Analysis and Design of Algorithms

Chapter 7: Transform and Conquer

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Transform and Conquer

- ***This group of techniques solves a problem based on transformation.***
  - Two stages:
  - 第一步：变。把问题的实例变得更容易。
  - 第二步：治。对实例进行求解。

Transform and Conquer

- ***Three variations of Transform and Conquer tech.***
  Differ by what we transform a given instance to:

  - *instance simplification*（实例化简）*:*

    to a simpler/more convenient instance of the same problem

  - *representation change*（改变表现）*:*

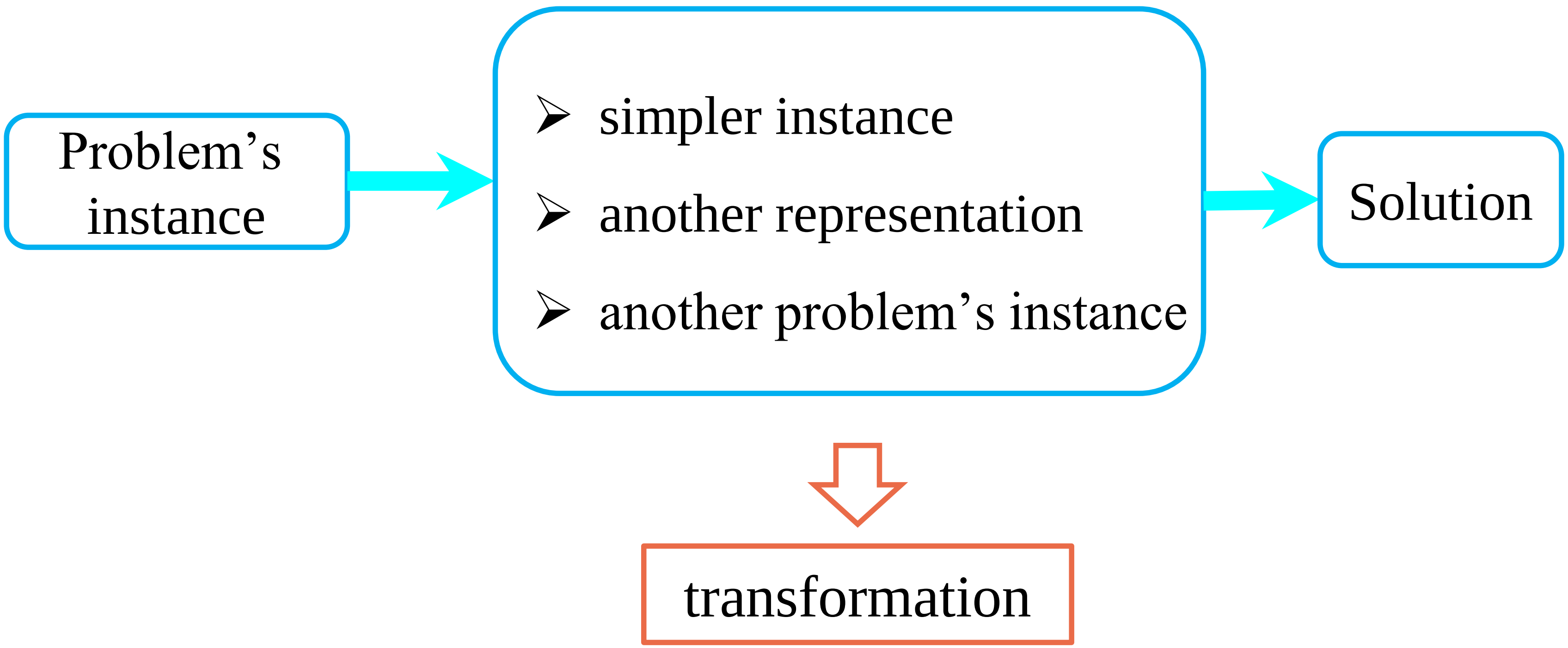
    to a different representation of the same instance

  - *problem reduction*（问题化简）*:*

    to a different problem for which an algorithm is already available

Transform and Conquer

- ***Three variations of Transform and Conquer tech.***
  - 变治法策略*:*



Presorting

- ***Presorting --- Instance simplification***
  - *Checking if all elements are distinct (element uniqueness)*
  - *Computing a mode*
  - *Searching*

Presorting

- **_Element Uniqueness with presorting_**
  - _Element Uniqueness problem --a brute-force method_
    - _compare all pairs of the array's elements until either two equal elements found or no more pairs left_

$$C_{worst}(n) = \frac{n(n-1)}{2} \in \Theta(n^2)$$

ALGORITHM *UniqueElements*$(A[0..n-1])$

```
//Determines whether all the elements in a given array are distinct
//Input: An array A[0..n − 1]
//Output: Returns "true" if all the elements in A are distinct
//            and "false" otherwise
for i ← 0 to n − 2 do
    for j ← i + 1 to n − 1 do
        if A[i] = A[j] return false
return true
```

Presorting

- **Element Uniqueness with presorting**
  - *Element Uniqueness problem --Presorting-based method*

- *Stage 1: sort by efficient sorting algorithm (e.g. mergesort)*
- *Stage 2: scan array to check pairs of adjacent elements*

```
ALGORITHM PresortElementUniqueness(A[0..n − 1])
    //Solves the element uniqueness problem by sorting the array first
    //Input: An array A[0..n − 1] of orderable elements
    //Output: Returns "true" if A has no equal elements, "false" otherwise
    sort the array A
    for i ← 0 to n − 2 do
        if A[i] = A[i + 1] return false
    return true
```

Presorting - Instance simplification

- 常见排序算法的时间效率：

- **Selection Sort ：** $\Theta(n^2)$
- **Bubble Sort ：** $\Theta(n^2)$
- **Insertion Sort ：** $C_{worst}(n) \in \Theta(n^2)$

  $C_{best}(n) \in \Theta(n)$

  $C_{avg}(n) \in \Theta(n^2)$
- **Mergesort：** $C_{worst}(n) \in \Theta(n \log n)$
- **Quicksort：** $C_{worst}(n) \in \Theta(n^2)$

  $C_{best}(n) \in \Theta(n \log n)$

  $C_{avg}(n) \in O(n \log n)$

没有一种基于比较的普通算法，在最坏情况下的效率能够超过n log n，平均效率也是。

Presorting

- ***Element Uniqueness with presorting***
  - *Element Uniqueness problem --Presorting-based method*

Efficiency Analysis：

- *time spent on sorting : at least n log n comparisons*
- *time spent on checking consecutive elements: no more than n-1 comparisons*
- ***use a good sorting alg.***

$$C(n)=C_{sort}(n)+C_{scan}(n)=\Theta(n \log n)+\Theta(n)=\Theta(n \log n)$$

If $t_1(n) \in O(g_1(n))$ and $t_2(n) \in O(g_2(n))$, then

$$t_1(n)+t_2(n) \in O(\max\{g_1(n), g_2(n)\})$$

Presorting

Computing a mode

Mode: a value that occurs most often in a given list of numbers

e.g. For {5, 1, 5, 7, 6, 5, 7} mode is 5

Brute-force method

- ***Idea:***

- Scan the list, compute the frequency of all its distinct values

- find the value with the largest frequency

Presorting

Computing a mode

Brute-force method (‘cont)

- ***implementation:***

- Store the values already encountered, along with their frequencies, in an auxiliary list (the values in this auxiliary list are all distinct);

- On each iteration, the *i*th element of the original list is compared with the values already encountered by traversing this an auxiliary list;

- If a matching value is found, its frequency is incremented;

- otherwise, the current element is added to the auxiliary list with frequency of 1.

Presorting

Computing a mode

- *Brute-force method (‘cont)*

- ***Worst case analysis***

  - When a list with no equal elements, the *i*th element is compared with *i*-1 elements of the auxiliary list.

  - number of comparisons in creating the frequency auxiliary list

$$C(n) = \sum_{i=1}^{n} (i-1) = \frac{(n-1)n}{2} \in \theta(n^2)$$

  - number of comparisons to find the largest frequency in the auxiliary list: n-1

  - The overall time efficiency: $\Theta(n^2)$

Presorting

- ***Computing a mode***
  - *Computing a mode with presorting*

Idea：

- sort the input firstly, then all equal values will be adjacent

- find the longest run of the adjacent equal values in the sorted array

Presorting

Computing a mode with presorting

```
ALGORITHM  PresortMode(A[0..n − 1])
    //Computes the mode of an array by sorting it first
    //Input: An array A[0..n − 1] of orderable elements
    //Output: The array's mode
    sort the array A
    i ← 0                          //current run begins at position i
    modefrequency ← 0              //highest frequency seen so far
    while i ≤ n − 1 do
        runlength ← 1;   runvalue ← A[i]
        while i + runlength ≤ n − 1 and A[i + runlength] = runvalue
            runlength ← runlength + 1
        if runlength > modefrequency
            modefrequency ← runlength;   modevalue ← runvalue
        i ← i + runlength
    return modevalue
```

Presorting

- ***Computing a mode***
  - ***Efficiency analysis:***

- *time spent on sorting : at least **n log n** comparisons –determine the overall efficiency*

- *time spent on checking longest run of the adjacent : **linear***

  - ***Conclusion:*** *using a good sorting algorithm* $\in \Theta(n \log n)$

Presorting

- ***Searching problem***

  --- *Search for a given K in A*[*0..n*-1]

  - *Brute-force method*

    - ***sequential search :***

$$T_{\text{worst}}(n) = n \qquad T_{\text{best}}(n) = 1$$

$$T_{avg}(n) = \frac{p(n+1)}{2} + n(1-p)$$

Presorting

- **Searching problem**
  - Searching with presorting

- *time spent on sorting: at least **n log n** comparisons*

- *time spent on binary search:*

$C_{worst}(n) = \lfloor \log_2 n \rfloor + 1 = \Theta(\log n)$; $C_{best}(n) = 1$

$C_{avg}(n) = \Theta(\log n)$;

如果在查找问题中，预先对数组进行排序，那么算法效率为：

$C(n)=C_{sort}(n)+ C_{search}(n)= \Theta(n \log n) + \Theta(\log n) = \Theta(n \log n)$

比顺序查找还要差

如果要在同一个列表中进行多次查找，可以考虑对列表进行预排序。（思考：为了使预排序的花费有价值，最少需要进行多少次查找？）

Presorting - Instance simplification

- 实例化简应用 --预排序 (*presorting*)
  - *Benefit from presorting：*

☆ *the benefits of a sorted list should be more than compensate for the time spent on sorting*

☆ *generally comparison-based sorting alg.* ***worst case, at least n log n***

Gaussian Elimination - Instance simplification

实例化简--高斯消去法（Gaussian Elimination ）

Problem: Given: *a system of n linear equations in n* unknowns with an arbitrary coefficient matrix.

Idea:

Stage1: ***Elementary operations*（初等变换）:** *Transform to an* ***equivalent*** *system of n linear equations in n unknowns with an* ***upper triangular*** *coefficient matrix.*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned} \quad \longrightarrow \quad \begin{aligned} a_{1,1}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \\ a_{nn}x_n &= b_n \end{aligned}$$

Gaussian Elimination - Instance simplification

- **Gaussian Elimination** *What's Elementary operations*

  To change from a system with an arbitrary coefficient matrix to an ***equivalent*** *system with an upper triangular coefficient matrix by*

  - exchanging two equations of the system

  - replacing an equation with its nonzero multiple

  - replacing an equation with a sum or difference of this equation and some multiple of the former

Gaussian Elimination - Instance simplification

Gaussian Elimination *What's Elementary operations*

Specifically:

- Use a_{11} as a pivot to make all x_1 coefficients zeros in the equations below the first one;
- Replace the second equation with the difference between it and **the first equation multiplied by a_{21}/ a_{11}** to get an equation with zero coefficient for $x_{1;}$
- Doing the same for the third, fourth, and finally nth equation – with the multiples $a_{31}/ a_{11}, a_{41}/ a_{11}, \ldots a_{n1}/ a_{11}$ of the first equation.

Gaussian Elimination - *Instance simplification*

Gaussian Elimination

*Stage2: Solve the latter by **backward substitutions** starting with the last equation and moving up to the first one.*

Specifically:

① Find the value of x_n from the last equation immediately;

② Substitute this value into the next to last equation to get $x_{n\text{-}1}$;

③ And so on, until we substitute the known values of the last n-1 variables into the first equation, to find the value of x_1.

Gaussian Elimination - Instance simplification

Gaussian Elimination

Solve
$$2x_1 - x_2 + x_3 = 1$$
$$4x_1 + x_2 - x_3 = 5$$
$$x_1 + x_2 + x_3 = 0$$

Gaussian elimination:

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 4 & 1 & -1 & 5 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{array}{l} \\ \text{row2} - (4/2)*\text{row1} \\ \text{row3} - (1/2)*\text{row1} \end{array} \Rightarrow \begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & 3/2 & 1/2 & -1/2 \end{bmatrix} \begin{array}{l} \\ \\ \text{row3} - (1/2)*\text{row2} \end{array}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

Backward substitution:

$$x_3 = (-2) / 2 = -1$$
$$x_2 = (3 - (-3)\, x_3) / 3 = 0$$
$$x_1 = (1 - x_3 - (-1)\, x_2)/2 = 1$$

Gaussian Elimination - Instance simplification

Gaussian Elimination

Stage1: *Elementary operations*

--前向消去法（forward elimination）

ALGORITHM *ForwardElimination*$(A[1..n, 1..n], b[1..n])$

//Applies Gaussian elimination to matrix A of a system's coefficients,
//augmented with vector b of the system's right-hand side values
//Input: Matrix $A[1..n, 1..n]$ and column-vector $b[1..n]$
//Output: An equivalent upper-triangular matrix in place of A with the
//corresponding right-hand side values in the $(n+1)$st column
for $i \leftarrow 1$ **to** n **do** $A[i, n+1] \leftarrow b[i]$ //augments the matrix
for $i \leftarrow 1$ **to** $n-1$ **do**
    for $j \leftarrow i+1$ **to** n **do**
        for $k \leftarrow i$ **to** $n+1$ **do**
            $A[j, k] \leftarrow A[j, k] - A[i, k] * A[j, i] / A[i, i]$

Gaussian Elimination - *Instance simplification*

Gaussian Elimination

- *Two considerations*

1. *If* $A[i, i] = 0$

   → exchange the *i*th row with some row below it with a nonzero coefficient in the *i*th column.

2. *If* $A[i, i]$ is so small that consequently the scaling factor $A[j, i]/A[i, i]$ so large that new $A[j, k]$ might distorted by a round-off error caused by a subtraction of two numbers of greatly different magnitudes.

   → look for a row in the largest absolute value of the coefficient in the *i*th column, exchange it with the *i*th row --- partial pivoting (保证比例因子的绝对值永远不会大于1)

Gaussian Elimination - Instance simplification

Gaussian Elimination

Stage1: Better Forward Elimination (改进之后)

ALGORITHM $BetterForwardElimination(A[1..n, 1..n], b[1..n])$

//Implements Gaussian elimination with partial pivoting
//Input: Matrix $A[1..n, 1..n]$ and column-vector $b[1..n]$
//Output: An equivalent upper-triangular matrix in place of A and the
//corresponding right-hand side values in place of the $(n+1)$st column
for $i \leftarrow 1$ **to** n **do** $A[i, n+1] \leftarrow b[i]$ //appends b to A as the last column
for $i \leftarrow 1$ **to** $n-1$ **do**
    $pivotrow \leftarrow i$
    for $j \leftarrow i+1$ **to** n **do**
        if $|A[j, i]| > |A[pivotrow, i]|$ $pivotrow \leftarrow j$
    for $k \leftarrow i$ **to** $n+1$ **do**
        $swap(A[i, k], A[pivotrow, k])$
    for $j \leftarrow i+1$ **to** n **do**
        $temp \leftarrow A[j, i] / A[i, i]$
        for $k \leftarrow i$ **to** $n+1$ **do**
            $A[j, k] \leftarrow A[j, k] - A[i, k] * temp$

Gaussian Elimination - Instance simplification

Gaussian Elimination

Time Efficiency：

最内层循环： $A[j,k] \leftarrow A[j,k] - A[i,k] * temp$

Basic operation: multiplication

$$
\begin{aligned}
C(n) &= \boxed{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=i}^{n+1} 1} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (n+1-i+1) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (n+2-i) \\
&= \sum_{i=1}^{n-1} (n+2-i)(n-(i+1)+1) = \sum_{i=1}^{n-1} (n+2-i)(n-i) \\
&= (n+1)(n-1) + n(n-2) + \cdots + 3 \cdot 1 \\
&= \sum_{j=1}^{n-1} (j+2)j = \sum_{j=1}^{n-1} j^2 + \sum_{j=1}^{n-1} 2j = \frac{(n-1)n(2n-1)}{6} + 2\frac{(n-1)n}{2} \\
&= \frac{n(n-1)(2n+5)}{6} \approx \frac{1}{3}n^3 \in \Theta(n^3).
\end{aligned}
$$

Gaussian Elimination - Instance simplification

Gaussian Elimination

Stage2: *Backward substitution* $\in \Theta(n^2)$

for $j \leftarrow n$ **downto** 1 **do**

  $t \leftarrow 0$

  for $k \leftarrow j+1$ **to** n **do**

    $t \leftarrow t + A[j, k] * x[k]$

  $x[j] \leftarrow (A[j, n+1] - t) / A[j, j]$

Gaussian Elimination - Instance simplification

Gaussian Elimination

- *Efficiency analysis*
  - *stage1:* Elementary operations $\Theta(n^3)$
  - *stage2:* Backward substitution $\Theta(n^2)$

Efficiency: $\Theta(n^3) + \Theta(n^2) = \Theta(n^3)$

Gaussian Elimination - Instance simplification

- Some discussions
  - ☆ *Gaussian Elimination either yields* ***an exact solution*** *to a system of linear equations when the system has a unique solution;*
  - ☆ *or discovers that no such solution exists, in this case, the system will have either* ***no solutions or infinitely many of them;***
  - ☆ *the principal difficulty lies in preventing the accumulation of* ***round-off error.***

Gaussian Elimination

Applications of Gaussian Elimination

- *LU decomposition*
- *Computing a matrix inverse*
- *Computing a determinant*

Heaps and Heapsort

- *实例化简 --堆排序*
- **Heaps**
  - *Heap is suitable for implementing priority queues.*

maintaining a set S of elements, each with an associated value called a key/priority. It supports the following operations:

- *Finding an item with the highest priority*
- *Deleting an item with the highest priority*
- *Adding a new item to the multiset*

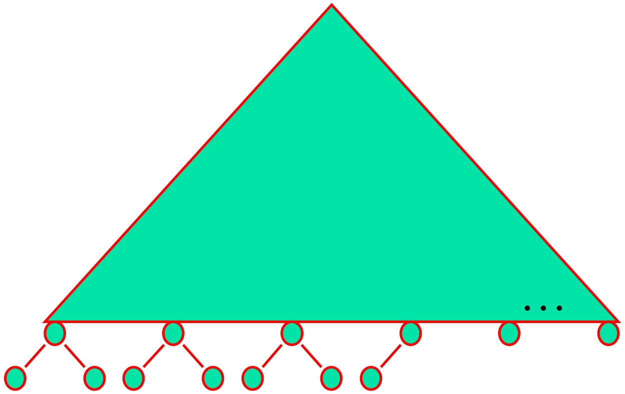
Heaps and Heapsort

Heaps

Notion of the Heap

- *A binary tree with keys assigned to its nodes, one key per node*
- *Shape requirement: the binary tree is essentially complete, i.e. all its levels are full except possibly the last level, where only some rightmost leaves may missing.*
- *Parental dominance requirement: for max-heap:*

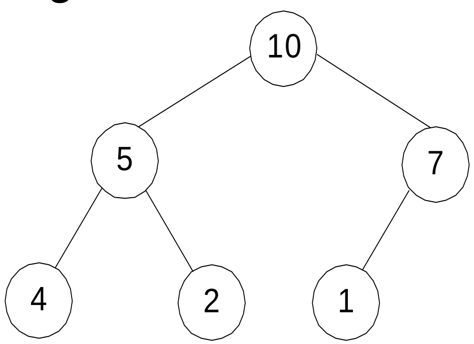
  key at each node ≥ keys at its children



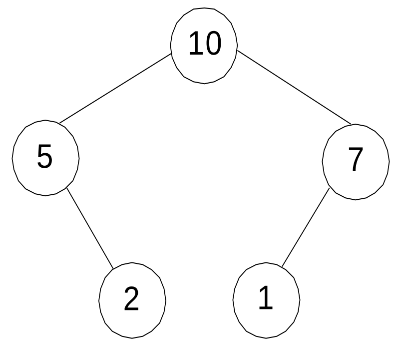
Heaps and Heapsort

Heaps

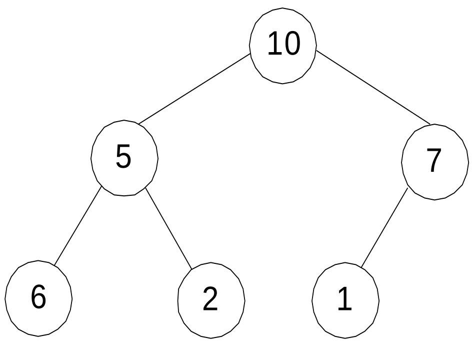
e.g.



a heap



not a heap



not a heap

- ☆ *Heap's elements are ordered top down (a sequence of values along any path down from its root is decreasing or non-increasing if equal keys are allowed)*
- ☆ *but they are not ordered left to right*

Heaps and Heapsort

- ***Heaps***
  - *Properties of Heaps*

- *There exits exactly one essentially complete binary tree with n nodes, its height is* $\lfloor \log_2 n \rfloor$.

- *Height of a node: the number of edges on the longest simple downward path from the node to a leaf.*
- *Height of a tree: the height of its root.*
- *level of a node: A node's level + its height = h, the tree's height.*

Heaps and Heapsort

Heaps

Properties of Heaps

- *The root of a heap always has the largest key (for a max-heap).*

- *A node of a heap considered with all its descendants is also a heap*

  (The subtree rooted at any node of a heap is also a heap)

- *Max-heap property and min-heap property*

  - *Max-heap: for every node other than root, A[PARENT(i)] >= A(i)*
  - *Min-heap: for every node other than root, A[PARENT(i)] <= A(i)*

Heaps and Heapsort

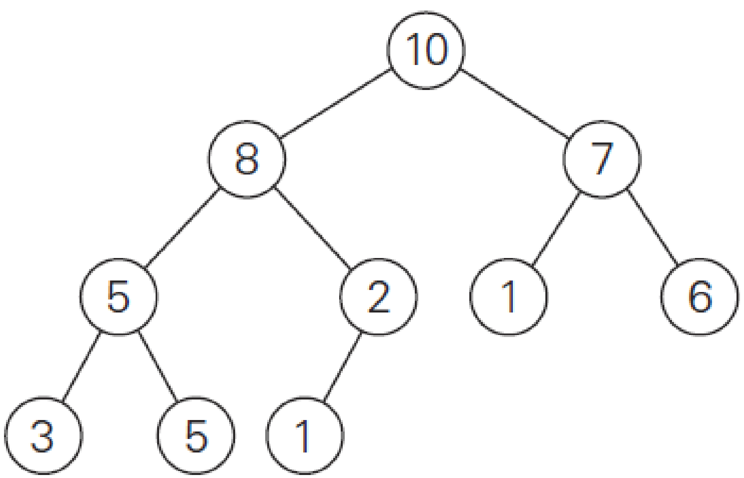
Heaps

Properties of Heaps

- *It is more efficient to implement a heap as an array, by storing the heap's elements in top-down left-to-right order.*

- *Parental nodes are represented in the first $\lfloor n/2 \rfloor$ locations of the array.*

- *Leaf keys occupy the last $\lceil n/2 \rceil$ locations.*



the array representation

index	0	1	2	3	4	5	6	7	8	9	10
value		10	8	7	5	2	1	6	3	5	1

parents — *the first $\lfloor n/2 \rfloor$*

leaves — *the last $\lceil n/2 \rceil$*

Heaps and Heapsort

Heaps

Properties of Heaps

- *Relationships between indexes of parents and children：*

The children of a key in the array's parental position i ($1 \le i \le \lfloor n/2 \rfloor$) will be in positions $2i$ and $2i + 1$, and, correspondingly, the parent of a key in position i ($2 \le i \le n$) will be in position $\lfloor i/2 \rfloor$.

Heaps and Heapsort

Heaps Construction

How to construct a heap with the given list of keys?

- *Bottom-up Heap construction*

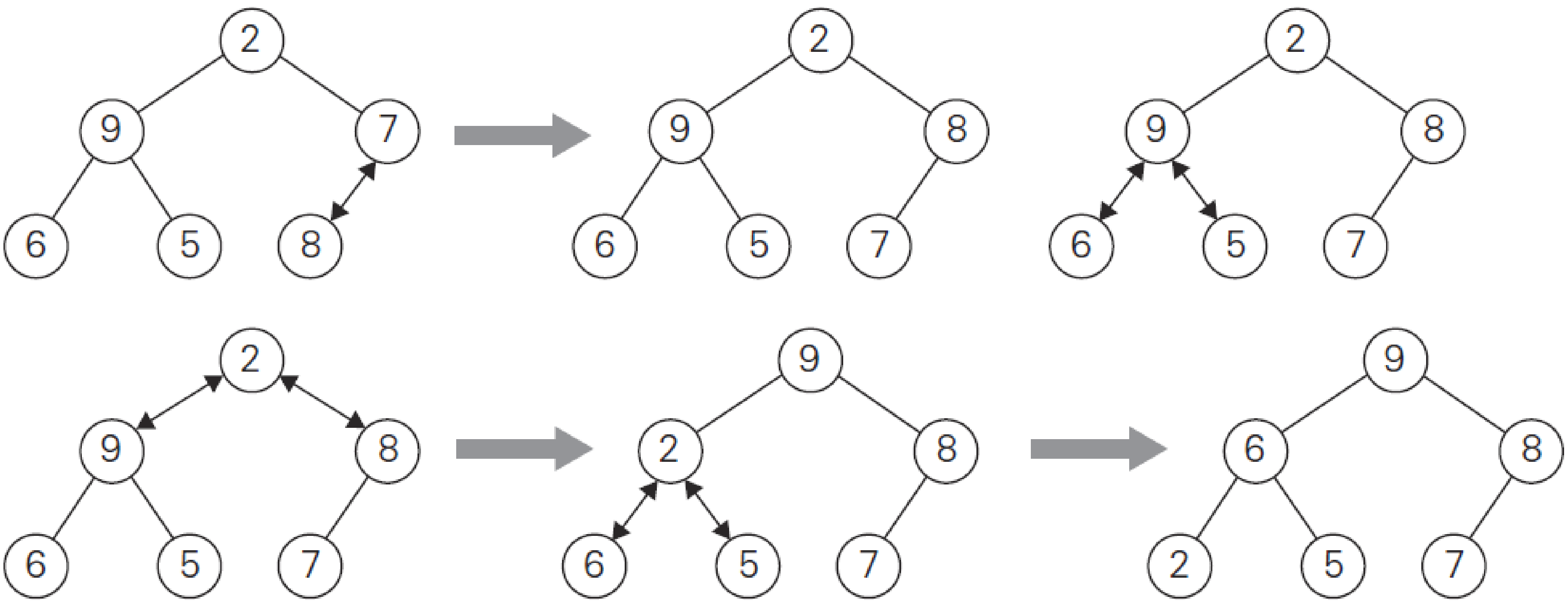
- *Build an essentially complete binary tree by inserting n keys in the given order.*
- ***Heapify** the tree.*

- *Starting with the last (rightmost) parental node, heapify/fix the subtree rooted at it; if the parental dominance condition does not hold for the key at this node:*
  - *exchange its key K with the key of its larger child*
  - *Heapify/fix the subtree rooted at the K's new position*
  - *until the parental dominance requirement for K is satisfied*
- *Proceed to do the same for the node's immediate predecessor.*
- *Stops after this is done for the tree's root.*

Heaps and Heapsort

Heaps Construction

- Bottom-up Heap construction('cont)
  - *Example 1:* Construct a heap for the list 2, 9, 7, 6, 5, 8



Heaps and Heapsort

Heaps Construction

- *Bottom-up Heap construction(‘cont)*
  - *Example 2:* Construct a heap for the list

    4 1 3 2 16 9 10 14 8 7

  - *Result*： 16 14 10 8 7 9 3 2 4 1

Heaps and Heapsort

Heaps Construction-*Bottom-up Heap construction (A Recursive version)*

ALGORITHM $HeapBottomUp(H[1..n])$

```
//Constructs a heap from elements of a given array
// by the bottom-up algorithm
//Input: An array H[1..n] of orderable items
//Output: A heap H[1..n]
for i <- ⌊n/2⌋ downto 1 do
    k <- i;   v <- H[k]
    heap <- false
    while not heap and 2 * k <= n do
        j <- 2 * k
        if j < n   //there are two children
            if H[j] < H[j + 1]  j <- j + 1
        if v >= H[j]
            heap <- true
        else H[k] <- H[j];   k <- j
    H[k] <- v
```

最后的父母节点

Heaps and Heapsort

Heaps Construction

Worst-Case Efficiency for Bottom-up

- *assume $n = 2^k - 1$, so the heap is full, the maximum number of nodes occurs on each level*
- *Worst case: each key on level i will travel to the leaf level h*
  - *height of the tree* $h = \lfloor \log_2 n \rfloor$
  - *moving to the level down needs two comparisons*
    - *one to find the larger child*
    - *one to determine whether the exchange is required*
  - *number of key comparisons for a key on level i: 2(h-i)*

$$C_{worst}(n) = \sum_{i=0}^{h-1} \sum_{\text{nodes at level i}} 2(h-i) = \sum_{i=0}^{h-1} 2(h-i)2^i = 2(n - \log_2(n+1))$$

$= O(n)$

Heaps and Heapsort

Heaps Construction

Top-down Heap Construction

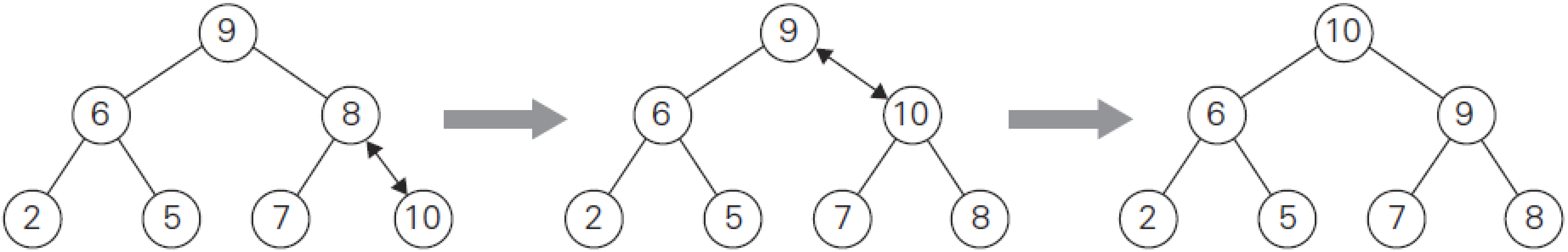
- *Successive insertions of new key into a previously constructed heap*
  - *Insertion of a new key K*
    - *Insert the new node with key K at the last position in heap, i.e. after the last leaf of the existing heap*
    - *Sift K up to its appropriate position*

Heaps and Heapsort

Heaps Construction

Top-down Heap Construction

- *sift K up to its appropriate position*
  - *Compare with its parent, and exchange them if it violates the parental dominance condition.*
  - *Continue comparing the element with its new parent, until K is not greater than its last parent or it reaches the root*



Heaps and Heapsort

Heaps Construction

- *Efficiency for Top-down*
  - *height of a heap with n node:* $h = \lfloor \log_2 n \rfloor$
  - *Inserting one new element to a heap with* n-1 *nodes requires no more comparisons than the heap's height*
  - *Time efficiency for Top-down insertion is* $O(\log n)$

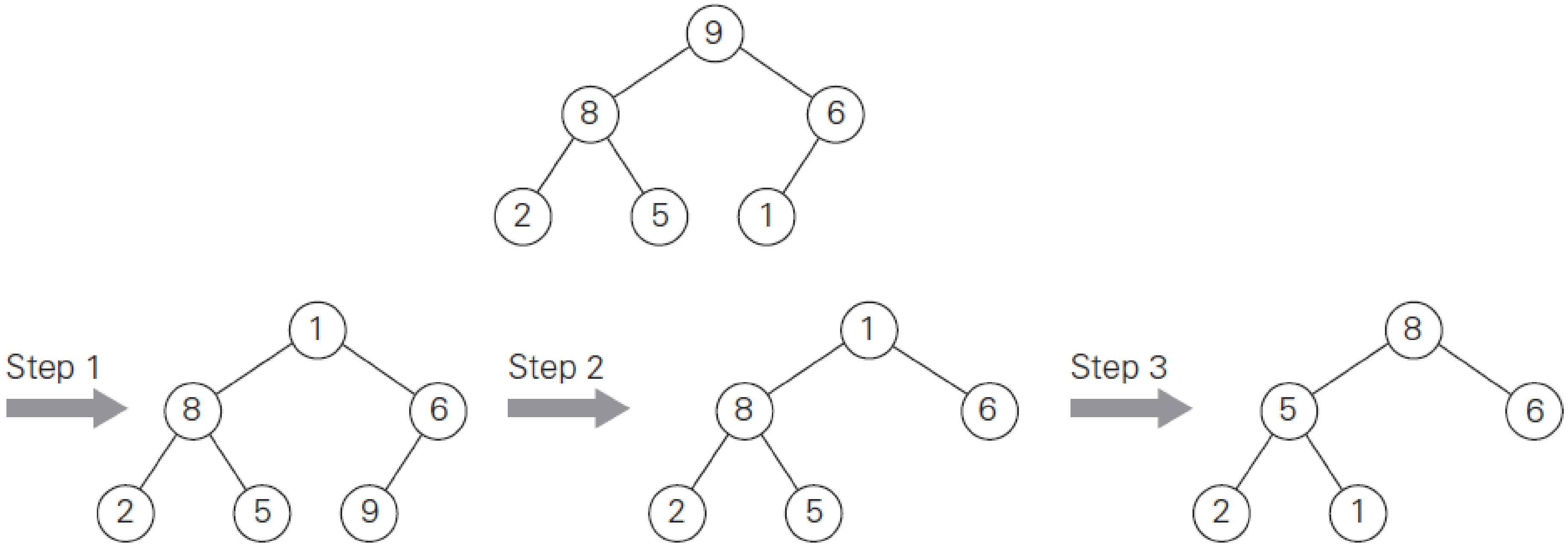
Heaps and Heapsort

- **Heaps Construction**
  - Root Deletion
    - *swap the root with the last leaf K*
    - *Decrease the heap's size by* 1
    - *Heapify the smaller tree by sifting K down the tree, in exactly the same way in Bottom-up Heap construction*
      - *verify the parental dominance for K,*
      - *if it holds, we done.*
      - *if not, swap K with the larger of its children and repeat this operation until parental dominance holds for K in its new position.*

Heaps and Heapsort

Heaps Construction

- *Root Deletion*



Heaps and Heapsort

- ## *Heaps Construction*
  - *Efficiency for Root Deletion*
    - *It can't make key comparison more than twice the heap's height*
    - *Efficiency:* $\Theta(\log n)$

Heaps and Heapsort

Heapsort

Steps of Heapsort

- *Stage 1: Bottom-up heap construction* （构造堆）
- *Stage 2: Root deletion, Repeat n-*1 *times until heap contains just one node* （删除最大键，假设构造的是最大堆）

  - 最终结果是按照降序删除了数组的元素。

Heaps and Heapsort

Heapsort

- *Stage 1: heap construction* （构造堆）
- *Stage 2: Root deletion* （删除最大键）

Stage 1 (heap construction)

2 9 **7** 6 5 8

2 **9** 8 6 5 7

2 9 8 6 5 7

9 **2** 8 6 5 7

9 6 8 2 5 7

Stage 2 (maximum deletions)

9 6 8 2 5 7

7 6 8 2 5 | **9**

8 6 7 2 5

5 6 7 2 | **8**

7 6 5 2

2 6 5 | **7**

6 2 5

5 2 | **6**

5 2

2 | **5**

2

Heaps and Heapsort

Analysis of Heapsort

- *Stage 1:* 构造堆，$C_1(n) = O(n)$
- *Stage 2:* 把堆的规模从n消减到2的过程中，为了删除根所需的键值比较次数记为*C*(*n*)

$$C_2(n) \le 2\lfloor \log_2(n-1) \rfloor + 2\lfloor \log_2(n-2) \rfloor + \ldots + 2\lfloor \log_2 1 \rfloor \le 2\sum_{i=1}^{n-1} \log_2 i$$

$$\le 2\sum_{i=1}^{n-1} \log_2(n-1) = 2(n-1)\log_2(n-1) \le 2n\log_2 n \in O(n\log n)$$

- *Analysis shows that* $C_1(n)+C_2(n)$ = ***O(n log n),*** *in both the worst and average cases, the same class as <u>mergesort</u>*
- *But <u>not require extra storage</u> ---implemented with arrays*
- *Experiments show that heapsort runs more <u>slowly than quicksort</u> but competitive with mergesort*

Horner's Rule- Representation change

- *改变表现* **--Horner's Rule For Polynomial Evaluation 霍纳法则**
  - *Polynomial Evaluation: Compute the value of a polynomial*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \quad (1)$$

  - *Two brute-force algorithms*

```
p ← 0
for i ← n downto 0 do
    power ← 1
    for j ← 1 to i do
        power ← power * x
        p ← p + a_i * power
return p
```

Horner's Rule- Representation change

Horner's Rule For Polynomial Evaluation

- *Horner's Rule --Representation change*

- *Obtained from* (1), *successively taking x as a common factor in the remaining polynomials of diminishing degrees*

$$p(x) = (\ldots (a_n x + a_{n-1})\, x + \ldots)\;\; x + a_0 \qquad (2)$$

e.g.

$$p(x) = 2x^4 - x^3 + 3x^2 + x - 5$$

$$= x(2x^3 - x^2 + 3x + 1) - 5$$

$$= x(x(2x^2 - x + 3) + 1) - 5$$

$$= x(x(x(2x - 1) + 3) + 1) - 5$$

Horner's Rule- Representation change

- **Horner's Rule For Polynomial Evaluation**
  - *Horner's Rule --Representation change*

$$p(x) = 2x^4 - x^3 + 3x^2 + x - 5$$

$$= x(x(x(2x - 1) + 3) + 1) - 5$$

To evaluate $p(x)$ at $x=3$:

coefficients	2	-1	3	1	-5
$x=3$	2	3*2+(-1)= 5	3*5+3=18	3*18+1=55	3*55+(-5)=160

x x x x

Horner's Rule- Representation change

Horner's Rule For Polynomial Evaluation

$$p(x) = (\ldots (a_n x + a_{n-1})\, x + \ldots)\ \ x + a_0$$

ALGORITHM *Horner*($P[0..n]$, x)

//Evaluates a polynomial at a given point by Horner's rule
//Input: An array $P[0..n]$ of coefficients of a polynomial of degree n,
// stored from the lowest to the highest and a number x
//Output: The value of the polynomial at x
$p \leftarrow P[n]$
for $i \leftarrow n - 1$ **downto** 0 **do**
    $p \leftarrow x * p + P[i]$
return p

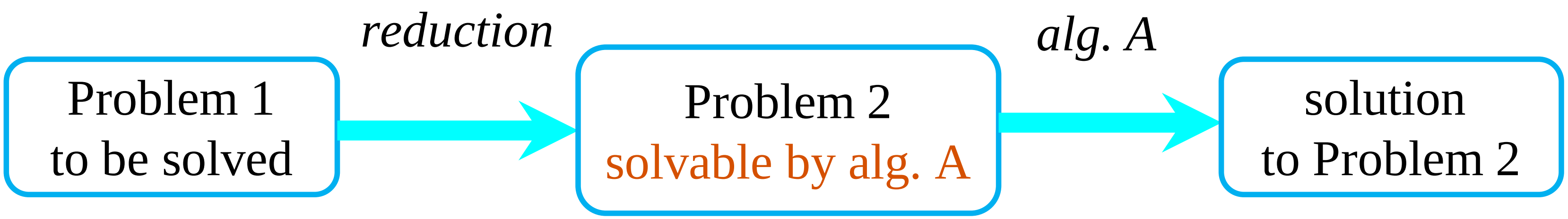
The number of multiplications and additions are given by the same sum:

$$M(n) = A(n) = \sum_{i=0}^{n-1} 1 = n.$$

Problem Reduction

Problem Reduction (问题化简)

- *To solve a problem, reduce it to another problem that you know how to solve*



Two points:

- *finding a problem to which the problem at hand should be reduced*
- *reduction-based algorithm to be more efficient than solving the original problem directly*

Problem Reduction

Example

In analytical geometry, for three arbitrary points in the plane, $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$, the determinant is positive if and only if the point p_3 is to the left of the directed line through points $p_1\ p_2$

$$\det\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1y_2 + x_2y_3 + x_3y_1 - x_3y_2 - x_2y_1 - x_1y_3$$

i.e. we *reduce* a geometric problem about the relative locations of three points to a problem about the sign of a determinant. (把关于三个点的相对位置的几何问题转化成关于行列式符号的问题。)

☆ *The entire idea of analytical geometry is based on reducing geometric problems to algebra ones.*

Problem Reduction

- *问题化简--线性规划* ***(Linear programming)***
  - *Linear programming:*

• *a problem of optimizing a linear function of several variables subject to constraints in the form of linear equations and linear inequalities.*

Maximize(or minimize) : $c_1x_1 + \ldots c_nx_n$

Subject to: $a_{i1}x_1 + \ldots + a_{in}x_n \leq (\text{or} \geq \text{or} =)\ b_i$, for $i = 1\ldots n$

$x_1 \geq 0, \ldots, x_n \geq 0$

Problem Reduction

- ***Linear programming***
  - *Algorithms for Linear programming:*
    - *Simplex method: worst-case efficiency is to be exponential*
    - *Ellipsoid algorithm: polynomial time.*
    - *Interior-point methods: polynomial time*
    - *Karmarkar's algorithm: polynomial worst-case efficiency*

Problem Reduction

- **Linear programming**
  - Algorithms for Linear programming:
    - Integer Linear programming: the variables of a Linear programming problem are required to be integers.
      - e.g., 0-1 knapsack problem
      - no known polynomial-time algorithm (NP-hard)
      - branch-and-bound method for solving Integer Linear programming

Problem Reduction

- ***Linear programming***
  - *Investment Problem:*
    - ***Scenario:***
      - *A university endowment needs to invest $100 million*
      - *Three types of investment:*
        - *Stocks (expected interest: 10%)*
        - *Bonds (expected interest: 7%)*
        - *Cash (expected interest: 3%)*
    - ***Constraints:***
      - *The investment in stocks is no more than 1/3 of the money invested in bonds*
      - *At least 25% of the total amount invested in stocks and bonds must be invested in cash*
    - ***Objective**:*
      - *An investment that maximizes the return*

Problem Reduction

- ***Linear programming***
  - *Investment Problem: (‘cont)*
    - ***mathematical model***

$$\text{Maximize} \qquad 0.10x + 0.07y + 0.03z$$

$$\text{Subject to} \qquad x + y + z = 100$$

$$x \leq (1/3)y$$

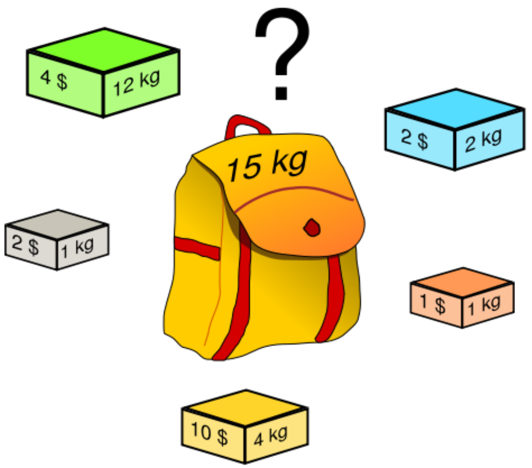
$$z \geq 0.25(x + y)$$

$$x \geq 0,\ y \geq 0,\ z \geq 0$$

optimal decision making problem ----→ linear programming problem

Problem Reduction

- ***Linear programming***
  - *Knapsack Problem (Discrete Version)*
    - ***Scenario***
      - *Given n items:*
        - *weights:* $w_1 \quad w_2 \ldots \quad w_n$
        - *values:* $v_1 \quad v_2 \ldots \quad v_n$
        - *a knapsack of capacity W*



    - ***Constraints***
      - *an item can either be put into the knapsack in its entirely or not be put into the knapsack.*
    - ***Objective:***

      Find the most valuable subset of the items

Problem Reduction

Linear programming

- Knapsack Problem (Discrete Version) ('cont)
  - ***mathematical model***

Maximize $$\sum_{i=1}^{n} v_i x_i$$

subject to $$\sum_{i=1}^{n} w_i x_i \le W$$

$$\boxed{x_i \in \{0,1\}} \quad \text{for } i = 1,\ldots,n$$

Problem Reduction

Linear programming

- Knapsack Problem (Continuous/Fraction Version):

  - **Scenario**
    - Given n items:
      - weights: $w_1 \quad w_2 \ldots \quad w_n$
      - values: $v_1 \quad v_2 \ldots \quad v_n$
      - a knapsack of capacity W
  - **Constraints**
    - Any fraction of any item can be put into the knapsack, x_i
  - **Objective**:
    - Find the most valuable subset of the items

Problem Reduction

- **Linear programming**
  - Knapsack Problem (Continuous/Fraction Version): ('cont)
    - ***mathematical model***

Maximize $$\sum_{i=1}^{n} v_i x_i$$

subject to $$\sum_{i=1}^{n} w_i x_i \leq W$$

$$\boxed{0 \leq x_i \leq 1} \quad \text{for } i = 1, \ldots, n$$

Problem Reduction

Reduction to Graph

- *many problems can be solved by reduction to one of the standard graph problems*
- *state-space graph: vertices of a graph represent possible states of the problem, edges indicate permitted transitions among such states*
- *one of the graph's vertices represents the initial state, another represents a goal state of the problem*
- *puzzles and games*
- *not always a straightforward task*

problem ----→ a path from the initial-state vertex to a goal-state vertex

Problem Reduction

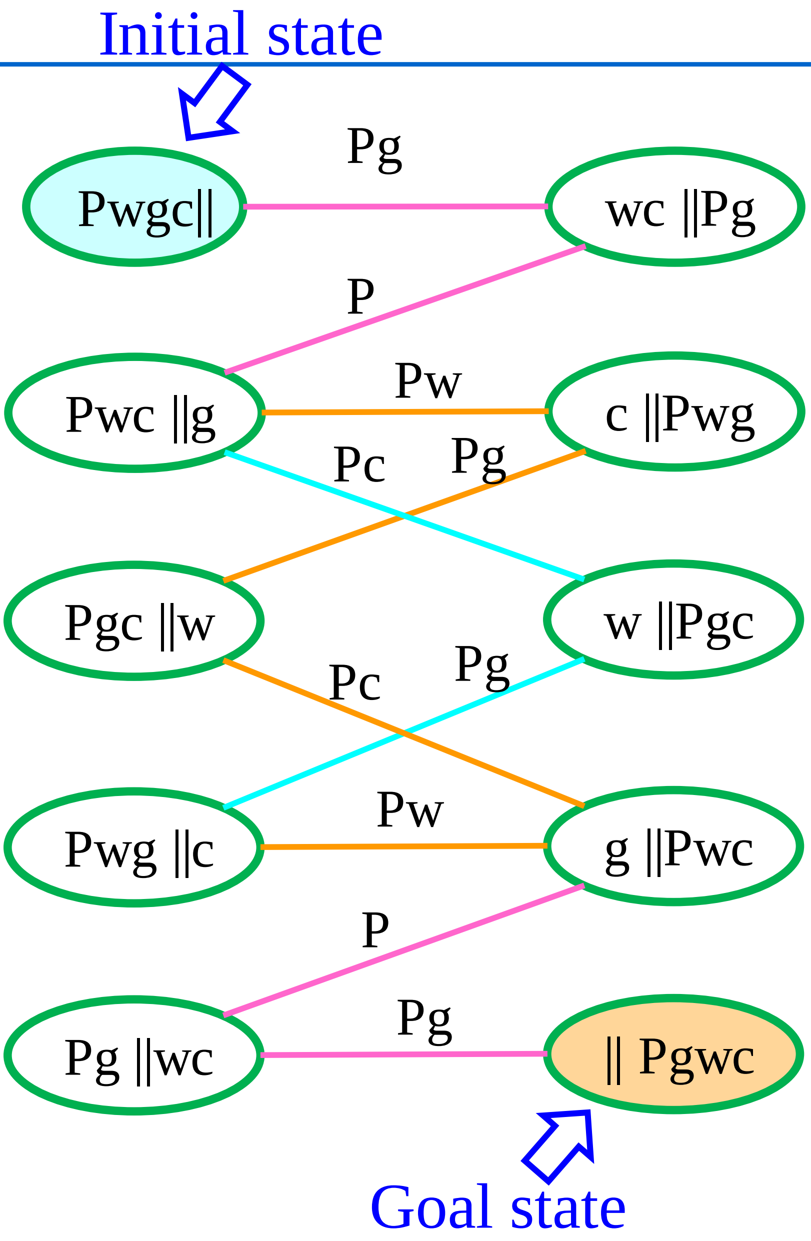
Reduction to Graph

- *River-crossing puzzle*
  - ***Problem**: The wolf, goat and bag of cabbage puzzle.*

- A peasant must transport a wolf, goat and bag of cabbage from one side of a river to another using a boat,
- the boat can only hold one item in addition to the peasant,
- subject to the constraints that the wolf cannot be left alone with the goat , and the goat cannot be left alone with the cabbage.

Problem Reduction

- **Reduction to Graph**
  - *River-crossing puzzle*
    - ***state-space graph***



Summary

1. 变治法是一种基于变换思想，把问题变换成一种更容易解决的类型。
2. 变治法的三种类型：实例化简，改变表现，和问题化简
3. 变治法三种类型对应的算法举例
4. 堆的概念，堆排序的思想：在排列好堆中的数组元素后，再从剩余堆中连续删除最大的元素。在最差以及平均情况下，该算法都属于在位的排序算法，时间复杂度⊙(nlogn)
5. 高斯消去法
6. 霍纳法则
7. 线性规划及整数线性规划