108A HW 11

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1 Problem 5.A.2

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that null S is invariant under T.

Solution Suppose that $v \in$ null S. Then $S(v) = 0$ and

$$\begin{aligned} ST(v) &= TS(v) \\ &= T(0) \\ &= 0. \end{aligned}$$

So we get that $T(v) \in$ null S. Therefore, null S is invariant under T.

2 Problem 5.A.7

Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Solution We want to find eigenvalues of T. Suppose that (x, y) is an eigenvector, thus $xy \neq 0$. So we let

$$\begin{aligned} T(x, y) &= \lambda(x, y) \\ &= (-3y, x). \end{aligned}$$

So we have

$$\begin{aligned} \lambda x &= -3y && (1) \\ \lambda y &= x. && (2) \end{aligned}$$

If we substitute (2) into (1), we get

$$\begin{aligned} \lambda^2 y &= -3y \\ \lambda^2 &= -3 \\ \lambda &= \sqrt{-3}. \end{aligned}$$

Since $T \in \mathcal{L}(\mathbb{R}^2)$, we get no real eigenvalues. Additionally, if we let $x = 0$, then by (1), $y = 0$ but since (x, y) is an eigenvector, we get a contradiction. Similarly, if we let $y = 0$, then by (2), it follows that $x = 0$ and we get a contradiction. Therefore, T has no real eigenvalues.

3 Problem 5.A.9

Define $T \in \mathcal{L}(\mathbb{F}^3)$ by

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3).$$

Find all eigenvalues and eigenvectors of T.

Solution We set a system of equations to find the eigenvalues.

$$\begin{aligned} T(z_1, z_2, z_3) &= \lambda(z_1, z_2, z_3) && (3) \\ \lambda z_1 &= 2z_2 && (4) \\ \lambda z_2 &= 0 && (5) \\ \lambda z_3 &= 5z_3 && (6) \end{aligned}$$

If $\lambda \neq 0$, this implies that $z_2 = 0$ in (5). It follows that in (4), $z_1 = 0$. Since eigenvectors can't be 0, then in (6), λ must equal 5. So 5 is the only non-zero eigenvalue of T. This leads us to our eigenvectors of the form $(0, 0, z_3)$ with $z \in \mathbb{F}$ when our eigenvalue is 5.

If $\lambda = 0$, then by (4), $z_2 = 0$. By (6), $z_3 = 0$. The last component of the vector must be nonzero, so our eigenvectors take the form $(z_1, 0, 0)$ with $z_1 \in \mathbb{F}$ when our eigenvalue is 0.

4 Problem 5.A.15

Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Solution Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible. Let λ be an eigenvalue of T, such that $Tv = \lambda v$ for every nonzero $v \in V$. So it follows that

$$\begin{aligned} S^{-1}TS(S^{-1}v) &= S^{-1}Tv \\ &= S^{-1}\lambda v \\ &= \lambda S^{-1}v \end{aligned}$$

Since S is invertible, then S^{-1} is invertible as well, so by definition of invertibility, $S^{-1}v \neq 0$ since v is nonzero. Thus λ is an eigenvalue of $S^{-1}TS$. Moreover, every eigenvalue of T is an eigenvalue of $S^{-1}TS$ and v is an eigenvector of T only if $S^{-1}v$ is an eigenvector of $S^{-1}TS$.

5 5.A.25

Suppose $T \in \mathcal{L}(V)$ and u, v are eigenvectors of T such that $u + v$ is also an eigenvector of T. Prove that u and v are eigenvectors of T corresponding to the same eigenvalue.

Solution Let λ_1 be the eigenvalue corresponding to u and λ_2 be the eigenvalue corresponding to v. Then $Tu = \lambda_1 u$ and $Tv = \lambda_2 v$. Let λ_3 be the eigenvalue corresponding to $u + v$. Then $\lambda_3(u + v) = T(u + v) = Tu + Tv$. Then

$$\begin{aligned}
\lambda_3(u+v) &= T(u+v) && (7)\\
&= Tu + Tv && (8)\\
&= \lambda_1 u + \lambda_2 v. && (9)
\end{aligned}$$

If we subtract (9) from the left-hand side of (7), we get

$$(\lambda_3 - \lambda_1)u + (\lambda_3 - \lambda_2)v = 0.$$

Since u and v are eigenvectors, they must be nonzero. Suppose $\lambda_1 \neq \lambda_2$. Then $\lambda_3 - \lambda_1$ and $\lambda_3 - \lambda_2$ cannot both equal zero. So λ_1 and λ_2 are linearly dependent since they are scalar multiples of each other. By 5.10 in the book, λ_1 and λ_2 cannot be distinct, so $\lambda_1 = \lambda_2$. In other words, $\lambda_1 = \lambda_2$ corresponds to both u and v.