HW-8
Math 117

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2.14

- Let E be a set with positive measure, that is

$$m^*(E) > 0$$

Now, let $A \subset E$ be bounded and define $A^* = E \setminus A$. Thus, we have that:

$$E = A \cup A^*$$

and therefore, we have

$$m^*(E) = m^*(A \cup A^*) > 0$$

but, by the property of subadditivity of the outer measure we have that:

$$0 < m^*(A \cup A^*) \leq m^*(A) + m^*(A^*)$$

Now, consider the case where $m^*(A) < 0$ and $m^*(A^*) < 0$, then we have that

$$0 < m^*(A) + m^*(A^*) < 0$$

which would be a contradiction. Now, without loss of generality assume that either one, or both of the measures are 0, and the other has negative measure therefore, we have that:

$$0 < m^*(A) + m^*(A^*) \leq 0$$

which is again a contradiction. Therefore, we must have that, at least one (or both) of A or A^* has a positive measure.

- Therefore, $m^*(A) > 0$ and/or $m^*(A^*) > 0$ and because $A \subset E$ and $A^* \subset E$ we know that, there must be a positive subset of E. if E has a positive measure.

2.17

- We know by the outer and inner approximations for the outer measure that for any $\frac{\epsilon}{2} > 0$, $\exists F \subseteq E \ni m^*(E \setminus F) < \frac{\epsilon}{2}$ where F is closed and also, $\exists O \supseteq E \ni m^*(O \setminus E) < \frac{\epsilon}{2}$ where O is open.
- Now, we need to show that $E \setminus F \cup O \setminus E = O \setminus F$.
- We will first show the forward direction. Let $x \in (E \setminus F) \cup (O \setminus E)$ thus, we know that $x \in (E \setminus F)$ or $x \in (O \setminus E)$ (or both). So, if $x \in (E \setminus F)$ we know that $x \in E$ and $x \notin F$. Because $x \in E$ and $E \subseteq O \Rightarrow x \in O$ and thus, we know that $x \in O \setminus F$. If $x \in (O \setminus E)$ then we know that $x \in O$ and $x \notin E$. Because $x \notin E$ and $F \subseteq E$ we know that $x \notin F$. Thus, we have that $x \in (O \setminus F)$. Therefore, if $x \in (E \setminus F) \cup (O \setminus E)$ then we have that $x \in (O \setminus F)$ and thus, $(E \setminus F) \cup (O \setminus E) \subseteq (O \setminus F)$.
- Now we will show the reverse direction. Let $x \in (O \setminus F)$. Now, we will use contradiction to prove the other direction. Assume that $x \notin (E \setminus F) \cup (O \setminus E)$. Thus, we know that, $x \notin (E \setminus F)$ AND, $x \notin (O \setminus E)$ (x cannot be in either as, if it is then x is in the union which is not what we assumed). If $x \notin O \setminus E$, we know that, either $x \notin O$ or, $x \in O$ and $x \in E$ as well. Thus, if $x \notin O$, then $x \notin O \setminus F$ which is a contradiction. Now, if $x \in O$, then for $x \notin O \setminus E$ we have that $x \in E$. Then, because $x \notin E \setminus F$, we have that $x \in F$ if $x \in E$. Therefore, $x \in O$ and $x \in F$ which implies that $x \notin O \setminus F$ which is a contradiction. Therefore, we reach a contradiction and our assumption must be wrong. Thus, if $x \in O \setminus F$ we must have that $x \in (E \setminus F) \cup (O \setminus E)$ and therefore, $O \setminus F \subseteq (E \setminus F) \cup (O \setminus E)$.

- Therefore, now that we have proven both directions, we have that for $F \subseteq E \subseteq O$ we have $O \setminus F = (E \setminus F) \cup (O \setminus E)$. Now, we have that:

$$m^*(O \setminus F) = m^*((E \setminus F) \cup (O \setminus E))$$

and, by the subadditivity of the outer measure we have that:

$$m^*(O \setminus F) = m^*((E \setminus F) \cup (O \setminus E)) \leq m^*(E \setminus F) + m^*(O \setminus E)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

2.20

- We will first show the forward direction. Let E be a measurable set and consider the interval $I = (a, b)$. Then, by the Caratheodory condition, we know that

$$m^*(I) = m^*(I \cap E) + m^*(I \cap E^c)$$

we can rewrite this using the properties of the setminus operation and the outer measure of an interval to get:

$$b - a = m^*((a, b) \cap E) + m^*(I \setminus E)$$

as, $I \cap E^c = I \setminus E$. Therefore, we have shown the forward direction.

- Now we will show the backward direction. Assume that for an interval $I = (a, b)$ we have that:

$$b - a = m^*(I \cap E) + m^*(I \setminus E)$$

We can find a countable collection of intervals which forms an open cover of E, $\{I_k\}_{k=1}^{\infty}$ where each I_k is an interval of the form (a, b) such that:

$$\sum_{k=1}^{\infty} m^*(I_k) < m^*(E) + \epsilon$$

for some $\epsilon > 0$.

- Furthermore, because each I_k is an interval, it is measurable and thus, by the Caratheodory condition we have that:

$$\sum_{k=1}^{\infty} m^*(I_k) = \sum_{k=1}^{\infty} [m^*(I_k \cap E) + m^*(I_k \setminus E)]$$

we can spread out the summation sign to get:

$$\sum_{k=1}^{\infty} m^*(I_k) = \sum_{k=1}^{\infty} m^*(I_k \cap E) + \sum_{k=1}^{\infty} m^*(I_k \setminus E)$$

and now, we can use the subadditivity of the outermeasure to get:

$$\sum_{k=1}^{\infty} m^*(I_k \cap E) + \sum_{k=1}^{\infty} m^*(I_k \setminus E) \geq m^*(\bigcup_{k=1}^{\infty}(I_k \cap E)) + m^*(\bigcup_{k=1}^{\infty} I_k \setminus E)$$

because the $\{I_k\}$ for an open cover of E, we know that $\bigcup_{k=1}^{\infty}(I_k \cap E) = E$ and thus, we now have:

$$\sum_{k=1}^{\infty} m^*(I_k \cap E) + \sum_{k=1}^{\infty} m^*(I_k \setminus E) \geq m^*(E) + m^*(\bigcup_{k=1}^{\infty} I_k \setminus E)$$

Now, by the properties of set minus and union, we can write $\bigcup_{k=1}^{\infty}(I_k \setminus E)$ as $(\bigcup_{k=1}^{\infty} I_k) \setminus E$. Furthermore, because each I_k is an open interval, we know that their union forms an open set (as the I_k are not nested we know that it doe snot form a closed set). Furthermore, because the union forms an open cover, we know that

$$\bigcup_{k=1}^{\infty} I_k \supseteq E$$

and thus, we now have that:

$$\sum_{k=1}^{\infty} m^*(I_k \cap E) + \sum_{k=1}^{\infty} m^*(I_k \setminus E) \geq m^*(E) + m^*((\bigcup_{k=1}^{\infty} I_k) \setminus E)$$

and now, including out original inference we have:

$$m^*(E) + \epsilon > \sum_{k=1}^{\infty} m^*(I_k \cap E) + \sum_{k=1}^{\infty} m^*(I_k \setminus E) \geq m^*(E) + m^*((\bigcup_{k=1}^{\infty} I_k) \setminus E)$$

thus,

$$\epsilon > m^*((\bigcup_{k=1}^{\infty} I_k) \setminus E)$$

for some $\epsilon > 0$ and since $(\bigcup_{k=1}^{\infty} I_k)$ is an open set such that $E \subseteq (\bigcup_{k=1}^{\infty} I_k)$ we have by Theorem 11 that:

$$m^*((\bigcup_{k=1}^{\infty} I_k) \setminus E) < \epsilon$$

and thus, E is a measurable set.

- Now since we have proven both directions, the equivalence statement holds.

2.26

- Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of disjoint and let measurable sets. Let A be any set. First of all, we know by Proposition 7 that, $\bigcup_{k=1}^{\infty} E_k$ is measurable as well. Thus, we know that for some arbitrary n we have that each I_k is an interval of the form (a, b):

$$\bigcup_{k=1}^{\infty} E_k \supseteq \bigcup_{k=1}^{n} E_k \Rightarrow A \cap \bigcup_{k=1}^{\infty} E_k \supseteq A \cap \bigcup_{k=1}^{n} E_k$$

and thus, by the properties of outer measure we know that

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) \geq m^*(A \cap \bigcup_{k=1}^{n} E_k)$$

and, by proposition 6, we know that:

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) \geq m^*(A \cap \bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

but, because n is arbitrary we know that:

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

- And now, we need to show the other direction. For some arbitrary n, we know that:

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) \geq \sum_{k=1}^{n} m^*(A \cap E_k)$$

and again, by proposition 6 we know that:

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) \geq \sum_{k=1}^{n} m^*(A \cap E_k) = m^*(A \cap \bigcup_{k=1}^{n} E_k)$$

but, because n is arbitrary, we have that this holds for all n and thus:

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) \geq m^*(A \cap \bigcup_{k=1}^{\infty} E_k)$$

Therefore, we have now shown both directions and thus:

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) = m^*(A \cap \bigcup_{k=1}^{\infty} E_k)$$

2.28

- Consider a measurable set E. Now, construct a countable collection of disjoint sets $\{I_k\}$ such that $\bigcup_{k=1}^{\infty} I_k = E$. It does not matter how these sets are constructed as long as they are disjoint.

- Now, consider another collection of sets $\{F_k\}$ such that, $F_1 = I_1$ and $F_n = \bigcup_{k=1}^{n} I_k$. Therefore, because of the way the sets are constructed, we know that

$$\bigcup_{k=1}^{n} I_k = \bigcup_{k=1}^{n} F_k$$

for any natural number n. Also note that, the F_k's are ascending as $F_k \subseteq F_{k+1}$. Furthermore, by the finite additivity of measure, we know that

$$m(\bigcup_{k=1}^{n} I_k) = \sum_{k=1}^{n} m(I_k)$$

and now, by the continuity of measure we have:

$$\begin{aligned} m(\bigcup_{k=1}^{\infty} I_k) &= m(\bigcup_{k=1}^{\infty} F_k) \\ &= \lim_{k\to\infty} m(F_k) \\ &= \lim_{k\to\infty} m(\bigcup_{n=1}^{k} I_n) \end{aligned}$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k} m(I_n)$$

$$= \sum_{k=1}^{\infty} I_k$$

- This proves the countable additivity of measure.

2.39

- If we were to construct a "generalized cantor set" by taking out portions of size $\frac{\alpha}{3^k}$ at each kth step, then at the first step, we would remove 1 interval of size

$$C_1 = \frac{\alpha}{3}$$

second time, we would remove 2 intervals of size

$$\frac{\alpha}{9}$$

and thus, we would remove a total of

$$C_2 = \frac{2\alpha}{9}$$

third time, we would remove

$$C_3 = \frac{\alpha 2^2}{3^3}$$

and the kth time, we would remove

$$C_k = \frac{\alpha(2^{k-1})}{3^k}$$

Therefore, in total, we would remove an interval of total size:

$$m(C_g) = \sum_{k=1}^{\infty} \frac{\alpha(2^{k-1})}{3^k}$$

which we can simplify and write as:

$$\frac{\alpha}{3} \sum_{k=0}^{\infty} (\frac{2}{3})^k = \frac{\alpha}{3} \cdot 3 = \alpha$$

Let C_g be the generalized cantor set which is the union of all C_k's . Furthermore, using the fact that $F = [0,1] \setminus C_g$ we know that:

$$m(F) = m([0,1]) - m(C_g) = 1 - \alpha$$

- In addition to that, because each set we remove are 2^{k-1} open intervals of size $\frac{\alpha}{3^k}$, we know that the complement of an open interval is a closed interval and therefore, what gets left behind will always be a closed interval.
- In addition to this, because the union of disjoint closed intervals is again a closed interval, we know that the set F is going to be a closed interval.
- We now need to show that $[0,1] \setminus F$ is dense. We know that F is the cantor set we achieved by continually removing the sets C_k from $[0,1]$. Therefore, at each stage let $F_1 = [0,1] \backslash C_1$ and then let $F_2 = F_1 \backslash C_2$ and then $F_k = F_{k-1} \backslash C_k$. Therefore, we have generated nested sets $\{F_k\}$. And furthermore, we know that $F = \bigcap F_k$. Furthermore, each time we remove sets we are left with 2^k sets of size at most $\frac{1}{2^k}$ (as the total size of the sets must be less than or equal

to 1. Therefore, we have that, the size of the union of intervals $F_k \leq \frac{1}{2^k}$.

- Now, for two numbers $b > a$, assume that $(a, b) \subseteq F$. Therefore, we have that, $(a, b) \subseteq F_k$ for all k's. But, we know that this is false as, we can always find some k such that $m(a, b) = b - a > \frac{1}{2^k}$ and thus, we must have that $(a, b) \nsubseteq F_k$ for some k. Therefore, $\exists x \in (a, b) \ni x \notin F_k$. Thus, we have that $x \notin F$. Therefore, for any two numbers $a, b \in [0, 1]$, we have that, there exists some number $x \ni a < x < b$ such that $x \notin F$ and therefore, by definition, $x \in [0, 1] \setminus F$. Thus, $[0, 1] \setminus F$ is dense in $[0, 1]$