HW-7
Math 117

Siddharth Deshpande

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2.4

- We will first show that the counting measure is countable additive. Let $\{I_n\}$ be a collection of disjoint sets. First assume that all the sets are finite with elements equal to m_n for each set. Therefore, we know because they are all finite sets we have that $\cup_{n=1}^{\infty} I_n$ is also a finite. Furthermore, because the sets are disjoint we know that the number of elements in $\cup_{n=1}^{\infty} I_n$ is $\sum_{n=1}^{\infty} m_n$. Therefore, we have that

$$\mu(\cup_{n=1}^{\infty} I_n) = \sum_{n=1}^{\infty} \mu(I_n)$$

Note that this would hold even if one or more of the sets were empty as the empty set is simply a set with 0 elements which is still finite.

- Now assume that one (or more) of the I_n is an infinite set, without loss of generality let is be I_{n_i}. Therefore, we have that

$$\infty = \mu(\cup_{n=1}^{\infty} I_n) = \sum_{n=1}^{n_1} \mu(I_n) + \mu(I_{n_i}) + \sum_{n=n_i}^{\infty} (I_n) = \sum_{n=1}^{\infty} \mu(I_n) = \infty$$

- Now we will show the transnational invariance. We will be using the translation invariance of the outer measure for an interval in this proof. (we know that an interval is transitional invariant as, the outer measure of an interval is just the length of the interval which is transitional invariant).

- Now, let $\{I_n\}$ be the smallest open cover of a set A such that each I_n is an interval. Therefore, we have that the open cover must also be an interval and because it is an interval, we can find the smallest such cover as, the length of an interval if well defined.

- Now, let $\{I_n + c\}$ for some $c \in \mathbb{R}$ be the translation of this open cover and similarly, let $A + c$ be the translation of the set A. Therefore, we first have that

$$m^*(A + c) \leq m^*(\{I_n + c\}) = m^*(I_n) = m^*(A)$$

- Similar to the last procedure, we will just reverse the process, let $\{I_n\}$ be a collection of intervals that is the smallest open cover for the set $A + c$. And then, let $\{I_n - c\}$ be a cover of A. Therefore, we have that

$$m^*(A) \leq m^*(\{I_n - c\}) = m^*(\{I_n\}) = m^*(A + c)$$

- Thus, we have that

$$m^*(A) = m^*(A + c)$$

for some $c \in \mathbb{R}$.

2.6

- We know that, the irrational numbers between $[0,1]$ can be written as:

$$\mathbb{Q}^c_{[0,1]} = [1,0] \setminus \mathbb{Q}$$

- Therefore, we know that the $\mathbb{Q}^c_{[0,1]} \subseteq [1,0]$ and therefore, we know that

$$m^*(\mathbb{Q}^c_{[0,1]}) \leq m^*([1,0]) = 1$$

- Now we shall show the reverse direction. First of all, we know that the set of rational numbers is countable and therefore, $m^*(\mathbb{Q}_{[0,1]}) = 0$. Furthermore, we from the sub-additivity of the outer measure that

$$m^*(\mathbb{Q}_{[0,1]} \cup \mathbb{Q}^c_{[0,1]}) = m^*([0,1]) = 1 \leq m^*(\mathbb{Q}_{[0,1]}) + m^*(\mathbb{Q}^c_{[0,1]}) \Rightarrow 1 \leq m^*(\mathbb{Q}^c_{[0,1]})$$

- Therefore, we have that

$$m^*(\mathbb{Q}^c_{[0,1]}) = 1$$

2.9

- First of all, we know by the sub-additive identity of the Outer measure that,

$$m^*(\bigcup_{i=1}^{\infty} E_k) \leq \sum_{i=1}^{\infty} m^*(E_k)$$

- Therefore, we know that

$$m^*(A \bigcup B) \leq m^*(A) + m^*(B) = m^*(B)$$

- Now we shall prove the reverse direction. We know that $B \subseteq A \bigcup B$ for any set A and B. Therefore, by the properties of outer measure, we know that

$$m^*(B) \leq m^*(A \bigcup B)$$

- Thus, we have now prove the reverse direction as well and therefore,

$$m^*(A \cup B) = m^*(B)$$

if $m^*(A) = 0$