108A HW 8

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1 Problem 3.C.1

Suppose V and W are finite dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least dim range T nonzero entries.

Solution Let $v_1, ..., v_n$ be a basis of V and $w_1, ..., w_m$ be a basis of W. Let $\mathcal{M}(T)$ be the matrix of T with respect to these bases.

For sake of contradiction, suppose $\mathcal{M}(T)$ has less than dim range T nonzero entries. In other words, $\mathcal{M}(T)$ has no more than dim range $T - 1$ nonzero entries. Then there are no more than dim range $T - 1$ nonzero vectors in $Tv_1, ..., Tv_n$. Since range $T =$ span $(Tv_1, ..., Tv_n)$, then it follows that

$$\text{dim range } T \leq \text{dim range } T - 1.$$

This is clearly a contradiction, so we can conclude that the matrix of T has at least dim range T nonzero entries.

2 Problem 3.C.3

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that, with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row j, column j are 1 for $1 \leq j \leq$ dim range T.

Solution Using the ideas in the Fundamental Theorem of Linear Maps, let $u_1, ..., u_m$ be a basis of null T. We can extend $u_1, ..., u_m$ to be a basis $u_1, ..., u_m, v_1, ..., v_n$ of V, so $Tv_1, ..., Tv_n$ is a basis for range T. From this, it follows that

$$\text{dim range } T = n.$$

Since $Tv_1, ..., Tv_n$ is a basis for range T, this list is linearly independent in W. Like for V, we can extend $Tv_1, ..., Tv_n$ to be a basis $Tv_1, ..., Tv_n, w_1, ..., w_p$ for W. It will be useful to rearrange the elements in our current basis of V to look like this: $v_1, ..., v_n, u_1, ..., u_m$. So we can construct a $\mathcal{M}(T)$ with respect to these bases of V and W.

With the reasoning that $Tv_j = 1 \cdot Tv_j$, for $1 \leq j \leq n = \dim \text{range } T$, we can say that all entries in the first n columns of the matrix are 0 except for the entries in row j, column j. Since $u_1, ..., u_m$ is a basis of null T, $Tu_k = 0$ for any $k \in \{1, ..., m\}$, so all entries in the remaining m columns are 0, completing our proof.

3 Problem 3.C.12

Give an example with 2-by-2 matrices to show that matrix multiplication is not commutative. In other words, find 2-by-2 matrices A and C such that $AC \neq CA$.

Solution Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

Then $AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$.

This is not equal to $CA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$.

Therefore, $AC \neq CA$.

4 3.D.1

Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Solution Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Then, by definition of invertible, there exists T^{-1} and S^{-1} such that $T \cdot T^{-1} = I$ and $S \cdot S^{-1} = I$.

Since T and S are invertible maps, it also follows that both T and S are injective and surjective. So in T, every unique $u \in U$ gets mapped to a unique $v \in V$. Likewise in S, every unique $v \in V$ gets mapped to a unique $w \in W$. So, in every unique $u \in U$ gets mapped to a unique $w \in W$. So we can conclude that $ST \in \mathcal{L}(U, W)$ is invertible.

Since we know that ST is invertible, then we can use the notation $(ST)^{-1}$ as its inverse. However, we must show that $(ST)^{-1} = T^{-1}S^{-1}$. To do this, let's show that $(ST)(T^{-1}S^{-1}) = I$, where I is the identity map on W, and that $(T^{-1}S^{-1})(ST) = I$ where this I is the identity map on U.

$$\begin{aligned}
(ST)(T^{-1}S^{-1}) &= S(TT^{-1})(S^{-1}) \\
&= S(I)S^{-1} \\
&= S(S^{-1}) \\
&= I.
\end{aligned}$$

Then,

$$\begin{aligned}(T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T \\ &= T^{-1}(I)T \\ &= T^{-1}(T) \\ &= I.\end{aligned}$$

Therefore, $(ST)^{-1} = T^{-1}S^{-1}$.

5 3.D.4

Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\text{null}T_1 = \text{null}T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.

Solution Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. It then follows that T_1 and T_2 are also finite-dimensional.

To prove the first part of the proof, let's assume $\text{null}T_1 = \text{null}T_2$. Then $\text{null}T_1 = \text{null}T_2 = \{v \in V : Tv = 0\}$. We must be weary to say that since $\text{null}T_1 = \text{null}T_2$, then $\text{range}T_1 = \text{range}T_2$ since this is not necessarily true. We can, however, deduce that $\dim \text{range}T_1 = \dim \text{range}T_2 \in V$. It follows that if we map a vector $v \in V$ from $\text{range}T_1 \to \text{range}T_2$, this map would be surjective. Additionally, this linear map from $\text{range}T_1 \to \text{range}T_2$ is an operator since $\text{range}T_1 \in V$ and $\text{range}T_2 \in V$. Let's call this operator $S \in \mathcal{L}(W)$. Note that since S is surjective and finite-dimensional, then S is also invertible. So, we found an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.

To prove the other side of the proof, assume there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.

Then since S is invertible, it follows that S is injective so null $ST_2 =$ null $T_2 =$ null T_1.