Problem Q4. *Solution.* W have the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. We can swap the first and second rows and the first and second columns to get

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \to \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}.$$

So we can write A as a sum of rank 1 matrices using CR factorization:

$$\begin{aligned} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} &= \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} &= \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \\ \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} &= \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} &= \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \end{aligned}$$

If we apply LU factorization, we get

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}.$$

Then we have

$$\begin{aligned} A - \begin{bmatrix} 0 & 0 \\ 2 & 6 \end{bmatrix} &= \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \\ A - \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}. \end{aligned}$$

Now we can write A as a sum of rank 1 matrices using CR factorization.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LU.$$