HW-4
Math 117

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1.4.27

- Let $x \in \mathbb{Q}$. Therefore, for any $r > 0$, we have the interval $(x - r, x + r)$ where $x + r, x - r \in \mathbb{R}$. Because we know that the Irrationals are dense in the Reals, we know that, $\exists y \in \mathbb{R} - \mathbb{Q} \ni y \in (x - r, x + r)$. Therefore, we have that $(x - r, x + r) \not\subset \mathbb{Q}$. Therefore, the Rationals are not an open set.
- Now, let $z \in \mathbb{R} - \mathbb{Q}$. Therefore, for any $\delta > 0$, we have the interval $(z - \delta, z + \delta) \ni z - \delta, z + \delta \in \mathbb{R}$. Therefore, because the $\mathbb{Q}$ are dense in the $\mathbb{R}$, we know that for any value of δ, there exsits some $x \in (z - \delta, z + \delta) \ni x \in \mathbb{Q}$. Therefore, z is a point of closure of $\mathbb{Q}$. But, because $z \in \mathbb{R} - \mathbb{Q}$, we have by definition that $z \notin \mathbb{Q}$. Therefore, the Rationals do not contain all of its points of closure and therefore, the Rationals are not closed.
- Thus, the Rationals are neither closed nor open.

1.4.30

(i)

- Let x be a point of closure of E'. Therefore, in every neighborhood of x, $\exists x' \in E'$. Furthermore, because $x' \in E'$, we know that in every neighborhood of x', $\exists y \in E \ni x' \neq y$ because E' is the collection of accumulation points of E. Let $\delta > 0 \ni x' \in (x - \delta, x + \delta)$ and $d(x, x') = r \ni 0 < r < \delta$. Therefore, we know that $(x' - \frac{min(\delta - r, r)}{2}, x' + \frac{min(\delta - r, r)}{2}) \subset (x - \delta, x + \delta)$ and, we also know by construction that, $x \notin (x' - \frac{min(\delta - r, r)}{2}, x' + \frac{min(\delta - r, r)}{2})$. But,

$(x' - \frac{min(\delta - r, r)}{2}, x' + \frac{min(\delta - r, r)}{2})$ is a neighborhood of x'. Therefore, we know by definition that, $\exists y \in E \ni y \in (x' - \frac{min(\delta - r, r)}{2}, x' + \frac{min(\delta - r, r)}{2})$ and we also know that $y \neq x'$, which also means that, $y \in (x - \delta, x + \delta)$. Therefore, for every neighborhood of x, $\exists y \in E \ni x \neq y$ as $y \in (x' - \frac{min(\delta - r, r)}{2}, x' + \frac{min(\delta - r, r)}{2})$ and $x \notin (x' - \frac{min(\delta - r, r)}{2}, x' + \frac{min(\delta - r, r)}{2})$. Therefore, x is an accumulation point of E and thus, $x \in E'$. Therefore, E' contains its closure points and therefore, is closed.

(ii)

- $\bar{A} \subseteq A \cup A'$

  Let $x \in \bar{A}$. Therefore, $x \in A$ or $x \notin A$. If $x \in A$, then, $x \in A \cup A' \Rightarrow \bar{A} \subseteq A \cup A'$ and we are done.

  If $x \notin A$. Then we know by definition that, $x \in \bar{A} \setminus A$. Therefore, for every neighborhood of x, $\exists y \in A$. Furthermore, we also know that $x \neq y$ as, $x \notin A$ but $y \in A$. Therefore, by definition, x is an accumulation point of A. Thus, $x \in A'$. Then, $x \in A \cup A' \Rightarrow \bar{A} \subseteq A \cup A'$

- $A \cup A' \subseteq \bar{A}$

  Let $x \in A \cup A'$. Therefore, either $x \in A$ or $x \in A'$ (or both). If $x \in A$, then $x \in \bar{A}$ as, $A \subset \bar{A}$ and by definition, $A \cup A' \subseteq \bar{A}$.

  If $x \in A'$, we have that, for every neighborhood of x, $\exists y \in A \ni x \neq y$. Therefore, by definition, x is a closure point of A as in every neighborhood we can find a point in A. Thus, $x \in \bar{A}$. Therefore, $A \cup A' \subseteq \bar{A}$.

- Therefore, we have proven that $\bar{A} = A \cup A'$

1.4.31

Let E be a set that consists of only isolated points. Let $x \in E$, therefore, we can find some neighborhood of x such that, $N_x(r) \cap E = \{x\}$. Furthermore, by construction we can describe $N_x(r)$ such that, for any other $N_{x'}(r')$ for $x' \in E \ni x' \neq x$, we have that $N_x(r) \cap N_{x'}(r') = \emptyset$. Therefore, we know that $(x - r, x + r) \subset \mathbb{R}$ and therefore, $\exists y \in (x - r, x + R) \ni y \in \mathbb{Q}$ as $\mathbb{Q}$ are dense in $\mathbb{R}$. Therefore, we can define an injective map $f : E \to \mathbb{Q}$ where $f(x) = y$. We know this is injective by the construction that $N_x(r) \cap N_{x'}(r') = \emptyset$. Furthermore, there may be more than 1 such rational number in the interval, we can choose any. Therefore, $\forall x \in E$, $\exists! y \in \mathbb{Q} \ni y \in N_x(r)$ and, we also know for any other $x' \in E, y \notin N_{x'}(r')$. Therefore, we have defined an injective map from $f : E \to \mathbb{Q}$. Thus, $E \subseteq \mathbb{Q}$ and therefore, E is countable as $\mathbb{Q}$ is countable

1.4.32

(i)

- Assume E is open. Therefore, $\forall x \in E$, we know by definition that, $\exists r > 0 \ni (x - r, x + r) \subset E$. Therefore, by definition we know that, $x \in int(E)$. Thus, $E \subseteq int(E)$. Furthermore, by construction we know that $int(E) \subset E$

as, $int(E)$ consists of the interior points of E which implies that all points in int(E) are in E. Thus, if E is open, we have shown that $E = int(E)$.

- Now, assume that $E = in(E)$. Therefore, we know that $\forall x \in E, x \in int(E)$ and therefore, $\forall x \in E, \exists r > 0 \ni (x - r, x + r) \subset E$. And, by definition, this makes E an open set. Therefore, if $E = int(E)$, then E is open.

- Therefore, because we have shown the implication in both direction, we have shown that E is open if and only if $E = int(E)$

(ii)

- Assume that E is dense in $\mathbb{R}$. We will use proof the forward direction (E is dense $\Rightarrow int(\mathbb{R} \setminus E) = \emptyset$) by contradiction. Assume that int($\mathbb{R} \setminus E) \neq \emptyset$. Therefore, by definition, $\exists x \in R \setminus E$, which is an interior point of the set, which means, $\exists r > 0 \ni (x-r, x+r) \subset R \setminus E$. But, this means that, E is not dense in the reals as, $\nexists y \in E \ni y \in (x - r, x + r)$. This is a contradiction to our very first assumption as $x - r, x + r \in \mathbb{R}$ and we have shown that E is not dense in reals (contradiction). Therefore, if E is dense in the $\mathbb{R}$, we must have that int($\mathbb{R} \setminus E) = \emptyset$.

- Now, assume that int($\mathbb{R} \setminus E) = \emptyset$. Now, let $a, b \in \mathbb{R}$. Without loss of generality, assume that $b > a$. Therefore, if $\nexists y \in \mathbb{R} \setminus E \ni y \in (a, b)$, then we have already proved our point and E is dense in the Reals. Now, assume that, $\exists y \in \mathbb{R} \setminus E \ni y \in (a, b)$. Then, we can define $r > 0 \ni r = min\{y - a, b - y\}$. Therefore, $(y - r, y + r) \subset (a, b) \subset \mathbb{R}$. But, because int($\mathbb{R} \setminus E) = \emptyset$, we know that, $\nexists h > 0 \ni (y - h, y + h) \subset \mathbb{R} \setminus E$. Therefore, by definition,

$\exists x \in E \ni x \in (y-r, y+r) \Rightarrow x \in (a,b)$. Therefore, E is dense in the $\mathbb{R}$.

- Therefore, we have proven that, E is dense if and only if int$(\mathbb{R} \setminus E) = \emptyset$

1.4.35

- We know by definition that, the collection of Borel sets the Borel σ-algebra is the smallest sigma algebra containing all the open sets (by definition.
- But, we also know by definition of a σ-algebra that, it is closed under complement. And, the complement of an open set is a closed set. Therefore, because the Borel σ-algebra contains all the open sets, it must contain all the closed sets as well (as it is closed under complement). Therefore, the Borel σ-algebra is the smallest σ-algebra containing the closed sets.
- In addition to that, to prove that it is the smalles, let Σ be another σ-algebra that contains all the closed sets. Therefore, by definition, Σ also contains all the open sets. But, we know that $\mathcal{B}$ is the smallest σ-algebra containing open sets and therefore, $\mathcal{B} \subset \Sigma$. Therefore, $\mathcal{B}$ (Borel σ-algebra) is the smallest sigma algebra containing the closed sets.

1.4.36

- Let $\mathcal{A}$ be a σ-algebra consisting of all sets [a,b) where $b > a$. Therefore, we know that, sets of the form $[a + \frac{1}{n}, b)$ are also in $\mathcal{A}$ for $n \in \mathbb{N}$. Furthermore, because we know by definition that all σ-algebras are closed under countable union, we know that, $\bigcup_{n\in\mathbb{N}}[a+\frac{1}{n},b)=(a,b)$ is contained in $\mathcal{A}$. Furthermore, Because (a,b) is contained in $\mathcal{A}$, we know that an interval of the form (a,n) is contained in $\mathcal{A}$ where $n\in\mathbb{N}$. Therefore, by the properties of σ-algebras we know that $\cup_{n\in\mathbb{N}}(a,n)=(a,\infty)$ is contained in $\mathcal{A}$ and similarly, $(-\infty,b)$ is also contained in $\mathcal{A}$. Thus, $(-\infty,\infty)$ is also contained. We can extend this argument and conclude that, $\mathcal{A}$ contains all the open sets as, an open set of $\mathbb{R}$ is the union of countably many open intervals. Therefore, we have proven that $\mathcal{A}$ is a σ-algebra containing all the open sets consisting of all intervals of the form [a,b). As we have previously proved, the smallest σ-algebra containing open sets is the collection of Borel sets ($\mathcal{B}$). Therefore, we now have that, $\mathcal{B}\subset\mathcal{A}$. Therefore, let the smallest σ-algebra containing all intervals of the form [a,b) be Σ. We know from what we proved earlier that, $\mathcal{B}\subset\Sigma$. Finally, because $b>a$, we know that $\exists x\ni a<x<b$ and therefore, we know that, $[a,x]\in\mathcal{B}$ and, $(x,b)\in\mathcal{B}$. Therefore, because $\mathcal{B}$ is a σ-algebra , we know that $[a,x]\cup(x,b)=[a,b)\in\mathcal{B}$ and therefore, we have shown that all intervals of the form $[a,b)$ are contained in $\mathcal{B}$.

- Therefore, going back, we have shown that $\mathcal{B}\subset\Sigma$ where Σ is the smallest σ-algebra containing the intervals of the form [a,b). But, we have also shown that $\mathcal{B}$ contain intervals of the form [a,b). Therefore, $\mathcal{B}$ which is the collection of all Borel Sets is the smalles σ-algebra containing intervals of the form [a,b)