Fullwave Computational Model of 2D EM Systems using Contour Integrals

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1 Solving the Singularity of $(\mathcal{L}X)(\rho)$

1.1 Brief Refresher

As shown in Fig. 1 when evaluating the contour integral equations one of the segments will lie on $\rho' = \rho$, which represents a singularity of the Green's function.

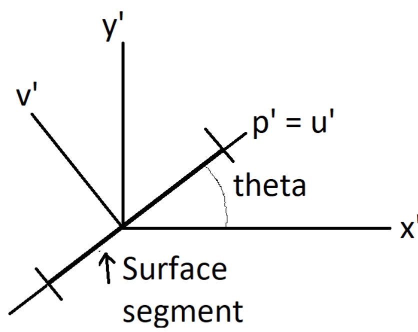

Figure 1: Rotation of the coordinate system for integration.

Because the variable ρ' is not along a singular variable a change of variables is performed. The goal is to make it so that the integrand is a simple function of the variable that is being integrated over. Following Fig. 1, the current variables are changed:

$$
\begin{aligned}
\rho' &= u' \\
x &= u\cos(\theta) - v\sin(\theta) \\
y &= u\sin(\theta) + v\cos(\theta) \\
x' &= u'\cos(\theta) - v'\sin(\theta) \\
y' &= u'\sin(\theta) + v'\cos(\theta)
\end{aligned}
$$

1.2 The New Stuff: Solving the New Singularity, $\frac{\partial^2}{\partial x^2}G(\rho,\rho')$

In the recent reformulation of the BEM, we are introduced to a new $\mathcal{L}$ operator. This necessitates solving for the new singularity that arises from the integral operator when evaluating near $\rho = \rho'$.

$$
\int_{\delta\ell} \frac{\partial^2}{\partial x^2} G(\rho,\rho') d\rho' = \int_{\delta\ell} \frac{\partial^2}{\partial x^2} \left(-\frac{i}{4} H_0^{(2)}(k|\rho - \rho'|) \right) d\rho' \tag{1}
$$

We begin with the second derivative of the Greens' function with respect to x. Because the integral is being evaluated very close to the Green's function singularity we once again need to use the small-argument approximation of the Green's function. While it is technically possible to first directly evaluate the second derivative of the Green's function and then substitute the appropriate small-order approximations, the math becomes a lot more complex and unintuitive.

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} \left(-\frac{i}{4} H_0^{(2)}(k|\rho - \rho'|) \right) &= \frac{\partial}{\partial x} \left(\frac{-(x-x')}{2\pi((x-x')^2 + (y-y')^2)} \right) \\
&= \frac{-1}{2\pi} \left(\frac{1}{(x-x')^2 + (y-y')^2} - \frac{2(x-x')^2}{((x-x')^2 + (y-y')^2)^2} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-1}{2\pi} \left(\frac{1}{(u-u')^2 + (v-v')^2} - \frac{2((u-u')\cos\theta - (v-v')\sin\theta)^2}{((u-u')^2 + (v-v')^2)^2} \right) \\
&= \frac{-1}{2\pi} \left(\frac{1}{(u-u')^2 + (v-v')^2} - \frac{2((u-u')^2\cos^2\theta - 2(u-u')(v-v')\sin\theta\cos\theta + (v-v')^2\sin^2\theta)}{((u-u')^2 + (v-v')^2)^2} \right)
\end{aligned}
$$

$$\int_{-\Delta l/2}^{\Delta l/2} \frac{-1}{2\pi}\left(\frac{1}{(u-u')^2+(v-v')^2} - \frac{2((u-u')^2\cos^2\theta - 2(u-u')(v-v')\sin\theta\cos\theta + (v-v')^2\sin^2\theta)}{((u-u')^2+(v-v')^2)^2}\right) du'$$

$$\begin{aligned}
=&\frac{-1}{2\pi}(\left(\frac{-\tan^{-1}(\frac{u-u'}{v-v'})}{(v-v')}\right) - \cos^2\theta\left(\frac{(u-u')}{(u-u')^2+(v-v')^2} - \frac{\tan^{-1}(\frac{u-u'}{v-v'})}{(v-v')}\right) + (odd) \\
&-\sin^2\theta\left(-\frac{(u-u')}{(u-u')^2+(v-v')^2} - \frac{\tan^{-1}(\frac{u-u'}{v-v'})}{(v-v')}\right))|_{-\Delta l/2}^{\Delta l/2} \\
=&\frac{1}{2\pi}\left(\cos^2\theta\left(\frac{(u-u')}{(u-u')^2+(v-v')^2}\right) - \sin^2\theta\left(\frac{(u-u')}{(u-u')^2+(v-v')^2}\right)\right)|_{-\Delta l/2}^{\Delta l/2}
\end{aligned}$$

Thankfully, a lot of the like-terms combine together and cancel each other out, especially since the value of $\tan^{-1}(1/v)/v$ does not converge. Similar to the previous singularities, we need to apply $u = 0, v = v' = 0$.

$$\begin{aligned}
\int_{-\Delta l/2}^{\Delta l/2} \frac{\partial^2}{\partial x^2} G(\rho,\rho')d\rho' &= \left(\cos^2\theta - \sin^2\theta\right)\frac{1}{2\pi}\left(\frac{(u-u')}{(u-u')^2+(v-v')^2}\right)|_{-\Delta l/2}^{\Delta l/2} \\
&= \left(\cos^2\theta - \sin^2\theta\right)\frac{1}{2\pi}\left(\frac{-1}{(u')}\right)|_{-\Delta l/2}^{\Delta l/2} \\
&= \left(\cos^2\theta - \sin^2\theta\right)\frac{1}{2\pi}\left(\frac{-2}{\Delta l} - \frac{2}{\Delta l}\right) \\
&= \left(\sin^2\theta - \cos^2\theta\right)\frac{2}{\pi\Delta l} \\
&= \left(2\sin^2\theta - 1\right)\frac{2}{\pi\Delta l}
\end{aligned}$$

Since from Fig. 1 we know that $\sin\theta = -\hat{n}_2\cdot\hat{x}$ and $\cos\theta = \hat{n}_2\cdot\hat{y}$ we can substitute the values into the above equation to get:

$$\int_{-\Delta l/2}^{\Delta l/2} \frac{\partial^2}{\partial x^2} G(\rho,\rho')d\rho' = \frac{2}{\pi\Delta l}\left(2(\hat{n}_2\cdot\hat{x})^2 - 1\right) \tag{2}$$

1.3 Solving $\frac{\partial^2}{\partial y^2}G(\rho,\rho')$

We do the exact same thing for the second derivative of the Greens' function with respect to y.

$$\begin{aligned}
\frac{\partial^2}{\partial y^2}\left(-\frac{i}{4}H_0^{(2)}(k|\rho-\rho'|)\right) &= \frac{\partial}{\partial y}\left(\frac{-(y-y')}{2\pi((x-x')^2+(y-y')^2)}\right) \\
&= \frac{-1}{2\pi}\left(\frac{1}{(x-x')^2+(y-y')^2} - \frac{2(y-y')^2}{((x-x')^2+(y-y')^2)^2}\right)
\end{aligned}$$

$$= \frac{-1}{2\pi}\left(\frac{1}{(u-u')^2+(v-v')^2} - \frac{2((u-u')\sin\theta + (v-v')\cos\theta)^2}{((u-u')^2+(v-v')^2)^2}\right)$$
$$= \frac{-1}{2\pi}\left(\frac{1}{(u-u')^2+(v-v')^2} - \frac{2((u-u')^2\sin^2\theta + 2(u-u')(v-v')\sin\theta\cos\theta + (v-v')^2\cos^2\theta)}{((u-u')^2+(v-v')^2)^2}\right)$$

$$\int_{-\Delta l/2}^{\Delta l/2} \frac{-1}{2\pi}\left(\frac{1}{(u-u')^2+(v-v')^2} - \frac{2((u-u')^2\sin^2\theta + 2(u-u')(v-v')\sin\theta\cos\theta + (v-v')^2\cos^2\theta)}{((u-u')^2+(v-v')^2)^2}\right) du'$$
$$= \frac{-1}{2\pi}(\left(\frac{-\tan^{-1}(\frac{u-u'}{v-v'})}{(v-v')}\right) - \sin^2\theta\left(\frac{(u-u')}{(u-u')^2+(v-v')^2} - \frac{\tan^{-1}(\frac{u-u'}{v-v'})}{(v-v')}\right) + (odd)$$
$$- \cos^2\theta\left(-\frac{(u-u')}{(u-u')^2+(v-v')^2} - \frac{\tan^{-1}(\frac{u-u'}{v-v'})}{(v-v')}\right))|_{-\Delta l/2}^{\Delta l/2}$$
$$= \frac{1}{2\pi}\left(\sin^2\theta\left(\frac{(u-u')}{(u-u')^2+(v-v')^2}\right) - \cos^2\theta\left(\frac{(u-u')}{(u-u')^2+(v-v')^2}\right)\right)|_{-\Delta l/2}^{\Delta l/2}$$

$$\int_{-\Delta l/2}^{\Delta l/2} \frac{\partial^2}{\partial y^2} G(\rho,\rho')d\rho' = \left(\sin^2\theta - \cos^2\theta\right)\frac{1}{2\pi}\left(\frac{(u-u')}{(u-u')^2+(v-v')^2}\right)|_{-\Delta l/2}^{\Delta l/2}$$
$$= \left(\sin^2\theta - \cos^2\theta\right)\frac{1}{2\pi}\left(\frac{-1}{(u')}\right)|_{-\Delta l/2}^{\Delta l/2}$$
$$= \left(\sin^2\theta - \cos^2\theta\right)\frac{1}{2\pi}\left(\frac{-2}{\Delta l} - \frac{2}{\Delta l}\right)$$
$$= \left(\cos^2\theta - \sin^2\theta\right)\frac{2}{\pi\Delta l}$$

$$\int_{-\Delta l/2}^{\Delta l/2} \frac{\partial^2}{\partial y^2} G(\rho,\rho')d\rho' = \frac{2}{\pi\Delta l}\left(2(\hat{n}_2\cdot\hat{y})^2 - 1\right) \tag{3}$$

1.4 Solving $\frac{\partial^2}{\partial x \partial y} G(\rho,\rho')$

$$\frac{\partial^2}{\partial x\partial y}\left(-\frac{i}{4}H_0^{(2)}(k|\rho-\rho'|)\right) = \frac{\partial}{\partial x}\left(\frac{-(y-y')}{2\pi((x-x')^2+(y-y')^2)}\right)$$
$$= \frac{(x-x')(y-y')}{\pi((x-x')^2+(y-y')^2)^2}$$
$$= \frac{((u-u')\cos\theta - (v-v')\sin\theta)((u-u')\sin\theta + (v-v')\cos\theta)}{\pi((u-u')^2+(v-v')^2)^2}$$

$$
\begin{aligned}
\int_{-\Delta l/2}^{\Delta l/2} \frac{\partial^2}{\partial y^2} G(\rho,\rho')d\rho' =& \int_{-\Delta l/2}^{\Delta l/2} \frac{(u-u')^2\sin(2\theta) + 2(u-u')(v-v')\cos(2\theta) - (v-v')^2\sin(2\theta)}{\pi((u-u')^2+(v-v')^2)^2} du' \\
=& \frac{1}{2\pi}\Bigg(\frac{\sin(2\theta)}{2}\left(\frac{(u-u')}{(u-u')^2+(v-v')^2} - \frac{\tan^{-1}(\frac{u-u'}{v-v'})}{v-v'}\right) + (odd) \\
& + \frac{\sin(2\theta)}{2}\left(\frac{(u-u')}{(u-u')^2+(v-v')^2} + \frac{\tan^{-1}(\frac{u-u'}{v-v'})}{v-v'}\right)\Bigg)\Big|_{-\Delta l/2}^{\Delta l/2} \\
=& \frac{\sin(2\theta)}{2\pi}\left(\frac{(u-u')}{(u-u')^2+(v-v')^2}\right)\Big|_{-\Delta l/2}^{\Delta l/2} \\
=& \frac{-2}{\pi\Delta l}\sin(2\theta) \\
=& \frac{-4}{\pi\Delta l}\sin\theta\cos\theta \\
\int_{-\Delta l/2}^{\Delta l/2} \frac{\partial^2}{\partial y^2} G(\rho,\rho')d\rho' =& \frac{4}{\pi\Delta l}(\hat{n}_2\cdot\hat{x})(\hat{n}_2\cdot\hat{y})
\end{aligned}
$$