HW-6
Math 117

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Problem 48

From the definition of continuous we know that, a function f is continuous at x if, for $\epsilon > 0$ $\exists \delta \ni |x - x'| < \delta$ for some x' implies that $|f(x) - f(x')| < \epsilon$. Because we are told that, for $x \notin \mathbb{Q}$, $f(x) = x$, for any $\epsilon > 0$ we have that $x - \epsilon \in \mathbb{R}$ and $x + \epsilon \in \mathbb{R}$. Furthermore, we also know that the irrationals are dense in the reals. Therefore, for any $x \notin \mathbb{Q}$ we can set $\delta = \epsilon$ and therefore, there exists x' such that $|x - x'| < \delta$ as the irrationals are dense in $reals$ and therefore, $|f(x) - f(x')| = |x - x'| < \delta = \epsilon$. And by definition, f is continuous at all $x \in \mathbb{R} \setminus \mathbb{Q}$.

Now assume that $x \in \mathbb{Q} \ni x = \frac{p}{q}$. Therefore, by the given function we know that $f(x) = p \sin(\frac{1}{q})$ and by the properties of sin, we know that for $x > 0$ we have that $x > sin(x)$. Furthermore, again, because the irrationals are dense in the $\mathbb{R}$ we have that there exists $y > x \ni |y - x| < \delta$ for some δ. Now focus on $|f(y) - f(x)|$. Because $y \notin \mathbb{Q}$ we have that $|f(y) - f(x)| = |y - f(x)|$. Furthermore, because $y > x$ we have that $|y - f(x)| > |x - f(x)|$. Finally, substituting in the values for x and $f(x)$ we have that $|x - f(x)| \geq |\frac{p}{q} - p \sin(\frac{1}{q})| > 0$. Now, if we let $\lambda = \frac{1}{q} - 1 \sin(\frac{1}{q}) > 0$ we have that

$$|f(x) - f(y)| = |y - f(x)| > |x - f(x)| = |\frac{p}{q} - p \sin(\frac{1}{q})| = |p|\lambda$$

for some $|y - x| < \delta$. Finally, if we let $\lambda = \frac{\epsilon}{|p|}$ we have that $|f(x) - f(y)| > \epsilon$ and therefore, by definition f is not continuous at x.

- Therefore, f is continuous at all the irrational numbers but it is not continuous at all the rational numbers.

49

(i)

Because f and g are continuous, the properties of continuous functions apply to these two. Now, let $x \in E$ be a number. Therefore, because both f and g are continuous, we know that, for some $\frac{\epsilon}{2} > 0$, we have some $\delta_f > 0$ such that for $x_f \in (x - \delta_f, x + \delta_f)$ we have that $|f(x) - f(x_f)| < \frac{\epsilon}{2}$ and similarly, there exists some $\delta_g > 0$ such that for $x_g \in (x - \delta_g, x + \delta_g)$ we have that $|g(x) - g(x_g)| < \frac{\epsilon}{2}$. Now let $\delta = max\{\delta_f, \delta_g\}$. Therefore, we have that for $x \in E$ and $x' \in (x - \delta, x + \delta)$ we have that:

$$\begin{aligned} |(f(x) + g(x)) - (f(x') + g(x'))| &= |(f(x) - f(x')) + (g(x) - g(x'))| \\ &\leq |f(x) - f(x')| + |g(x) - g(x')| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

- And therefore, by definition of continuous we have that $f + g$ is continuous at x and by extension, it is continuous wherever f and g are continuous.

(ii)

- Let $x, x' \in E$ such that $h(y) = x'$ and $h(y') = x'$. For $\epsilon > 0$, there exists some $\delta_f > 0$ such that for $x' \in (x - \delta_f, x + \delta_f)$ implies that $f(x') \in (f(x) - \epsilon, f(x) + \epsilon)$. But, we also know that h is continuous. Therefore, for $\delta_f > 0$ we have that there exist some $\delta_h > 0$ such that for $y' \in (y - \delta_h, y + \delta_h)$ we have that $h(y') \in (h(y) - \delta_f, h(y) + \delta_f)$. Therefore, $|f \circ h(y) - f \circ h(y')| =$

$|f(x) - f(x')| < \epsilon$ for $|y - y'| < \delta_h$ and therefore, by definition $f \circ h$ is continuous

(iii)

- Let the difference of the two function $g(x) - f(x) = d(x)$. Now, let us first define the max function:

$$max(f(x), g(x)) = \begin{cases} f(x), & d(x) < 0 \\ g(x), & d(x) > 0 \end{cases}$$

- Because we know that $d(x)$ can only take 3 possible range of values, $d(x) > 0$, $d(x) < 0$ or $d(x) = 0$, we have covered two possible ranges here.
- Now, for an invterval or value such that $d(x) > 0$, we have that the $max(x) = g(x)$. But, g is continuous and therefore, $max(f(x), g(x)$ is also continuous at this point. For any point or range such that $d(x) < 0$ we have that $max(f(x), g(x)) = f(x)$. But, f is a continuous function and therefore, for these values we have that $max(f(x), g(x))$ is a continuous function. Now we only have to prove continuity for the $d(x) = 0 \Rightarrow f(x) = g(x)$ case.
- Let $y \in E \ni g(y) = f(y)$. Then, by definition of continuous, for f, for some $\epsilon > 0$, there exists aδ_f such that for $y' \in (y - \delta_f, y + \delta_f$, we have that $|f(y) - f(y')| < \epsilon$. Similarly, there exists some δ_g such that for $y' \in (y - \delta_g, y + \delta_g)$ we have that $|g(y) - g(y')| < \epsilon$. Now let $\delta = max(\delta_f, \delta_g)$. Now assume without loss of generality that $g(y') > f(y')$, then we have that for $|y - y'| < \delta$,

we have that $|max(f(y), g(y)) - max(g(y'), f(y')| = |g(y) - g(y')| < \epsilon$ as $|y - y'| < \delta_g \leq \delta$ and therefore by definition $max(f(x), g(x))$ is continuous wherever f and g are continuous.

(iv)

- Let $h(x) = -f(x)$. Then, we know that the absolute value function entails:

$$|f| = \begin{cases} f(x), & f(x) > 0 \\ h(x), & h(x) < 0 \end{cases}$$

- Therefore, this is the same as, $|f| = max\{f(x), h(x)\}$ as, if $f(x) < 0$, then $h(x) = -f(x) > 0 > f(x)$ which implies $max\{f(x), h(x)\} = h(x)$ and conversely, if $f(x) > 0 \Rightarrow h(x) = -f(x) < 0 < f(x)$ which implies $max\{f(x), h(x)\} = f(x)$.
- Furthermore, we know that, because f is continuous, then $-f$ must also be continuous. Therefore, we have two continuous functions $f(x)$ and $h(x) = -f(x)$ and a maximum function $|f| = max\{f(x), h(x)\}$. Therefore, by what we have shown earlier (part iii), $|f| = max\{f(x), h(x)\}$ is continuous wherever f and $-f$ are continuous.

50

- Let f be a Lipshitz function on some set E. Therefore, we know that:

$$|f(x) - f(x')| < c|x - x'|$$

for all x, x' and some fixed value of c. Now, for some $\epsilon > 0$, let $\delta_1 = \frac{\epsilon}{c}$. Thus for $|x - x'| < \delta_1$ we have that

$$|f(x) - f(x')| < c|x - x'| < c\delta_1 = c \cdot \frac{\epsilon}{c} = \epsilon$$

and therefore by definition f is uniformly continuous. - Thus a Lipshitz function is uniformly continuous.

- Now, let $f(x) = \sqrt{c}$. Therefore, we have:

$$\begin{aligned}
|f(x) - f(x')| &= |\sqrt{x} - \sqrt{x'}| \\
&= |\frac{(\sqrt{x} - \sqrt{x'})(\sqrt{x} + \sqrt{x'})}{(\sqrt{x} + \sqrt{x'})}| \\
&\leq |\frac{1}{(\sqrt{x} + \sqrt{x'})}||x - x'|
\end{aligned}$$

- And therefore, because $|f(x) - f(x')| \leq |\frac{1}{(\sqrt{x}+\sqrt{x'})}||x - x'|$ we know that, there cannot be a fixed value c such that

$$|f(x) - f(x')| \leq c|x - x'|$$

- Thus, $\sqrt{x}$ is not Lipshitz.

- Now, for $\epsilon > 0$, let $\delta = \epsilon^2$ and $|x - x'| < \delta$. Therefore, we have that:

$$|\sqrt{x}-\sqrt{x'}|^2 = |\sqrt{x}-\sqrt{x'}||\sqrt{x}-\sqrt{x'}| \leq |\sqrt{x}-\sqrt{x'}||\sqrt{x}+\sqrt{x'}| \leq |x-x'| \leq \delta = \epsilon^2$$

Therefore, $|\sqrt{x} - \sqrt{x'}|^2 < \epsilon^2 \Rightarrow |\sqrt{x} - \sqrt{x'}| < \epsilon$

52

- We will first prove the forward direction. We know that f is a continuous function defined on a closed and bounded set. Therefore, by the Extreme Value theorem, we know that f will attain its maximum somewhere on the set E.
- We will now prove the reverse direction. Assume that, all continuous real-values functions on E attain their maximum somewhere on E. Now, consider the function $f(x) = x$ on the set E. Because we know that $f(x)$ attains its maximum , we know that x attains its maximum on E and therefore, E attains its maximum. Therefore, we know that E must be bounded as it has a maximum.
- Now finally, we need to show that E is closed. Let $\bar{E}$ be the closure set of E. Let $x \in \bar{E}$. Furthermore, let M be the maximum for set E. Now, if $x > M$, then we know that $\exists r > 0 \ni M \notin (x-r, x+r)$. But, if this was true, then $\nexists y \in E \ni y \in (x-r, x+r)$, which would be a contradiction to the definition of a limit point. therefore, we know that $x \leq M$. Furthermore, if $x \notin E$, then this is a contradiction to the definition of maximum (M) as then, $y < x < M$ for all $y \in E$. Therefore we must have that $x \in E$. If $x = M$, then we already know that $x \in E$ as, we know that E has a maximum. Thus, because E contains its limit points, E is closed.

- We will prove the forward direction first. Let E be a closed and bounded set that has an open cover $\{I_x\}$. Then, by the Heine-Borel theorem, we know that there exists a finite subcover of E.

- We will now prove the reverse direction. Assume that every open cover of E has a finite subcover. Then, we know that there exists an enumeration of the finite subcover $\{I_n\}_{n=1}^{m}$. Now, assume that E is unbounded above. That is, E is infinite. But, we know that E has a finite subcover and thus, if E was unbounded, then there cannot exist a finite subcover. Therefore, E must be bounded. Now, we need to show that E is closed. We know that, for any sequence $\{x_n\}$ in E, there is a convergent subsequence (according to the Bolzano-Weierstrass theorem) as $\{x_n\}$ is bounded. Now, define such a sequence such that $\{x_n\} \to x$ where $x \in \bar{E}$. Therefore, we know that there is a convergent subsequence such that $\{x_{n_k}\} \to x$. But, because this sequence must converge in E as $\{x_n\}$ is bounded. We know that $x \in E$. And thus, E contains its limit points and therefore, is closed.

54

- We will first show the forward direction. Let E be a nonempty set that is an interval. Let f be any continuous real-valued function on E. For any two values $f(a), f(b) \in f(E)$, define the closed interval $[a, b]$. By construction $[a, b] \subset E$ and thus, f is continuous on $[a, b]$ and by IVT, for any $f(a) < y < f(b)$, we have that, there exists a $c \in [a, b]$ such that $f(c) = y$. Thus, because between any two points in the image of f, we have that all point between those two points are mapped, we have that $f(E)$ is an interval as well.
- Now we will show the reverse direction. Assume that every real-calued continuous function on E has an interval as its image. That is, for every continuous function on E, $f(E)$ is an interval. Then we have to show that E is an interval. Take the function $f(x) = x$ defined on a set E. We know that $f(E)$ is an interval and therefore, if we have $a, b \in f(E)$, then we know that $a, b \in E$ and because $f(E)$ is an interval, we know that for any value $a < y < b$, $\exists y \in E$. Therefor,e for any two elements in $f(E)$, we have that there exist a corresponding interval in E. Therefore, we have that E must be an interval as, if it were not then that means there are elements in $f(E)$ that do not correspond to an interval which could be a contradiction to what we just showed. Therefore, E must be an interval.

59

- We want to show hat, for a sequence of continuous functions $\{f_n\}$ which converges to f, that f is itself continuous, that is, for some $\epsilon > 0$ there is a δ such that for $|x - x'| < \delta$ we have that $|f(x) - f(x')| < \epsilon$.
- By the definition of uniform convergence, we know that, for some $\lambda > 0$, there exists a number N such that $|f_m(x) - f(x)| < \lambda$ for all $m \geq N$ and for all $x \in E$.
- Furthermore, because $f_m(x)$ is continuous, we know that for some $\lambda > 0$, there exists some $x' \in (x-\delta, x+\delta)$ such that $|f_m(x) - f_m(x')| < \lambda$. Therefore, from this we can write:

$$\begin{aligned}
|f(x) - f(x')| &= |f(x) - f_m(x) + f_m(x) - f_m(x') + f_m(x') - f(x')| \\
&\leq |f(x) - f_m(x)| + |f_m(x) - f_m(x')| + |f_m(x') - f(x')| \\
&< \lambda + \lambda + \lambda = 3\lambda
\end{aligned}$$

- Now, we can substitute $\lambda = \frac{\epsilon}{3}$ to get:

$$|f(x) - f(x')| < 3\lambda = 3 \cdot \frac{\epsilon}{3} = \epsilon$$

for $|x - x'| < \delta$.
- Therefore, by definition f is continuous.