HW #3

1) Let $U = \{(x,y) \mid x=0 \text{ or } y=0\}$.

An example of U would be $(x,0)$.

It would be closed under scalar multiplication:

Consider $c(x,0) = (cx, 0)$, where $c \in \mathbb{F}$.

It would not be closed under addition:

consider $(1,0)$ and $(0,1)$. Both belong to U, however, their sum $(1,1)$ does not.

2) To prove that $U_1 \cap U_2$ is a subspace of V, we must show it is closed under addition AND scalar multiplication.

Let $u, v \in U_1 \cap U_2$. Then $u, v \in U_1$ and $u, v \in U_2$. Since U_1 & U_2 are subspaces, then we can say $u+v \in U_1$ and $u+v \in U_2$. So, $u+v \in U_1 \cap U_2$ thus proving that $U_1 \cap U_2$ is closed under addition.

Similarly, for any $c \in \mathbb{F}$, $cu \in U_1 \cap U_2$ and $cv \in U_1 \cap U_2$. Thus it is closed under scalar multiplication.

Finally, since U_1 & U_2 are both subspaces, $0 \in U_1$ and $0 \in U_2$. Therefore $0 \in U_1 \cap U_2$.

Based on these conditions, we can conclude that $U_1 \cap U_2$ is a subspace of V.

3. If U is a subspace of V, then $U + U$ is another subspace of V.

4. $U = \{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}$

$U + W$ is a direct sum iff $U \cap W = \{0\}$

Consider $W = \{(0, 0, a, b, c) : a, b, c \in \mathbb{F}\}$

Now, for any $(x, y, a, b, c) \in \mathbb{F}^5$,

$$\begin{pmatrix} x \\ y \\ a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y \\ x-y \\ 2x \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a-x-y \\ b-x+y \\ c-2x \end{pmatrix}$$

We now have that $\mathbb{F}^5 = U + W$.

Suppose that $m = (x, y, a, b, c) \in U \cap W$. Since $m \in W$, $x = 0 = y$. Since $m \in U$, we have $a = x+y = 0+0 = 0$. Additionally, we have $b = x - y = 0 - 0 = 0$. Finally, $c = 2 \cdot x = 2 \cdot 0 = 0$. Therefore, $m = 0 \in \mathbb{F}^5$ so $U \cap W = \{0\}$. We have now proven that $\mathbb{F}^5 = U \oplus W$, where $W = \{(0, 0, a, b, c) : a, b, c \in \mathbb{F}\}$.

5. $U_e = \{ \forall\ f(-x) : f(-x) = f(x) \}$

$U_o = \{ \forall\ f(-x) : f(-x) = -f(x) \}$

Let $a = f(x)$. Clearly for any function a on $\mathbb{R}$, $a \neq -a$ unless $a = 0$. So the only element $a \in U_e \cap U_o = \{0\}$.

Thus, $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.