7th Sep 15 Chp4

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YOUR GENEROSITY, OUR GRATITUDE

Title
Important
Own Analogy
All Others

MH1812 Week 5 Lecture 1 Proof Techniques 1/1

What are proof techniques?

Proof Techniques
- ↳ Proof an argument to be valid.
- ↳ Conclusion follows from given assumptions { premise, premise } Hypothesis; ∴ conclusion

What are the different techniques in proof techniques?

Techniques of proof
- → Direct Prove
- → Prove by Induction → Mathematical Induction; → Complete Induction
- → Prove by Contradiction
- → Prove by Contrapositive

What is direct proof?

Direct Prove
→ Analyzing an equation and making logical assumptions that proves both equations.

Example
→ Prove $\forall n \in \mathbb{N}, \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ (Natural Numbers)

Define $\sum_{i=0}^{n} i = 0+1+\dots+n-1+n$ as S ①

Inverse $S = n+n-1+\dots+1+0$ ②

①+②: $2S = \underbrace{n+n+\dots+n+n}_{(n+1)\text{terms}}$ If $n=2$, $S = \underbrace{0,1,2}_{3\text{ terms}} = (n+1)$ terms

Rewritten as: $2S = n \times (n+1)$ Common Factor

$\therefore S = \frac{n(n+1)}{2}$

What is prove by induction?

Prove by Induction
- → Proving by assumptions that some terms are True and thus the overall argument is valid.
- → Consists of ~~two~~ 2 steps
  - → Basis Step : Show $P(1)$ is True
  - → Inductive Step: Assume $P(k)$ is True, prove $P(k+1)$ is True
- → It works as $P(k)$ & $P(k+1)$ will prove the next elements in the set to be True.

$[\underbrace{P(1)}_{\text{Basis Step}} \wedge \underbrace{\forall k(P(k) \rightarrow P(k+1))}_{\text{Inductive Step}}] \rightarrow \underbrace{\forall n\, P(n)}_{\text{Valid Argument}}$

Mathematical Induction
→ Same as Prove by Induction explanation

~~Ex~~ Example
→ Prove $\forall n \in \mathbb{N}, \sum_{i=0}^{n} \frac{n(n+1)}{2}$

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Proof Techniques Pg 2

What is proof by Contrapositive?

Proof by Contrapositive — Proof the Opposite

→ ~~If $P(n) \Rightarrow Q(n)$~~ is ~~then proof its contrapositive~~
~~$\neg Q(n) \Rightarrow \neg P(n)$~~

→ Proof $P(n) \Rightarrow Q(n)$ is true when its contrapositive
$\neg Q(n) \Rightarrow \neg P(n)$ is True

Example

→ Prove n^2 is even, then n is even

$P(n) =$ "n^2 is even", $Q(n) =$ "n is even"

$\neg Q(n) \Rightarrow \neg P(n)$

(~~then~~ If n is $\underbrace{\text{not even}}_{\text{odd}}$, then n^2 is $\underbrace{\text{not even}}_{\text{odd}}$

Then $n^2 = (2k+1)^2$ where $n = 2k+1$ is odd
$= 2(2k^2+2k)+1$ (odd)

$\therefore \neg Q(n) \Rightarrow \neg P(n)$ is True and thus
$P(n) \Rightarrow Q(n)$ is True.

Proof Techniques → Direct Proof
↳ Proof by Induction → Mathematical Induction
↳ Complete Induction
↳ Proof by Contradiction
↳ Proof by Counterpositive

What is Proof by Contradictions?

What are the assumptions made?

What do we need to prove first?

Question?

Proof by Contradiction

Proof the ~~Opposite~~ Inverse

→ Proof by assuming argument is False and it's Contradiction is True

→ The argument is valid when C = True is a Contradiction

Example

→ Prove $P(n) \to Q(n)$ is True

Assume by Contradiction, ~~$P(n)$~~ $\neg(P(n) \to Q(n))$ is True ①

It is True when $P(n)$ is True & $Q(n)$ is False

$$\neg(P(n) \to Q(n)) \to \underbrace{C}_{P(n)\ \text{Contra}} \wedge \underbrace{\neg C}_{Q(n)\ \text{Contra}}$$ by Direct Proof

$\therefore \neg(P(n) \to Q(n)) \to C \wedge \neg C$ is True by ①

Check $S(n) = $ "$P(n) \to Q(n)$" is equivalent by truth table ??

S	C	$\neg S$	$C \wedge \neg C$	$(\neg S) \to (C \wedge \neg C)$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F
Same				Same

$\therefore P(n) \to Q(n)$ is True

Example

~~Prove th~~ → Prove that n^2 is even, then n is even, for n integer

$P(n) = $ "n^2 is even", $Q(n) = $ "n is even"

~~PC~~ Prove $P(n) \to Q(n) = T$

Assuming $\neg(P(n) \to Q(n)) = T$

It is True when $P(n) = $ "n^2 is even" and $\neg Q(n) = $ "n is not even" or "n is odd"

Then $n^2 = (2k+1)^2$ where $n = 2k+1$ representing a odd number

$= 4k^2 + 4k + 1$

$= \underbrace{2(2k^2 + 2k)}_{\text{Even}} + \underbrace{1}_{\text{odd}}$

~~$\neg(P(n) \to Q(n)) \to C \wedge \neg C$~~

This forms a contradiction, $C \wedge \neg C$

$\neg(P(n) \to Q(n)) \to C \wedge \neg C$ is True

$\therefore P(n) \to Q(n)$ is True

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Chp 5

14th Sep 2015

MH812 Week 6 Lec 1 Combinatorics 1/~~2~~1

— Title
— Important
— Own Definition
— All Others

What is Principles of Counting?

Principle of Counting

↳ Total number of choices to fill ~~n~~ slots with n elements

choices = $n_1, n_2, \dots, n_r$

What are ways to determine cardinality of Power Set?

Cardinality of Power Set (Set Theory)

↳ Set of subsets of a set A

↳ $|P(A)| = |\{\emptyset, \{1\}, \{2\}, \{1,2\}\}| = 4$, where $A = \{1,2\}$

$$\left.\begin{aligned} a_0 &= \emptyset = 00 \\ a_1 &= \{1\} = 01 \\ a_2 &= \{2\} = 10 \\ a_3 &= \{1,2\} = 11 \end{aligned}\right\}$$ Binary vectors, 0,1 denotes membership

↳ Can be determined by 2^n, Binomial Theorem

Finding size of Power set

What is the formula for choices when elements cannot be repeated?

Elements cannot be repeated

↳ Once used, ~~the e~~ to fill a slot, element is taken out from the set.

↳ ~~for~~ 1st Slot : n choices
2nd Slot : n − 1 choices
last Slot : n − (r − 1) choices

eg. n = 5, r = 3, c = 5
c = ~~5~~ 4
c = ~~4~~ 3 = (5) − [(3) − 1]

choices = $n(n-1)(n-2)\cdots(n-(r-1))$

choosing people for sub-committee

What is Permutation?

Permutation

↳ Arrangement of a set with no repetition with respect to order.

$P(n,r) = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$

n: elements, r: slots

Handling Repetition

↳ Where elements repeats itself k times, for each element,

$P(n,n) = \frac{n!}{(k_1!\,k_2!\cdots k_n)!}$; eg. MISSISSIPPI

What is a Combination?

Combination $C(n,r)$ or $\binom{n}{r}$

↳ Same as Permutation except without order constraint

Matters when new elements comes into a set

$\{0,1\}$, $\{1,0\}$ → 1
$\{1,2\}$ → 1

$r!\,C(n,r) = P(n,r)$

$C(n,r) = \frac{n!}{r!(n-r)!}$

Principles of Counting → Permutation, Combination

Ternary = $\{0,1,2\}$

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Summary
→ Definition of Recurrence Relation
→ Methods of solving Recurrence Relation → Back-tracking
→ Characteristic Equation

21 Sep 15 Chp 6

— Title
— Important
— Own Understandings
— Other notes.

MH1812 Week 7 Elearning Linear Recurrence

What is Recurrence Relation?

~~Linear Recurrence~~

Recurrence Relation

↳ An equation that recursively defines a sequence accompanied with initial conditions

eg. Fibonacci Sequence

$f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0, f_1 = 1$

Useful for Sorting Algorithms like Divide & Conquer

What are the techniques to solve it?

Techniques to solve Recurrence Relation

→ Backtracking

→ Characteristic Equation

What is backtracking?

Backtracking

↳ Finding an explicit ~~formula~~ based on patterns → Does not

eg. $a_n = 2a_{n-1} - a_{n-2}$ with $a_1 = 3, a_0 = 0$

$= 3a_{n-2} - 2a_{n-3}$

$= 4a_{n-3} - 3a_{n-4}$ $(n-3)+4 = 0$

$\vdots$ $1 + (n-3+4-1) = 0$

$= (n)a_1 - (n-1)a_0$ $1 + (n) = 0$

$a_n = (n)a_1 - (n-1)a_0$ ~~So~~

$a_n = 3n$ Sub a_1 & a_0

What is characteristic equation?

Characteristic Equation

↳ Using equations to solve but firstly by some rules

↳ Must be Linear Homogeneous Relation of degree d.

↳ Fibonacci Sequence is

↳ $a_n = 2a_{n-1}$ is (degree 1) $a_n \Rightarrow x$, $x^1 = 2x^0$

↳ $a_n = 2a_{n-1} + 1$ is invalid

What is Linear Homogeneous Relation?

Linear Homogeneous Relation of degree d

↳ $a_n = c_1 a_{n-1} + c_2 a_{n-2} \Rightarrow \underbrace{x^2 = c_1 x^1 + c_2 x^0}_{\text{Characteristic Equation}}$ (Degree 2)

What are the formulas to determine the roots

~~Formula~~

Formula

↳ Finding roots of characteristic equation

$S_1, S_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

↳ If roots are distinct, the explicit formula is

$u \cdot S_1^n + v \cdot S_2^n$ → power n

↳ If roots are equal or single root, explicit formula is,

$u \cdot S_1^n + v \cdot n \cdot S_1^n$ where u, v determined by initial conditions

What is a Power Set?	**Power Set** ↳ All subsets of a given set S ↳ $P(S)$ ↳ $P(S) = \{A \mid A \subseteq S\}$ (subset of S) ↳ $P(S) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ when $S = \{1,2\}$ ↳ $\lvert S\rvert = n$, $\lvert P(S)\rvert = 2^n$, prove by Binomial Theorem Set that contains all subsets of S
Difference between Union & Intersection?	**Union** ~~& Intersection~~ ↳ Elements either in A or B ↳ $A \cup B \triangleq \{x \mid x \in A \vee x \in B\}$ **Intersection** ↳ Elements in A and B ↳ $A \cap B \triangleq \{x \mid x \in A \wedge x \in B\}$ Union is Or, Intersection is And
What is a disjoint set?	**Disjoint Sets** ↳ All elements in A does not ~~belong~~ belong to B. Vice versa ↳ $A \cap B = \emptyset$, $\lvert A \cap B\rvert = 0$ Sets differ from each other
What to take note in Cardinality of Union?	**Cardinality of Union** ↳ Number of elements in $A \cup B$ ↳ $\lvert A \cup B\rvert = \lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$ $\lvert A\rvert + \lvert B\rvert$ creates an extra $\lvert A \cap B\rvert$
What are the predicate logic of set difference & Complement?	**Set Difference & Complement** ↳ $A - B \triangleq \{x \mid x \in A \wedge x \notin B\}$ ↳ $\bar{A} = U - A \triangleq \{x \mid x \notin A\}$ Universe Set (Encompasses all sets)
What is a Cartesian Product and its cardinality?	**Cartesian Product** ↳ Set of ordered pair elements (a,b) ↳ $A \times B \triangleq \{(a,b) \mid a \in A \wedge b \in B\}$ Collection of Ordered elems. In General $A_1 \times A_2 \times \dots \times A_n \triangleq \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1,2,\dots,n\}$ $A = \{1,2\}$, $B\{x,y\}$ Order is important $A \times B = \{(1,x), (1,y), (2,x), (2,y)\}$ $B \times A = \{(x,1), (x,2), (y,1), (y,2)\}$ $\lvert A \times B\rvert = \lvert A\rvert \cdot \lvert B\rvert$
What is a Partition?	**Partition** ↳ Non empty sets that are mutually disjoint Next to each other but disjoint $A = A_1 \cup A_2 \cup A_3$ $A_i \cap A_j = \emptyset$

- Sets
  - Membership
  - Null Set
  - Equality
  - Subset
- Venn Diagram
- Set Operations
  - Union
  - Intersection
  - Complement
  - Difference
  - Cartesian Product
- Partition

Title
Very Important
Analogy / Own Understanding
Everything Else

Chp 7

5th Oct 2015

CA2
→ Proof Techniques
→ Combinatorics
→ Set Theory
→ Linear Recurrence

MH1812 Week 8 lec1 Set Theory #1/2

What is a set?	**Set** ↳ Collection of abstract objects → Defined by { } ↳ Distinct → Represents distinct elements / members ↳ Order not important Collection of distinct elements
How to specify a set?	**Specifying a set** ↳ Explicit → $A = \{2, 3\}$ ↳ Implicit → $A = \{x \mid x \text{ is a prime number}\}$ → Predicate Logic $\mid$ ← such that
What is Membership and Subset?	**Membership** ↳ An element (x) in set S ↳ $x \in S$ **Subset** ↳ A set A within Set B ↳ $A \subseteq B$ ↳ $\subsetneq$ strict inclusion but not equal ↳ $A \subseteq B \triangleq \forall x\,(x \in A \rightarrow x \in B)$
What are the differences between Membership & Subset?	**Membership vs. Subset** rock $\in S$ $R = \{\text{rock}\}, R \subseteq S$ where $S = \{\text{rock, paper, scissors}\}$ Single vs Plural
What is an empty set?	**Empty Set** ↳ No element in a set ↳ Referred also as null set ↳ $\emptyset \neq \{\emptyset\}$ $\emptyset$: empty set; $\{\emptyset\}$: set with empty set Set is empty
When is a set equal to another?	**Set Equality** – By Definition → Sets are equal if both have the same elements ↳ $A = B \triangleq \forall x\,(x \in A \leftrightarrow x \in B)$ $\{1,2,3\} = \{3,1,2\} = \{1,1,2,2,3,3\}$ Order not impt distinct elements in set Same elements in both sets
What is Cardinality	**Cardinality** ↳ Number of elements in a given set S ↳ $\lvert S \rvert$ ↳ If $\lvert S \rvert$ is finite, $\lvert S \rvert < \infty$, else, $\lvert S \rvert = \infty$ ↳ $\lvert \emptyset \rvert = 0$ Size of set

~~ists~~ when $x \in C$ and $x \notin A$, $(x \in C) \wedge (x \notin A) = (C - A)$
$= (B-A) \cup (C-A)$

$\therefore$ RHS $\subseteq$ LHS

With LHS $\subseteq$ RHS and RHS $\subseteq$ LHS, we can conclude LHS = RHS

What does Set Identities do?

Using Set Identities

→ Using laws to prove Set Identity.

Show that $(A-B)-(B-C) = A-B$

$(A-B)-(B-C) = (A \cap \overline{B}) \cap \overline{(B \cap \overline{C})}$

$= (A \cap \overline{B}) \cap (\overline{B} \cup C)$ De Morgan's Law

$= [(A \cap \overline{B}) \cap \overline{B}] \cup [(A \cap \overline{B}) \cap C]$ Distributive Law

$= [A \cap (\overline{B} \cap \overline{B})] \cup [A \cap (\overline{B} \cap C)]$ Associative & Indempotent Law

$= A \cap [\overline{B} \cup (\overline{B} \cap C)]$ Absorption

$= A \cap \overline{B} = \underline{A-B}$

What is membership tables like?

Membership Tables (Like Truth Table)

↳ "1" = membership (exists), "0" = non-membership (does not exist)

Prove $(A \cup B) - B = A - B$

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

→ Set Identities
→ Prove Set Identities
→ Each other subset
→ Using Set Identities
→ Membership Tables

— Title
— Own Notes
— Important
— All Others

8th October 2015

MH1812 Week 8 Lec 2 Set Theory 2/2

What is Set Identities for?

Set Identities
↳ Formulas similar to predicate logic
↳ Draw Venn Diagrams for clearer picture

What are the important laws?

~~Formulas~~ Laws
→ Identity $A \cup \emptyset = A$ $A \cap U = A$
→ Domination $A \cap \emptyset = \emptyset$ $A \cup U = U$
→ Idempotent $A \cup A = A \cap A = A$
→ Double Complement $\overline{\overline{A}} = A$
→ Commutative $A \cup B = B \cup A$, same for $\cap$
→ Associative $A \cup (B \cup C) = (A \cup B) \cup C$, same for $\cap$
→ Distributive $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
→ De Morgan $\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$
→ Absorption Bigger Set $A \cup (A \cap B) = A$ Smaller Set $A \cap (A \cup B) = A$ Bigger Set Smaller Set
* Similar to Identity & Domination Repectively
→ Set Difference $A - B = A \cap \overline{B}$
Similar to Predicate Logic laws

Proving Set Equality
→ Both sets are equal, $A \subseteq B$ and $B \subseteq A$

What are methods to prove set equality?

Methods to Prove
→ Show each set is a subset of the other (Each other's subset)
→ Apply set identities laws
→ Use Membership Table (Like Truth Table)

What is Each others subset?

Each others subset
→ Prove LHS $\subseteq$ RHS and RHS $\subseteq$ LHS by proving elements
For any $x \in$ LHS, $x \in (B-A)$ or $x \in (C-A)$ [or both] $B \cup C > B$, $x \in B \cup C$
when $x \in (B-A)$, $(x \in B) \wedge (x \notin A) = (x \in B \cup C) \wedge (x \notin A)$
element comparator $= (B \cup C) - A$
Show that $(B-A) \cup (C-A) = (B \cup C) - A$
when $x \in (C-A)$, $(x \in C) \wedge (x \notin A) = (x \in B \cup C) \wedge (x \notin A)$
$= (B \cup C) - A$

$\therefore$ LHS $\subseteq$ RHS

for any $x \in$ RHS, $x \in (B \cup C)$ and $x \notin A$
when $x \in B$ and $x \notin A$, $(x \in B) \wedge (x \notin A) = (B - A)$
$= (B-A) \cup (C-A)$
Exists in $(B-A)$ Smaller set; Exists in this bigger set

What is the Matrix repr of R^{-1}

R^{-1} Matrix Repr

R^{-1} = transpose of R

$a_i R b_j$ = true $\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & F & T \\ a_2 & T & F \end{array}$

$b_i R^{-1} a_j$ = True $\begin{array}{c|ccc} & a_1 & a_2 & a_3 \\ \hline b_1 & F & T & T \\ b_2 & T & F & F \end{array}$

flip rows & cols

What are Composition of relations?

Composition of Relations

$R = \cancel{A} \in A \times B$

$S = B \times C$

1st Rel^n 2nd Rel

$R \circ S = \{(a,c) \in A \times C \mid \exists b \in B, aRb \text{ and } bSc\}$

Example

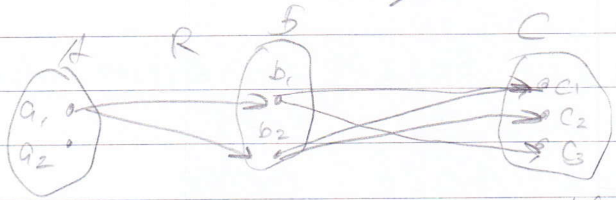
$A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$

$\cancel{R \in A \times B} = \{(a_1, \boxed{b_1}), (a_1, \boxed{b_2})\}$

$S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$

Ignored because sets are defined

$R \circ S = \{(a_1, c_1), (a_1, c_3), \cancel{(a_1, c_1)}, (a_1, c_2)\}$



$(a_1, c_2) \neq (a_1, b_1)(b_1, c_2)$ Not exist.

$= (a_1, b_2)(b_2, c_2)$

*At least one root

What is reflexivity? and its def^n?

Reflexivity

*Only defined within the set itself

$\forall x \in A, xRx$

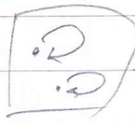
Matrix

$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & T & & \\ x_2 & & T & \\ x_3 & & & T \end{array}$ } Reflexivity

Examples

$xRx \iff x = x$

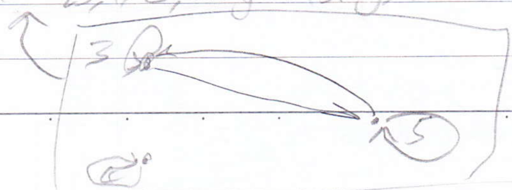
$A = \mathbb{Z}$, $xRy \iff x = y$: reflexive



→ self relation

$A = \mathbb{Z}$, $xRy \iff x > y$: not reflexive

$A = \{3, 4, 5\}$, $xRy \iff (x - y)$ is even : ~~not re~~ reflexive



Binary Relations
→ Relational Databases
→ Algorithms

Date No.

Relation btw 2 sets

Binary relation R from A ~~to~~ to B is a subset of $A \times B$ (Cartesian product) $R \subset A \times B$

Involving a pair of elements

Given (x,y) in $A \times B$ x is related to y by R $(xRy) \Leftrightarrow (x,y) \in R$

$A = \{1,2\}$, $B = \{1,2,3\}$, $(x,y) \in R \overset{iff}{\Leftrightarrow} (x-y)$ is even

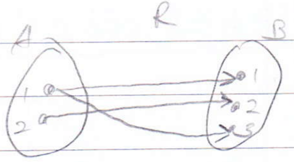
$A \times B = \{(1,1), \dots (2,3)\}$

$\underset{1-1=0}{(1,1) \in R}$, $\underset{1-3=-2}{(1,3) \in R}$, $\underset{2-2=0}{(2,2) \in R}$ $R = \{(1,1), (1,3) (2,2)\}$

~~$ARB \neq BRA$~~

$\boxed{xRy \neq yRx}$

Graphically



What is an inverse of a relation?

Inverse of Relation

R: Flipping the order of the pairs

$\boxed{R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}}$

such that

Example

$A = \{2,3,4\}$, $B = \{2,6,8\}$, $(x,y) \in R \Leftrightarrow x$ divides y

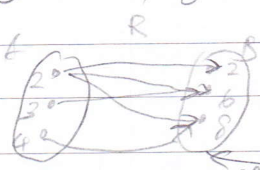
$A \times B = \{(2,2), (2,6), \dots (4,8)\}$

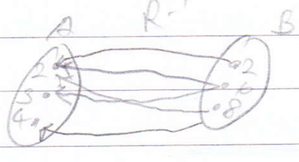
$(2,2) \in R$, $(2,6) \in R$ .. $(4,8) \in R$

flip

$(2,2) \in R^{-1}$ $(6,2) \in R^{-1}$ ... $(8,4) \in R^{-1}$

$(y,x) \in R^{-1}$, y is a multiple of x





Difference, arrows change direction.

What is a matrix representation?

Matrix Representation

$a_i = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3, b_4)$

$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$

$\underset{(rows)}{a_i} R \underset{(col)}{b_j}$ repr by T, F

$\boxed{\begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & F & T & F & F \\ a_2 & T & F & F & F \\ a_3 & T & F & F & T \end{array}}$

Example

$A = \{2,3,4\}$, $B = \{2,6,8\}$,

$(x,y) \in R \Leftrightarrow x$ divides y

$a_i R b_j$

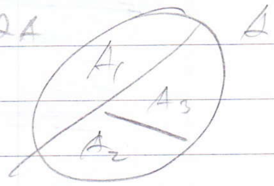
$\begin{array}{cc|ccc} & & & B & \\ & & 2 & 6 & 8 \\ \hline A & 2 & T & T & T \\ & 3 & F & T & F \\ & 4 & F & F & T \end{array}$

What is eq class?

Equivalence class

Equivalence classes of A form a partition of A

$A_i \cap A_j = \emptyset$ when $i \neq j$



Examples

$[a_i] \cap [a_j] = \begin{cases} [a_i] = [a_j] & \text{eg. } [3] = [5] \\ a_i \cap a_j = \emptyset & \text{eg. } [5] \cap [4] = \emptyset \end{cases}$

$a \underset{\text{relation}}{\equiv} b \pmod n \Leftrightarrow a = qn + b$

reflexive: $\forall x \in A \; x \equiv x \bmod n$ $\rightarrow x = \overset{0}{q}n + x$

Symmetric: $\forall x \forall y \; x \equiv y \pmod n \rightarrow y \equiv x \pmod n$ $\rightarrow x = 1, y = 10$: $1 \equiv 10 \bmod n$, $10 \equiv 1 \bmod n$

Transitive: $\forall x \forall y \forall z \; x \equiv y \pmod n \wedge y \equiv z \pmod n \rightarrow x \equiv z \pmod n$

$x = qn + y \qquad y = q'n + z$

diff

very imp.

$x = qn + y = qn + q'n + z = n(q + q') + z$

$\therefore$ Shows transitivity

$\therefore$ mod n $\rightarrow$ equivalence relation

Equivalence class of $[0] = \{0, n, 2n, 3n, \ldots, -n, -2n, -3n, \ldots\}$

" $[1] = \{1, n+1, 2n+1, \ldots, -n+1, -2n+1\}$

mod 4 $a = [0], [1], [2], [3]$

mod n represented by 0 to $n-1$

What is antisymmetry & its defn?

Anti Symmetry

$\rightarrow$ Not related to Symmetry

$\forall x \forall y \; xRy \wedge yRx \rightarrow x = y$

Examples

$(x = y) \wedge (y = x) \rightarrow x = y$

1. $A = \mathbb{Z}$ $xRy \Leftrightarrow x = y$: Antisymmetric
2. $A = \mathbb{Z}$, $xRy \Leftrightarrow x > y$: Vacuously True $\underbrace{(x > y) \wedge (y > x)}_{\text{Can never happen}} \rightarrow x = y$ $\therefore$ True
3. $BRC \Leftrightarrow B \subseteq C$: Antisymmetry

$(B \subseteq C) \wedge (C \subseteq B) \rightarrow B = C$

What is symmetry & its defn?

Symmetry

$(x,y) \in R$ implies $(y,x) \in R$ ※ Not necessary have arrow

$\forall x \forall y \; xRy \to yRx$ — If there is, there must be an arrow back.

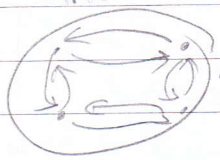
Examples

$A = \mathbb{Z}$ $xRy \Leftrightarrow x=y$: symmetric : $\forall x \forall y \; x=y \to y=x$

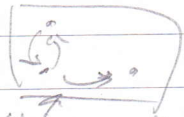
$A = \mathbb{Z}$ $xRy \Leftrightarrow x>y$: not sym. $\forall x \forall y \; x>y \to y>x$ → wrong

Twitter follows but doesn't follow back. Not sym

Facebook friends is mutual. Symmetry



every relation returns to itself symmetry



Not symmetric

2/3

What is transitivity? and its defn?

Transitivity

3 elems. 1st R 2nd, 2nd R 3rd implies 1st R 3rd

※ $\forall x \forall y \forall z \; xRy \wedge yRz \to xRz$

Examples

$A = \mathbb{Z}$, $xRy \Leftrightarrow x=y$: transitive → $\forall x \forall y \forall z \; xRy \wedge yRz \to x=z$; $x=y=z$

$A = \mathbb{Z}$, $xRy \Leftrightarrow x>y$: transitive

$\underset{\text{if true}}{x>y} \wedge \underset{\text{if true}}{y>z}$ thus $x>z$

What defines an Equivalence Relation?

Equivalence Relation

when R is reflexive, symmetric, transitive

Very similar relations

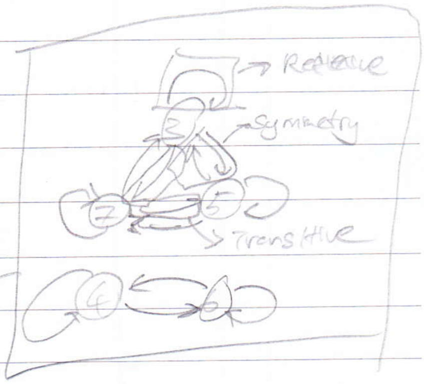
※ Equivalence class of a in A: $[a] = \{x \in A \mid aRx\}$

for R is a equivalence Relation

Example

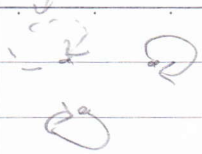
$A = \{3,4,5,6,7\}$ $xRy \Leftrightarrow (x+y)$ is even

R is Reflexive, symmetric, transitive



Equivalence class: $[3] = \{3,5,7\}$, $[4] = \{4,6\}$

$[5] = \{3,5,7\}$ ← no connection on graph closure relation



What is closure?

Closure

A relation off the leftover ordered pairs to satisfy all conditions

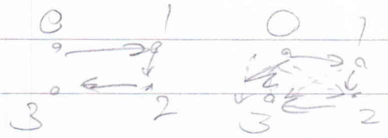
Transitive closure

1. R^+ is transitive
2. $R \subseteq R^+$
3. If S is ~~any~~ any other transitive closure $R^+ \subseteq S$

Example

$A = \{0, 1, 2, 3\}$

$R = \{(0,1), (1,2), (2,3)\}$



$R^+ = \{(0,1), (1,2), (2,3), (0,2), (1,3), (0,3)\}$

smallest set

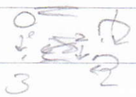
$(0,1)(1,2) \to (0,2)$
$(1,2)(2,3) \to (1,3)$
$(0,1)(1,3) \to (0,3)$

$\forall x,y,z \in A$ if $(xRy \land yRz \land \cancel{xRz})$ then add (x,z)

→ Repeat

→ ordering does not matter

S



$R^+ \subseteq S$

What are non-binary relations?

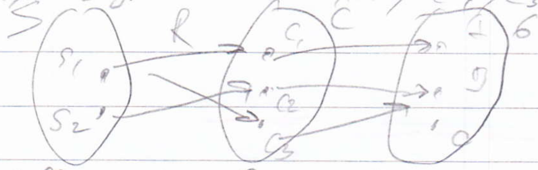
Non-binary relations

n-ary relation R subset of $A_1 \times \ldots \times A_n$, $a_1, \ldots, a_n$, $(a_1, \ldots, a_n) \in R$

Example

$S = \{S_1, S_2\}$ stud. $C = \{C_1, C_2, C_3\}$ courses

$G = \{A, B, C\}$ grades (S_1, C_1, A), (S_1, C_3, B), (S_2, C_2, B)



$R = \{(S_1, C_1, A) \ldots (S_2, C_2, B)\} \subseteq S \times C \times G$

What is compl. of a relation & its def^n?

Complement of a Relation → Relational Databases (through Set Theory)

Relational complement of R is $\overline{R}$

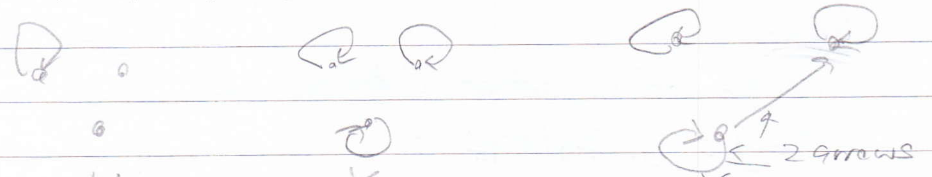
$(a_1, \ldots, a_n) \in \overline{R} \iff (a_1, \ldots, a_n) \notin R$, $\overline{R} = A \times B - R$

Example

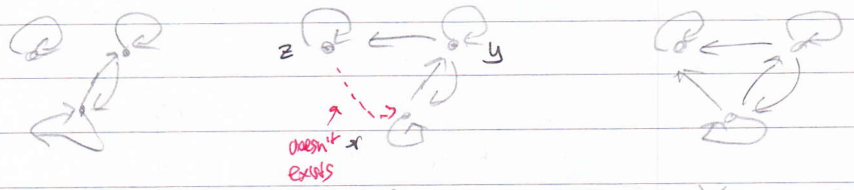
$A = \{1, 2\}$, $B = \{3, 5\}$ $R = \{(1,3), (2,5)\}$

$\overline{R} = (A \times B) - R = \{(1,5), (2,3)\}$

Date No.

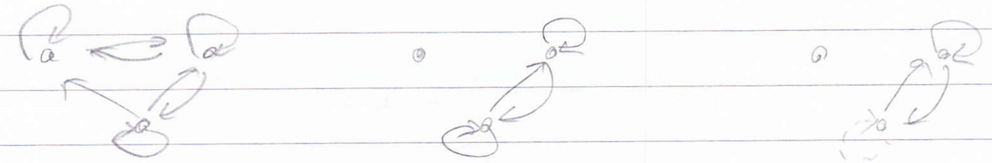


Reflexive	N	Y	Y
Symmetric	Y	Y	N
AntiSymmetric	Y	Y	Y
Transitive	Y	Y	Y



Reflexive	Y	Y	Y	→ Self Loop
Symmetric	Y	N	N	→ Both Direction Relation
Antisymmetric	N (Go & Back but not the same)	N	N	→ (Here & Back and must be same)
Transitive	Y	~~Y~~ N (No connection from $x \to z$)	Y	→ Relation btw $x \to y$, $y \to z$, $x \to z$

3.3



R	Y	N	N
S	~~Y~~ N	Y	Y
A	N	N	N
T	N	Y	~~Y~~ N (Because no self-loop)

What is Partial Order?

Partial Order

→ Reflexive, Antisymmetric, transitive

↳ Equivalence relation takes symmetry

→ For sorting, ~~algo~~ scheduling problems → order may be not obvious

Example

$A = \mathbb{Z}$, $xRy \Leftrightarrow x \le y$

reflexive : $\forall x \in \mathbb{Z}, x \le x$ Yes

antisymmetry : $\forall x \forall y, x \le y \land y \ge x \to x = y$ Yes

Transitive : $\forall x \forall y \forall z, x \le y \land y \le z \to x \le z$ Yes

Date No.

What are union and intersection of relations?

Union of Relations

→ $(a_1, \dots a_n) \in R \cup S \Leftrightarrow (a_1, \dots, a_n) \in R \vee (a_1, \dots a_n) \in S$

→ Normal Union

Intersection of Relations

→ $(a_1, \dots a_n) \in R \cap S \Leftrightarrow (a_1, \dots a_n) \in R \wedge (a_1, \dots, a_n) \in S$

→ Normal Intersection

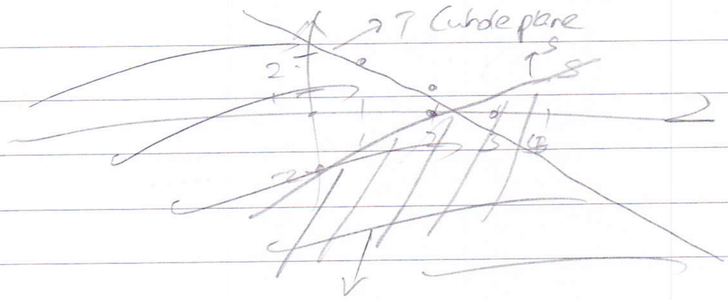
Example

such that

$T = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x+y \leq 3\}$

$S = \{(u,v) \in \mathbb{R} \times \mathbb{R} \mid u - v \leq 2\}$

$T \cap S = \{(x, y \in R \times R \mid (x+y \leq 3) \wedge (x-y \leq 2)\}$



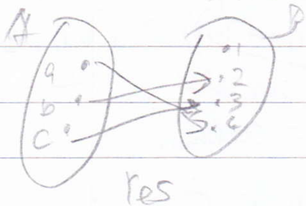
What is one-to-one & its defⁿ?

One-to-one (injective) function ← All elems in domain is mapped to different elems in co-domain

f is one-to-one ⟺ $f(x)=f(y)$ is $x=y$

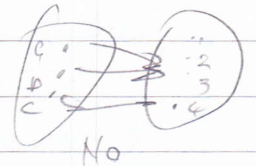
→ $f: A \to B$ is one-to-one ⟺ $\forall x_1, x_2 \in A\ (f(x_1) = f(x_2) \to x_1 = x_2)$

→ $f: A \to B$ is one-to-one ⟺ $\forall x_1, x_2 \in A\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$ ← Contrapositive



Yes

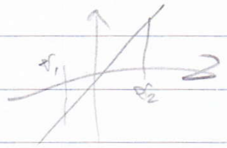
One-to-one



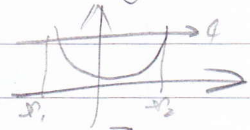
No

a & b has the same image

$f: \mathbb{R} \to \mathbb{R},\ f(x) = 4x-1$



$g: \mathbb{R} \to \mathbb{R},\ g(x) = x^2$



two diff points

Not Injective

$\forall x_1, x_2 \in \mathbb{R}\ f(x_1) = f(x_2) \to x_1 = x_2)$

Take some $x_1, x_2 \in \mathbb{R}$ with $f(x_1) = f(x_2)$

$4x_1 - 1 = 4x_2 - 1$, $4x_1 = 4x_2$, ∴ $x_1 = x_2$

∴ Yes, is one-to-one

Take $x_1 = 2$, $x_2 = -2$

$g(x_1) = 2^2 = 4 = g(x_2)$

and $x_1 \neq x_2$

$g(x_1) = g(x_2)$

$x_1^2 = x_2^2$

$x_1 = \pm\sqrt{x_2^2}$

$x_1 = \pm x_2$

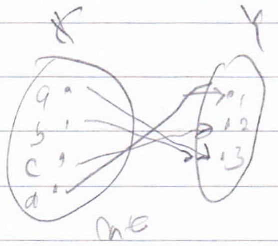
∴ No one to one

What is onto and its defⁿ?

Onto function (Surjective)

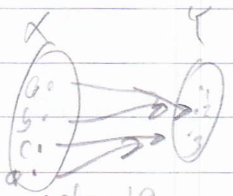
$f: X \to Y$ is onto ⟺ $\forall y \in Y\ \exists x \in X,\ f(x) = y$

Every element in codomain has a preimage



Onto

all $\forall y \in Y$ has preimage



not onto

1 has no pre-image

Chap 9

What are functions and its defn?	

Functions

X, Y are sets. function f from X to Y is a rule
→ assigns every element of x of X to a unique y in Y

$f: X \to Y$ and $f(x) = y$

$\underbrace{(\forall x \in X, \exists y \in Y, y = f(x))}_{\text{assigns every element of } x} \wedge \underbrace{(\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2}_{\text{unique } y}$

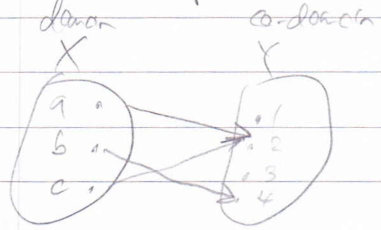
X = domain Y = codomain
Start

y = image of x under f

x = preimage of y under f

range = subset of Y with preimages

Example



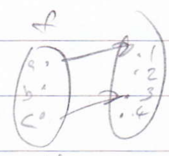
Domain $X = \{a, b, c\}$
Co-domain $Y = \{1, 2, 3, 4\}$
$f = \{(a,2), (b,4), (c,2)\}$
preimage of $\underbrace{2}_{y}$ is $\underbrace{\{a, c\}}_{x}$ → look at co-domain
range $= \{2, 4\}$ ← Belongs to preimage

Func from $\mathbb{Z}$ to $\mathbb{Z}$

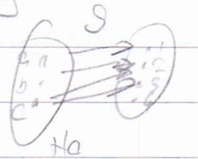
$f: \mathbb{Z} \to \mathbb{Z}, f(x) = x^2$

Domain & co-domain of f: $\mathbb{Z}$

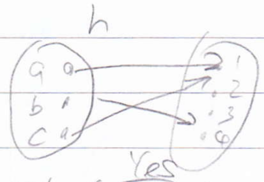
Range(f): $\underbrace{\{0, 1, 4, 9, 16, 25, \ldots\}}_{y}$, $\{0, 1, 2, 3, 4, 5\}$
$\downarrow$ 0 1 4 9 16 25



No
b ~~has~~ no image



No
C has two images



Yes
each elem in X has one image
Can $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

What is an image of a set?	

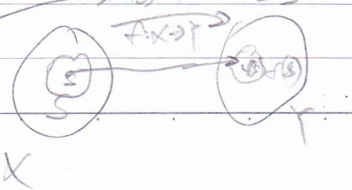
Image of a Set

→ f be ~~set~~ function from X to Y and $S \subseteq X$.

→ Image of S is a ~~set~~ subset of Y that consists images of elements of S

$f(S) = \{ ~~f(x)~~ f(s) \mid s \in S\}$

Another subset



What is an inverse function and its defⁿ?

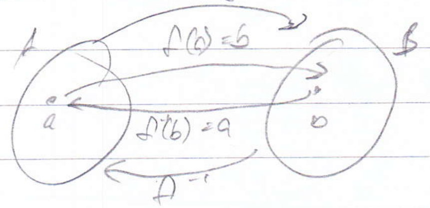
must be

Inverse function (Bijective)

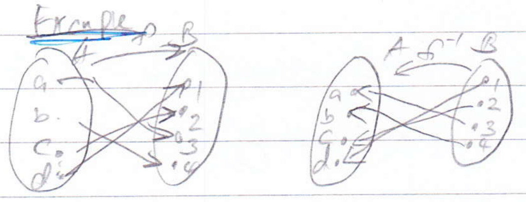
$f: A \to B$, inverse: $f^{-1}: B \to A$

$f^{-1}(b)$ = unique elem $c \in A$ such that $f(c) = b$

Thm: f is invertible



Example



Let $f: A \to B$ be bijective and $f^{-1}: B \to A$

~~Let~~ Then $\forall b \in B\ \forall a \in A\ (f^{-1}(b) = a \iff b = f(a))$

Inverse of $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 4x - 1$

Let $y \in \mathbb{R}$, find x s.t. $f(x) = y$

$y = 4x - 1 \iff x = \frac{y+1}{4}$

Hence $f^{-1}(y) = \frac{y+1}{4}$

Inverse of $g: \mathbb{R} \to \mathbb{R}$

$g(x) = x^2$

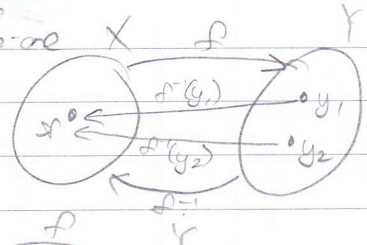
∴ No inverse because

$g(x)$ is not bijection

~~proof~~ how to prove inverse ~~func~~? func?

~~Direct Proofs~~ Must RMB ☆

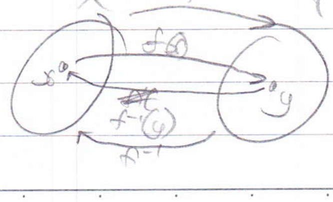
$f: X \to Y$ is onto one, ~~&~~ then $f^{-1}: Y \to X$ → Prove onto one-to-one



Proof one-to-one

$y_1, y_2 \in Y$ such that $f^{-1}(y_1) = f^{-1}(y_2) = x$

Then $f(x) = y_1$ and $f(x) = y_2$, thus $y_1 = y_2$



Proof onto

Take some $x \in X$, and let $y = f(x)$

then $f^{-1}(y) = x$

Onto Example

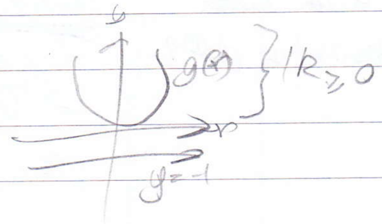
$g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$

Important to check where the elem is defined

$\exists y \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$ $g(x) \neq y$

Take $y = -1$

Then any $x \in \mathbb{R}$ holds $g(x) = x^2 \neq -1 = y$



But $g: \mathbb{R} \to \mathbb{R}_{\geq 0}$ is onto!

Summary 1/3

Definitions

→ Domain, co-domain → Injectivity (one-to-one)

→ Image, pre-image → Surjectivity (onto)

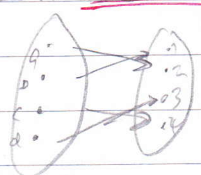
→ Range

Functions 2/3

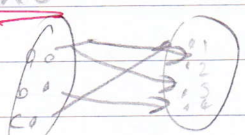
What is one-to-one correspondence?

One to one correspondence (Bijection) → Not one-to-one

↳ Both one-to-one & onto



Bijection



(Not a function, a has to 2 images)

What is Identity Function?

Identity Function

$i_A: A \to A$, $i_A(x) = x$

↳ Any identity function is a bijection

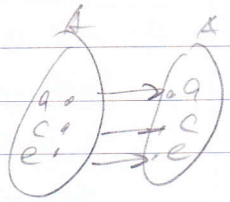
Example

$A = \{a, c, e\}$

Injective: $i_A(x_1) = i_A(x_2) \to x_1 = x_2$

Surjective: $y \in A$, $\exists x \in A$

$i_A(x) = y$



∴ Bijective

What is onto propagation?

Onto Propagation

Let $f: X \to Y$ & $g: Y \to Z$ be onto funcs.

Then $g \circ f$ is also onto

Proof: $\forall z \in Z, \exists x \in X$ such that $(g \circ f)(x) = z$

Let $z \in Z$

Since g is onto $\exists y \in Y$ with $g(y) = z$

Since f is onto $\exists x \in X$ with $f(x) = y$

Hence, with $(g \circ f)(x) = g(f(x)) = g(y) = z$

??

2/3

Functions → Bijectivity, Identity, Inverse

Composition of funcs & properties

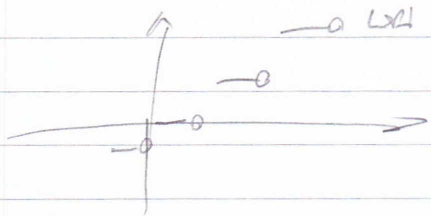
Functions 3/3

~~Ceiling~~ Floor → input $x \in \mathbb{R}$, output $\lfloor x \rfloor = \mathbb{Z}$

$\lfloor x \rfloor \leq x$

Ceiling → Top

$\lceil x \rceil \geq x$



What is pigeonhole principle?

Pigeonhole Principle

→ k pigeonholes, n pigeons $n > k$

↳ at least one pigeonhole ~~has~~ contains at least two pigeons

$\lceil 4/3 \rceil = 2$ (4 pigeons, 3 pigeonhole, at least)

↳ func from 1 finite set to a smaller finite set ≠ one-to-one

2 elems in domain has same image in co-domain.

What is countable sets and how to prove the set is countable?

Countable Sets

↦ though ∞ → diff. forms

Countable → finite set or $|\mathbb{Z}^+|$ is called countable

Uncountable → else

Example

Set of ~~all pos~~ odd $\mathbb{Z}^+$ is a countable set

1 2 3 4 5 ...

1 3 5 7 9 → func. ∃ one-to-one

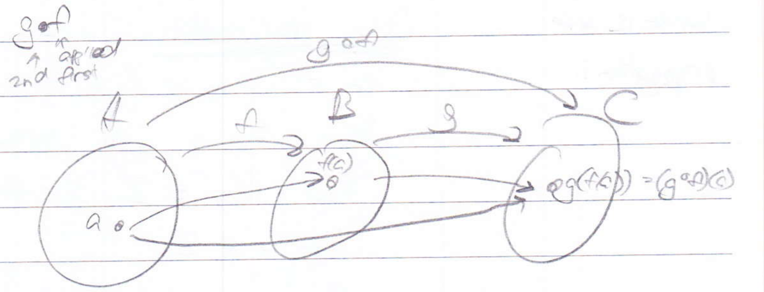
What are composition of functions?

Composition of functions

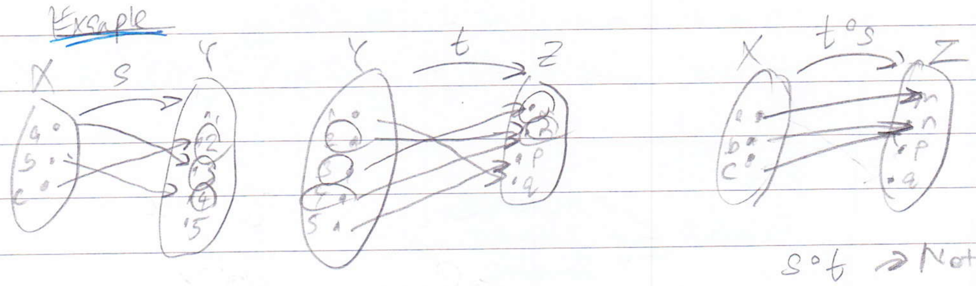
$f: A \to B$ & $g: B \to C$

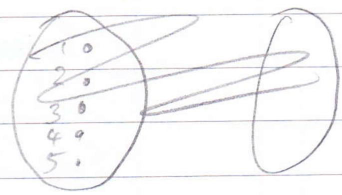
$g \circ f: A \to C$, $(g \circ f)(a) = g(f(a))$

$g \circ f$ — 2nd applied first



Example





$t: Y \to Z$

$s: X \to Y$ ✗ $Z \to \ldots$

$f: \mathbb{Z} \to \mathbb{Z}$ $f(n) = 2n+3$, $g: \mathbb{Z} \to \mathbb{Z}$ $g(n) = 3n+2$, $g \circ f$ ($f \circ g$)

$(f \circ g)(n) = f(g(n)) = f(3n+2) = 2(3n+2)+3 = 6n+7$

$(g \circ f)(n) = g(f(n)) = g(2n+3) = 3(2n+3)+2 = 6n+11$

$f \circ g \neq g \circ f$ ordering is very impt

What is one-to-one propagation?

One-to-one propagation (prove)

Let $f: X \to Y$ & $g: Y \to Z$ be both one-to-one func

Then $g \circ f$ is also one to one

How to prove one-to-one propagation?

Proof: $\forall x_1, x_2 \in X \; ((g \circ f)(x_1) = (g \circ f)(x_2)) \to x_1 = x_2$

Suppose $x_1, x_2 \in X$ w/ $(g \circ f)(x_1) = (g \circ f)(x_2)$

Then $g(f(x_1)) = g(f(x_2))$

Since g is one-to-one, $f(x_1) = f(x_2)$ ← Bring in assumptions

Since f is one-to-one, $x_1 = x_2$

→ one-to-one
onto

→ To show countable, find one-to-one correspondence (bijection) btw set & $\mathbb{Z}^+$ set

→ Cons $f(n) = 2n-1$ } odd integer

Prove one-to-one : suppose $f(n) = f(m)$, then $2n-1 = 2m-1$ and must be $n=m$

Prove onto : take m as odd pos int. Then m ~~is~~ $< 2k$, even integer (k is natural no.)

~~Thus k = ... ∈ ℤ~~

Thus $m = 2k-1 = f(k)$

What are uncountable sets?

Uncountable Sets → $\mathbb{R}$

↳ Proof by "Cantor diagonalization argument"

Cantor ~~Diagon~~ Diagonalization

→ Proof by Contradiction

Suppose $\mathbb{R}$ is countable

If $\mathbb{R}$ is countable, then set of ~~the~~ $0 < R < 1$ is also countable.
set of

Since one-to-one correspondence, we can label them

$r_1, r_2, r_3 \ldots$

In dec. repr. → Looking at the ~~dig~~ diagonal.

$r_1 = 0.d_{11} d_{12} d_{13}$

$r_2 = 0.d_{21} d_{22} d_{23}$ } d_{ij} belongs to $\{0, 1, 2 \ldots 9\}$

$r_3 = 0.d_{31} d_{32} d_{33}$

d_{11}
$r_1 = 0.134721 \ldots$

A new real number r with dec expansion

$r = 0.d_1 d_2 d_3 \ldots$

$r_2 = 0.1473$
d_{22}

} $r = 0.45$

where d_i is 5 if $d_{ii} = 4$ and 4 otherwise

$d_1 = 5$ if $d_{11} = 4$

$d_1 = 4$ if $d_{11} \neq 4$

This r is diff from all other real no. in [0, 1]

all r_i position is going to be different

$d_2 = 5$ if $d_{22} = 4$

$d_2 = 4$ if $d_{22} \neq 4$

∴ A contradiction since r does not belong

→ ceiling & floor 3/3

→ Pigeonhole principle

→ Countable set. * Show that ... is countable

↳ Uncountable, use the proof $\mathbb{R}$ is uncountable

~~Bridges of Königsberg~~

What are Euler Path & Euler Circuit? what are the differences?

Euler Path (Eulerian Trail) If have exactly 2 vertices of odd degree.

~~Walk on edges~~

→ use each edge of graph visited exactly once } Beginning end of walk must be on same vertex

Euler Circuit (Eulerian Cycle)

Similar to Euler path } end w/ on the same start vertex

degree = 3

Example

→ Suppose beginning & end are the same node i.e.

→ Graph must be connected.

→ Every vertex node $v \neq u$, they must return

thus (degree of v) is even.

→ thus odd degrees, no solution

Euler Theorem

① If Euler ~~path~~ Circuit, starts & ends same node, all nodes have an even degree

② Euler path, two nodes have an odd degree

Definitions of Vertex, ~~Edge~~, Adjacent & Incident

Simple, ~~Multi~~ graph, Multigraph, directed (multi) graph

Subgraph

Euler path & Euler Theorem

Graph Theory 2/3

Example (wolf, Goat, Cabbage) → Shows how graphs are useful. keeps all stages.

→ f = ferryman, g = goat, w = wolf, c = cabbage

(wgcf/ -) → (wc/gf) → (wcf/g) → (w/cfg) → (wgf/c) → (g/wfc)

(wcf/g) → (c/wfg) → (cfg/w) → (g/wfc)

left right

(g/wfc) ↓ (gf/wc) ↓ (-/wcfg)

All possible steps

Chp 10

Graph Theory ⇐ Networks, Distributed systems, Coding Theory

Date No.

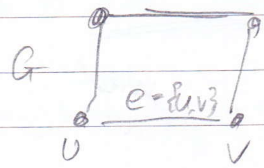
What are graphs?

Graph

$G = (V, E)$ structure consisting set V of vertices (nodes) and set E of edges (lines joining vertices)

sets

✱ No arrows



Adjacent → u and v

in G if {u, v} is an edge

Incident

$e = \{u, v\}$, edge e is called incident with u & v

What are the different types of graphs?

What are subgraphs?

Subgraphs

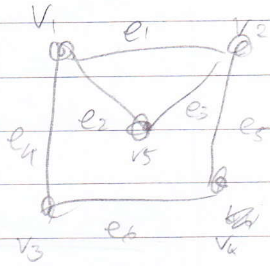
Like subsets

$H = (V_H, E_H)$ is subgraph of $G = (V_G, E_G)$

$V_H \subset V_G$, $E_H \subset E_G$

$V_H = \{v_1, v_2, v_5\}$ is a subset of V_G

$E_H = \{e_1, e_2, e_3\}$ is a subset of E_G



What are simple graphs?

Simple Graphs

→ Has no loop (edge {u, v} with u = v)

→ No parallel edges o═o X

What are multigraphs?

Multigraphs

→ No loop

→ At least 2 parallel edges

✱ X Directed

Undirected graph

What are directed (multi) graphs?

Directed (Multi)graphs

→ are ordered. {u, v} ~~≠~~ (u, v)

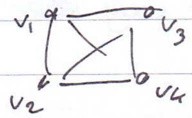
→ Parallel edges & loops — No Directed graph — has order

✱ Directed ~~graph~~ graph is a Simple Graph

A'ZONE

How does an adjacency matrix look like?	**Adjacency matrix of a complete graph**

$$A = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 1 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 1 & 1 & 0 \end{array}$$



Complete, Bipartite graph
Handshaking Theorem

Graph Theory 3/3

Hamiltonian Circuit (Cycle)

- ↳ Flattening of 3-D object to 2-D
- ↳ Every vertex is visited once → Not needed to go through every edge
- ↳ Starts & ends at the same vertex

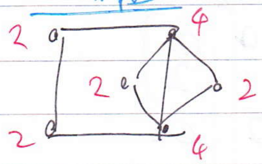
Hamiltonian Path (Trail)

- ↳ ~~Every~~ Same as Circuit
- ↳ Not needed to be start & end on same vertex

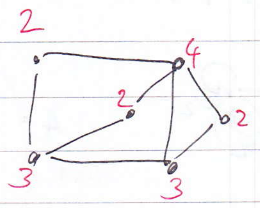
what are differences between Hamilton vs Eulerian?	**Hamiltonian vs. Eulerian**

Hamiltonian	Eulerian
↳ Walk through every Vertice	↳ Walk through every ~~edge~~ edge
↳ No Direct Proof ↳ Proof by finding a path ↳ No path doesn't determine it does not exist.	↳ Euler Theorem

Examples



Euler Circuit
Hamiltonian Path



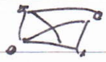
No Euler circuit, but Euler Path ↗ Exactly 2 odd edge(v.)

Date No.

What are Complete Graph?

Complete Graph

↳ every vertex connected to every other vertex (simple graph)

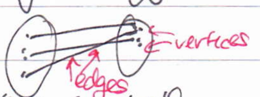


What are bipartite graphs?

Bipartite Graphs - they be drawn differently.

↳ Partitioned into 2 (disjoint) subsets V & W sets

↳ each edge only connects $v \in V$ & $w \in W$



* No connections in itself.

What is node degree?

Node Degree

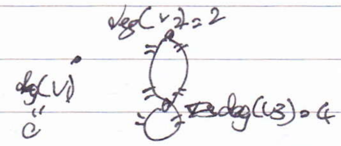
↳ $\deg(v)$

↳ ~~≠~~ undirected / directed

↳ ~~Directed~~

In-degree & out-degree

edges coming in ↑ edges going out ↓



Total Degree

$$\sum_{v \in V} \deg(v) = \deg(v_1) + \deg(v_2) + \deg(v_3) + \dots$$

What is handshaking theorem?

Handshaking Theorem

↳ Relations between Edges & Vertices

$$2e = \sum_{v \in V} \deg(v)$$

$$2|E|$$

If $e \in E(G)$ has endpoints $v, w \in V$. e contributes 1 from $\deg(v)$ & $\deg(w)$. Then $2|E|$

What is an adjacency matrix?

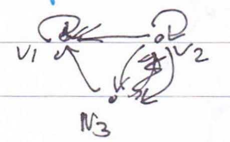
Adjacency Matrix

↳ Graph represented as a matrix $A = (a_{ij})$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

↳ a_{ij} = the number of arrows from v_i to v_j

↳ For simple graph, there is a symmetry end

Example



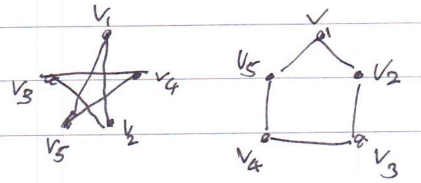
$$A = \begin{array}{c|ccc} \text{start} \backslash \text{to} & v_1 & v_2 & v_3 \\ \hline v_1 & 1 & 0 & 0 \\ v_2 & 1 & 1 & 2 \\ v_3 & 1 & 0 & 0 \end{array}$$

Date No.

What are graph isomorphism?

Graph Isomorphism

↳ Same graph with different pictorial representation



Isomorphism Test ~~→ Same graph~~

↳ Graph $G = (V_G, E_G)$ is Isomorphic to another graph $H = (V_H, E_H) \Leftrightarrow$ 2 bijection mapping ~~vertex & edge sets~~ one to one correspondence

$g: V_G \to V_H$ $\quad h: E_G \to E_H$

<Copy Def^n> edge $e \in E_G$ is incident on $v, w \in V_G$

$\Leftrightarrow$ edge $h(e) \in E_H$ is incident on $g(v), g(w) \in V_H$

$$
\begin{array}{c|ccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 \\
\hline
v_1 & & & & & \\
v_2 & & & & & \\
v_3 & & & & & \\
v_4 & & & & & \\
v_5 & & & & &
\end{array}
\left[\begin{array}{ccccc}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{array}\right]
$$

Example $\quad v \overset{e}{\text{———}} w \quad g(v) \overset{h(e)}{\text{———}} g(w)$

$g = \{(v_1, w_2), (v_2, w_3), (v_3, w_1), (v_4, w_5), (v_5, w_4)\}$

$h = \{(e_1, f_3), (e_2, f_2), (e_3, f_1), (e_4, f_7), (e_5, f_6), (e_6, f_5), (e_7, f_4)\}$

<Copy Graph>

Exam Tips

↳ If to be proven, means no disprove is needed unless prove or disprove

↳ Write what you need to prove.
Use Assumptions → They're to be used.

↳ To describe distinct elements, use distinct symbols
↳ e.g. $x = 2k+1$, $y = 2k+1$
2 unique Numbers

↳ know your symbols
↳ $A \times B = \{(a,b), a \in A, b \in B\}$ ← Cartesian Product
$\{0,1\} \times \{2,3\} = \{(0,2), (0,3), (1,2), (1,3)\}$