HW-2
Math 117

Siddharth Deshpande

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Problem 1.1.2

(i)

Proof:

Assume a > 0, we can multiply both sides by a again to get $a \cdot a > a \cdot 0 \Rightarrow a^2 > 0$.

Now assume that $a < 0$, we can multiply both sides by -1 to get, $-a > 0$. Then, we can multiply both sides by a. But notice that, $a < 0$, therefore, our inequality becomes $-a^2 < 0$. Again multiplying by -1 we get $a^2 > 0$.

- Therefore, for $a \in \mathbb{R} : a \neq 0$, we have that, $a^2 > 0$

(ii)

Proof:

Let $a \in \mathbb{R} : a > 0$, assume that, the multiplicative inverse of a $a^{-1} = b < 0$. If we multiply both sides of $a > 0$ by a's multiplicative inverse, we get $ab < 0$. But, by definition, $ab = 1$, therefore, $1 < 0$, which is a contradiction. Therefore, $b = a^{-1} > 0$

(iii)

Let $a, b \in \mathbb{R} : a > b$. Assume that, $c \in \mathbb{R} : c > 0$. We know by definition that, because $a >$, $a - b > 0$. Multiplying both sides by c we get, $(a - b)c >$

$0 \Rightarrow ac - bc > 0 \Rightarrow ac > bc$

Now assume that, $c \in \mathbb{R} : c < 0$. We know by definition that, because $a > b$, $a - b > 0$. Multiplying both sides by c gives us the inequality, but with the inequality flipped as $c < 0$, $(a - b)c < 0 \Rightarrow ac - bc < 0 \Rightarrow ac < bc$

Problem 1.1.4

(i)

Proof:

Let $ab = 0$. Without loss of generality assume that $a \neq 0$. Therefore, if we can prove that, $b = 0$ assuming that $a \neq 0$, our proof is complete as, the same can be applied assuming b is non-zero ($b \neq 0 \Rightarrow a = 0$). Therefore, we have $ab = 0$. Let a^{-1} be the multiplicative inverse of a. Therefore, multiplying both sides of the equation with the multiplicative inverse we get, $aba^{-1} = 0a^{-1} = 0$. Furthermore, by the commutativity of multiplication we have, $aba^{-1} = aa^{-1}b = 0$. But, we know that, $aa^{-1} = 1$, therefore, $1 \cdot b = 0$. However, by the multiplicative identity know as 1, we know that, $1 \cdot b = b$. Therefore, $1 \cdot b = b = 0$

(ii)

Let $a, b \in \mathbb{R}$, then we can use the multiplicative axioms to prove that, $a^2 - b^2 = (a-b)(a+b)$.

$$
\begin{aligned}
a^2 - b^2 &= (a-b)(a+b) \\
&= a(a-b) + b(a-b) \quad \text{(distributive axiom)} \\
&= a^2 - ab + ab - b^2 \\
&= a^2 - b^2 \quad \text{(as -ab and ab are additive inverses)}
\end{aligned}
$$

- To complete this proof, we can prove that, for a number $a \in \mathbb{R}, -a$ is the additive inverse:

$$
\begin{aligned}
a + (-1)a &= 1(a) + (-1)a \\
&= (-1+1)a \\
&= 0a \\
a + (-a) &= 0
\end{aligned}
$$

- Therefore, a and -a are additive inverses. Therefore, our proof is complete.
- Now, assume that, $a^2 = b^2$. Then, we know that, $a^2 - b^2 = 0 = (a-b)(a+b)$. Looking at our result from part (i), we know that, either (a-b) = 0 or (a+b) = 0. Therefore, assume that $a - b = 0 \Rightarrow a = b$. Now assume $a + b = 0 \Rightarrow a = -b$. Therefore, if $a^2 = b^2 \Rightarrow a = b$ or $a = -b$

(iii)

Non-Emptyness:

Because c is a positive number, $c > 0$. If x = 0, we have that $x^2 = 0 < c$. Therefore, $0 \in E, \forall c \in \mathbb{R} : c > 0$. Thus, $E \neq \emptyset$.

Sup(E) = $\sqrt{c}$:

Assume that, $\exists x_1 \in \mathbb{R} > 0 : x_1^2 = c$ therefore, by definition of E, $x_1 > x, \forall x \in E$. Furthermore, let $x_0 = Sup(E)$. Therefore, by definition of a supremum, $x_0 > x, \forall x \in E$. Then, if we assume $x_0 > x_1$, then by definition $x_1 = Sup(E), x_0 \neq Sup(E)$ which is a contradiction. Therefore, we have the case that, $x_0 \leq x_1$.

If we consider that, $x_0 = x_1$, then we have already proven what we wanted to show that, $\exists x \in \mathbb{R} : x^2 = c$.

Assume that $x_0 < x_1$. Therefore, we know that, $\exists y \in \mathbb{R} : x_0 \leq y < x_1$. Therefore, because $y < x_1$, we know that, $y^2 < x_1^2 = c$. This implies that, $y \in E$. But, $x_0 < y$ and $x_0 = Sup(E)$. This is therefore a contradiction. Thus, $\nexists y \in \mathbb{R} : x_0 \leq y < x_1$. Thus, because $x_0 \not> x_1$ and $x_0 \not< x_1$, it stands to reason that $x_1 = x_0$. Therefore, $x_0 = Sup(E) = x_!$ and, $x_1^2 = c = x_0^2$. Therefore, we showed that, $\exists x_0 \in \mathbb{R} : x_0^2 = c$. This x_0 is denoted by $x_0 = \sqrt{c} = Sup(E)$.

Uniqueness:

Furthermore, now assume that $\exists x_2 \in \mathbb{R} > 0 : x_2^2 = c$. Therefore, we have that $x_1^2 = x_2^2$, therefore, by the identity proven in (ii), we know that, $x_1 = x_2$ or $x_1 = -x_2$. But, because $x_1, x_2 > 0$, $x_1 \neq -x_2$ as that would be a contradiction. Therefore, we must have that $x_1 = x_2$ and therefore the square root of a positive real number is also unique

1.1.6

- Because E has a lower bound, we know that it has an infimum by the Completeness Axiom. Let $y = inf(E) \Rightarrow y < x, \forall x \in E$ and, let x_1 be the lower bound for E, that is, $x_1 \leq x : \forall x \in E$, therefore, $y < x_1 \leq x : \forall x \in E$ which implies, $-y > -x_1 \geq -x : \forall x \in E$.
- Let $-E = \{-x \mid x \in E\}$. Therefore, $-y > -x_1 \geq z \ \forall z \in -E$. Assume that, $\exists y_2 : -y > -y_2 > -x_1 \Rightarrow y < y_2 < x_1$. Therefore, by definition, $y \neq inf(E)$ which is a contradiction. Therefore, no such y_2 exists such that $-y > -y_2 > -x_1 \geq z \forall z \in -E$. Thus, by definition $-y = Sup(-E) \Rightarrow y = -Sup(-E)$. Therefore, $y = inf(E) = -Sup(\{-x \mid x \in E\})$

1.2.8

- We shall prove the statement that, for any natural number n, there are no natural numbers between (n, n+1).

Proof:

- We shall prove the base case for 1. (that is from (1,2)). We know that, the natural numbers occur in intervals of 1. So, assume that, there is a natural number on the interval (1,2) called a. Therefore, because there exists a natural number on (1,2), there exists a natural number a' such that, $a' = a - 1$. We know that, $a < 2 \Rightarrow a - 1 = a' < 1$. But, there cannot be a natural number less than 1. Therefore, a cannot be a natural number. Thus, there is no natural number on the interval (1,2).

- Assume that, there is no natural number on the interval (n,n+1). So, we shall prove that there is no natural numbers on the interval ((n+1), (n+1)+1) = (n+1, n+2) either. Assume that, there is a natural number $b \in (n+1, n+2)$. Therefore, by the definition of natural numbers, $\exists a \in (n, n+1) : b = a + 1, a \in \mathbb{N}$. But, this contradicts our original assumption that there are no natural numbers on the interval (n, n+1). Thus, if there are no natural numbers on the interval (n, n+1), then there are no natural numbers on the interval (n+1, n+2).

- Therefore, we have completed our proof by induction

1.2.10

- We can prove this by using induction for integers. So, we shall prove that, the property holds for 0, then prove that if it holds for n then it holds for n+1 and n-1 as well.

Proof:

- Let n = 0. Therefore, the theorem states that, there exists an integer on the interval $[0, 1) \Rightarrow \exists x \in \mathbb{Z} : 0 \leq x < 1$. We can see that, by definition this is true as, if x = 0, then $0 \leq 0 < 1$ is true and, $0 \in \mathbb{Z}$.

- Assume $\exists x \in \mathbb{Z} : n \leq x < n+1$, then, we have to show that, $\exists y \in \mathbb{Z} : n + 1 \leq y < n + 2$. Assume that, $\nexists y in \mathbb{Z} : n + 1 \leq y < n + 2$, then, by the definition of integer, $\nexists (y-1) \in \mathbb{Z} : n \leq y-1 < n+1$. But, this is a contradiction to our original assumption that, $\exists x \in \mathbb{Z} : n \leq x < n+1$. Therefore, $\exists y \in \mathbb{Z} : n + 1 \leq y < n + 2$. And we have proven that, if the statement holds for n, it holds for n+1.

- Assume $\exists x \in \mathbb{Z} : n \leq x < n + 1$, then we also have to show that, $\exists z \in \mathbb{Z} : n - 1 \leq z < n$. Assume that, $\nexists z \in \mathbb{Z} : n - 1 \leq z < n$, therefore, by the definition of integers, $\nexists (z + 1) \in \mathbb{Z} : n \leq (z + 1) < n + 1$. But, this is a contradiction to our original assumption that, $\exists x \in \mathbb{Z} : n \leq x < n+1$. Therefore, $\exists z \in \mathbb{Z} : n - 1 \leq z < n$. Therefore, we have also shown that, if the statement holds for n, it holds for n-1 as well.

- This completes our proof by induction for integers as, the statement holds for n = 0, and, we have also shown that, if it holds for n, it holds for n+1 and n-1 as well and thus holds for all integers.

1.2.12

For this problem, let $x, y \in \mathbb{R} : y > x$. So, we have to show that, there exists an irrational number between x and y to prove that the irrationals are dense in the reals.

Thus, we know that $x + \sqrt{3} < y + \sqrt{3}$. Furthermore, by teh ardimedian principle, $\exists m \in \mathbb{N} : m > \frac{1}{y-x} \Rightarrow m(y-x) > 1$. Because we have chosen an m such that, $m(y-x) > 1$, we know that, $\exists n \in \mathbb{N} : my > n > mx$. So, if we were to substitute our previous inequality for y and x, we get, $\exists n \in \mathbb{N} : m(y - \sqrt{3}) > n > m(x - \sqrt{3})$. Therefore, after simplification we get:

$$x < \frac{n}{m} + \sqrt{3} < y$$

- Thus, we have shown that there exists an irrational number between x and y and therefore, the irrationals are dense in $\mathbb{R}$

1.2.14

- We can prove this by induction.

Proof:

- The base step is to prove that this identity works with 1. Thus, if we let n = 1, we get, $1 + r \geq 1 + r$ which is true as they are equal.
- Assume that $(1+r)^n \geq 1 + nr$ for n. Therefore, if we can prove it is

true for n+1, then we have completed our proof by induction.

$$(1+r)^{n+1} \geq 1+(n+1)r$$

$$(1+r)^n(1+r) \geq 1+nr+r$$

$$(1+r)^n + r(1+r)^n \geq 1+nr+1$$

$$\because r > 0 \Rightarrow r+1 > 1$$

$$(r+1)^n > 1^n = 1$$

$$r(r+1)^n > r \quad \text{(because r > 0)}$$

$$\therefore r(1+r)^n > r, (1+r)^n \geq 1+nr$$

$$\therefore (1+r)^n + r(1+r)^n > 1+nr+r$$

$$\therefore (1+r)^{n+1} > 1+(n+1)r$$

- Therefore, using mathematical induction, we have proven that, $(1+r)^n > 1+nr$ for $r > 0$