CE1012

Engineering Mathematics Part 1

*Only notes of critical points

Date No.

Combination of Functions * Remember to state domain

→ $(f+g)(x) = f(x) + g(x)$

→ $(f-g)(x) = f(x) - g(x)$

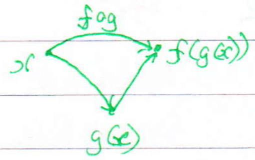
→ $(fg)(x) = f(x)g(x)$

→ $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

Composition of Functions

$(f \circ g)(x) = f(g(x))$

This is circle



Domain of g: all x

Domain of f: all g(x)

→ Function g applied first.

→ Use domain of 1st function

Properties of Logarithms

→ $\log_a 1 = 0$

→ $\log_a a = 1$

→ $a^{\log_a x} = x$ → $y = \log_a x$

$a^y = x$

$a^{\log_a x} = x$

Laws of Logarithms

→ $\log_a x = \frac{\log_b x}{\log_b a}$

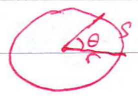
→ $\log_a x = \frac{1}{\log_x a}$ } Change of Base

→ $\log_a \frac{1}{y} = -\log_a y$ } Reciprocal Rule

Trigonometric Radian

→ $\theta = \frac{s}{r}$

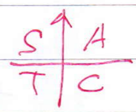
→ $360° = \frac{2\pi r}{r} = 2\pi$ → Diameter

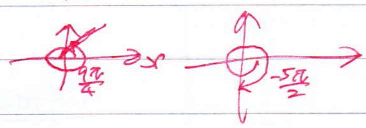


Rotations

→ Clockwise
↳ Positive

→ Anticlockwise
↳ Negative



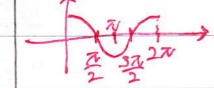


Special Angles

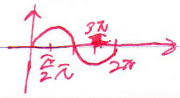
→ 30°, 60°, 45°

Trigonometric Functions

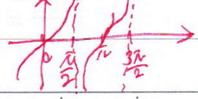
$y = \cos x$



$y = \sin x$



$y = \tan x$



Key Formulas

→ $\sin^2\theta + \cos^2\theta = 1$

→ $1 + \tan^2\theta = \sec^2\theta$

→ $1 + \cot^2\theta = \csc^2\theta$

A'ZONE

Rate of change

$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$

→ Usually not measured easily & have to use related rates problem

Related Rates Problem

→ Draw a Diagram

→ Find an eqn to relate variables

→ Differentiate with Chain Rule

Integration → Rmb (+C) after Integration

Trigonometric * Memory Work

→ $\int \tan x\,dx = \ln|\sec x| + C$, $\int \cot x\,dx = \ln|\sin x| + C$

→ $\int \sec x\,dx = \ln|\sec x + \tan x| + C$, $\int \csc x\,dx = \ln|\csc x - \cot x| + C$

Exponential

→ $\int a^x\,dx = \frac{a^x}{\ln a} + C$, $\int f'(x)\,a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + C$

Logarithmic

→ $\int \frac{1}{x} = \ln|x| + C$

By Substitution

~~$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$~~

→ $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$, setting a "u"

By Parts

→ $\int u\,dv = uv - \int v\,du$ u based on LIATE rule

Log Inv Trig. Alg. Trig Exp.

$\int \underbrace{x}_{u}\,\underbrace{\sin x\,dx}_{dv}$

Riemann Sum *

→ $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\boxed{x_i})\boxed{\Delta x}$

$\Delta x = \frac{b-a}{n}$

$x_i = a + i\Delta x$

Date No.

Theorem that links sequence & function?

Sequences

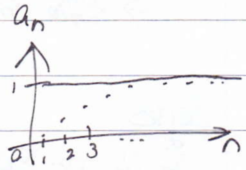
list of numbers in definite order $a_1, a_2, a_3 \dots a_n$

$\{a_1, a_2, a_3, \dots\} = \{a_n\} = \{a_n\}_{n=1}^{\infty} = a_n = \frac{n+1}{n}$

* n doesn't has to start from 1

* plotted by points

Definition of Convergence & Divergence

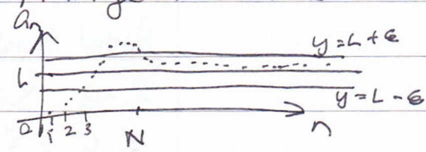


→ $\lim_{n\to\infty} a_n$ exists, the sequence converges

→ if not, the sequence diverges

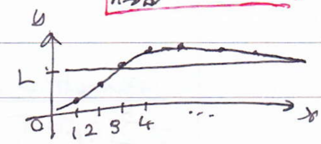
→ $\lim_{n\to\infty} a_n = L$ exists, if for every $\epsilon > 0$, there is a ~~#~~ integer N such that if $N < n$ then $|a_n - L| < \epsilon$

y − (limit)



Theorem for Limits on Sequences *

→ $\boxed{\lim_{x\to\infty} f(x) = L}$ and $\boxed{f(n) = a_n}$, where n is an integer, then $\boxed{\lim_{n\to\infty} a_n = L}$



~~→ $\lim_{n\to\infty} a_n = \infty$ means that every pos M~~

Squeeze Theorem for Sequences *

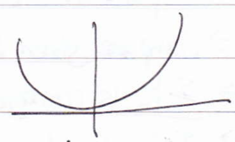
→ $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$

~~if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$~~

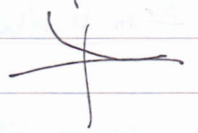
~~If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\lim_{n\to\infty} f(a_n) = f(L)$~~

Monotonic

Either decreasing or increasing



X monotonic

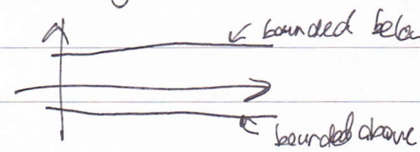


Decreasing monotonic

Increasing Monotonic

Bounded Sequence

bounded by m & n above & below



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Comparison Tests Example

→ $\sum \frac{5}{2n^2+4n+3}$

→ $\frac{\boxed{5}}{\boxed{2n^2}+4n+3} < \frac{5}{2n^2}$

→ $\sum \frac{5}{2n^2} = \frac{5}{2}\boxed{\sum \frac{1}{n^2}}$

p-series

→ $\sum \frac{1}{n^2}$ converge as $p>1$

∴ $a_n < b_n$, a_n is convergent

convergent

Absolute Convergence

→ $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent

↳ If positive terms, $|a_n| = a_n$

→ Conditionally convergent if not ~~a~~ absolutely convergent

→ $\sum a_n$ is absolutely convergent, $\sum a_n$ is convergent

~~The Ratio~~ ~~Test~~ Test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ *

→ $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

→ $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or ∞, $\sum a_n$ is divergent

→ $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, no outcome is given

Similar

~~Root~~ Root Test $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ *

→ $L<1$, absolutely convergent

→ $L>1$, $L=\infty$, divergent

→ $L=1$, inconclusive

Power Series $\sum_{n=0}^{\infty} c_n x^n$

→ watch lecture video

→ Converges & diverges based on x

→ $\sum_{n=0}^{\infty} c_n (x-a)^n$ = power series centred around a

Power Series Theorems $\sum_{n=0}^{\infty} c_n (x-a)^n$

→ radius of converges

area if converges

∞, all x

→ ~~at~~ $x=a$, converge

→ for all x, converge

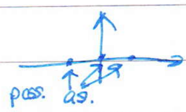
→

Date No.

Taylor Series → Ratio test used for test of convergence & Divergence

→ If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad |x-a| < R$

↳ $c_n = \frac{f^{(n)}(a)}{n!}$



→ by substitution, $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n}$ Taylor series of function f at a

→ Maclaurin Series, $a=0$

Theorem

→ $f(x) = T_n(x) + R_n(x)$, $\lim_{n\to\infty} R_n(x) = 0$, $|x-a| < R$, $f(x) = T_n(x)$ on interval $|x-a| < R$

↳ Taylor polynomial ↳ Remainder

Useful Expressions

→ $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all $x \in \mathbb{R}$

→ $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x

→ $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

CE1012 Engin Math II

Date No.

Signal & System

- How signals are applied
  - Def^n of ~~System~~
  - How signal detector function
    - Math Formulation in Sig. processing Theory
    - Sampling process in an dimensional signal

Signal

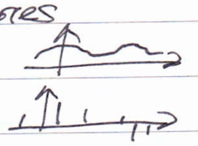
- Signal changes —— to convey information
  - Position, Orientation, Amplitude, etc.
  - e.g. Stock prices

System

- Processes Signals
- Extract useful inform^n & convert to other signals.
  - e.g. Detect signals in noise, MRZ ~~det~~ detector, Edge Detection

Mathematical Formulation ~~sig~~ Signal processing theory

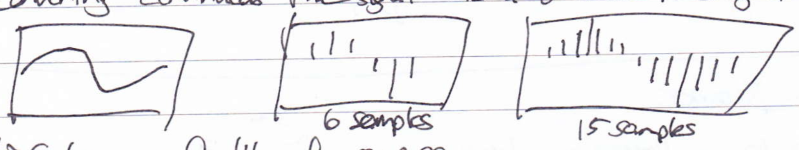
- Signals ~~are~~ in two categories
  - Continuous-time (analog)
  - Discrete time signal



Both can have a
→ Continuous amplitude
→ Discrete amplitude

Sampling

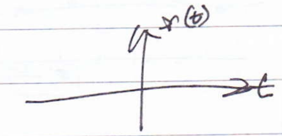
- Converting continuous-time signal into a discrete-time signal



- ~~Costs vs. Quality of samples~~
- Quality vs. Complexity.

Continuous Time Signals

- Function of var. time t, $x(t)$ that assumes all values $-\infty < t < \infty$



~~Transforma~~ Transformation *

- Transform $x(t)$ to $g(t)$
- Time Reversal → $x(-t)$
- Time Scaling → $x(at)$
- Time Shifting → $x(t-t_0)$
- Amplitude Transformation → $Ax(t)+B$

Date No.

Addition

$s = (2+j2) + (3-j1) = 5 + j1$

Multiplication

$s = (2+j2)(3-j1) = 6 - j2 + j6 - \cancel{j^2}\,2$

$= (6+2) + j(6-2) = 8 + j4$

Conjugate

$s^* = (a+jb)^* = a - jb$

$(s)(s^*) = (a+jb)(a-jb) = a^2+b^2$ [real]

Division

↳ Multiply by Conjugate

↳ $\frac{a+jb}{c+jd} = \frac{a+jb}{c+jd}\left(\frac{c-jd}{c-jd}\right) = \frac{ac+bd}{c^2+d^2} + j\,\frac{bc-ad}{c^2+d^2}$

Euler's Relation *

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

Power Series

Take Note

$e^{jy} = 1 + jy + \frac{(jy)^2}{2!} + \frac{(jy)^3}{3!} + \frac{(jy)^4}{4!} + \frac{(jy)^5}{5!} + \dots$

$= \underbrace{\left[1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right]}_{\cos y} + j\underbrace{\left[y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right]}_{\sin y}$

$e^{jy} = \cos y + j\sin y$ *

$e^{-jy} = \cos y - j\sin y$

$\cos y = \frac{e^{jy}+e^{-jy}}{2}$

$\sin y = \frac{e^{jy}-e^{-jy}}{2j}$

Euler's Relation for complex Numbers

Signals

↳ Periodic if $\boxed{x(t) = x(t+T)}$, $T > 0$ (in seconds), $x(t) = x(t+nT)$ for n is any integer (any integer)

↳ aperiodic → Not Periodic

↳ Fundamental period of Signal $= T_o$ → base value of T

f_o * $\boxed{f_o = \frac{1}{T_o},\quad \omega_o = 2\pi(f_o) = \frac{2\pi}{T_o}}$

freq. — f_o: 1/(Period/sec) Hz; fundamental freq — ω_o: rad/s

Periodic Signals

↳ T_o = seconds, f_o = (number of periods per sec) $= \frac{1}{T_o}$ Hz (not fundamental frequency)

↳ ω_o = fundamental freq $= 2\pi f_o$ (radians per sec) $= \frac{2\pi}{T_o}$ rad/s

Date No.

Approximating Periodic Functions

↳ Solving complex problems by dividing into simple parts
↳ Divide & Conquer

Signal Representation

↳ Representing all signals with a small set of signals
↳ Using Fourier Series & Fourier Transform

Example

$x(t)$ is periodic, if $x(t) = x(t+T)$ is satisfied for all t

$\cos w(t+T) = \cos(wt + wT) = \cos(wt + 2\pi) = \cos wt$

$\boxed{w = 2\pi = \frac{2\pi}{T}}$

Properties of Periodic functions

↳ $x(t) = x(t+T)$

↳ $x(t) = x(t+T) = x(t+nT)$

↳ $x(t) = x(t+T)$ when Fund. period T_0 is the min val of period $T>0$

$\boxed{w_0} = 2\pi f_0 = \boxed{\frac{2\pi}{T_0}}$

(rad/s) Hz sec

Determine Error

~~$B_1 \sin w t$~~

↳ $\underbrace{e(t)}_{\text{error function}} = x(t) - \underbrace{B_1 \sin w_0 t}_{\text{approx. signal repr.}}$

$\boxed{B_1 = \frac{2}{T_0}\int_0^{T_0} x(t) \sin w_0 t \, dt}$

Derive B_1 constant

Mean Square approximation

↳ $x(t) = \begin{cases} 1, & 0 < t < \frac{T_0}{2} \\ -1, & \frac{T_0}{2} < t < T_0 \end{cases}$

↳ $B_1 = \frac{2}{T_0}\int_0^{T_0} x(t) \sin w_0 t \, dt$ ← (Given)

$= \frac{2}{T_0}\int_0^{T_0/2} (1) \sin w_0 t \, dt + \frac{2}{T_0}\int_{\frac{T_0}{2}}^{T_0} (-1) \sin w_0 t \, dt$

$= \frac{2}{T_0}\left[\left[\frac{-\cos w_0 t}{w_0}\right]_0^{T_0/2} + \left[\frac{\cos w_0 t}{w_0}\right]_{T_0/2}^{T_0}\right]$

$= \frac{1}{\pi}(-\cos\pi + \cos 0 + \cos 2\pi - \cos\pi) = \boxed{\frac{4}{\pi}}$ ← Amplitude for B1 that makes the wave smaller

$w_0 t = \left(\frac{2\pi}{T_0}\right)\left(\frac{T_0}{2}\right) = \pi$, $w_0 T_0 = 2\pi$

Date No.

Fourier Series

↳ Representing signals with sinusoidal functions — a fundamental freq or harmonic

↳ $\boxed{x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}}$; $C_k = C^*_{-k}$ $k\omega_0$ is k^{th} harmonic

Exponential

$\boxed{C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\,dt}$ ← Coefficient

Example

$x(t) = 10 + 3\cos\omega_0 t + 5\cos(2\omega_0 t + 30°) + 4\sin 3\omega_0 t$

Through Euler's relation

$x(t) = 10 + \frac{3}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) + \frac{5}{2}(e^{j(2\omega_0 t + 30°)} + e^{-j(2\omega_0 t + 30°)})$
$+ \frac{4}{2j}(e^{j3\omega_0 t} - e^{-j3\omega_0 t})$

$= 10 + 1.5e^{j\omega_0 t} + 1.5e^{-j\omega_0 t} + (2.5e^{j\pi/6})e^{j2\omega_0 t} + (2.5e^{-j\pi/6})e^{-j2\omega_0 t}$
$+ (2e^{-j\pi/2})e^{j3\omega_0 t} + (2e^{j\pi/2})e^{-j3\omega_0 t}$ (Expand)

$= (2e^{j\pi/2})e^{-j3\omega_0 t} + (2.5e^{-j\pi/6})e^{-j2\omega_0 t} + 1.5e^{-j\omega_0 t}$
$+ 10 + 1.5e^{j\omega_0 t} + (2.5e^{j\pi/6})e^{j2\omega_0 t} + (2e^{-j\pi/2})e^{j3\omega_0 t}$ (Rearrange)

$\frac{4}{2j}[e^{j3\omega_0 t} - e^{-j3\omega_0 t}]$

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$\frac{4}{2j} = \frac{4j}{-2} = -2j$

$2j = 2(0) + 2j1$

$= 2\cos\frac{\pi}{2} + 2j\sin\frac{\pi}{2} = 2e^{j\frac{\pi}{2}}$

$-2j = 0 - 2j1$

where does the 2 come from

$= \cos\frac{\pi}{2} - 2j\sin\frac{\pi}{2} = 2e^{-j\frac{\pi}{2}}$

$-2j(e^{j3\omega_0 t} - e^{-j3\omega_0 t}) = 2e^{-j\frac{\pi}{2}}e^{j3\omega_0 t} + 2e^{j\frac{\pi}{2}}e^{-j3\omega_0 t}$

In compact form

$x(t) = C_{-3}e^{-j3\omega_0 t} + C_{-2}e^{-j2\omega_0 t} + C_{-1}e^{-j\omega_0 t} + C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t}$
$+ C_3 e^{j3\omega_0 t} = \sum_{k=-3}^{3} C_k e^{jk\omega_0 t}$

Date No.

Fourier Transform

$\rightarrow F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$

↳ Fourier Transform

$\rightarrow f(t)$

↳ Fourier Series