HW-5
Math 117

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1.5.38

- We will first have to show that lim $\sup\{a_n\} = s$ and $\liminf\{a_n\} = i$ are indeed cluster points. We will first show that $\limsup\{a_n\}$ is a cluster point and the argument for lim inf will follow similarly. To do so, we will have to show that a subsequence $a_{n_k} \to s$ as $n_k \to \infty$. We can define $\sup\{a_m \mid m \geq n\} = s_n$. Therefore, by the definition of lim sup, we know that for any $\epsilon > 0$, $\exists N \ni |s_n - s| < \frac{\epsilon}{2}, \forall n \geq N$. Furthermore, for each s_n, by the definition of supremum, we know that, $\exists l \geq n \ni s_n > a_l > s_n - \frac{\epsilon}{2}$ and therefore, $|a_l - s_n| < \frac{\epsilon}{2}$. For our subsequence $\{a_{n_k}\}$, if we let $n_k = l$ for s_n for each n, then we have defined an entire subsequence and:

$$|a_{n_k} - s| \leq |a_{n_k} - s_n| + |s_n - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

And by definition, $\{a_{n_k}\}$ converges to s and therefore, $s = \limsup\{a_n\}$ is a cluster point.

- Similarly, we can define $\inf\{a_m \mid m \geq n\} = i_n$. Therefore, by the definition of lim sup, we know that for any $\epsilon > 0$, $\exists N \ni |i_n - i| < \frac{\epsilon}{2}, \forall n \geq N$. Furthermore, for each i_n, by the definition of infimum, we know that, $\exists l \geq n \ni i_n < a_l < i_n + \frac{\epsilon}{2}$ and therefore, $|a_l - i_n| < \frac{\epsilon}{2}$. For our subsequence $\{a_{n_k}\}$ we can similarly let $n_k = l$ for i_n for each n. Thus:

$$|a_{n_k} - i| \leq |a_{n_k} - i_n| + |i_n - i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and by definition $\liminf\{a_n\}$ is a cluster point.

- We will now show that lim sup is the largest cluster point by definition by contradiction. Let us assume that there exists some subsequence of $\{a_n\}$ called $\{b_{n_k}\}$ such that $b_{n_k} \to t > s$. Furthermore, because $t > s$, $\exists \epsilon > 0 \ni |t - s| > \epsilon$ and furthermore, $|b_{n_k} - t| < \epsilon$ for some $n \geq N$ which implies that $t - \epsilon < b_{n_k} < s_n \leq s$. But, because $|t - s| > \epsilon$ we know that $t - \epsilon > s$. Therefore, we now have that $s > t - \epsilon > s$ which is a contradiction. Therefore, the $\limsup\{a_n\}$ must be the largest cluster point for that sequence.

- Similarly, if we assume that there exists a subsequence of $\{a_n\}$ called $\{c_{n_k}\}$ such that $c_{n_k} \to u < i$. Because $u < i$, we know that there exists some $\epsilon > 0$ such that $|i - u| > \epsilon$. Furthermore, because u is the cluster point of the series, we know that there exists some N such that $|b_{n_k} - u| < \epsilon, \forall n_k \geq N$. Thus, we can see that $b_{n_k} < u + \epsilon$. Furthermore, because $|i - u| > \epsilon$, we know that $u + \epsilon < i$. Therefore, we have that $b_{n_k} < u + \epsilon < i$, and by the definition of $\liminf$, we know that $i \leq i_n < b_{n_k} < u + \epsilon < i$ which is a contradiction. Therefore, $i = \liminf\{a_n\}$ must be the smallest cluster point.

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(i)

- We can prove the first part using contradiction. Let us assume that there are infinitely many indices n such that $a_n > l + \epsilon$. Then, we have that $\forall m \in \mathbb{N}, \exists z \in \{a_n \mid n \geq m\} \ni z > l + \epsilon$ for some $\epsilon > 0$. Furthermore, if we let $s_m = \sup\{a_n \mid n \geq m\}$, then we know that $s_m > s + \epsilon \ \forall m \in \mathbb{N}$. Therefore, $|s - s_m| > \epsilon, \forall m \in \mathbb{N}$. But, this is a contradiction as the definition of lim sup tells us that for all $\epsilon > 0$, there exists some indices m such that $|s - s_m| < \epsilon$. Thus, our original assumption is false and, we know that there are only finitely many indices n such that $a_n > l + \epsilon$.

- Now, assume that there are only finitely many indices such that $l - \epsilon < a_n$. Then, because there are finitely many, we know that for some $m \in \mathbb{N} \ni n \geq m$, $a_n \leq l - \epsilon$. Therefore, we know that $s_m = \sup\{a_n \mid n \geq m\} < l - \epsilon$. Which further implies that $|s_m - l| > \epsilon$. But, this is a contradiction to the definition of lim sup. Thus, we know that our original assumption is wrong and that, there must be infinitely many indices such that $a_n > l - \epsilon$.

(ii)

- ($\Rightarrow$): We will prove the forward direction using contradiction. First assume that $\limsup\{a_n\} = \infty$. Then, Assume that $\{a_n\}$ is bounded above. Let the upper bound for the sequence be c. Therefore, we know that, for any number N, we know that $s_N = \sup\{a_n \mid n \geq N\} \leq c$. Furthermore, because $|\limsup\{a_n\} - s_N| < \epsilon$ for some $\epsilon < 0$, we can choose ϵ arbitrarily small so that $\limsup\{a_n\} \leq c$. But this is a contradiction to our original assumption that $\limsup\{a_n\} = \infty$. Therefore, we must have that $\{a_n\}$ is not bounded above.

- ($\Leftarrow$): We will prove the reverse direction using contrapositive. First assume that $\limsup\{a_n\} = c < \infty$. Furthermore, if we let $s_n = \sup\{a_m | m \geq n\}$ $m \geq n$, then by definition of lim sup, we have that for some $\epsilon > 0$, $\exists N \ni |s_n - c| < \epsilon$, $\forall n \geq N$. Therefore, from this we can see that, the sequence is bounded above by $max\{c+\epsilon, s_1, s_2, \dots s_N\}$. We can find such an upper bound for any $\epsilon > 0$. Therefore, we can see that there is a finite upper bound for the entire sequence. Therefore, by contrapositive, if $\{a_n\}$ has no upper bound then $\limsup\{a_n\} = \infty$.

- Therefore, we have that $\limsup\{a_n\} = \infty$ if and only if $\{a_n\}$ is not bounded above.

(iii)

- To prove this, we will use the fact that for some set A, we have that $\sup(A) = -\inf(-A)$. So, we know by the definition of $\limsup\{a_n\}$ that,

$$\limsup\{a_n\} = \lim_{n\to\infty} \sup\{a_m \mid m \geq n\}$$

$$\limsup\{a_n\} = \lim_{n\to\infty} -\inf\{-a_m \mid m \geq n\}$$

$$\limsup\{a_n\} = -\lim_{n\to\infty} \inf\{-a_m \mid m \geq n\}$$

$$\limsup\{a_n\} = -\liminf\{-a_n\}$$

(iv)

- ($\Leftarrow$) We will first show the reverse direction and therefore, assume that $(\limsup\{a_n\} = i) = (\liminf\{a_n\} = s) = a$. Let $\sup\{a_m | m \geq m\} = s_n$ and , $\inf\{a_m \mid m \geq n\} = i_n$. Then, by the definition of supremum and infimum, we know that, $i_n \leq a_n \leq s_n$. And furthermore, by definition of lim sup and lim inf, we can see that

$$\lim_{n\to\infty} i_n = i = a \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} s_n = s = a$$

Therefore, we can see that $a \leq \lim_{n\to\infty} a_n \leq a$ and therefore, we can see that $\lim_{n\to\infty} a_n = a$.

- ($\Rightarrow$) We will now show the opposite direction. Assume that the series $\{a_n\}$ converges to a, that is, $a_n \to a$ as $n \to \infty$. Therefore, by definition we know that, for any $\epsilon > 0$, $\exists N \ni |a_n - a| < \epsilon$, $\forall n \geq N$. From this we can see that,

$$a - \epsilon < a_n < a + \epsilon \,, \forall n \geq N$$

And furthermore, by the definition of infimum and supremum, we can see that,

$$a - \epsilon \leq \inf\{a_m \mid m \geq n\} \leq \sup\{a_m \mid m \geq m\} \leq a + \epsilon$$

Thus, we can see that $|i_n - a| \leq \epsilon$ and $|s_n - a| \leq \epsilon$. And, by definition, we can see that $\lim_{n\to\infty} s_n = a$ and, $\lim_{n\to\infty} i_n = a$. Therefore, $\limsup\{a_n\} = \liminf\{a_n\} = a$

(v)

- Let $\inf\{a_m \mid m \geq n\} = A_n$ and, let $\inf\{b_m \mid m \geq n\} = B_n$. Furthermore, because $a_n \leq b_n$, $\forall n$, we can see by definition that $A_n \leq B_n$, $\forall n$.
- Finally, from the Monotonity of Convergence, we can see that $\lim_{n\to\infty} A_n \leq \lim_{n\to\infty} B_n$, which by definition is:

$$\liminf\{a_n\} \leq \liminf\{b_n\}$$

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- Let $\{a_n\}$ be a sequence and let $\sup\{a_m \mid m \geq n\} = s_n$ and let $\inf\{a_m \mid m \geq n\} = i_n$. Therefore, $\forall a_m \in \{a_m \mid m \geq n\}$ by the definition of supremum and infimum, we know that $i_n \leq a_m \leq s_n$, $\forall m \geq n$. And therefore, $i_n \leq s_n$, $\forall n$. As this is true for all n it follows that

$$\lim_{n\to\infty} i_n \leq \lim_{n\to\infty} s_n$$

and therefore,

$$\liminf\{a_n\} \leq \limsup\{a_n\}$$

1.5.43

We first need to show that a monotone subsequence exists. Therefore, let $\{a_n\}$ be a sequence of $\mathbb{R}$. We know that there exists a monotone subsequence of a sequence, when there is one value such that all values after it are either lower than it, in other words, if we can find a local maximum we can construct a monotone subsequence.

- Let the set $A = \{n \in \mathbb{N} \mid \forall m > n, a_m \leq a_N\}$ therefore, we now have a set of local maximum. If $\{a_n\}$ itself is monotone decreasing, then we have infinite number of such indices. If $\{a_n\}$ itself is monotone increasing, then we have no such indices. Therefore, A ranges from $\emptyset$ to infinite.

- Now, if A is infinite, then we have indices $n_1, n_2, n_3, \ldots$ such that for eahc n_k we know that $a_{n_k} > a_{n_{k+1}}$. Therefore, for all the indices in A, we know that $a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq \ldots$ and we have found our monotone (increasing) subsequence.

- In the case where A is finite. Let a_{n_k} be the last element of this finite set, that is, there are no more such local extremum in the entire $\{a_n\}$ sequence after a_{n_k}. Therefore, if we let $b_1 = n_k + 1$, then we have that $a_{b_1} \notin A$. Therefore, because a_{b_1} is not a local maximum, we know that there exists some indice b_2 such that $b_2 > b_1$ and, $a_{b_2} > a_{b_1}$. Now, we know that a_{b_2} is not in A either, therefore, we can find a $b_3 > b_2$ such that $a_{b_3} > a_{b_2}$ and we can continue this process. Therefore, $a_{b_1} \leq a_{b_2} \leq a_{b_3} \leq \ldots$ is now our monotone increasing subsequence.

- Therefore, we have shown that every sequence in $\mathbb{R}$ has a monotone subsequence. Note that this also works in the case where $\{a_n\}$ itself is either monotone decreasing or monotone increasing.
- Now that we know every sequence in $\mathbb{R}$ has a monotone subsequence, we can use this to prove the Bolsano-Weiestrass theorem. Let $\{a_n\}$ be our bounded sequence. Then, we know that there exists a monotone subsequence (by what we proved earlier). Let this monotone subsequence be $\{x_n\}$. Because $\{a_n\}$ is bounded, we know that $\{x_n\}$ will be bounded as well.

  - Assume $\{x_n\}$ is monotone increasing. Then we know that there is a bound for $\{x_n\}$. Let $x = \sup\{x_n\}$. Therefore, we know that $x > x_n, \forall n$. Furthermore, by definition of supremum, we know that for some $\epsilon > 0$, exists some number N such that $x - \epsilon < x_k, \forall k \geq N$. And furthermore, we also know that $x_n \leq x < x + \epsilon$. Therefore, we have that $x - \epsilon < x_n < x + \epsilon$, $\forall n \geq N$ which implies that $|x - x_n| < \epsilon$. Therefore, we have shown that $x_n \to x$.
  - Now assume that $\{x_n\}$ is monotone decreasing. Similar to the increasing subsequence, let $y = \inf\{x_n\}$. Therefore, by definition, we know tha $y < x_n$, $\forall n$. Furthermore, by definition of infimum, we know that, for some $\epsilon > 0$ there exists a number N such that $x_k < y + \epsilon$, $\forall k \geq N$. And, we also know that $y - \epsilon < y < x_n$. Therefore, we have that $y - \epsilon < x_k < y + \epsilon$, $\forall k \geq N$. and therefore, $|y - x_n| < \epsilon$ and we have show convergence.

- This completes the proof of the Bolsano-Weierstrass theorem.

1.5.45

(i)

($\Rightarrow$) Assume that the series $\sum_{k=1}^{\infty} a_k$ is summable and it is equal to s. Let $\sum_{k=1}^{n} a_k = s_n$. Therefore, we know that $\lim_{n\to\infty} s_n = s$. Thus, for any $\epsilon > 0$, we know that, $\exists N$ such that $|s - s_n| < \frac{\epsilon}{2}$, $\forall n \geq N$. Now, let $b > n \geq N$, therefore, we have by the triangle inequality that,

$$|s_b - s_n| < |s_b - s| + |s - s_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Furthermore, we can rewrite $|s_b - s_n|$ as, $\sum_{k=1}^{b} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n}^{b}$. Furthermore, because $b > n$, we can rewrite $b = n + m$ for some natural number m. Therefore, we have proven that

$$|\sum_{k=n}^{n+m} a_k| < \epsilon$$

($\Leftarrow$) Assume that for some $\epsilon > 0$, there exists some number N such that $|\sum_{k=n}^{n+m} a_k| < \epsilon$, $\forall n \geq N$. We can rewrite $\sum_{k=n}^{n+m} a_k$ as $\sum_{k=1}^{n+m} a_k - \sum_{k=1}^{n} a_k$. We can write this as the partial sum, and therefore, $|s_{n+m} - s_n| < \epsilon$. We can further rewrite $n+m=b$ and we get , for some $\epsilon > 0$, there exists s number N such that, for some number $n+m=b>n \geq N$ we have that

$$|s_b - s_n| < \epsilon$$

By definition, this make the sequence of partial sums $\{s_n\}$ Cauchy. And, we have proven earlier that, a sequence that is Cauchy converges. Therefore we have shown that $\{s_n\}$ converges to some number (let us call is s). Therefore, $\sum_{k=1}^{\infty} a_k$ is summable as $\lim_{n\to\infty} s_n = s$

(ii)

- Assume that the series $\sum_{k=1}^{\infty} |a_k|$ is summable and is equal to s. We will now use contradiction to show that $\sum_{k=1}^{\infty} a_k$ must be summable as well. Assume that $Sum_{k=1}^{\infty} a_k$ is not summable. Now, define the partial sums as, $\sum_{k=1}^{n} a_k = s_n$. Finally, we know that $a_k \leq |a_k|$, $\forall k$, thus $s_n \leq |s_n|$, $\forall n$. Furthermore, by the linearity fo convergence, we know that $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} |s_n|$. But, by our assumption that $Sum_{k=1}^{\infty} a_k$ is not summable, we know that $\lim_{n\to\infty} s_n = \infty \leq \lim_{n\to\infty} |s_n|$. But this implies that $\sum_{k=1}^{\infty} |a_k|$ is not summable. But, this contradicts our original assumption. Therefore, if $\sum_{k=1}^{\infty} |a_k|$ is summable, then $\sum_{k=1}^{\infty} a_k$ is summable

(iii)

- (⇒) Assume that $\sum_{k=1}^{\infty} a_k$ is summable and is equal to s. Furthermore, define each partial sums as $\sum_{k=1}^{n} a_k = s_n$. Therefore, we know that $\lim_{n\to\infty} s_n = s$. Therefore because each a_k is non-negative, we know that $s_n < s_n + 1,\ \forall n$. Thus, the sequence of partial sums $\{s_n\}$ is monotone (increasing). Then, by Theorem 15, because $\{s_n\}$ converges to s and it is a monotone sequence, we know that it is bounded.

- (⇐) Assume that the sequence of partial sums $\{s_n\}$ is bounded. Because each a_k is non-negative, we know that $\{s_n\}$ is a monotone increasing sequence. Therefore, by Theorem 15 again, because $\{s_n\}$ is bounded and monotone, it must converge. Therefore, $\lim_{n\to\infty} s_n = s = \sum_{k=1}^{\infty} a_k$. Therefore, $\sum_{k=1}^{\infty}$ is summable.

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- Consider a bounded interval $I_n = [a, b]$ such that they are nested. Therefore, for each interval, we have that $a_1 < b_n < b_1$. Thus, if we have a sequence $\{b_n\}$, by the Bolzano-Weierstrass Theorem, we know that there exists a subsequence of $\{b_n\}$ that converges to b. Furthermore, because the subsequence b_{n_k} is decreasing, we know that, the point of convergence $b \leq b_n$ for all n. In addition to that, because we have that $a_n < b_n \forall n$, we can still see that $a_n < b$. Therefore, we have that $a_n \leq b \leq b_n$. for all n. Therefore,

$b \in [a_n, b_N]$, $\forall n$ and therefore, $\bigcap_{n=1}^{\infty} I_n = b \neq \emptyset$. Therefore, if we have a set A that is non-empty and bounded above, there are elements of A that are not an upper-bound, let oen such element be a_1. Let b_1 be an upper bound for A. Now, define $m_1 = \frac{a_1+b_1}{2}$. If m_1 is an upper bound let $b_2 = m_1$ and let $a_2 = a_1$. If m_1 is not an upper bound, then let $a_2 = m_1$ and let $b_2 = b_1$ and then define $I_2 = [a_2, b_2]$. We will repeat this process such that $m_n = \frac{a_n+b_n}{2}$ and, if m_n is an upper bound then $b_{n+1} = m_n$ and $a_{n+1} = a_n$, and if m_n is not na upper bound let $a_{n+1} = m_n$ and let $b_{n+1} = b_n$ and then define $I_{n+1} = [a_{n+1}, b_{n+1}]$. By what we showed earlier, we know that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Therefore, let $\bigcap_{n=1}^{\infty} I_n = b$. To show that b is an upper bound, assume that it is not. Then, for some a, we know that $a > b$ and, $a - b = \epsilon$ for some $\epsilon > 0$. But, this is impossible as, we chose ϵ arbitrarily and, the length of the interval $[a_N, b_N]$ for some N implies that, this is not possible as $b \in [a_n, b_n]$, $\forall n$. Therefore, b must be an upper bound. Now, let $\epsilon > 0$ and let the number N be such that $N > \frac{d}{\epsilon}$ where d is the interval length. Therefore, if we have another upper bound, we know that $a_N < a \leq b_N$. Therefore, we have that $|b - a| < \frac{d}{N} < \epsilon$. Therefore, we know that $b - \epsilon < a$ for some arbitrary ϵ and therefore, $b < a$ which implies that there exists some lowest upper bound, which the Completeness Axiom calls the supremum. A similar argument can be made for the infimum, but replacing all upper bounds with lower bounds.