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Midterm 2 B

1. Every linear functional is either surjective or the zero map. Consider $T: V \to W$, where T is not the zero map. Then there exists $v \in V$ such that $Tv \neq 0$.

Let $Tv = a$ where $a \neq 0$. If we have $T(a^{-1}wv)$, then for every $w \in W$, we have

$$
\begin{aligned}
T(a^{-1}wv) &= a^{-1}w\,(Tv) \\
&= a^{-1}w\,(a) \\
&= w
\end{aligned}
$$

Since $w \in \operatorname{range} T$ and w was an arbitrary element in W, then we have proven that $\operatorname{range} T = W$, and therefore is surjective. Therefore, if a linear functional is not the zero map, then it is surjective.

2. a) The Fundamental Theorem of Algebra states that every nonconstant polynomial with <u>complex</u> coefficients has at least 1 zero.

b) You cannot conclude anything since the Fundamental Theorem of Algebra only covers polynomials $\in \mathbb{C}$, not $\in \mathbb{R}$.

3) Suppose $T: \mathbb{R}^7 \to \mathbb{R}^7$. Then $T \in \mathcal{L}(\mathbb{R}^7, \mathbb{R}^7)$. By the Fundamental Theorem of Linear Maps,

$$\dim \mathbb{R}^7 = \dim \text{range } T + \dim \text{null } T$$
$$7 = 7 + \dim \text{null } T$$
$$0 = \dim \text{null } T$$

Since $\dim \text{range } T (=7) \neq \dim \text{null } T$, then range $T \neq$ null T.

4) Suppose U is a subspace of V, st $\dim V/U = 1$. We want to prove that there exists $\varphi \in \mathcal{L}(V, \mathbb{F})$ s.t. null $\varphi = U$.

We know that $V/U = \{v + U \mid v \in V\}$. By definition of $\tilde{\varphi}$, $\tilde{\varphi}: V/\text{null } \varphi \to \mathbb{F}$, where $\tilde{\varphi}(v + \text{null } \varphi) = \varphi(v)$. Therefore, there exists a $\varphi \in \mathcal{L}(V, \mathbb{F})$ s.t. null $\varphi = U$.

5) Suppose V & W are finite-dimensional & $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that range$(T_1)$ = range(T_2) iff there exists an invertible operator $S \in \mathcal{L}(V)$ s.t. $T_1 = T_2 S$

$\Rightarrow$. Suppose that range (T_1) = range (T_2). Then null (T_1) = null (T_2). Let $S \in \mathcal{L}(V)$. This does not necessarily mean that $T_1(v) = T_2(v)$. However, there is a way for it to equal each other if we apply S to either T_1 or T_2 since their ranges are the same. Let $f \in \text{range } T_1$ and $g \in \text{range } T_2$. Then $T_1(v) = f$ and $T_2(v) = g$ for some $f \in \text{range } T_1$ and $g \in \text{range } T_2$. Then $f = S(g)$ since $f, g \in \text{range } (T_1 \cup T_2)$.

$\Leftarrow$: Suppose there exists $S \in \mathcal{L}(V)$ s.t. $T_1 = T_2 S$. Then, for any $f \in \text{range } T_1$ and $g \in \text{range } T_2$, $f = S(g)$. Since f and g was arbitrary, then $\text{range } T_1 = \text{range } T_2$.