Rightarrow \ \Phi_{[x \to n]}\big)\,.$$

Definition 95 (Order for $\mathbb{N}$):
(1) $x < y \ \Leftrightarrow \ x \in y$.
(2) $x \leq y \ \Leftrightarrow \ (x \in y \vee x = y)$.

Theorem 96 (Essence of Order and Succession):
(1) $x < \operatorname{next}(x)$.
(2) $x < \operatorname{next}(y) \ \Leftrightarrow \ x \leq y$.

Theorem 97 (About 0): If $n \in \mathbb{N}$, then
(1) $0 < \operatorname{next}(n)$.
(2) $0 \leq n$.
(3) $0 \neq \operatorname{next}(n)$.

Theorem 98 (Predecessor): If $n \in \mathbb{N}$ and $n \neq 0$, then $n = \operatorname{next}(m)$ for some $m \in \mathbb{N}$.

Theorem 99 (Succession Preserves Order): Assume that $n \in \mathbb{N}$.
(1) If $m < n$, then $\mathrm{next}(m) \leq n$.
(2) If $m < n$, then $\mathrm{next}(m) < \mathrm{next}(n)$.

Theorem 100 (Transitivity for Order on $\mathbb{N}$): Assume that $n, m, k \in \mathbb{N}$.
(1) If $n < m$ and $m < k$, then $n < k$.
(2) If $n \leq m$ and $m < k$, then $n < k$.
(3) If $n < m$ and $m \leq k$, then $n < k$.
(4) If $n \leq m$ and $m \leq k$, then $n \leq k$.

Theorem 101 : If $n < m$ and $m \in \mathbb{N}$, then $n \in \mathbb{N}$.

Theorem 102 (Irreflexivity for Order on $\mathbb{N}$): If $n \in \mathbb{N}$, then $\neg(n < n)$

Theorem 103 (Successor Cancels): Assume that $n, m \in \mathbb{N}$.
(1) If $\mathrm{next}(m) < \mathrm{next}(n)$, then $m < n$
(2) If $\mathrm{next}(m) = \mathrm{next}(n)$, then $m = n$.

Theorem 104 (Trichotomy for Order on $\mathbb{N}$): Assume that $n, m \in \mathbb{N}$. Then either $m < n$, $n < m$, or $m = n$, and only *one* of these conditions holds. That is,
(1) $(m = n) \ \vee \ (m < n) \ \vee \ (n < m)$.
(2) $\neg(m = n \ \wedge \ m < n)$.
(3) $\neg(m = n \ \wedge \ n < m)$.
(4) $\neg(m < n \ \wedge \ n < m)$.

Definition 105 (Minimum): Given $A \subseteq \mathbb{N}$, we say that n *is a minimum of* A if and only if $n \in A$ and $(\forall a \mid a \in A \ \Rightarrow \ n \leq a)$.

Theorem 106 (Well-Ordering of $\mathbb{N}$): Every nonempty set of natural numbers has a minimum. That is, if $A \subseteq \mathbb{N}$ then

$$A \neq \{\} \Rightarrow (\exists n \mid n \in A \ \wedge \ (\forall a \mid a \in A \Rightarrow n \leq a)).$$

Theorem 107 (Definition by Recursion - Existence):
If $s \in X$ and $r : X \to X$, then there exists an $f : \mathbb{N} \to X$ such that $f(0) = s$ and

$$(\forall n \mid n \in \mathbb{N} \ \Rightarrow \ f(\mathrm{next}(n)) = r(f(n))).$$

Theorem 108 (Definition by Recursion - Uniqueness):
Assume that $s \in X$ and $r : X \to X$. If
(1) $f, g : \mathbb{N} \to X$,
(2) $f(0) = g(0) = s$,
(3) $(\forall n, x \mid n \in \mathbb{N} \Rightarrow f(\mathrm{next}(n)) = r(f(n)))$, and
(4) $(\forall n, x \mid n \in \mathbb{N} \Rightarrow g(\mathrm{next}(n)) = r(g(n)))$,
then $f = g$.

Definition 109 (Adding One): $\ddagger = \{\, (n, \mathrm{next}(n)) \mid n \in \mathbb{N} \,\}$.

Definition 110 (Adding n): Define $\mathcal{ADD} : \mathbb{N} \to \{\, f \mid f : \mathbb{N} \to \mathbb{N} \,\}$ to be the unique function that satisfies

- $\mathcal{ADD}(0) = \mathrm{id}_{\mathbb{N}}$ and
- $\mathcal{ADD}(\mathrm{next}(n)) = \ddagger \circ (\mathcal{ADD}(n))$ for all $n \in \mathbb{N}$.

Definition 111 (Addition): Assume that $a, b \in \mathbb{N}$. Then we define

$$a + b = \mathcal{ADD}(b)(a).$$

Theorem 112 (Essence of $+$): Assume that $m, n \in \mathbb{N}$. Then

(1) $m + n \in \mathbb{N}$.
(2) $m + 0 = m$.
(3) $m + \mathrm{next}(n) = \mathrm{next}(m + n)$.
(4) $\mathrm{next}(n) = n + 1$.
(5) $m + (n + 1) = (m + n) + 1$.

Theorem 113 (Properties of $+$): Assume that $m, n, k \in \mathbb{N}$. Then

(1) $m + (n + k) = (m + n) + k$ (associativity for $+$)
(2) $n + 0 = n$ (0 is a right-identity for $+$)
(3) $0 + n = n$ (0 is a left-identity for $+$)
(4) $1 + n = n + 1$ (preliminary to commutativity)
(5) $m + n = n + m$ (commutativity for $+$)
(6) $m < n \;\Rightarrow\; m + k < n + k$ ($+$ respects $<$)
(7) $m + k = n + k \;\Rightarrow\; m = n$ ($+$ cancels under $=$)
(8) $m + k < n + k \;\Rightarrow\; m < n$ ($+$ cancels under $<$)
(9) $m + n = 0 \;\Rightarrow\; (m = 0 \wedge n = 0)$
(10) $m + n = 1 \;\Rightarrow\; ((m = 1 \wedge n = 0) \vee (m = 0 \wedge n = 1))$
(11) $m \leq m + k$
(12) $m \leq n \;\Rightarrow\; (\,\exists j \mid j \in \mathbb{N} \wedge m + j = n\,)$ (solvability of certain equations)

Definition 114 (Cardinality):

(1) $A \approx B \;\Leftrightarrow\; (\,\exists f \mid f : A \to B \wedge f$ is a bijection onto $B\,)$.
(2) $A \preceq B \;\Leftrightarrow\; (\,\exists f \mid f : A \to B \wedge f$ is injective$\,)$.
(3) $A \prec B \;\Leftrightarrow\; (A \preceq B \wedge \neg(A \approx B))$.

Theorem 115 :

(1) $A \approx A$ (reflexivity for $\approx$)
(2) If $A \approx B$ then $B \approx A$. (symmetry for $\approx$)
(3) If $A \approx B$ and $B \approx C$, then $A \approx C$. (transitivity for $\approx$)

Theorem 116 :

(1) $A \preceq A$ (reflexivity for $\preceq$)

(2) If $A \preceq B$ and $B \preceq C$, then $A \preceq C$. (transitivity for $\preceq$)

Theorem 117 (Preliminary to Cantor-Schroeder-Bernstein): If $C \subseteq A$ and $A \preceq C$, then $A \approx C$.

Theorem 118 (Cantor-Schroeder-Bernstein, AKA Antisymmetry for $\preceq$): If $A \preceq B$ and $B \preceq A$, then $A \approx B$.

Theorem 119: If $A \preceq B$, then there exists a surjection $g : B \to A$.

Definition 120 (Finiteness):

(1) We say that A *has* n *elements* iff $n \in \mathbb{N}$ and $A \approx n$.

(2) A set is *finite* iff it has n elements for some $n \in \mathbb{N}$.

(3) A set is *infinite* iff it is not finite.

Theorem 121 (Element Swap): If $p, q \in A$, then there is a bijection $f : A \to A$ such that $f(p) = q$ and $f(q) = p$.

Theorem 122 ($\preceq$ and $\leq$): If $n, m \in \mathbb{N}$, then

$$n \preceq m \;\Leftrightarrow\; n \leq m.$$

Theorem 123 ($\approx$ and $=$ for Natural Numbers): If $n, m \in \mathbb{N}$, then

$$n \approx m \;\Leftrightarrow\; n = m.$$

Theorem 124 (Finite Choice - Preliminary Version): If $n \in \mathbb{N}$ and $f : A \to n$ surjectively, then there exists $g : n \to A$ such that $f \circ g = \mathrm{id}_n$.

Theorem 125 (Finite Choice): If X is finite and $f : A \to X$ surjectively, then there exists $g : X \to A$ such that $f \circ g = \mathrm{id}_X$.

Theorem 126 : If X is finite and $f : A \to X$ surjectively, then $X \preceq A$.

Theorem 127 (Adding One Element):

(1) If X has n elements and $a \notin X$, then $X \cup \{a\}$ has $n + 1$ elements.

(2) If X is finite, then $X \cup \{a\}$ is finite.

Theorem 128 (Subsets Inherit Finiteness - Preliminary Version): If $n \in \mathbb{N}$ and $S \subseteq n$, then S is finite.

Theorem 129 (Images Inherit Finiteness): If $f : A \to B$, $X \subseteq A$, and X is finite, then $f[X]$ is finite.

Theorem 130 (Subsets Inherit Finiteness): If X is finite and $S \subseteq X$, then S is finite.

Theorem 131 ($\cup$ and $+$): If the set X has n elements, the set Y has m elements, and $X \cap Y = \{\}$, then $X \cup Y$ has $n + m$ elements.

Theorem 132: Assume that $n \in \mathbb{N}$ and $f : n \to n$. If f is injective, then f is surjective onto n.

Theorem 133 (Infinitude of $\mathbb{N}$): $\mathbb{N}$ is infinite.

Definition 134 (Countability): A set A is said to be *countably infinite* iff $A \approx \mathbb{N}$.

Theorem 135 (Hilbert's Hotel): $\text{next}(\mathbb{N}) \approx \mathbb{N}$.

Definition 136 (Even and Odd):

(1) n is *even* iff $n = k + k$ for some $k \in \mathbb{N}$.
(2) n is *odd* iff $n = k + k + 1$ for some $k \in \mathbb{N}$.

Theorem 137 : Every natural number is either odd or even.

Theorem 138 : A natural number cannot be both odd and even.

Theorem 139 : If $k, \ell \in \mathbb{N}$ and $k + k = \ell + \ell$ then $k = \ell$.

Theorem 140 (Doubling Countable Infinity): $2 \times \mathbb{N} \approx \mathbb{N}$.

Theorem 141 (Squaring Countable Infinity): $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.

Theorem 142 (Cantor's Theorem): There does not exist a surjection $f : A \to \mathcal{P}(A)$.

Definition 143 (Uncountability): A set A is said to be *uncountably infinite* iff it is infinite, but it is not countably infinite.

Definition 144 (Relations):

(1) R is a *relation on A* if and only if $R \subseteq A \times A$.
(2) When R is a relation on A, we write "xRy" to mean "$(x, y) \in R$."

Definition 145 (Adjectives for Relations): Assume that $\sim \subseteq A \times A$.

(1) $\sim$ is *reflexive* on A iff $a \sim a$ for all $a \in A$.
(2) $\sim$ is *symmetric* iff $a \sim b \;\Rightarrow\; b \sim a$ for all a, b.
(3) $\sim$ is *transitive* iff $((a \sim b) \;\wedge\; (b \sim c)) \;\Rightarrow\; (a \sim c)$ for all a, b, c.
(4) $\sim$ is *antisymmetric* iff $((a \sim b) \;\wedge\; (b \sim a)) \;\Rightarrow\; (a = b)$ for all a, b.
(5) $\sim$ is *total* on A iff $(a \sim b) \;\vee\; (b \sim a)$ for all $a, b \in A$.

Definition 146 (Special Kinds of Relations): Assume that $\sim \subseteq A \times A$.

(1) $\sim$ is a *preorder* on A iff it is reflexive on A and transitive.
(2) $\sim$ is a *partial order* on A iff it is an *antisymmetric* preorder on A.
(3) $\sim$ is a *total order* on A iff it is a *total* partial order on A.
(4) $\sim$ is an *equivalence relation* on A iff it is a *symmetric* preorder on A.

Definition 147 (Equivalence Class): Assume that $\sim$ is an equivalence relation on A and $a \in A$. Then we define

$$[a]_\sim = \{\, x \mid x \sim a \,\}$$

and we call this the *equivalence class of* a, or the *equivalence class of* a *modulo* $\sim$. If the relation involved is clear from context, then we may simply write $[a]$ instead of $[a]_\sim$.