108A HW 9

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1 Problem 3.E.2

Suppose $V_1, ..., V_m$ are vector spaces such that $V_1 \times ... \times V_m$ is finite-dimensional. Prove that V_j is finite-dimensional for each $j = 1, ..., m$.

Solution Suppose that $V_1, ..., V_m$ are vector spaces and $V_1 \times ... \times V_m$ is finite-dimensional.

By Example 3.74 in the book, it follows that $V_1 \times ... \times V_m$ is isomorphic to $V_1 + ... + V_m$. By Theorem 3.59 in the book, they have the same dimension. So, it follows that

$$\text{dim } V_1 \times ... \times V_m = \text{dim } V_1 + ... + \text{dim } V_m.$$

It follows that for each V_j, $\text{dim } V_j \leq \text{dim } V_1 + ... + \text{dim } V_m$. Therefore, each V_j is finite-dimensional.

Another solution By the definition of product of vector spaces, every $V_j \in V_1, ..., V_m$ is a subspace of $V_1 \times ... \times V_m$, so by Theorem 2.26 in the book, each V_j is finite-dimensional.

2 Problem 3.E.9

Suppose that A_1 and A_2 are affine subsets of V. Prove that the intersection $A_1 \cap A_2$ is either an affine subset of V or the empty set.

Solution Suppose that A_1 and A_2 are affine subsets of V. Let U be a subspace of V and $v, w \in V$. Let $A_1 = v + U$ and $A_2 = w + U$. Then the following are equivalent:

- $v - w \in U$
- $v + U = w + U$
- $(v + U) \cap (w + U) \neq \varnothing$.

Assume that $A_1 \cap A_2$ is nonempty. Then $A_1 = A_2 = A_1 \cap A_2$, an affline subset of V. We know this to be true since A_1 and A_2 are parallel, then they must be the same subset if the intersection is nonempty. Suppose we invert the equivalence: if $v + U \neq w + U$ then $A_1 \cap A_2$ is the empty set. Therefore, we have proven that $A_1 \cap A_2$ is either an affline set of V or the empty set.

3 3.E.15

Suppose that $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $\varphi \neq 0$. Prove that $\dim(V/(\text{null}(\varphi))) = 1$.

Solution Suppose that $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $\varphi \neq 0$.

By Theorem 3.91, we have that range $\tilde{\varphi} =$ range φ which is a subspace of $\mathbb{F}$, so dim range φ may be 0 or 1. Since $\varphi \neq 0$, then dim range φ must equal 1. We also have that $V/(\text{null } \varphi)$ is isomorphic to range φ. By Theorem 3.59, we know that $\dim V/(\text{null } \varphi) = \dim \text{range } \varphi = 1$.

4 3.F.3

Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

Solution Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Let $\varphi_1, ..., \varphi_n$ be a basis of V'. Extend v to a basis $v_1, ..., v_n$ of V. Then by definition of a dual basis,

$$\varphi_j(v_k) = 1 \text{ if } k = j.$$

Therefore, we have proven that there $\varphi \in V'$ such that $\varphi(v) = 1$.

5 Problem 3.F.6

Suppose V is finite-dimensional and $v_1, ..., v_m \in V$. Define a linear map $\Gamma : V' \to \mathbb{F}^m$ by

$$\Gamma(\varphi) = (\varphi(v_1), , ..., \varphi(v_m)).$$

(a) Prove that $v_1, ..., v_m$ spans V if and only if Γ is injective.

(b) Prove that $v_1, ..., v_m$ is linearly independent if and only if Γ is surjective.

Solution

(a) $\Rightarrow$: Assume that $v_1, ..., v_m$ spans V. Then we can write every $v \in V$ uniquely as

$$v = a_1 v_1 + ... + a_m v_m$$

for $a_1, ..., a_m \in \mathbb{F}$.

Now suppose that $0 \in \text{null } \Gamma$. Let $\Gamma(\varphi) = 0 = (\varphi(v_1), ..., \varphi(v_m))$. We then have $\varphi(v_1) = ... = \varphi(v_m) = 0$. Since each $v \in V$ is arbitrary, it can be inferred that φ is the zero map, so $\varphi = 0$. It follows then that $\text{null } \Gamma \subset \{0\}$. Additionally, since $\text{null } \Gamma$ is a subspace of V', then it contains the additive identity, or $\{0\}$, by definition of a subspace. So, $\{0\} \subset \text{null } \Gamma$, completing the equality $\text{null } \Gamma = \{0\}$. By Theorem 3.16, Γ is injective if $\text{null } \Gamma = \{0\}$.

$\Leftarrow$: Assume that Γ is injective. Then $\text{null } \Gamma = \{0\}$. So we can take $\Gamma(\varphi) = 0 = (\varphi(v_1), ..., \varphi(v_m))$. Since $\varphi(v_i) = 0$ for all $v_1, ..., v_m \in V$, it follows that $V^0 = \{\varphi \in V' : \varphi(v) = 0 \text{ for all } v \in V\}$. So, $v_1, ..., v_m$ spans V.

(b) $\Rightarrow$: Assume that $v_1, ..., v_m$ is linearly independent. We need to prove that Γ is surjective. In other words, we want to prove that $\text{range } \Gamma = \mathbb{F}^m$. Since $v_1, ..., v_m$ is linearly independent then $a_1 v_1 + ... + a_m v_m = 0$ only if $a_1 = ... = a_m = 0$ for any $a \in \mathbb{F}$. It follows that $\dim\ (\varphi(v_1), ..., \varphi(v_m)) = \dim \text{ range } \Gamma = m = \dim\ \mathbb{F}^m$. Therefore, Γ is surjective.

$\Leftarrow$: Assume that Γ is surjective. We want to prove that $v_1, ..., v_m$ is linearly independent. In other words, we want to prove that $a_1 v_1 + ... + a_m v_m = 0$ only if $a_1 = ... = a_m = 0$ for any $a \in \mathbb{F}$. Since Γ is surjective, then range $\Gamma = \mathbb{F}^m$. Then $\dim\ \mathbb{F}^m$ must equal $\dim\ (\varphi(v_1), ..., \varphi(v_m)) = m$. Therefore, it is clear that $v_1, ..., v_m$ is linearly independent.