CV Assignment 2 - Math Part(1-3)

1.

For the infinity point $\mathbf{p}_\infty$ on the line, let its homogeneous coordinates be $(x, y, 0)$. Then, in the coordinate system O_1 in the plane, the vector with coordinates (x, y) is parallel to the line in the coordinate system O corresponding to $\mathbf{p}_\infty$ (whose direction vector is also $(x, y, 0)$).

Therefore, by appending a 0 to the direction vector of the line, we obtain the homogeneous coordinates of the infinity point. Obviously, for the line x − 3 y + 4 = 0, the directional vector is $(3, 1)$.

Thus the homogeneous coordinates of the infinity point $\mathbf{p}_\infty$ on the line is $(3, 1, 0)^T$.

2.

According to the derivative formula of vector function with respect to vector variable:

$$\frac{\mathrm{d}\mathbf{p}_d}{\mathrm{d}\mathbf{p}_n^T} = \begin{bmatrix} \frac{\partial x_d}{\partial x} & \frac{\partial x_d}{\partial y} \\ \frac{\partial y_d}{\partial x} & \frac{\partial y_d}{\partial y} \end{bmatrix} \tag{1}$$

Next, calculate the value of each element in the matrix separately.

Substitude $r^2 = x^2 + y^2, r^4 = x^4 + 2x^2y^2 + y^4, r^6 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6$ into the definition of x_d and y_d:

$$x_d = x + k_1(x^3 + xy^2) + k_2(x^5 + 2x^3y^2 + xy^4) + \rho_1 \cdot 2xy + \rho_2(3x^2 + y^2) + k_3(x^7 + 3x^5y^2 + 3x^3y^4 + xy^6) \tag{2}$$

$$y_d = y + k_1(x^2y + y^3) + k_2(x^4y + 2x^2y^3 + y^5) + \rho_2 \cdot 2xy + \rho_1(x^2 + 3y^2) + k_3(x^6y + 3x^4y^3 + 3x^2y^5 + y^7) \tag{3}$$

Then calculate each partial derivative:

$$\frac{\partial x_d}{\partial x} = 1 + k_1(3x^2 + y^2) + k_2(5x^4 + 6x^2y^2 + y^4) + \rho_1 \cdot 2y + \rho_2 \cdot 6x + k_3(7x^6 + 15x^4y^2 + 9x^2y^4 + y^6) \tag{4}$$

$$\frac{\partial x_d}{\partial y} = k_1 \cdot 2xy + k_2(4x^3y + 4xy^3) + \rho_1 \cdot 2x + \rho_2 \cdot 2y + k_3(6x^5y + 12x^3y^3 + 6xy^5) \tag{5}$$

$$\frac{\partial y_d}{\partial x} = k_1 \cdot 2xy + k_2(4x^3y + 4xy^3) + \rho_2 \cdot 2y + \rho_1 \cdot 2x + k_3(6x^5y + 12x^3y^3 + 6xy^5) \tag{6}$$

$$\frac{\partial y_d}{\partial y} = 1 + k_1(x^2 + 3y^2) + k_2(x^4 + 6x^2y^2 + 5y^4) + \rho_2 \cdot 2x + \rho_1 \cdot 6y + k_3(x^6 + 9x^4y^2 + 15x^2y^4 + 7y^6) \tag{7}$$

Note that $\frac{\partial x_d}{\partial y} = \frac{\partial y_d}{\partial x}$, the matrix is a symmetric matrix.

Thus, we obtained the Jacobian matrix of $\mathbf{p}_d$ with respect to $\mathbf{p}_n$.

3.

Expand the matrix R to get its complete expression:

$$\mathbf{R} = \beta\mathbf{I} + \gamma\mathbf{n}\mathbf{n}^T + \alpha\mathbf{n}\hat{} = \begin{bmatrix} \gamma n_1n_1 + \beta & \gamma n_1n_2 - \alpha n_3 & \gamma n_1n_3 + \alpha n_2 \\ \gamma n_2n_1 + \alpha n_3 & \gamma n_2n_2 + \beta & \gamma n_2n_3 - \alpha n_1 \\ \gamma n_3n_1 - \alpha n_2 & \gamma n_3n_2 + \alpha n_1 & \gamma n_3n_3 + \beta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{8}$$

According to the relationship between θ, $\mathbf{d}$ and $\mathbf{n}$: $\mathbf{d}=\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}=\theta\mathbf{n}=\theta\begin{bmatrix}n_1\\n_2\\n_3\end{bmatrix}$ and that $\mathbf{n}$ is a unit vector, we know $\theta=||d||=\sqrt{d_1^2+d_2^2+d_3^2}$.

And obviously, $n_1=\frac{d_1}{\theta}, n_2=\frac{d_2}{\theta}, n_3=\frac{d_3}{\theta}$.

According to the derivative formula of vector function with respect to vector variable:

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\mathbf{d}^T}=\begin{bmatrix}\frac{\partial r_{11}}{\partial d_1} & \frac{\partial r_{11}}{\partial d_2} & \frac{\partial r_{11}}{\partial d_3}\\ \frac{\partial r_{12}}{\partial d_1} & \frac{\partial r_{12}}{\partial d_2} & \frac{\partial r_{12}}{\partial d_3}\\ \vdots & \vdots & \vdots\\ \frac{\partial r_{33}}{\partial d_1} & \frac{\partial r_{33}}{\partial d_2} & \frac{\partial r_{33}}{\partial d_3}\end{bmatrix} \tag{9}$$

Each element in $\mathbf{R}$ is a function of α,β,γ and θ (since $n_1=\frac{d_1}{\theta}, n_2=\frac{d_2}{\theta}, n_3=\frac{d_3}{\theta}$). In order to find their derivatives with respect to d_1,d_2 and d_3, let's do some preparation:

- Calculate the derivatives of n_1,n_2 and n_3 with respect to d_1,d_2 and d_3.

$$\frac{\partial n_1}{\partial d_1}=\frac{\partial(\frac{d_1}{\sqrt{d_1^2+d_2^2+d_3^2}}))}{\partial d_1}=\frac{\sqrt{d_1^2+d_2^2+d_3^2}-\frac{d_1}{\sqrt{d_1^2+d_2^2+d_3^2}}}{d_1^2+d_2^2+d_3^2}=\frac{\theta-\frac{\theta^2 n_1^2}{\theta}}{\theta^2}=\frac{1-n_1^2}{\theta} \tag{10}$$

Similarly, we get

$$\frac{\partial n_2}{\partial d_2}=\frac{1-n_2^2}{\theta} \tag{11}$$

$$\frac{\partial n_3}{\partial d_3}=\frac{1-n_3^2}{\theta} \tag{12}$$

$$\frac{\partial n_1}{\partial d_2}=\frac{\partial n_2}{\partial d_1}=\frac{-n_1 n_2}{\theta} \tag{13}$$

$$\frac{\partial n_2}{\partial d_3}=\frac{\partial n_3}{\partial d_2}=\frac{-n_2 n_3}{\theta} \tag{14}$$

$$\frac{\partial n_1}{\partial d_3}=\frac{\partial n_3}{\partial d_1}=\frac{-n_1 n_3}{\theta} \tag{15}$$

- Calculate the derivatives of θ with respect to d_1,d_2 and d_3.

$$\frac{\partial\theta}{\partial d_1}=\frac{\partial(\sqrt{d_1^2+d_2^2+d_3^2}))}{\partial d_1}=\frac{d_1}{\sqrt{d_1^2+d_2^2+d_3^2}}=\frac{d_1}{\theta}=n_1 \tag{16}$$

Similarly, we get

$$\frac{\partial\theta}{\partial d_2}=n_2 \tag{17}$$

$$\frac{\partial\theta}{\partial d_3}=n_3 \tag{18}$$

- Calculate the derivatives of α with respect to d_1,d_2 and d_3.

$$\frac{\partial\alpha}{\partial d_1}=\frac{\partial(\sin\theta)}{\partial d_1}=\cos\theta\frac{\partial\theta}{\partial d_1}=\beta n_1 \tag{19}$$

Similarly, we get

$$\frac{\partial \alpha}{\partial d_2} = \beta n_2 \tag{20}$$

$$\frac{\partial \alpha}{\partial d_3} = \beta n_3 \tag{21}$$

- Calculate the derivatives of β with respect to d_1, d_2 and d_3.

$$\frac{\partial \beta}{\partial d_1} = \frac{\partial(\cos\theta)}{\partial d_1} = -\sin\theta \frac{\partial \theta}{\partial d_1} = -\alpha n_1 \tag{22}$$

Similarly, we get

$$\frac{\partial \beta}{\partial d_2} = -\alpha n_2 \tag{23}$$

$$\frac{\partial \beta}{\partial d_3} = -\alpha n_3 \tag{24}$$

- Calculate the derivatives of γ with respect to d_1, d_2 and d_3.

$$\frac{\partial \gamma}{\partial d_1} = \frac{\partial(1-\cos\theta)}{\partial d_1} = \sin\theta \frac{\partial \theta}{\partial d_1} = \alpha n_1 \tag{25}$$

Similarly, we get

$$\frac{\partial \gamma}{\partial d_2} = \alpha n_2 \tag{26}$$

$$\frac{\partial \gamma}{\partial d_3} = \alpha n_3 \tag{27}$$

After these calculations, we calculate the derivatives of $r_{ij}, 1 \leq i, j \leq 3$ with respect to d_1, d_2 and d_3 according to *Chain Rule*.

- r_{11}

$$\begin{gathered} \frac{\partial r_{11}}{\partial d_1} = \frac{\partial r_{11}}{\partial \gamma}\frac{\partial \gamma}{\partial d_1} + \frac{\partial r_{11}}{\partial n_1}\frac{\partial n_1}{\partial d_1} + \frac{\partial r_{11}}{\partial \beta}\frac{\partial \beta}{\partial d_1} = \\ n_1^2 \cdot \alpha n_1 + 2\gamma n_1 \cdot \frac{1-n_1^2}{\theta} + 1 \cdot (-\alpha n_1) = \\ \alpha n_1^3 + \frac{2\gamma n_1(1-n_1^2)}{\theta} - \alpha n_1 \end{gathered} \tag{28}$$

Similarly, we get

$$\frac{\partial r_{11}}{\partial d_2} = \alpha n_1^2 n_2 - \frac{2\gamma n_1^2 n_2}{\theta} - \alpha n_2 \tag{29}$$

$$\frac{\partial r_{11}}{\partial d_3} = \alpha n_1^2 n_3 - \frac{2\gamma n_1^2 n_3}{\theta} - \alpha n_3 \tag{30}$$

Noticed that r_{11}, r_{22}, r_{33} have the same structure, we observe that their derivatives can be obtained easily by exchanging the subscript of $n_i, 1 \leq i \leq 3$.

- r_{22}

$$\frac{\partial r_{22}}{\partial d_1} = \alpha n_1 n_2^2 - \frac{2\gamma n_1 n_2^2}{\theta} - \alpha n_1 \tag{31}$$

$$\frac{\partial r_{22}}{\partial d_2} = \alpha n_2^3 + \frac{2\gamma n_2(1-n_2^2)}{\theta} - \alpha n_2 \tag{32}$$

$$\frac{\partial r_{22}}{\partial d_3} = \alpha n_2^2 n_3 - \frac{2\gamma n_2^2 n_3}{\theta} - \alpha n_3 \tag{33}$$

- r_{33}

$$\frac{\partial r_{33}}{\partial d_1} = \alpha n_1 n_3^2 - \frac{2\gamma n_1 n_3^2}{\theta} - \alpha n_1 \tag{34}$$

$$\frac{\partial r_{22}}{\partial d_2} = \alpha n_2 n_3^2 - \frac{2\gamma n_2 n_3^2}{\theta} - \alpha n_2 \tag{35}$$

$$\frac{\partial r_{33}}{\partial d_3} = \alpha n_3^3 + \frac{2\gamma n_3(1 - n_3^2)}{\theta} - \alpha n_3 \tag{36}$$

Repeat the above process of calculating derivatives using the *Chain Rule* and quickly expanding by exchanging subscripts by using the same structure, we get the following results.

- r_{12}

$$\frac{\partial r_{12}}{\partial d_1} = \alpha n_1^2 n_2 - \beta n_1 n_3 + \frac{\gamma n_2(1 - 2n_1^2) + \alpha n_1 n_3}{\theta} \tag{37}$$

$$\frac{\partial r_{12}}{\partial d_2} = \alpha n_1 n_2^2 - \beta n_2 n_3 + \frac{\gamma n_1(1 - 2n_2^2) + \alpha n_2 n_3}{\theta} \tag{38}$$

$$\frac{\partial r_{12}}{\partial d_3} = \alpha n_1 n_2 n_3 - \beta n_3^2 + \frac{\alpha(n_3^2 - 1) - 2\gamma n_1 n_2 n_3}{\theta} \tag{39}$$

- r_{23}

$$\frac{\partial r_{23}}{\partial d_2} = \alpha n_1 n_2 n_3 - \beta n_1^2 + \frac{\alpha(n_1^2 - 1) - 2\gamma n_1 n_2 n_3}{\theta} \tag{40}$$

$$\frac{\partial r_{23}}{\partial d_2} = \alpha n_2^2 n_3 - \beta n_1 n_2 + \frac{\gamma n_3(1 - 2n_2^2) + \alpha n_1 n_2}{\theta} \tag{41}$$

$$\frac{\partial r_{23}}{\partial d_3} = \alpha n_2 n_3^2 - \beta n_1 n_3 + \frac{\alpha n_1 n_3 + \gamma n_2(1 - 2n_3^2)}{\theta} \tag{42}$$

- r_{31}

$$\frac{\partial r_{31}}{\partial d_1} = \alpha n_1^2 n_3 - \beta n_1 n_2 + \frac{\alpha n_1 n_2 + \gamma n_3(1 - 2n_1^2)}{\theta} \tag{43}$$

$$\frac{\partial r_{31}}{\partial d_2} = \alpha n_1 n_2 n_3 - \beta n_2^2 + \frac{\alpha(n_2^2 - 1) + 2\gamma n_1 n_2 n_3}{\theta} \tag{44}$$

$$\frac{\partial r_{31}}{\partial d_3} = \alpha n_1 n_3^2 - \beta n_2 n_3 + \frac{\alpha n_2 n_3 + \gamma n_1(1 - 2n_3^2)}{\theta} \tag{45}$$

- r_{13}

$$\frac{\partial r_{13}}{\partial d_1} = \alpha n_1^2 n_3 + \beta n_1 n_2 + \frac{\gamma n_3(1 - 2n_1^2) - \alpha n_1 n_2}{\theta} \tag{46}$$

$$\frac{\partial r_{13}}{\partial d_2} = \alpha n_1 n_2 n_3 + \beta n_2^2 + \frac{\alpha(1 - n_2^2) - 2\gamma n_1 n_2 n_3}{\theta} \tag{47}$$

$$\frac{\partial r_{13}}{\partial d_3} = \alpha n_1 n_3^2 + \beta n_2 n_3 + \frac{\gamma n_1(1 - 2n_3^2) - \alpha n_1 n_3}{\theta} \tag{48}$$

- r_{21}

$$\frac{\partial r_{21}}{\partial d_1} = \alpha n_1^2 n_2 + \beta n_1 n_3 + \frac{\gamma n_2(1 - 2n_1^2) - \alpha n_1 n_3}{\theta} \tag{49}$$

$$\frac{\partial r_{21}}{\partial d_2} = \alpha n_1 n_2^2 + \beta n_2 n_3 + \frac{\gamma n_1(1-2n_2^2) - \alpha n_2 n_3}{\theta} \tag{50}$$

$$\frac{\partial r_{21}}{\partial d_3} = \alpha n_1 n_2 n_3 + \beta n_3^2 + \frac{\alpha(1-n_3^2) - 2\gamma n_1 n_2 n_3}{\theta} \tag{51}$$

- r_{32}

$$\frac{\partial r_{32}}{\partial d_1} = \alpha n_1 n_2 n_3 + \beta n_1^2 + \frac{\alpha(1-n_1^2) - 2\gamma n_1 n_2 n_3}{\theta} \tag{52}$$

$$\frac{\partial r_{32}}{\partial d_2} = \alpha n_2^2 n_3 + \beta n_1 n_2 + \frac{\gamma n_3(1-2n_2^2) - \alpha n_1 n_2}{\theta} \tag{53}$$

$$\frac{\partial r_{32}}{\partial d_3} = \alpha n_2 n_3^2 + \beta n_1 n_3 + \frac{\gamma n_2(1-2n_3^2) - \alpha n_1 n_3}{\theta} \tag{54}$$

Now we know the structure of the Jacobian matrix $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\mathbf{d}^T} = \begin{bmatrix} \frac{\partial r_{11}}{\partial d_1} & \frac{\partial r_{11}}{\partial d_2} & \frac{\partial r_{11}}{\partial d_3} \\ \frac{\partial r_{12}}{\partial d_1} & \frac{\partial r_{12}}{\partial d_2} & \frac{\partial r_{12}}{\partial d_3} \\ \vdots & \vdots & \vdots \\ \frac{\partial r_{33}}{\partial d_1} & \frac{\partial r_{33}}{\partial d_2} & \frac{\partial r_{33}}{\partial d_3} \end{bmatrix}$ and each element of it(equation 28-54).