Fullwave Computational Model of 2D EM Systems using Contour Integrals (Updated by tjs)

February 9, 2019

1 General Math

In electromagnetics, the Maxwell's equations can be used in the frequency domain to solve for the field in an entire volume for a given source. However, for large-scale simulation domains, such as in the case of mm-Waves on the scale of a room, it becomes very computationally intensive to determine all of the fields in the given domain. This is where the surface equivalence theorem comes in.

From the book "The Method of Moments in Electromagnetics", the surface equivalence theorem "states that every point on an advancing wavefront is itself a source of radiating waves" [**?**]. The surface equivalence can transform volumetric sources into equivalent surface sources that produce the exact same scattered fields. What this means is that instead of modeling a region using the entire volume we can instead model it using the impressed surface current densities on the enclosed surface of the volume.

Fig. 1 shows an arbitrary volume V_2 lying in free-space V_1. An incident electromagnetic field, E and H, interacts with the surface of the volume to produce a surface current $J_s(r)$. This surface current then produces a "scattered" field, which is denoted as E^s and H^s, which can further interact with the volume. Once the surface currents are known, the scattered field everywhere can be solved for.

In general, when an incident electromagnetic field interacts with a boundary between two media it generates an electric *and* magnetic surface current density, J_s and K_s[1] respectively. Both of these currents affect the electromagnetic field and need to be taken into account in the general equations.

If both the electric and magnetic surface currents are known everywhere on the surface of a volume, the scattered fields can be solved for using the following equations [**?**][**?**]:

[1]It should be noted that in most literature M_s is used to represent the magnetic surface current density. K_s is used here instead in order to not confuse between the magnetic surface current induced on a regular surface and the magnetic polarization density of metasurfaces.

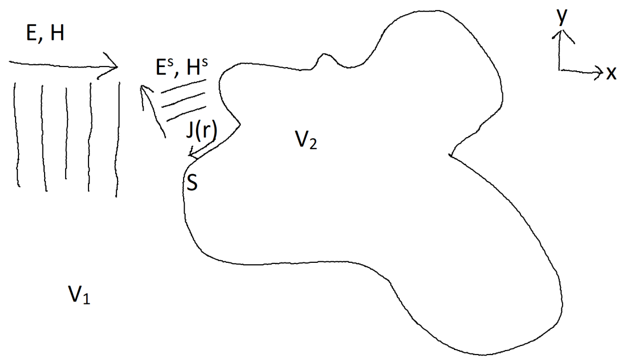

Figure 1: The general 3D problem, showing an incident field on the volume V_2 producing a surface electric current density $J_s(r)$, which generates a scattered electric and magnetic field.

$$E^s(r) = -i\omega\mu \iint_S G(r,r')[1 + \frac{1}{k^2}\nabla'\nabla'\cdot]J_s(r')\,dr' - \nabla\times\iint_S G(r,r')K_s(r')\,dr' \tag{1a}$$

$$H^s(r) = -i\omega\epsilon \iint_S G(r,r')[1 + \frac{1}{k^2}\nabla'\nabla'\cdot]K_s(r')\,dr' + \nabla\times\iint_S G(r,r')J_s(r')\,dr' \tag{1b}$$

where ∇' is an operator with respect to r' and $G(r,r')$ is the Green's function.

Because this is a scattering problem, the surface currents are not known. Therefore, the scattered fields have to be solved for in two steps:

1. J_s and K_s have to be solved using Eqs.(1) with the boundary conditions and the incident electromagnetic field
2. The scattered fields are evaluated by integrating the induced currents J_s and K_s

In most electromagnetic problems the only present surface current is the electric current J_s. This reduces the scattered field equations into:

$$E^s(r) = -i\omega\mu \iint_S G(r,r')[1 + \frac{1}{k^2}\nabla'\nabla'\cdot]J_s(r')\,dr' \tag{2a}$$

$$H^s(r) = \nabla\times\iint_S G(r,r')J_s(r')\,dr' \tag{2b}$$

These equations are commonly used by themselves to solve for the scattered electromagnetic fields, and are referred to as the Electric Field Integral Equation (EFIE) and the Magnetic Field Integral Equation (MFIE), respectively[**?**].

1.1 Boundary Conditions

Eqs. (1) only provide the effects of the scattered fields. In order to take into account the incident fields we have to use the boundary conditions of the surface. Fig. 2 shows the boundary between two different regions. The generalized electromagnetic boundary conditions can be written as:

$$-\hat{n}_2 \times (E_2 - E_1) = K_s \tag{3}$$
$$\hat{n}_2 \times (H_2 - H_1) = J_s \tag{4}$$
$$\hat{n}_2 \cdot (D_2 - D_1) = q_e \tag{5}$$
$$\hat{n}_2 \cdot (B_2 - B_1) = q_m \tag{6}$$

where $\hat{n}_2$ is the normal vector on the surface pointing from region 2 to region 1. In most cases, Eqs. (3) and (4) are sufficient to solve the problem.

Fig. 2 shows the boundary between the two regions graphically.

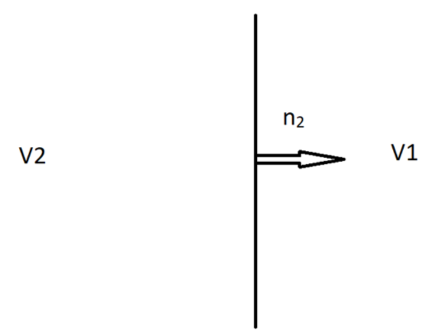

Figure 2: Boundary between two regions of space, where $\hat{n}_2$ is the unit vector normal to region V_2.

1.1.1 Perfect Electric Conductor

Consider the simplest case of a perfect electric conductor (PEC) placed between region 1 and 2 with an incident field coming from region 2. Because it is a PEC there are no magnetic surface currents at the boundary and the field in region 1 is zero. Therefore, the boundary conditions can be written as:

$$-\hat{n}_2 \times E_2 = 0 \tag{7}$$
$$\hat{n}_2 \times H_2 = J_s \tag{8}$$

1.2 Green's Function

The Green's function is the solution to the scalar Helmholtz equation and is required in order to solve for the scattered fields. The form of the Green's function depends on if you are solving a 2D solution or a 3D solution. Although the initial focus is in providing a solution to a 2D scattering problem, both forms will be listed.

1.2.1 Two-Dimensional Green's Function

The scalar Helmholtz equation in two dimensions can be written as [**?**]:

$$\nabla^2 G(\rho, \rho') + k^2 G(\rho, \rho') = -\delta(\rho, \rho') \tag{9}$$

where ρ is the position vector in two dimensions. The solution to this equation, assuming outgoing waves only, is:

$$G(\rho, \rho') = -\frac{i}{4} H_0^{(2)}(k|\rho - \rho'|) \tag{10}$$

where $H_0^{(2)}$ is the Hankel function of the second kind of order zero.

1.2.2 Three-Dimensional Green's Function

$$G(r, r') = \frac{e^{-ik|r-r'|}}{4\pi|r - r'|} \tag{11}$$

1.3 Solving the Scattered Fields

Now that we have the equations for the scattered field and the boundary conditions it is possible to formulate the equations that are needed to solve for the surface currents. We will use Fig. 2 to represent the problem.

1.3.1 Electric Field Integral Equation

Consider an incoming field from region 2 incident on a PEC. Using Eq. (7) the fields can be represented as:

$$-\hat{n}_2(r) \times E_2(r) = 0$$
$$\hat{n}_2(r) \times (E^i(r) + E^s(r)) = 0$$
$$\hat{n}_2(r) \times E^i(r) = -\hat{n}_2(r) \times E^s(r)$$

where $E^i(r)$ is the incident electric field at position r. Substituting in Eq. (2a) we get:

$$\hat{n}_2(r) \times E^i(r) = \hat{n}_2(r) \times i\omega\mu \iint_S G(r, r')[1 + \frac{1}{k^2}\nabla'\nabla'\cdot]J_s(r')\,dr'$$

$$-\frac{i}{\omega\mu}\hat{n}_2(r) \times E^i(r) = \hat{n}_2(r) \times \iint_S G(r, r')[1 + \frac{1}{k^2}\nabla'\nabla'\cdot]J_s(r')\,dr' \tag{12}$$

1.3.2 Magnetic Field Integral Equation

There is a lot to go over here in order to get to the final equation due to solving the singularity, which will be covered. In three dimensions at least, the final solution is found to be:

$$\hat{n}_2(r) \times H^i(r) = \frac{J_s(r)}{2} - \hat{n}_2(r) \times \iint_{S-\delta S} J_s(r') \times \nabla' G(r, r')\, dr' \tag{13}$$

where the contribution from the small area δS, which is found at $r = r'$, is combined in the $J_s(r)/2$ term.

2 Solving the Field Integral Equations in Two Dimensions

Eq. (1) can instead be represented by two linear operators $\mathcal{L}$ and $\mathcal{R}$ on the surface current densities. This gives us:

$$E^s(\rho) = -i\omega\mu(\mathcal{L}J_s)(\rho) - (\mathcal{R}K_s)(\rho), \tag{14a}$$

$$H^s(\rho) = -i\omega\epsilon(\mathcal{L}K_s)(\rho) + (\mathcal{R}J_s)(\rho) \tag{14b}$$

where the operators $\mathcal{L}$ and $\mathcal{R}$ are written as:

$$(\mathcal{L}X)(\rho) = \int_\ell G(\rho, \rho')[1 + \frac{1}{k^2}\nabla'\nabla' \cdot]X(\rho')\, d\rho' \tag{15}$$

$$(\mathcal{R}X)(\rho) = \nabla \times \int_\ell G(\rho, \rho')X(\rho')\, d\rho' \tag{16}$$

The advantage of splitting up the equations into separate linear operators is that if a solution is found for a linear operator it can be applied to both of the field equations.

2.1 $(\mathcal{R}X)(\rho)$

We being with Eq. (16) in 2D coordinates

$$(\mathcal{R}X)(\rho) = \nabla \times \int_\ell G(\rho, \rho')X(\rho')\, d\rho'$$

We wish to move the $\nabla\times$ operator so

$$\begin{aligned}(\mathcal{R}X)(\rho) &= \nabla\times\int_\ell G(\rho,\rho')X(\rho')\,d\rho' \\ &= \int_\ell \nabla\times G(\rho,\rho')X(\rho')\,d\rho'\end{aligned}$$

we can use,

$$\nabla\times(G(\rho,\rho')X(\rho')) = \nabla G(\rho,\rho')\times X(\rho') + X(\rho')(\nabla\times X(\rho'))$$

but X is a current and $\nabla\times J = 0$ and $\nabla\times K = 0$ (I gather?) so,

$$\nabla\times(G(\rho,\rho')X(\rho')) = \nabla G(\rho,\rho')\times X(\rho')$$

as $A\times B = -B\times A$,

$$\nabla\times(G(\rho,\rho')X(\rho')) = -X(\rho')\times\nabla G(\rho,\rho')$$

and $\nabla' G = -\nabla G($we have,

$$(\mathcal{R}X)(\rho) = \int_\ell X(\rho')\times\nabla' G(\rho,\rho')\,d\rho'$$

Ok.

$$\begin{aligned}(\mathcal{R}X)(\rho) &= \nabla\times\int_\ell G(\rho,\rho')X(\rho')\,d\rho' \\ &= \int_\ell X(\rho')\times\nabla' G(\rho,\rho')\,d\rho'\end{aligned}$$

$$= \int_\ell\left(\left(-X_{z'}(\rho')\frac{\partial}{\partial y'}G(\rho,\rho')\right)\hat{x'} + \left(X_{z'}(\rho')\frac{\partial}{\partial x'}G(\rho,\rho')\right)\hat{y'} + \left(X_{x'}\frac{\partial}{\partial y'}G(\rho,\rho') - X_{y'}\frac{\partial}{\partial x'}G(\rho,\rho')\right)\hat{z'}\right)\,d\rho'$$

Because of how the equation is layed out, we can solve each individual equation segment.

$$(\mathcal{R}X)(\rho)\cdot\hat{x'} = \int_\ell -X_{z'}\frac{\partial}{\partial y'}G(\rho,\rho')\,d\rho'$$

For these equation segments we are going to need to determine the derivative of the Green's function. Recall from Eq. (10) that the 2D Green's function is:

$$G(\rho,\rho') = -\frac{i}{4}H_0^{(2)}(k|\rho-\rho'|)$$

$$G(x,y,x',y') = -\frac{i}{4}H_0^{(2)}(k\sqrt{(x-x')^2+(y-y')^2})$$

$$\frac{\partial}{\partial y'}G(x,y,x',y') = -\frac{i}{4}\frac{k(y-y')H_1^{(2)}(k\sqrt{(x-x')^2+(y-y')^2})}{\sqrt{(x-x')^2+(y-y')^2}} \tag{17}$$

$$\frac{\partial}{\partial y'}G(\rho,\rho') = -\frac{i}{4}\frac{k(y-y')H_1^{(2)}(k|\rho-\rho'|)}{|\rho-\rho'|}$$

Similarly, the derivative of the Green's function with respect to x' is:

$$\frac{\partial}{\partial x'}G(\rho,\rho') = -\frac{i}{4}\frac{k(x-x')H_1^{(2)}(k|\rho-\rho'|)}{|\rho-\rho'|} \tag{18}$$

2.1.1 Integral Evaluation

First, let us solve the integral involving the derivative of the Green's function with respect to x'. As we should be aware, the Green's function has a singularity at $|\rho-\rho'|=0$. Therefore when integrating over the contour of the 2D area special care has to be taken for this region. We will break the integral intwo two parts as follows:

$$\int_\ell X_{z'}(\rho')\frac{\partial}{\partial x'}G(\rho,\rho')\,d\rho' = \int_{\ell-\delta\ell} X_{z'}(\rho')\frac{\partial}{\partial x'}G(\rho,\rho')\,d\rho' + \int_{\delta\ell} X_{z'}(\rho')\frac{\partial}{\partial x'}G(\rho,\rho')\,d\rho' \tag{19}$$

where $\delta\ell$ is a very small linear region of segment length Δl where $|\rho-\rho'|\to 0$. The singularity will be solved in the next section.

Because the problem is being solved computationally the above integral can be evaluated as a sum of small segments of finite length. If we have N segments along the contour each of length Δl_n and at position $\rho_n = (x_n, y_n)$ the contour can be evaluated as:

$$\int_{\ell-\delta\ell} X_{z'}(\rho')\frac{\partial}{\partial x'}G(\rho,\rho')\,d\rho' = \sum_{n'=1,\rho'_{n'}\neq\rho}^{N} X_{z'}(\rho'_{n'})\frac{\partial}{\partial x'}G(\rho,\rho'_{n'})\Delta l'_n \tag{20}$$

Assuming a segmented surface of elements $\delta\ell_{n'}$:

$$\sum_{n'=1,\rho'_{n'}\neq\rho}^{N}\int_{\delta\ell_{n'}} X_{z'}(\rho')\frac{\partial}{\partial x'}G(\rho,\rho')\,d\rho'$$

Using a surface normal for each segment of,

$$\hat{n}_2 = n_{x2}\hat{x}' + n_{y2}\hat{y}'$$

we integrate over the line segment from $l' = 0$ to $l' = \Delta\ell$ with

$$\begin{aligned} n_{y2}x - n_{x2}y &= b \\ x' &= \alpha_x + n_{y2}l' \\ y' &= \alpha_y + n_{x2}l' \end{aligned}$$

and for each n' segment we need to evaluate

$$\int_0^{\Delta\ell} X_{z'}(\rho') \frac{\partial}{\partial x'} G(\rho, \rho')\, d\rho'$$

if over the small segment $X_{z'}(\rho')\frac{\partial}{\partial x'}G(\rho, \rho')$ is almost constant we do not need the parameterization and we have,

$$\begin{aligned} &X_{z'}(\rho'_{n'}) \frac{\partial}{\partial x'} G(\rho, \rho'_{n'}) \int_0^{\Delta\ell} d\rho' \\ &X_{z'}(\rho'_{n'}) \frac{\partial}{\partial x'} G(\rho, \rho'_{n'}) \Delta\ell \end{aligned}$$

as Scott had in (20)!

2.1.2 Solving the Singularity of $(\mathcal{R}X)(\rho)$

As shown in Fig. 3 when evaluating the contour integral equations one of the segments will lie on $\rho' = \rho$, which represents a singularity of the Green's function.

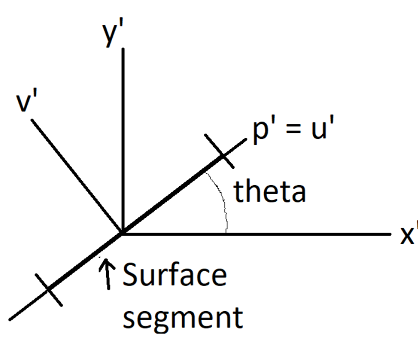

Figure 3: Rotation of the coordinate system for integration.

Because the line segment is very small, we can assume that the field $X(\rho')$ is constant along the length. This simplifies the equation to an integration of the derivative of the Green's function.

$$\int_{\delta\ell} X_{z'}(\rho') \frac{\partial}{\partial x'} G(\rho, \rho')\, d\rho' = X_{z'}(\rho') \int_{\delta\ell} \frac{\partial}{\partial x'} G(\rho, \rho')\, d\rho' \tag{21}$$

Because the integral is being evaluated very close to the Green's function singularity we need to use the small-argument approximation of the Green's function [?]. The small-argument approximation of the Green's function is:

$$\begin{aligned} G(\rho, \rho') &= -\frac{i}{4} H_0^{(2)}(k|\rho - \rho'|) \\ \lim_{\rho \to \rho'} G(\rho, \rho') &= -\frac{i}{4} - \frac{1}{2\pi} ln \left(\frac{\gamma k}{2} |\rho - \rho'| \right) \end{aligned} \tag{22}$$

where γ is the Euler-Mascheroni constant. Differentiating the Green's function with respect to x' then yields us:

$$\frac{\partial}{\partial x'} G(\rho, \rho') = \frac{1}{2\pi} \frac{(x - x')}{(x - x')^2 + (y - y')^2} \tag{23}$$

Combining this equation with Eq. (21) and integrating from $-\Delta l/2$ to $\Delta l/2$ yields us the following integral:

$$\int_{\delta \ell} \frac{\partial}{\partial x'} G(\rho, \rho') \, d\rho' = \int_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \frac{1}{2\pi} \frac{(x - x')}{(x - x')^2 + (y - y')^2} \, d\rho' \tag{24}$$

Because the variable ρ' is not along a singular variable a change of variables is performed. The goal is to make it so that the integrand is a simple function of the variable that is being integrated over. Following Fig. 3, the current variables are changed:

$$\begin{aligned} \rho' &= u' \\ x &= u \cos(\theta) - v \sin(\theta) \\ y &= u \sin(\theta) + v \cos(\theta) \\ x' &= u' \cos(\theta) - v' \sin(\theta) \\ y' &= u' \sin(\theta) + v' \cos(\theta) \end{aligned}$$

where u and v is the new coordinate system such that the line segment lies on the $v = 0$ line and $\hat{n}$ is in the direction of $\hat{v}$. Our point ρ will be chosen so that it is an infinitesimal amount above the surface at $\rho = (u, v)$. Applying this change of variables to Eq. (24) yields:

$$\begin{aligned} \int_{\delta \ell} \frac{\partial}{\partial x'} G(\rho, \rho') \, d\rho' &= \frac{1}{2\pi} \int_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \frac{\cos\theta[u - u'] - \sin\theta[v - v']}{(\cos\theta[u - u'] - \sin\theta[v - v'])^2 + (\sin\theta[u - u'] + \cos\theta[v - v'])^2} \, du' \\ &= \frac{1}{2\pi} \int_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \frac{\cos\theta[u - u'] - \sin\theta[v - v']}{(u - u')^2 + (v - v')^2} \, du' \end{aligned}$$

We know that $v' = 0$ since we are integrating over the surface, and we will assume $u = 0$ for simplicity.

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \frac{\cos\theta[-u'] - \sin\theta[v]}{(u')^2 + (v)^2}\, du' \\
&= -\frac{1}{2\pi}\int_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \frac{\cos\theta[u']}{(u')^2 + (v)^2}\, du' - \frac{1}{2\pi}\int_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \frac{\sin\theta[v]}{(u')^2 + (v)^2}\, du'
\end{aligned}
$$

The first integral is 0 because the integrand is an odd function of u' over a symmetric range. This leaves us with:

$$
\begin{aligned}
&= -\frac{1}{2\pi}\int_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \frac{\sin\theta[v]}{(u')^2 + (v)^2}\, du' \\
&= -\frac{1}{2\pi}\Big[\sin(\theta) tan^{-1}\left(\frac{u'}{v}\right)\Big]_{-\frac{\Delta l}{2}}^{\frac{\Delta l}{2}} \\
&= -\frac{1}{\pi}\sin(\theta) tan^{-1}\left(\frac{\Delta l}{2v}\right)
\end{aligned}
$$

The final simplification is we assume that because the point ρ is very close to the surface $v \to 0^+$. And because $\lim_{v\to 0^+} tan^{-1}(1/v) = \frac{\pi}{2}$ and $\sin(\theta) = -\hat{n}_2 \cdot \hat{x}$ we finally get:

$$
\int_{\delta\ell} \frac{\partial}{\partial x'} G(\rho, \rho')\, d\rho' = \frac{1}{2}\hat{n}_2 \cdot \hat{x} \tag{25}
$$

We can make an argument based on symmetry that the following is also true:

$$
\int_{\delta\ell} \frac{\partial}{\partial y'} G(\rho, \rho')\, d\rho' = \frac{1}{2}\hat{n}_2 \cdot \hat{y} \tag{26}
$$

2.1.3 Full Evaluation of $(\mathcal{R}X)(\rho)$

Now that the singularity is solved for, it is a relatively simple matter to solve Eq. (19).

$$
\int_\ell X_{z'}(\rho')\frac{\partial}{\partial x'} G(\rho, \rho')\, d\rho' = \frac{\hat{n}_2 \cdot \hat{x}}{2} X_{z'}(\rho) + \sum_{n'=1, \rho'_{n'} \neq \rho}^{N} X_{z'}(\rho'_{n'})\frac{\partial}{\partial x'} G(\rho, \rho'_{n'})\Delta l'_n \tag{27}
$$

$$
\int_\ell X_{z'}(\rho')\frac{\partial}{\partial y'} G(\rho, \rho')\, d\rho' = \frac{\hat{n}_2 \cdot \hat{y}}{2} X_{z'}(\rho) + \sum_{n'=1, \rho'_{n'} \neq \rho}^{N} X_{z'}(\rho'_{n'})\frac{\partial}{\partial y'} G(\rho, \rho'_{n'})\Delta l'_n \tag{28}
$$

2.2 $(\mathcal{L}X)(\rho)$

We begin with Eq. (15) in 2D coordinates

$$(\mathcal{L}X)(\rho) = \int_\ell G(\rho,\rho')[1 + \frac{1}{k^2}\nabla'\nabla'\cdot]X(\rho')\,d\rho'$$

Since this is slightly tricky to do all at once we can instead break up the operator into two parts:

$$(\mathcal{L}_1X)(\rho) = \int_\ell G(\rho,\rho')[\frac{1}{k^2}\nabla'\nabla'\cdot]X(\rho')\,d\rho' \tag{29}$$

$$(\mathcal{L}_2X)(\rho) = \int_\ell G(\rho,\rho')X(\rho')\,d\rho' \tag{30}$$

Because Eq. (29) is more complicated than Eq. (30) we shall evaluate it first.

2.2.1 Expansion of $(\mathcal{L}_1X)(\rho)$ in 2D

Although the equation is complicated it is a simple enough process to evaluate everything properly. We know that $\nabla' = \frac{\partial}{\partial x'}\hat{x}' + \frac{\partial}{\partial y'}\hat{y}' + \frac{\partial}{\partial z'}\hat{z}'$, which is important because $\frac{\partial}{\partial z}$ is 0 in 2D (the fields and currents do not vary with respect to z).

$$(\mathcal{L}_1X)(\rho) = \int_\ell G(\rho,\rho')\left(\frac{1}{k^2}\nabla'\nabla'\cdot X(\rho')\right)d\rho'$$

$$(\mathcal{L}_1X)(\rho) = \int_\ell G(\rho,\rho')\left(\frac{1}{k^2}\nabla'\left(\frac{\partial}{\partial x'}X_x(\rho') + \frac{\partial}{\partial y'}X_y(\rho')\right)\right)d\rho'$$

$$(\mathcal{L}_1X)(\rho) = \int_\ell G(\rho,\rho')\left(\frac{1}{k^2}\left(\left(\frac{\partial^2}{\partial x'^2}X_x(\rho') + \frac{\partial^2}{\partial x'\partial y'}X_y(\rho')\right)\hat{x}' + \left(\frac{\partial^2}{\partial x'\partial y'}X_x(\rho') + \frac{\partial^2}{\partial y'^2}X_y(\rho')\right)\hat{y}'\right)\right)d\rho'$$

where X_i is the component of the field in the x direction. In order to simplify the above expression for a discrete evaluation, we shall assume that along an infinitesimally small segment of the surface the term $\nabla'\nabla' X(\rho')$ does not change. If we have N segments of surface, the evaluation of the operator at position $\rho = \rho_n$ is:

$$(\mathcal{L}_1X)(\rho_n) = \sum_{m=1}^{N}\left(\frac{1}{k^2}\left(\left(\frac{\partial^2}{\partial x^2}X_{m_x} + \frac{\partial^2}{\partial x\partial y}X_{m_y}\right)\hat{x} + \left(\frac{\partial^2}{\partial x\partial y}X_{m_x} + \frac{\partial^2}{\partial y^2}X_{m_y}\right)\hat{y}\right)\right)\int_{\delta\ell_m} G(\rho_n,\rho'_m)\,d\rho' \tag{31}$$

where X_m is the segment of current at $\rho = \rho_m$ and $\int_{\delta\ell_m}$ indicates the integral of a small segment around $\rho' = \rho'_m$. For the sake of simplicity, the integral of the small segment of the Green's function will be represented as:

$$\int_{\delta\ell_m} G(\rho_n, \rho'_m)\, d\rho' = IG_{n,m} \tag{32}$$

To further simplify, we will introduce one more sub-operator:

$$(\mathcal{L}_{1_{uv}} X)(\rho_n) = \sum_{m=1}^{N} \left(\frac{1}{k^2} \left(\frac{\partial^2}{\partial u \partial v} X_m \right) \right) IG_{n,m} \tag{33}$$

$$\begin{aligned}(\mathcal{L}_1 X)(\rho_n) = &\left((\mathcal{L}_{1_{xx}} X_x)(\rho_n) + (\mathcal{L}_{1_{xy}} X_y)(\rho_n) \right) \hat{x} \\ &+ \left((\mathcal{L}_{1_{xy}} X_x)(\rho_n) + (\mathcal{L}_{1_{yy}} X_y)(\rho_n) \right) \hat{y}\end{aligned}$$

If we were to discritize the double derivative, we can get the following:

$$\frac{\partial^2}{\partial u \partial v} X_m = \frac{X_{m+1} - 2X_m + X_{m-1}}{\Delta u_m \Delta v_m} \tag{34}$$

where Δu_m and Δv_m is the length of the segment at $\rho = \rho_m$ in the u and v directions, respectively.

As an aside, if the length of segment in a particular direction is 0 (say, if the segment is only in the x-direction and we are trying to evaluate δy) we assume that $\frac{\partial^2}{\partial u \partial v} X_m = 0$ for that segment. In the case of the segment only being in the x-direction, the function would not depend on y, and therefore $\frac{\partial}{\partial y} = 0$.

Using Eq. (34) and shifting the indexes around, Eq. (33) finally becomes:

$$(\mathcal{L}_{1_{uv}} X)(\rho_n) = \sum_{m=1}^{N} \left(\frac{1}{k^2} \left(\frac{IG_{n,m-1}}{\Delta u_{m-1} \Delta v_{m-1}} - 2\frac{IG_{n,m}}{\Delta u_m \Delta v_m} + \frac{IG_{n,m+1}}{\Delta u_{m+1} \Delta v_{m+1}} \right) X_m \right) \tag{35}$$

Why use $IG_{n,m-1}$ and $IG_{n,m+1}$? And the derivative is not of X_m. Should it not be,

$$(\mathcal{L}_{1_{uv}} X)(\rho_n) = \sum_{m=1}^{N} \frac{IG_{n,m}}{k^2} \left(\frac{X_{n,m-1}}{\Delta u_{m-1} \Delta v_{m-1}} - 2\frac{X_{n,m}}{\Delta u_m \Delta v_m} + \frac{X_{n,m+1}}{\Delta u_{m+1} \Delta v_{m+1}} \right)$$

Looking at the code it looks like this is what is done.

2.2.2 Solving the Singularity of $(\mathcal{L}X)(\rho)$

As with the previous operator, there exists a singularity when evaluating the contour integral along the segment where $\rho' = \rho$. Because the Green's function is symmetric around the singularity in all directions, we can perform the integral along a given direction and have it work for all singularities.

$$IG_{n,m=n} = \int_{\delta\ell} \frac{-i}{4} H_0^{(2)}(k|\rho - \rho'|)\, d\rho'$$

Using the small-order approximation, assuming that it is shifted such that $\rho = 0$ and that the small segment has a length of Δl:

$$IG_{n,n} = \frac{-i}{4} \int_{-\Delta l/2}^{\Delta l/2} \left(1 - \frac{i2}{\pi} ln\left(\frac{\gamma k}{2}\sqrt{\rho'^2}\right)\right) d\rho'$$

$$IG_{n,n} = \frac{-i}{4} \left(\Delta l + \frac{i2}{\pi} - \frac{i2}{\pi}\Delta l * ln\left(\frac{\gamma k}{4}\Delta l\right)\right)$$