Math 108 a Ver. B TJ SIPIN

1) If $\dim U = 4$ and $\dim W = 4$, then we know

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$
$$7 = 4 + 4 - \dim(U \cap W)$$
$$1 = \dim(U \cap W).$$

Since $\dim U \cap W = 1$, then $U \cap W \neq \{0\}$.

2) Suppose that m is a nonnegative integer such that there exist distinct real numbers $x_0, x_1, \ldots, x_m$ such that $p(x_j) \in \mathbb{R}$ for $j = 0, 1, \ldots, m$. We want to prove that all the coefficients are real.

For sake of contradiction, assume all the coefficients are not real. Then, since $x_0, x_1, \ldots, x_m$ are distinct real numbers, we would get

$$p(x) = (a_0 + b_0 i) + (a_1 + b_1 i)x + \ldots + (a_m + b_m i)x^m,$$

for all $x \in x_0, x_1, \ldots, x_m$ and where $a_0 + b_0 i, a_1 + b_1 i, \ldots, a_m + b_m i \in \mathbb{C}$. Clearly $p(x) \notin \mathbb{R}$, so we get a contradiction. Thus, it must be that all the coefficients are real.

3) $\Rightarrow$: Suppose $U \cup W$ is a subspace of V. Let $u \in U$ and $w \in W$. Since $U \cup W$ is a subspace, then it is closed under addition and there exists an additive inverse for any $u, w \in U \cup W$. Thus, we get

$$① \quad u + w - w \in U \cup W$$

as well as

$$② \quad u + w - u \in U \cup W.$$

This means that for ①, $u \in U$ or $u \in W$. Similarly, for ② $w \in U$ or $w \in W$. Therefore, either U is a subspace of W or W is a subspace of U, completing this side of the proof.

$\Leftarrow$: Suppose one subspace U of V is contained in another subspace W of V. So we have $U \subseteq W$. It follows that $U \cup W = W$, which is a subspace of V. Similarly, if $W \subseteq U$, then $U \cup W = U$, which is a subspace of V. In either case, we have proven that if a subspace of V is contained in another subspace of V, the union of the two subspaces are another subspace of V.

4. A basis of a vector space is a list of linearly independent vectors that spans the vector space.

Suppose we have a basis $v_1, \ldots, v_n$ of V. Then, for any $a_1, \ldots, a_n \in \mathbb{R}$,

$$a_1 v_1 + \cdots + a_n v_n = 0,$$

where $a_1 = \cdots = a_n = 0$. Thus, we have proven linear independence.

Additionally, we can get any $v \in V$ as a linear combination of $v_1, \ldots, v_n$. So,

$$v = a_1 v_1 + \cdots + a_n v_n,$$

proving that $v_1, \ldots, v_n$ spans V.

5. Define $T \in \mathcal{L}(\mathbb{F}^2)$ by

$$T(w, z) = (-z, w)$$

$T(w,z) = (-z, w) = \lambda(w, z)$

$-z = \lambda w$ $\qquad\qquad\qquad\qquad$ $w = \lambda z$

$-z = \lambda^2 z$

$\sqrt{-1} = \sqrt{\lambda^2}$

$\lambda = i, -i$

When $\lambda = i$, our eigenvectors are of the form (iz, z). When $\lambda = -i$, our eigenvectors are of the form $(-iz, z)$. It should be added that $z \neq 0$ in order for it to be an eigenvector.

6. Suppose $T \in \mathcal{L}(V)$ and $\dim \text{range } T = k$. We want to prove that T has at most $k+1$ distinct eigenvalues.

Since $\dim \text{range } T = k$, then T has at most k distinct nonzero eigenvalues. However, T can also have 0 as an eigenvalue, so T has at most $k+1$ distinct eigenvalues

7. Suppose $T \in \mathcal{L}(V)$ is invertible. ~~Then it is either injective or surjective~~ Then, there exists T^{-1} such that $TT^{-1} = I$
Since $E(\lambda, T) = \text{null } (T - \lambda I) = T = \lambda I$, it follows that

$$T \frac{1}{\lambda} = I$$
$$\frac{1}{\lambda} = T^{-1} I = T^{-1} = \frac{1}{\lambda} I$$
$$T^{-1} - \frac{1}{\lambda} I = 0.$$

From $T^{-1} - \frac{1}{\lambda} I$, we get $\text{null}(T^{-1} - \frac{1}{\lambda} I) = E(\frac{1}{\lambda}, T^{-1})$. Therefore, we have proven that if $T \in \mathcal{L}(V)$ is invertible, then $E(\lambda, T) = E(\frac{1}{\lambda}, T^{-1})$ for every $\lambda \in \mathbb{F}$ w/ $\lambda \neq 0$.

8. Suppose that V and W are finite-dimensional, $T \in \mathcal{L}(V, W)$, and there exists $\varphi \in W'$ such that $\text{null}(T') = \text{span}(\varphi)$. We want to prove that $\text{range}(T) = \text{null}(\varphi)$.

Since $(\text{range } T)^0 = \text{null}(T') = \text{span}(\varphi)$, then $\text{span}(\varphi) = \{\varphi \in W' : \varphi(v) = 0 \text{ for all } v \in \text{range} T\}$.
So, $\text{span}(\varphi) = 0 = (\text{range } T)^0$. It follows that $\text{null}(\varphi)$ is equal to everything not equal to 0, thus $\text{null}(\varphi) = \text{range}(T)$.