CV Assignment 1 - Math Part(1-3)

1.

(1)Closure

$\forall M_i, M_j \in \{M_i\}$,

let $M = M_i \times M_j = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \times \begin{bmatrix} R_j & \mathbf{t}_j \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_iR_j & R_i\mathbf{t}_j + \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix}$.

To prove closure is to prove that $M \in \{M_i\}$.

Obviously, $M \in \mathbb{R}^{4\times4}, R_i \in \mathbb{R}^{3\times3}, R_j \in \mathbb{R}^{3\times3}, R_iR_j \in \mathbb{R}^{3\times3}, R_i\mathbf{t}_j + \mathbf{t}_i \in \mathbb{R}^{3\times1}$.

$det(R_iR_j) = det(R_i)det(R_j) = 1 \times 1 = 1$.

$(R_iR_j)(R_iR_j)^T = (R_iR_j)(R_j^TR_i^T) = R_i(R_jR_j^T)R_i^T = R_iIR_i^T = R_iR_i^T = I$, where $I \in \mathbb{R}^{3\times3}$ indicates identity matrix. That means matrix R_iR_j is an orthonormal matrix.

In summary, matrix M satisfies that $M \in \mathbb{R}^{4\times4}, R_iR_j \in \mathbb{R}^{3\times3}, R_i\mathbf{t}_j + \mathbf{t}_i \in \mathbb{R}^{3\times1}, R_iR_j$ is orthonormal , and $det(R_iR_j) = 1$. So matrix $M \in \{M_i\}$.

Thus $\{M_i\}$ satisfies the property of closure.

(2)Associativity

$\forall M_i, M_j, M_k \in \{M_i\}$,

$$(M_iM_j)M_k = (\begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_j & \mathbf{t}_j \\ \mathbf{0}^T & 1 \end{bmatrix}) \begin{bmatrix} R_k & \mathbf{t}_k \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_iR_j & R_i\mathbf{t}_j + \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_k & \mathbf{t}_k \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_iR_jR_k & R_iR_j\mathbf{t}_k + R_i\mathbf{t}_j + \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$M_i(M_jM_k) = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} (\begin{bmatrix} R_j & \mathbf{t}_j \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_k & \mathbf{t}_k \\ \mathbf{0}^T & 1 \end{bmatrix}) = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_jR_k & R_j\mathbf{t}_k + \mathbf{t}_j \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_iR_jR_k & R_iR_j\mathbf{t}_k + R_i\mathbf{t}_j + \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix}$$

So $(M_iM_j)M_k = M_i(M_jM_k)$.

Thus $\{M_i\}$ satisfies the property of associativity.

(3)Existence of an identity element

Let identity matrix $E = I_{4\times4} = \begin{bmatrix} I_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$, where $\mathbf{0} \in \mathbb{R}^{3\times1}$.

Firstly, we prove that $E \in \{M_i\}$.

Obviously, $I_{3\times3}I_{3\times3}^T = I_{3\times3}^TI_{3\times3} = I_{3\times3}, det(I_{3\times3}) = 1$.

So E is an element of set $\{M_i\}$.

Because E is an identity matrix, we know that

$\forall M \in \{M_i\}, EM = ME = M$

Thus $\{M_i\}$ satisfies the property of existence of an identity element.

(4)Existence of an inverse element for each group element

For $\forall M \in \{M_i\}, M = (\begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$, R is orthonormal, so $RR^T = I$, where I indicates identity matrix.

Let $M^{-1} = \begin{bmatrix} R^T & -R^T\mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$.

Firstly, we prove that $M^{-1} \in \{M_i\}$.

Obviously, $R^T \in \mathbb{R}^{3\times3}, -R^T\mathbf{t} \in \mathbb{R}^{3\times1}$,

$R^T(R^T)^T = R^TR = I$, where I indicates identity matrix,

and $det(R^T) = det(R) = 1$.

So M^{-1} is an element of set $\{M_i\}$.

To prove that M^{-1} is an inverse element for each group element, we compute:

$$MM^{-1} = \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R^T & -R^T\mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} RR^T & -RR^T\mathbf{t} + \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} I_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = I_{4\times4}$$

$$M^{-1}M = \begin{bmatrix} R^T & -R^T\mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R^TR & R^T\mathbf{t} - R^T\mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} I_{3\times3} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = I_{4\times4}$$

So $M^{-1} \in \{M_i\}$ is an an inverse element for each group element.

2.

a)

To prove that the $n\times n$ real symmetric matrix M is semi-positive definite is to prove that its eigenvalues satisfy that $\lambda_1 \geq 0, \lambda_2 \geq 0$.

We know that:

$\lambda_1\lambda_2 = det(M) = \sum_{(x_i,y_i)\in w}(I_x)^2 \sum_{(x_i,y_i)\in w}(I_y)^2 - (\sum_{(x_i,y_i)\in w}(I_xI_y))^2$

$\lambda_1 + \lambda_2 = trace(M) = \sum_{(x_i,y_i)\in w}(I_x)^2 \sum_{(x_i,y_i)\in w}(I_y)^2$.

Obviously, $\sum_{(x_i,y_i)\in w}(I_x)^2 \geq 0, \sum_{(x_i,y_i)\in w}(I_y)^2 \geq 0$, so $\lambda_1 + \lambda_2 \geq 0$.

Let's apply Cauchy-Schwarz Inequality:

$\sum_{(x_i,y_i)\in w}(I_x)^2 \sum_{(x_i,y_i)\in w}(I_y)^2 \geq (\sum_{(x_i,y_i)\in w}(I_xI_y))^2$,

so $\lambda_1\lambda_2 \geq 0$.

$\lambda_1\lambda_2 \geq 0 \wedge \lambda_1 + \lambda_2 \geq 0 \iff \lambda_1 \geq 0 \wedge \lambda_2 \geq 0$.

Thus M is positive semi-definite.

b)

For simplicity, we note $\sum_{(x_i,y_i)\in w}(I_x)^2 = A, \sum_{(x_i,y_i)\in w}(I_xI_y) = B, \sum_{(x_i,y_i)\in w}(I_y)^2 = C$.

So $M = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$.

$[x \quad y]M\begin{bmatrix} x \\ y \end{bmatrix} = [x \quad y]\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = Ax^2 + 2Bxy + Cy^2 = 1$.

According to the theory of conic sections, to prove that $Ax^2 + 2Bxy + Cy^2 = 1$ represents an ellipse is to prove that $A > 0, C > 0$ and the discriminant $4AC - (2B)^2 > 0$.

M is positive definite, so its eigenvalues satisfy $\lambda_1 > 0, \lambda_2 > 0$.

M satisfies that $det(M) = AC - B^2 = \lambda_1\lambda_2 > 0$, so the discriminant $4AC - (2B)^2 = 4(AC - B^2) > 0$.

$trace(M) = AC = \lambda_1 + \lambda_2 > 0$, and obviously $A = \sum_{(x_i,y_i)\in w}(I_x)^2 \geq 0, C = \sum_{(x_i,y_i)\in w}(I_y)^2 \geq 0$, so $A > 0, C > 0$.

In summary, $Ax^2 + 2Bxy + Cy^2 = 1$ satisfies that $A > 0, C > 0$ and the discriminant $4AC - (2B)^2 > 0$.

Thus $[x \quad y]M\begin{bmatrix}x\\y\end{bmatrix}$ represents an ellipse.

c)

We note $\alpha_1, \alpha_2(||\alpha_1||_2^2 = 1, ||\alpha_2||_2^2 = 1)$ the eigenvectors of M, corresponding to λ_1, λ_2 respectively.

According to the condition that eigenvalues of the $n \times n$ real symmetric matrix M satisfy $\lambda_1 > \lambda_2 > 0$, its eigenvectors are linearly independent, so M can be diagonalized.

Diagonalize M:

$$M = P^T \Lambda P, P = [\alpha_1, \alpha_2], \Lambda = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}$$

Rewrite $[x \quad y]M\begin{bmatrix}x\\y\end{bmatrix} = 1$:

$$[x \quad y]M\begin{bmatrix}x\\y\end{bmatrix} = [x \quad y]P^T \Lambda P\begin{bmatrix}x\\y\end{bmatrix} = (P\begin{bmatrix}x\\y\end{bmatrix})^T \Lambda (P\begin{bmatrix}x\\y\end{bmatrix}) = 1$$

Note $X = \begin{bmatrix}u\\v\end{bmatrix} = P\begin{bmatrix}x\\y\end{bmatrix}$, we have

$$X^T \Lambda X = X^T \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix} X = \lambda_1 u^2 + \lambda_2 v^2 = \frac{u^2}{\frac{1}{\lambda_1}} + \frac{v^2}{\frac{1}{\lambda_2}} = 1.$$

We know $\lambda_1 > \lambda_2 > 0$, so $\frac{1}{\lambda_2} > \frac{1}{\lambda_1} > 0$

Thus for the ellipse, the length of its semi-major axis is $\frac{1}{\sqrt{\lambda_2}}$, the length of its semi-minor axis is $\frac{1}{\sqrt{\lambda_1}}$.

3.

Firstly, let's prove that $rank(A^T A) = rank(A) = n$, for which we prove that the homogeneous linear equation system $A\mathbf{x} = 0$ and $A^T A\mathbf{x} = 0$ have the same solution

Obviously, the solutions of $A\mathbf{x} = 0$ are also solutions of $A^T A\mathbf{x} = 0$

Let vector α be the solution of $A^T A\mathbf{x} = 0$, then $\alpha^T A^T A\alpha = (A\alpha)^T (A\alpha) = 0$. The inner product of vector $A\alpha$ is 0, then $A\alpha = \mathbf{0}$. So the solutions of $A^T A\mathbf{x} = 0$ are also solutions of $A\mathbf{x} = 0$.

Now we know that $A\mathbf{x} = 0$ and $A^T A\mathbf{x} = 0$ have the same solution, so $rank(A^T A) = rank(A) = n$.

Obviously, $A^T A \in \mathbb{R}^{n \times n}$. Since $rank(A^T A) = n$, $A^T A$ is invertible.