HW-3
Math 117

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1.3.16

- If we can find a one-to-one function $f : \mathbb{Z} \to \mathbb{N}$ then we have proven that, $\mathbb{Z}$ is equipotent to $\mathbb{N}$ and we have proved that $\mathbb{Z}$ is countable.

- Let us define a function $f : \mathbb{Z} \to \mathbb{N}$ such that:

$$f = \begin{cases} f(0) = 1 & \\ f(x) = 2^x & x > 0 \\ f(x) = 3^{-x} & x < 0 \end{cases}$$

- We can now prove that f is one-to-one.

Proof:

> Assume f(n) = f(m) = 1. Then, we know by the definition of the function that n = m = 0.
>
> - Assume f(n) = f(m). Without loss of generality, assume that $n > 0$ and $m < 0$. Therefore, $2^n = 3^{-m}$. But, by the prime factorization theorem, we know that this is not possible. Therefore, it is a contradiction.
>
> - Assume that $n > 0$ and $m > 0$. Then, we know that, $2^n = 2^m$. Therefore, $n = m$. Similarly, if $n < 0$ and $m < 0$, then, $3^{-n} = 3^{-m}$. Therefore, $-n = -m \Rightarrow n = m$.
>
> - Therefore, we have shown that f is one-to-one.

- Furthermore, we have shown that, a one-to-one funciton exists between $\mathbb{Z}$

and $\mathbb{N}$. Therefore, $\mathbb{Z}$ is equipotent to $\mathbb{N}$. But, we know that $\mathbb{N}$ is a countable set therefore, we have proven that $\mathbb{Z}$ is a countable set.

1.3.18

- We can prove this identity using induction. Let n = 2 be the base case (as if n = 1, then the product just becomes $\mathbb{N}$). We know that $\mathbb{N}\times\mathbb{N}$ is countable as the one-to-one function $g : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ can be defined as $g = (m+n)^2+n$. Therefore, we known that the case holds for $n = 2$.

- Now assume that the product of $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \overset{n}{\ldots} \mathbb{N}$ is countable . We need to show that $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \overset{n+1}{\cdots} \mathbb{N}$ is also countable. Because of our assumption we know that there exists a function $h : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \overset{n}{\ldots} \mathbb{N} \to \mathbb{N} = b$ that is a one-to-one correspondence. Furthermore, define a function $j : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \overset{n+1}{\cdots} \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ such that, $j(x_1, x_2, \ldots x_n, x_{n+1}) = (b, x_{n+1})$ We know that this is a one-to-one function as, by our assumption h is one-to-one and, because we are keeping the last element as it is, it is without an explaination going to be unique (as if $f(x) = x$ and $f(x) = f(y) \Rightarrow x = y$). Therefore, we know that, the function $j : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \overset{n+1}{\cdots} \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a one-to-one correspondence. But, we have already proven that $\mathbb{N} \times \mathbb{N}$ is a countable set. Therefore, by Theorem, we know that if a set is equipotent to a countable set, then it its countable and therefore, $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \overset{n+1}{\cdots} \mathbb{N}$ is countable .

-Therefore, we have proven by induction that, for any Natural number n, we have that $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \overset{n}{\cdots} \mathbb{N}$ is countable

1.3.20

To go about proving both, we can first demonstrate that $g \circ f$:

- We can first show that the function is one-to-one. Assume that, $\exists x, y : g(f(x)) = g(f(y))$. Because g is a bijective function, we know that g has a inverse, $g^{-1} : g(g^{-1}(n)) = g^{-1}(g(n)) = n$. Now, we can apply the inverse function to both sides to get, $g^{-1}(g(f(x))) = g^{-1}(g(f(y))) \Rightarrow f(x) = f(y)$. By definition, because f is one-to-one, we have that $x = y$. Therefore, we have proven that $g \circ f$ is one-to-one

- Now we can show that the function is onto. First of all, because g is surjective, for $z \in C, \exists y \in B : g(y) = z$. But, because f is surjective, for $y \in B, \exists x \in A : f(x) = y$. Therefore, $g \circ f(x) = g(f(x)) = g(y) = c$. Therefore, $g \circ f$ is surjective.

- Therefore we have proven that $g \circ f$ is bijective. Now, because both f and g are bijective, we know that both functions have an inverse that we can label f^{-1} and g^{-1} as inverses of f and g respectively. Thus, $f^{-1} : B \to A$ and $g^{-1} : C \to B$. Thus, we need to prove that $f^{-1} \circ g^{-1}$ is also bijective:

- Similar to the argument we used last time, we can first show the function is one-to-one. Assume $\exists x, y : f^{-1}(g^{-1}(x)) = f^{-1}(g^{-1}(y))$. Because f^{-1} is the inverse of f, we know that $f(f^{-1}(x)) = x$. Therefore, applying the f transformation to both sides we get, $f(f^{-1}(g^{-1}(x))) =$

$f(f^{-1}(g^{-1}(y))) \Rightarrow g^{-1}(x) = g^{-1}(y)$. We can similarly apply the function to g now to get, $g(g^{-1}(x)) = g(g^{-1}(y)) \Rightarrow x = y$ therefore, $f^{-1} \circ g^{-1}$ is one-to-one.

- Now we need to show that the function is surjective. We know that $f^{-1} \circ g^{-1} : C \to A$ therefore, we need to show that, $\forall a \in A, \exists c \in C : f^{-1}(g^{-1}(c)) = a$. To prove this, we need to recall the fact that, the inverse of a bijective function is bijective itself. Therefore, we know that f^{-1} and g^{-1} are both bijective. Because we know this, we need to show that, the image of the composite function is the entire codomain (A) :

$$\begin{aligned}
(f^{-1} \circ g^{-1})(C) &= f^{-1}(g^{-1}(C)) \\
&= f^{-1}(B) \quad \text{(because } g^{-1} \text{ is bijective)} \\
&= A \quad \text{(beause } f^{-1} \text{ is bijective)}
\end{aligned}$$

- Therefore, we have shown that the image of the composite function is the entire codomain and thus, we have proved that $f^{-1} \circ g^{-1}$ is surjective.

1.3.22

- We can prove that the power set of $\mathbb{N}$ is also uncountable using a similar decimal representation technique used to show that $\mathbb{R}$ is uncountable.

Proof:

- Assume that, $2^{\mathbb{N}}$, the power set of $\mathbb{N}$ is countable. Thus, we know that there exists a bijective function $f : \mathbb{N} \to 2^{\mathbb{N}}$

- We can take the binary approach to writing out the power set of a given set. Therefore, let $A \subseteq \mathbb{N}$. We can represent A as a binary number like $x_1 x_2 x_3 \ldots x_n$ where each $x_n = 1 : n \in A$ and $x_n = 0 : n \notin A$. Thus, if $A = \{2, 3, 6, 7\} = 01100110000\ldots$. Because $\mathbb{N}$ is countable and, we have assumes that $2^{\mathbb{N}}$ is also countable, we know that there exist enumerations for both such sets. Therefore, we can simply assign the function f between two values from each set such as:

$$\begin{array}{ll} 1 & x_{11}x_{12}x_{13}\ldots x_{1n}\ldots \\ 2 & x_{21}x_{22}x_{23}\ldots x_{2n}\ldots \\ \vdots & \\ n & x_{n1}x_{n2}x_{n3}\ldots x_{nn}\ldots \end{array}$$

- We can define a new number $\mathcal{N} = x_{N1}x_{N2}x_{N3}\ldots x_{Nn}\ldots$. Such that, each $x_{Nn}, \forall n \in \mathbb{N}$ of $\mathcal{N}$ can be defined like:

$$x_{Nn} = \begin{cases} 1, & x_{nn} = 0 \\ 0, & x_{nn} = 1 \end{cases}$$

- For all $n \in \mathbb{N}$. Therefore, $x_{N1} = 1$ if $x_{11} = 0$ and $x_{N1} = 0$ if $x_{11} = 1$. Similarly, $x_{N2} = 1$ if $x_{22} = 0$ and $x_{N2} = 0$ if $x_{22} = 1$ and the same applies $\forall n \in \mathbb{N}$. By this method, we know that, there exists no such $n \in \mathbb{N}$ such that $f(n) = \mathcal{N}$ as each digit in $\mathcal{N}$ is different than any binary number (and therefore any subset) we have mapped from $\mathbb{N}$. Therefore, f is not a bijection. But this is a contradiction.

- Therefore, our original assumption that $2^{\mathbb{N}}$ is countable is wrong and therefore, $2^{\mathbb{N}}$ is uncountable and by definition, it is infinite.

1.3.24

Let I be a non-degenrate interval of reals. We will prove by contradiction that I fails to be finite. Suppose that I is finite (countable). Then we know that there exists an enumeration of I, let $\{x_n \mid n \in \mathbb{N}\}$ be an enumeration of I. By Zorn's Lemma and Axiom of choice, we can choose an interval in I $[a_1, b_1]$ which is a closed bounded non-degenerate sub-interval such that $x_1 \notin [a_1, b_2]$. Furthermore, we can choose a subinterval of $[a_1, b_1]$ namely $[a_2, b_2]$ such that $x_2 \ /in[a_2, b_2]$. We can repeat this process for $[a_n, b_n]_{n=1}^{\infty}$ where for each n we have that, $x_n \notin [a_n, b_n]$. Therefore, we can define a new set $E = \{a_n \mid n \in \mathbb{N}\}$. Because E has an upper bound b_1 because we know by definition that $b_1 > a_n \forall n \in \mathbb{N}$, by the Completeness Axiom we know that E has a supremum. Let $\sup(E) = x_i$. Therefore, by definition we have that, $a_n < x_i, \forall n \in \mathbb{N}$. But, because $\sup(E) = x_i$, we know that $x_i < b_n, \forall n \in \mathbb{N}$. Because if this wasnt the case, then, $\exists j \in \mathbb{N} : b_j < x_i \Rightarrow x_i \neq \sup(E)$ which would be a contradiction. Therefore $a_n < x_i < b_n, \forall n \in \mathbb{N}$. Therefore, $x_i \in [a_n, b_n], \forall n \in \mathbb{N}$ by definition. But, if $x_i \in [a_1, b_1]$ then we know that there exists some interval $[a_i, b_i], i \in \mathbb{N}$ such that $x_i \notin [a_1, b_1]$. But this is a contradiction to $x_i \in [a_n, b_n], \forall n \in \mathbb{N}$. Therefore, our original assumption must be wrong that I is finite, as that assumption allowed us to have an enumeration of I. Therefore I must be infinite.

1.3.26

- We have to make an important note here. Because we know that, there exist one-to-one correspondence from $(0,1) \to \mathbb{R}$ and $(0,1) \times (0,1) \to \mathbb{R}^2$, we only need to show that there exists one-to-one correspondence between $(0,1) \times (0,1)$ and $(0,1)$. Therefore, if we can find a one-to-one correspondence between these two sets, then we have proven our result as they are respectively equipotent to $\mathbb{R}^2$ and $\mathbb{R}$.
- To prove that $(0,1) \times (0,1)$ and $(0,1)$ have the same cardinality we need to define one-to-one correspondence between the two sets. We can first define a one-to-one function $f : (0,1) \times (0,1) \to (0,1)$.One thing we can

do is, we can write down any real number as a string of digits, that is, $x \in (0,1) : x = 0.x_1x_2x_3\ldots$. So, given $x, y \in (0,1)$, we can define f such that, $f(x,y) = 0.x_1y_1x_2y_2x_3y_3\ldots$. Because we have defined the function in such a way, we know that the function is one-to-one.

- Now, we can define a function $g : (0,1) \to (0,1) \times (0,1)$. We need to define g such that it is one-to-one and we will have proven our theorem. We can similarly decompose each real into its decimal components. For example, if $x \in (0,1)$, we can display $x = 0.x_1x_2x_3\ldots$. And now, we can define g as, $g(x) = (0.x_1x_3x_5\ldots x_{2n-1}\ldots, 0.x_2x_4x_6\ldots x_{2n}\ldots), \forall n \in \mathbb{N}$. By definition, we know that g is also a one-to-one function. Because, if $g(x) = g(y)$ that means $(0.x_1x_3x_5\ldots x_{2n-1}\ldots, 0.x_2x_4x_6\ldots x_{2n}\ldots) = (0.y_1y_3y_5\ldots y_{2n-1}\ldots, y_2y_4y_6\ldots y_{2n}\ldots)$, which implies that, $x_1x_3x_5\ldots x_{2n-1}\ldots = y_1y_3y_5\ldots y_{2n-1}\ldots$ and $x_2x_4x_6\ldots x_{2n}\ldots = y_2y_4y_6\ldots y_{2n}\ldots$ and because each x_i is a digit, we must have that $x_1 = y_1$, $x_2 = y_2$ and so on. Therefore we have shown that $x = y$ and thus, g is one-to-one.

- Thus, we have proven that $(0,1) \times (0,1)$ and $(0,1)$ have the same cardinality (are equipotent). But, we also know that $(0,1)$ and $\mathbb{R}$ are equipotent and, $(0,1) \times (0,1)$ and $\mathbb{R}^2$ are equipotent. Furthermore, we also know that euipotency is a equivalence relationship. Therefore, we have proven that $\mathbb{R}^2$ and $\mathbb{R}$ are equipotent.