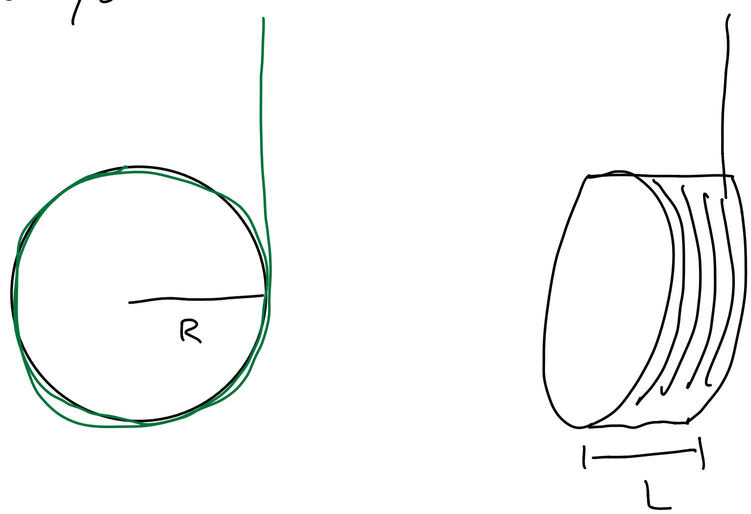
Ex: A yo-yo



How fast is the yo-yo spinning when it is unspooled?

(60 cm)

Idea:

Gravity exerts a torque on the yo-yo which increases $\vec{L}$

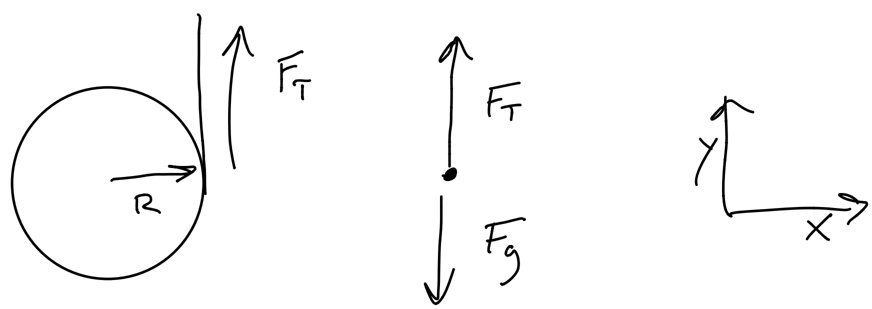
Find $\vec{\tau}$

Find Δt (time to unspool)

then $\vec{L} = \vec{\tau}\,\Delta t$

What force exerts the torque?

String tension



Momentum principle:

$$\frac{dp_y}{dt} = F_T - F_g = F_T - mg$$

$$\frac{dp_y}{dt} = -ma = F_T - mg$$

$$F_T = mg - ma$$

Angular momentum principle

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{r} = R\,\hat{x}$$

$$\vec{F} = F_T\,\hat{y}$$

$$\vec{\tau} = R F_T\,\hat{z}$$

$$\vec{\tau} = \frac{d}{dt}\vec{L} = \frac{d}{dt}\left(I\vec{\omega}\right)$$

$$= I\,\frac{d\omega}{dt}\,\hat{z}$$

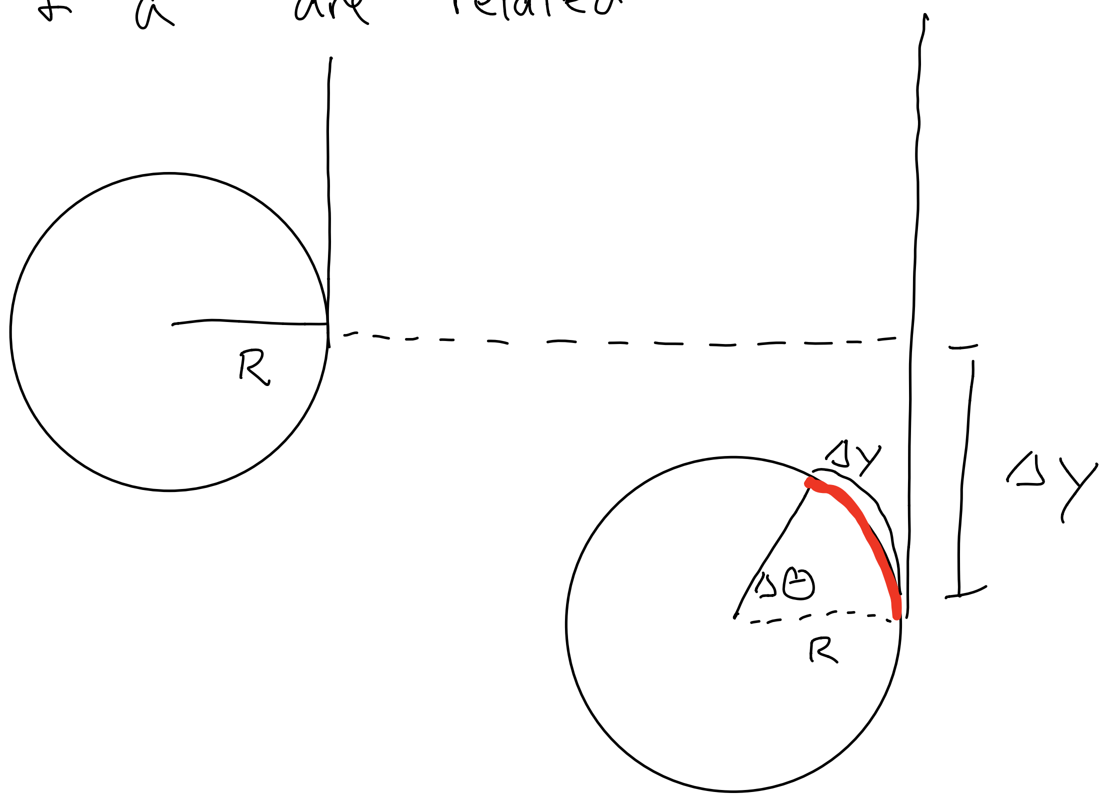
$$I\,\frac{d\omega}{dt} \equiv I\alpha = R F_T$$

$$F_T = \frac{I}{R}\alpha$$

$F_T = mg - ma$

$F_T = \frac{I}{R} \alpha$

α + a are related



$R\Delta\theta = \Delta y$

$R\Delta\theta = v\Delta t$

$R\frac{\Delta\theta}{\Delta t} = v$

$v = R\omega$

$\boxed{a = R\alpha}$

$\frac{dv}{dt} = R\frac{d\omega}{dt} = R\alpha$

$$F_T = mg - ma$$

$$F_T = \frac{I}{R}\alpha$$

$$a = R\alpha$$

so:

$$F_T = \frac{I}{R}\left(\frac{a}{R}\right) = \frac{I}{R^2}a$$

$$F_T = mg - ma = \frac{I}{R^2}a$$

$$a = \frac{mgR^2}{I + mR^2}$$

$$a = g\left(\frac{1}{1 + \frac{I}{mR^2}}\right)$$

constant accel

$a \leq g$, $a = g$ if $I = 0$

How fast is the yo-yo spinning after 60 cm of string unspool?

$$\frac{dL}{dt} = \tau$$

$$I \frac{d\omega}{dt} = \tau$$

$$\frac{d\omega}{dt} = \frac{\tau}{I} \qquad \left(\frac{dv}{dt} = \frac{F}{m} \right)$$

$$\omega = \omega_i + \frac{\tau}{I} t \qquad \left(v = v_i + \frac{F}{m} t \right)$$

$$\theta = \theta_i + \omega_i t + \frac{1}{2}\frac{\tau}{I} t^2 \qquad \left(y = y_i + v_i t + \frac{1}{2}\frac{F}{m} t^2 \right)$$

$$\Delta\theta = \omega_i t + \frac{1}{2} \frac{\tau}{I} t^2$$

$$\omega_i = 0$$

$$\Delta\theta = \frac{1}{2} \frac{\tau}{I} t^2$$

$$R\,\Delta\theta = \Delta y \implies \Delta\theta = \frac{\Delta y}{R}$$

$$\Delta\theta = \frac{\Delta y}{R}$$

$$\tau = R F_T = R \frac{I}{R^2} a = \frac{I}{R} a \, ; \quad a = g\left(\frac{1}{1 + \frac{I}{mR^2}}\right)$$

$$\frac{\Delta y}{R} = \frac{1}{2} \frac{a}{R} t^2$$

$$\Delta y = \frac{1}{2} a t^2 \quad (!)$$

$$t^2 = \frac{2\Delta y}{a}$$

$$t = \sqrt{\frac{2\Delta y}{a}}$$

$$\omega = \omega_i + \frac{\tau}{I} t$$

$$\omega = \frac{a}{R}\left(\frac{2\Delta y}{a}\right)^{\frac{1}{2}}$$

$$\omega = \left(2a \frac{\Delta y}{R^2}\right)^{\frac{1}{2}}$$

$$a = g\left(\frac{1}{1 + \frac{I}{mR^2}}\right)$$

$$\omega = \sqrt{g\left(\frac{1}{1 + \frac{I}{mR^2}}\right) \cdot \frac{2\Delta y}{R^2}}$$

$$I = \frac{1}{2} m R^2$$

$$\omega = \sqrt{\frac{4}{3} \frac{g \Delta y}{R^2}}$$