Last lecture:

$$K_{rot} = \frac{1}{2} I \omega^2$$

I: moment of inertia $= (m_1 r_{\perp 1}^2 + m_2 r_{\perp 2}^2 + \ldots)$

ω: angular velocity $\left(\frac{\text{rad}}{\text{sec}}\right)$

$\frac{1}{2}$: $1 \div 2$

We found I for a bicycle wheel: $I = MR^2$

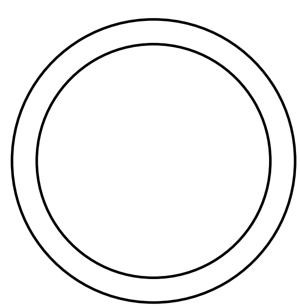
Our ultimate goal: Use energy principle to predict the final velocity of a rolling ball

- Several steps to this, but the first is calculating I

- This is a pretty difficult calculation for a sphere, so let's do some easier shapes first

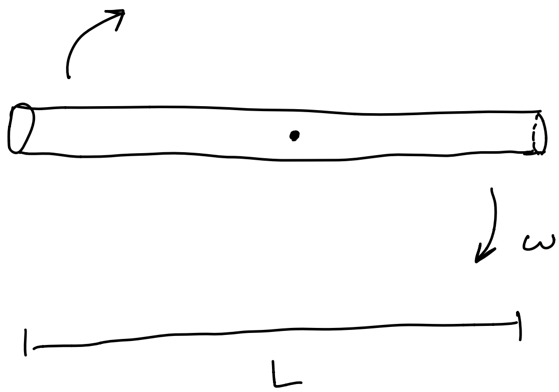
We already did the wheel

This was simple because $r_\perp$ is the same for every point on the rim



$$I = (m_1 R^2 + m_2 R^2 + \ldots) = MR^2$$

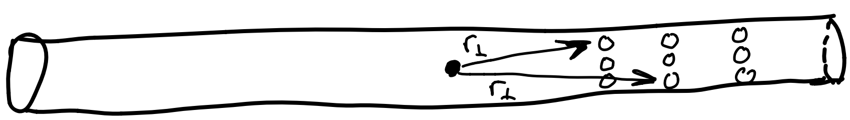
Let's look at a different shape



Thin rod of mass M

What we need to do:

Sum over every piece of mass in the rod + find $\sum m_i r_\perp$

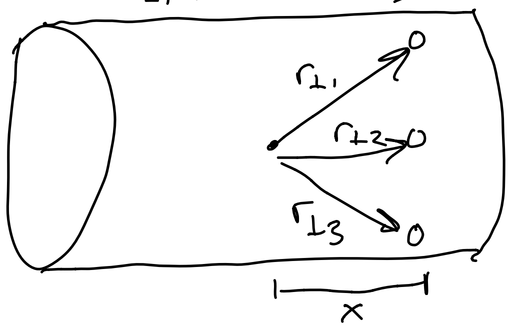


unlike the wheel, $r_\perp$ is different for every piece

Assumption:

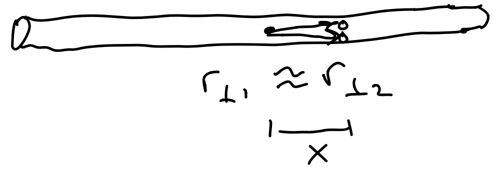
The rod is very thin $(R << L)$

$r_{\perp 1} \neq r_{\perp 2} \neq r_{\perp 3}$

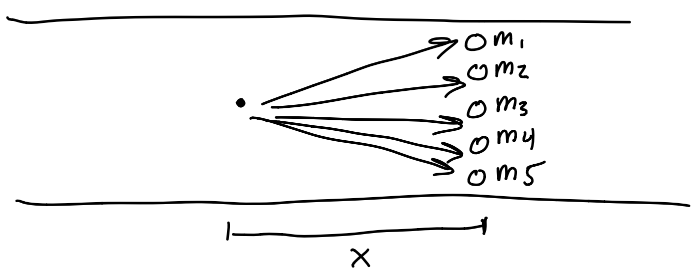


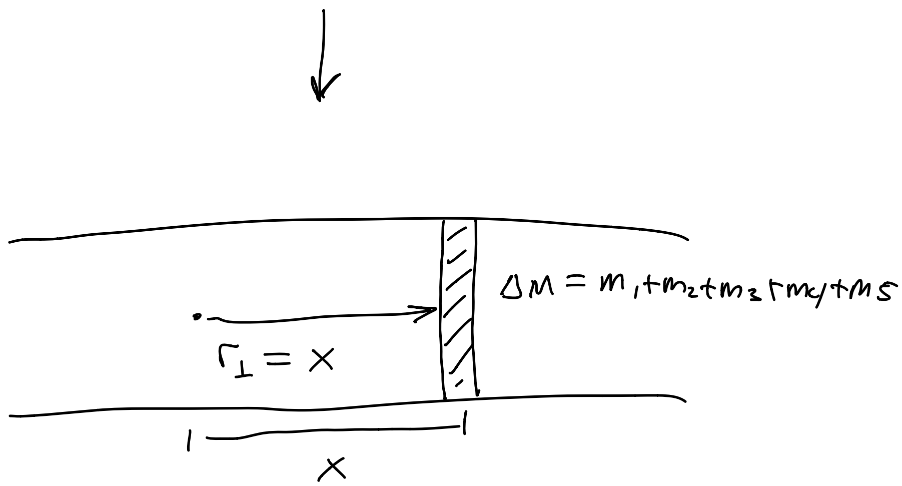
Approximation

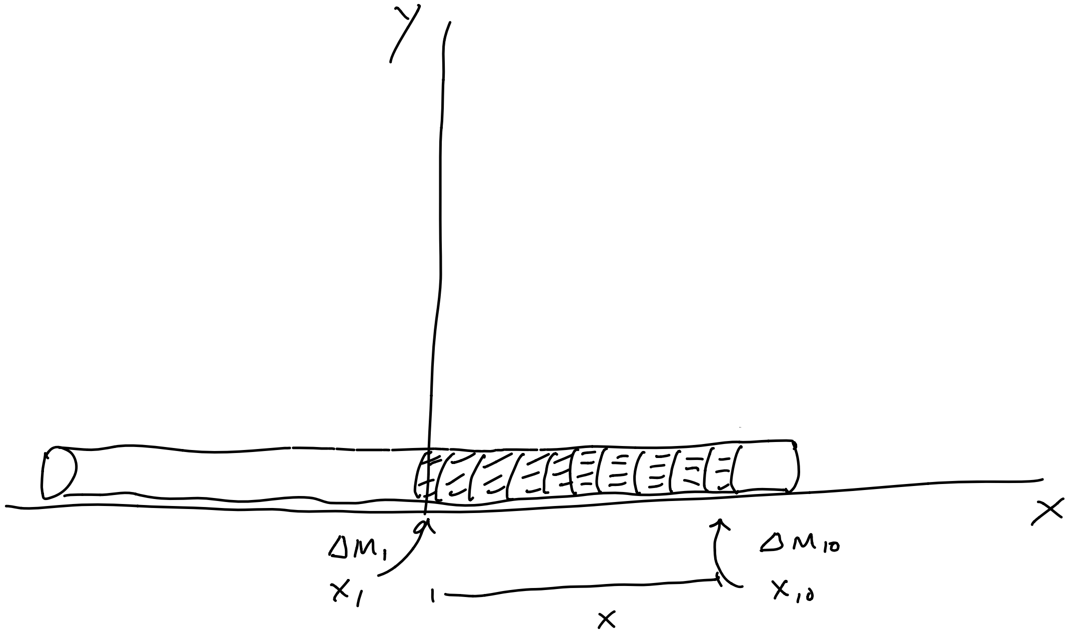
Horizontal distance from center of rod $x \approx r_\perp$



This means we can lump all molecules that are a distance "X" from the center into a single piece





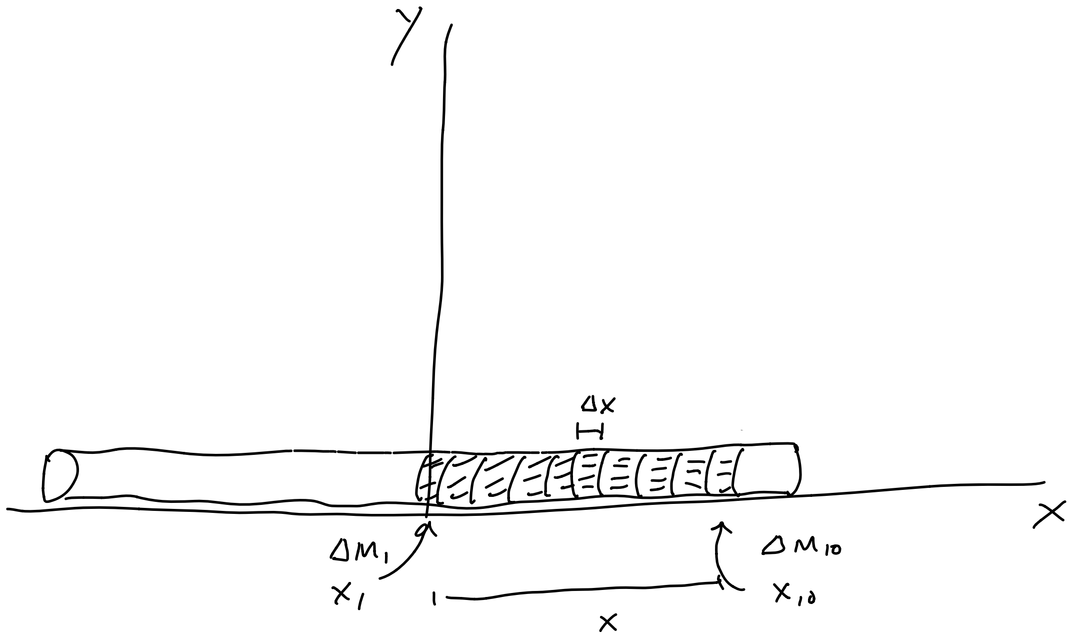


$$I = \Delta M_1 x_1^2 + \Delta M_2 x_2^2 + \Delta M_3 x_3^2 + \cdots$$

Technically, each vertical strip is just a stack of molecules & there are almost ∞ many sections!

The diameter of a single molecule is very small compared to L!

Approx: divide the rod into some n number of equal width vertical sections



To simplify things further, let's say that each vertical section has the same mass

Number of slices $= n$

Width of slice $= \Delta x$

Mass of slice $= \Delta M$

But the mass of the whole rod is M,

and $\Delta M \cdot n = M$, so $\Delta M = \frac{M}{n}$

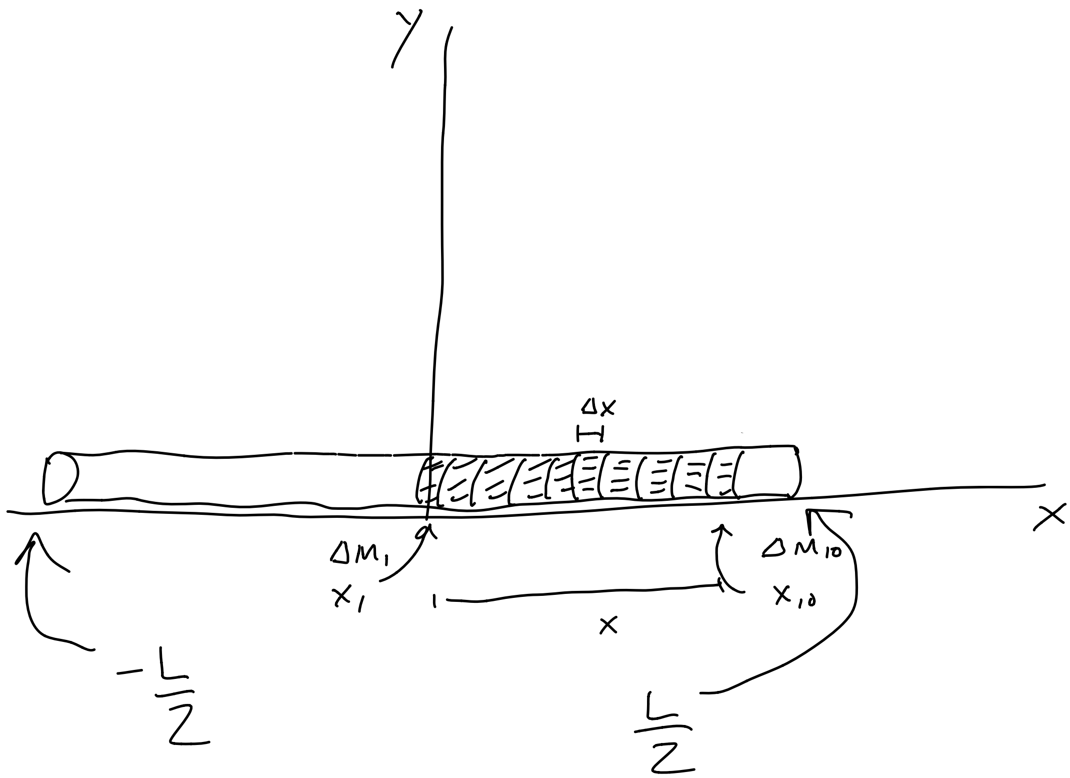
But $n = \frac{L}{\Delta x}$, so $\Delta M = M \frac{\Delta x}{L}$

So $I = \Delta M_1 x_1^2 + \Delta M_2 x_2^2 + \Delta M_3 x_3^2 + \dots$

$$= \frac{M \Delta x}{L}\left(x_1^2 + x_2^2 + x_3^2 + \dots\right)$$

$$I = \frac{M}{L} \sum_{i=0}^{n} x_i^2 \Delta x$$

$$\lim_{\substack{n \to \infty \\ \Delta x \to 0}} \frac{M}{L} \sum_{i=0}^{n} x_i^2 \Delta x = \frac{M}{L} \int_{x_i}^{x_f} x^2 \, dx$$



$$I = \frac{M}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \, dx = \frac{1}{3}\frac{M}{L}\left(\frac{L^3}{8} - \frac{(-L)^3}{8}\right)$$

$$= \frac{1}{24} \frac{M}{L} (2L^3) = \frac{ML^2}{12}$$

$$I = \frac{1}{12} ML^2$$

So K_{rot} for a rod rotating around its center is

$$K_{rot} = \frac{1}{2}\left(\frac{1}{12}ML^2\right)\omega^2$$
$$= \frac{1}{24}ML^2\omega^2$$

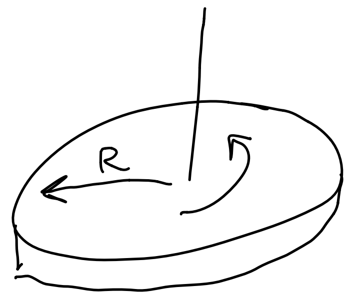
In general, calculating I requires setting up & evaluating an integral

- For most common shapes, this requires a 2D or 3D integral, which you haven't learned yet

$\downarrow$

Common moments of inertia

Disk:



$$I = \frac{1}{2} M R^2$$

Sphere (solid):

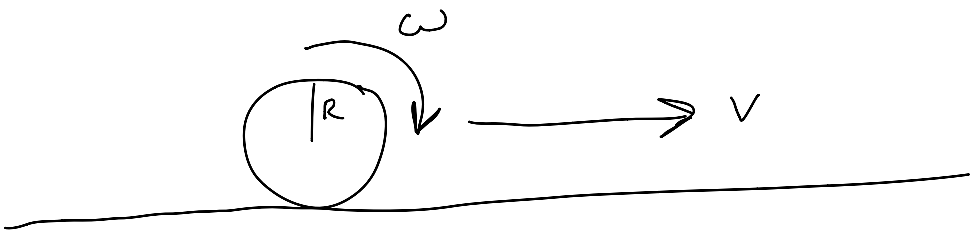
$$I = \frac{2}{5} M R^2$$

Sphere (hollow):

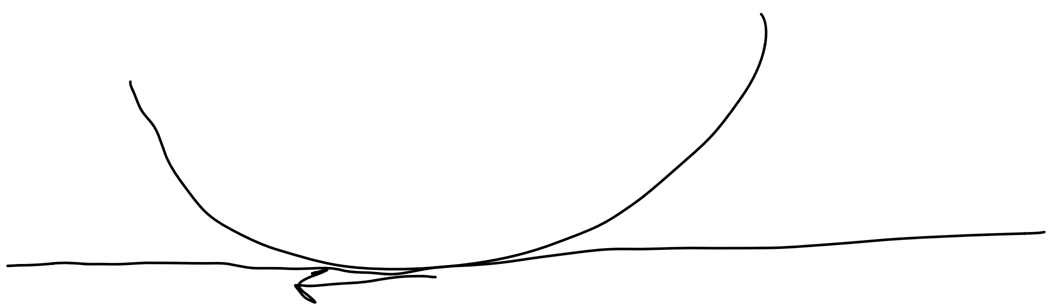
$$I = \frac{2}{3} M R^2$$

Note: For 2 spheres of the same mass, "I" is greater for the hollow sphere, since the mass is concentrated farther out

Ex: Rolling ball (R, M)



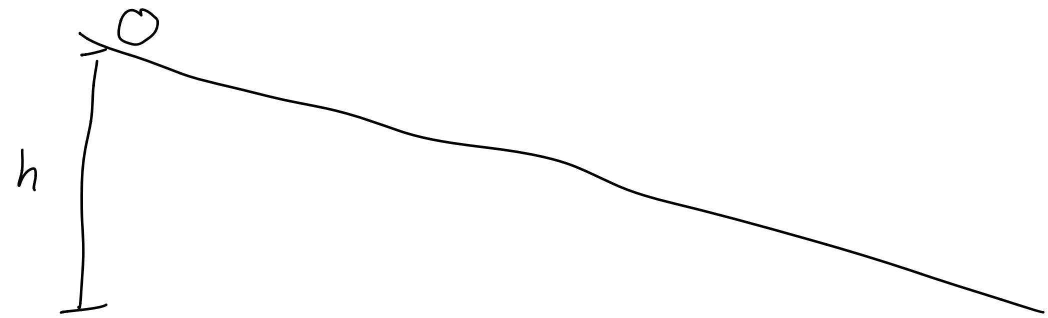
(Rolling is only possible due to friction)



$$K = K_{trans} + K_{rot}$$

$$K = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$$

Ex: Ball rolling down a ramp



$m = 535\ g$

$h = 25.5\ mm$

$R = 2.5\ cm$

In class, we used

$$mgh = \frac{1}{2}mv^2$$

to predict $v = .71\ m/s$

We measured $v = 0.5\ m/s$

How fast was the ball rotating?

sys = ball + Earth

$$\Delta E_{sys} = 0$$

$$E_i = mgh$$

$$E_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$\sqrt{\frac{mgh - \frac{1}{2}mv^2}{\frac{1}{2}I}} = \omega$$

$$I = \frac{2}{5}mR^2$$

$$\omega = \sqrt{\frac{gh - \frac{1}{2}v^2}{\frac{1}{5}\ R^2}}$$

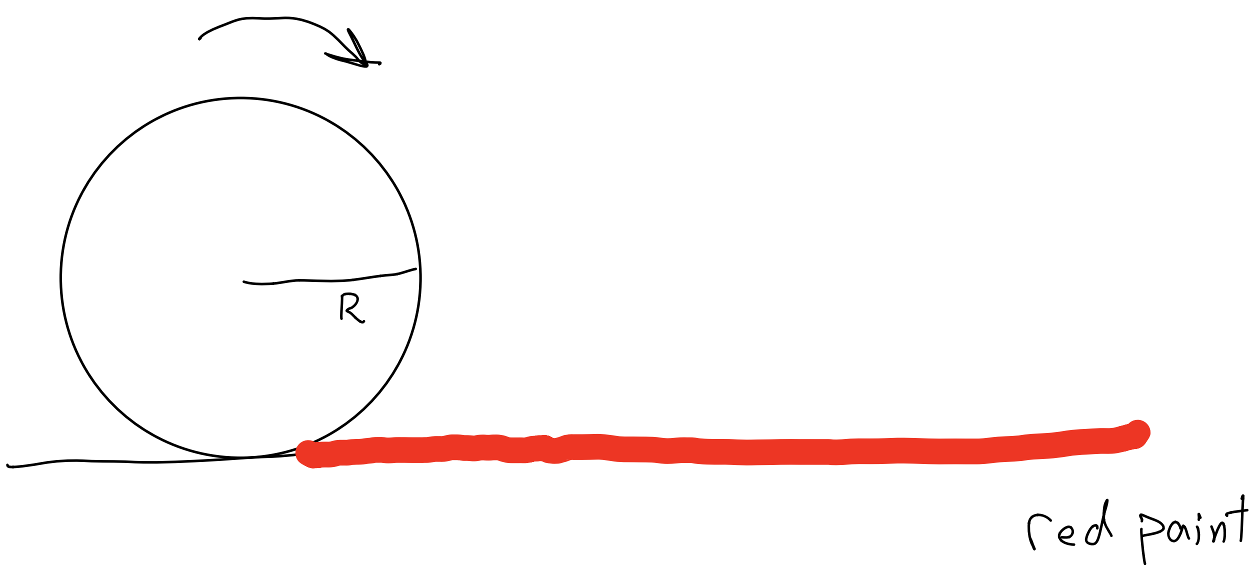
$$= \sqrt{\frac{(9.8)(0.025) - \frac{1}{2}(0.5)^2}{(0.2)(0.025)^2}}$$

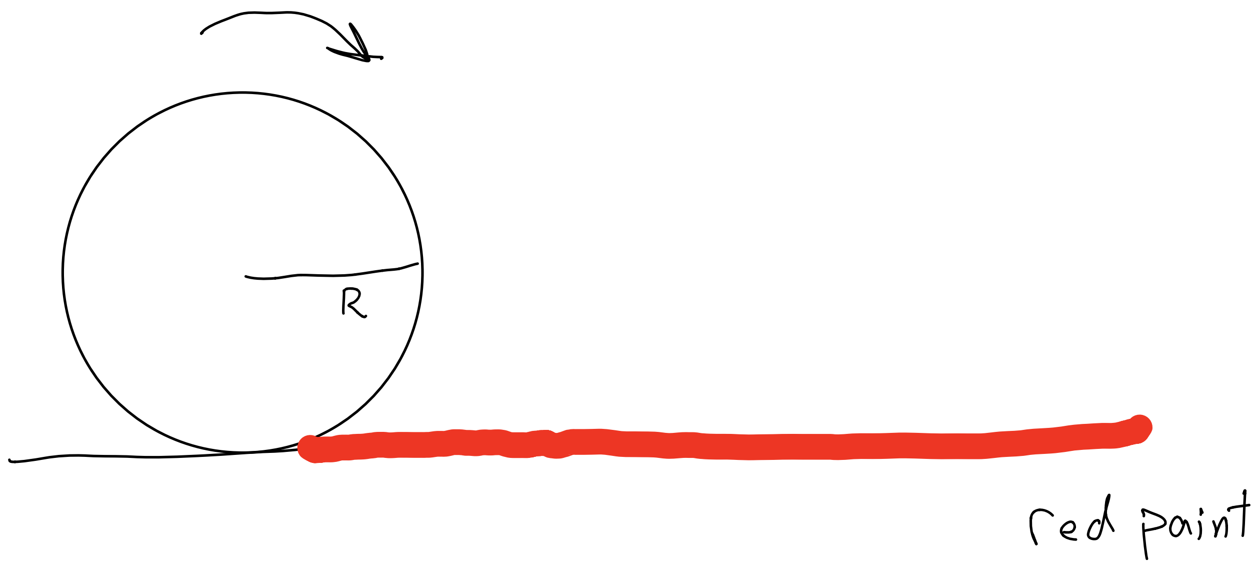
$$\omega = 31\ \frac{rad}{sec} \qquad f = \frac{\omega}{2\pi} \cong 5\ \frac{rot}{sec}$$

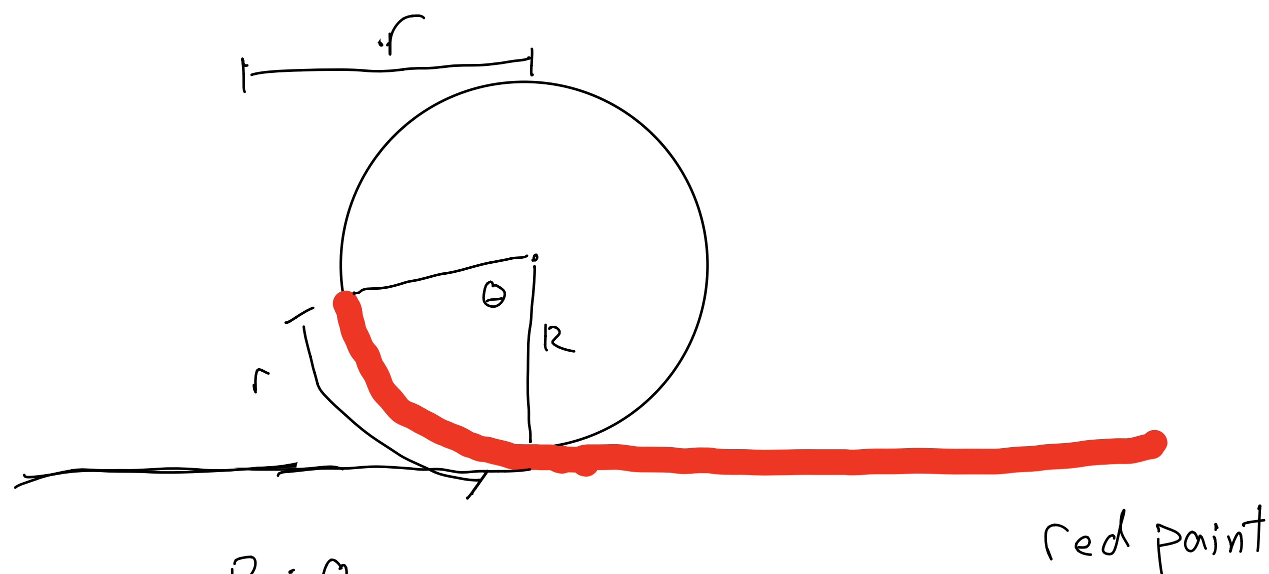
Common Special case:
rolling w/o slipping

If the ball does not slip when it makes contact with the ground, v + ω will be related

- silly thought experiment







$$r = R \cdot \theta$$

$$\frac{d}{dt} r = \frac{d}{dt} (R\theta)$$

$$\frac{dr}{dt} = v = R \frac{d}{dt}\theta = R\omega \; ; \quad \omega = \frac{v}{R}$$

so

$$K = K_{trans} + K_{rot}$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2$$

$$mgh = \frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2$$

$$v = \sqrt{\frac{2mgh}{m + \frac{I}{R^2}}}$$