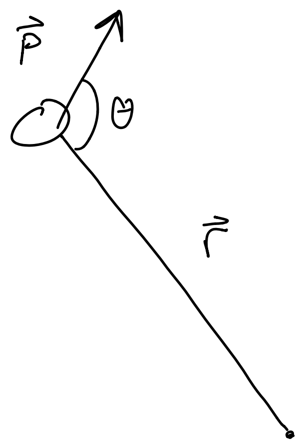
Angular momentum

orbital (translational)

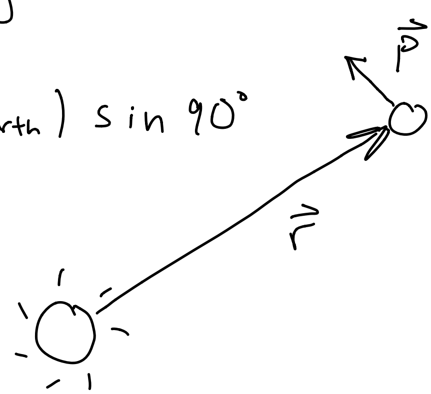
angular momentum



$$|\vec{L}_{orb}| = |\vec{r}||\vec{p}| \sin\theta$$

Ex: Earth orbiting the Sun

$$|\vec{L}_{orb}| = |\vec{r}_{earth-sun}||\vec{p}_{Earth}| \sin 90^\circ$$



$|\vec{r}| = 1.5 \times 10^{11}$ m

$|\vec{p}_{earth}| = m_E |\vec{v}_E|$

To find $|\vec{V}_E|$: Earth travels around circle of radius $2\pi r$ every year

$$|\vec{v}| = \frac{2\pi(1.5\times10^{11}\,m)}{1\,yr \times 365\,\frac{da}{yr} \times 24\,\frac{hr}{da} \times 3600\,\frac{s}{hr}}$$

$$|\vec{v}| = 3\times10^{4}\,\frac{m}{s}$$

$$|\vec{P}_E| = M_E\left(3\times10^{4}\,\frac{m}{s}\right)$$

$$= \left(6\times10^{24}\,kg\right)\left(3\times10^{4}\,\frac{m}{s}\right)$$

$$= 18\times10^{28}$$

$$= 1.8\times10^{29}\,\frac{kg\,m}{s}$$

$$|\vec{L}_{orb}| = \left(1.5\times10^{11}\,m\right)\left(1.8\times10^{29}\,\frac{kg\,m}{s}\right)(1)$$

$$|\vec{L}_{orb}| = 2.7\times10^{40}\,\underbrace{kg\,\frac{m^2}{s}}_{\text{unit}}$$

$$|\vec{L}_{orb}| = |\vec{r}|\,|\vec{p}|\sin\theta$$

what is the significance?

Chapter 1:

An object wants to continue moving at constant velocity unless acted on by an external force

Chapter 2:

How does an external force change an objects velocity?

Through the momentum principle

$$|\Delta\vec{p}| = |\vec{F}_{net}|\,\Delta t$$

$$\Delta v = \frac{\Delta p}{m}$$

- momentum changes in response to a net force
- ~~total~~ momentum is conserved

If my system is rotating rather than translating, angular momentum is what changes in response to a rotational force

Total angular momentum is constant

Just like we needed to understand $p = mv$ before getting to $\Delta p = F \Delta t$, we need to understand $L = p r \sin\theta$ before getting to our last fundamental principal

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So the earth has orbital angular momentum

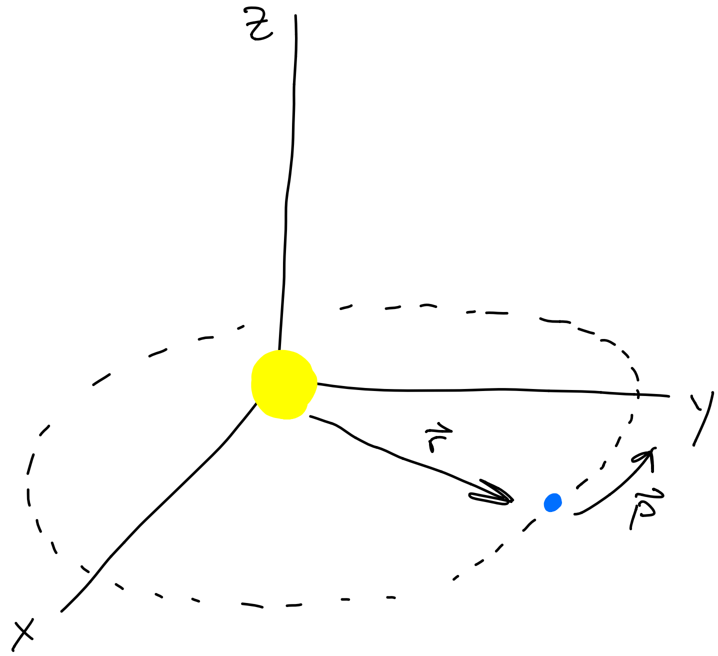
$$|\vec{L}_{orb}| = 2.7 \times 10^{40} \ \frac{kg\, m^2}{s}$$

Just like $\vec{p}$, $\vec{L}$ is a vector

Direction of $\vec{p}$ is easy

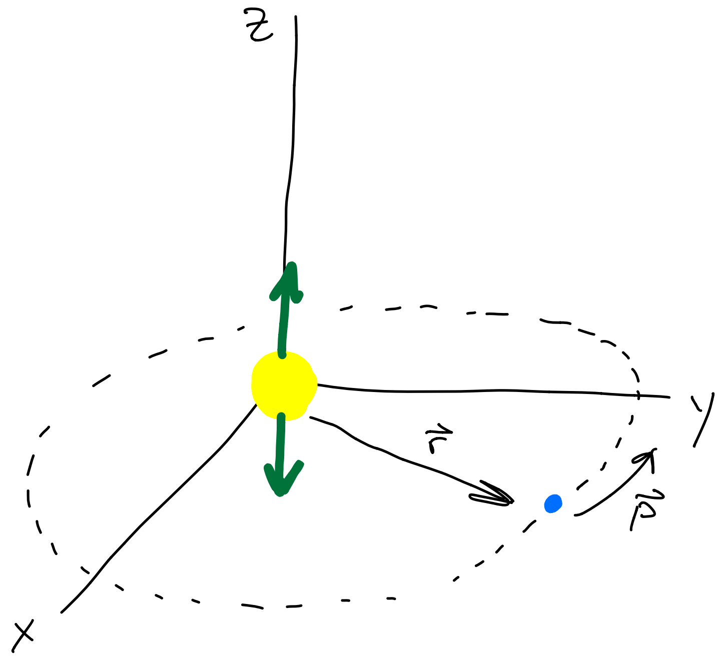
Direction of $\vec{L}$ is more complicated

1) Direction of $\vec{L}$ is along the axis of rotation



z axis is axis of rotation

$\hat{L}$ points "up" (+z) or "down" (−z)



How to decide?

The "right hand rule"

- Align thumb of right hand with axis of rotation

- Curl fingers in direction of rotation

- Direction of thumb = direction of $\vec{L}$

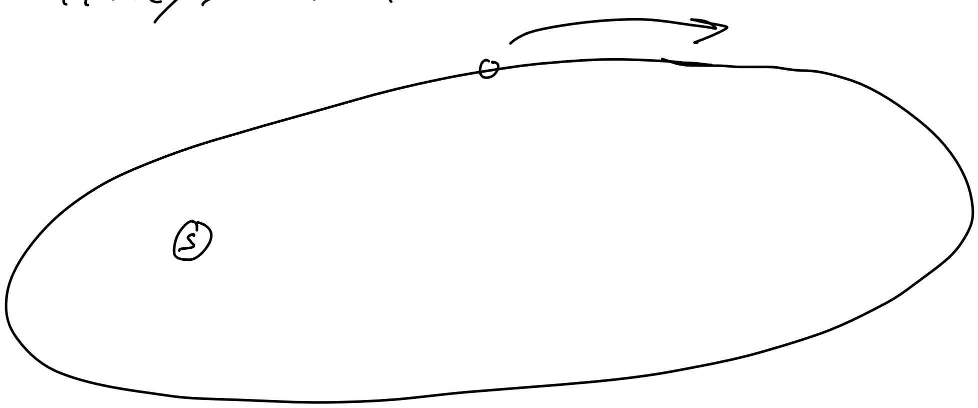
So $\vec{L}$ is in the z direction, &

$$\vec{L}_{orb} = (0, 0, 2.7\times 10^{40})\ \mathrm{kg}\,\frac{m^2}{s}$$

It seems obvious that $\vec{L}$ is constant for the Earth - Sun system

Since $|\vec{r}|$, $|\vec{p}|$, & θ are all constant

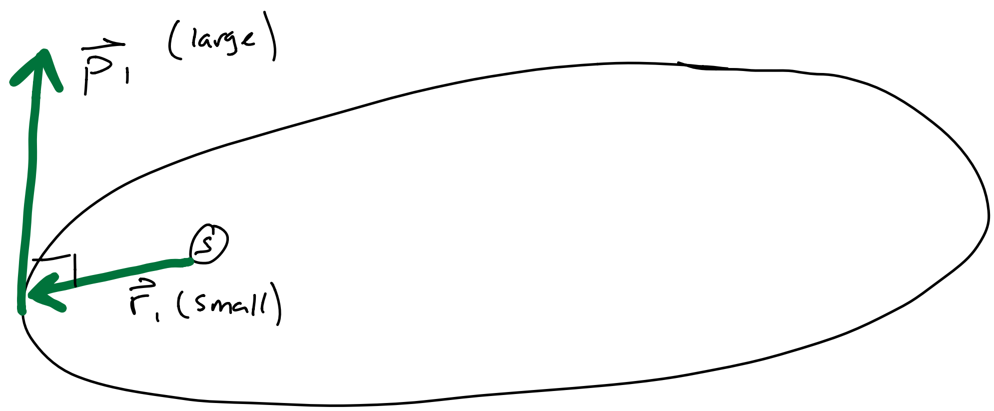
Ex: Halley's Comet



Calculate $\vec{L}$ @ two different points in the orbit

$(m = 2.2\times 10^{14}\ \mathrm{kg})$

$\vec{L}_1$: "perihelion" (closest approach)



$|\vec{r}_1| = 8.77 \times 10^{10}$ m

$|\vec{v}_1| = 5.46 \times 10^4$ m/s

$\theta = 90°$

$$|\vec{L}_1| = |\vec{r}_1||\vec{v}_1| \sin 90°$$

$$= (8.77\times10^{10}\text{ m})(2.2\times10^{14}\text{ kg})(5.46\times10^{4}\tfrac{\text{m}}{\text{s}})$$

$$= 1.1\times10^{30}\text{ kg m}^2/\text{s}$$

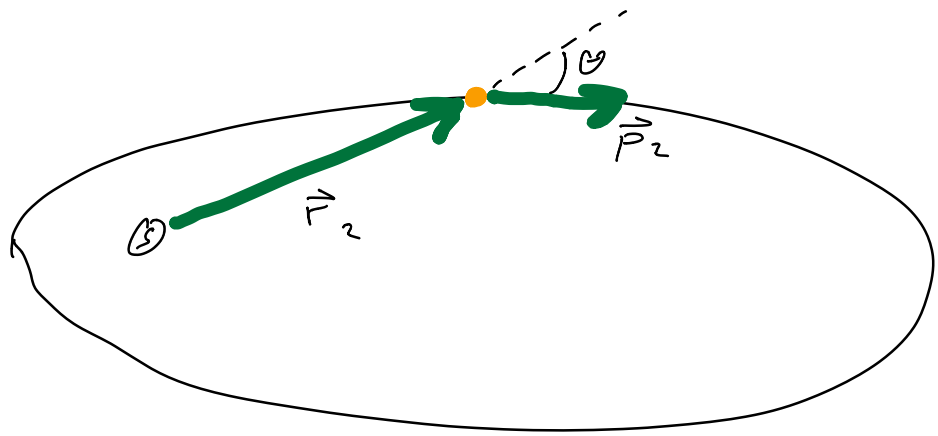
Direction?

Into the page/screen

Let's say that $+\hat{z}$ comes out at us

then $\vec{L}_1 = \langle 0, 0, -1.1\times10^{30}\rangle \text{ kg}\,\frac{\text{m}^2}{\text{s}}$

At some other point:



$$|\vec{r}_2| = 1.19 \times 10^{12}\ \text{m}$$

$$|\vec{v}_2| = 1.32 \times 10^{4}\ \frac{\text{m}}{\text{s}}$$

$$\theta = 17.81^\circ$$

$$|\vec{L}_2| = (1.19\times10^{12}\,\text{m})(2.2\times10^{14}\,\text{kg})(1.32\times10^{4}\,\text{m/s})\sin(17.81^\circ)$$

$$|\vec{L}_2| = 1.1\times10^{30}\ \text{kg}\,\frac{\text{m}^2}{\text{s}}$$

$$\vec{L}_2 = \langle 0, 0, -1.1\times10^{30}\rangle\ \frac{\text{kg m}^2}{\text{s}}$$

$$= \vec{L}_1$$

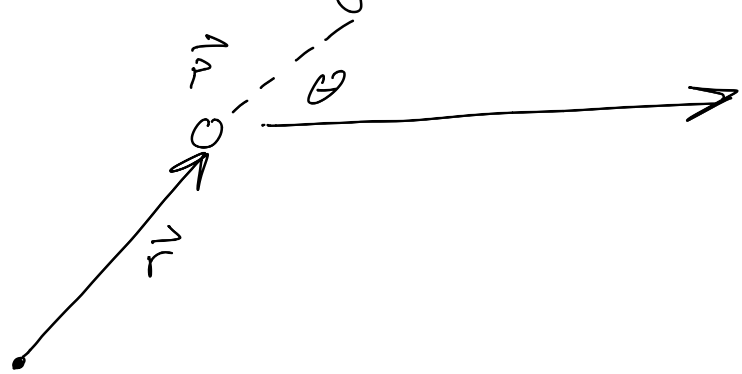
Even though $\vec{r}$, $\vec{p}$, + θ are continually changing,

$\vec{L}$ is constant

Some notes on $\vec{L}$:

- object does not need to be rotating or curving in order to have an angular momentum

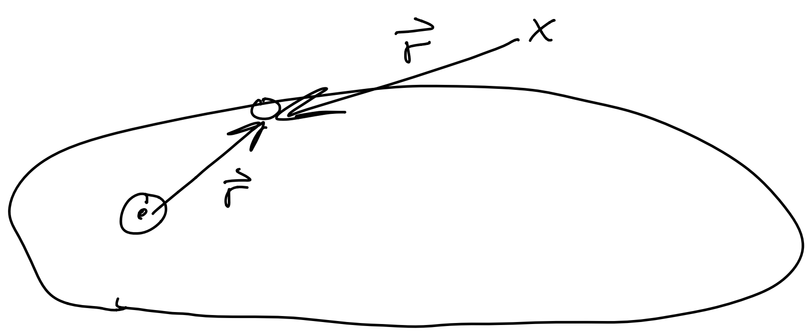
- An object moving in a straight line can have angular momentum



As long as we can define $\vec{p}$ & $\vec{r}$ & θ, we can define $\vec{L}$

What point do we choose to measure $\vec{r}$ from?

In previous examples, $\vec{r}$ was measured relative to axis of rotation



But we can pick any point we want!

The value we calculate will be different, but it will still be constant

Thus we say "the angular momentum relative to position A is ..."

Clicker Questions

"Let's talk more about the direction of $\vec{L}$"

- We know how to calculate the magnitude of $\vec{L}$

- We have a trick to figure out its direction

(RHR)

This isn't very "mathematical"

(How do you put the RHR in an equation?)

For regular momentum, this wasn't a problem

Direction of $\vec{v} = \hat{p}$

We want a mathematical operation to give us the direction of $\vec{L}$, given $\vec{r}$ & $\vec{p}$

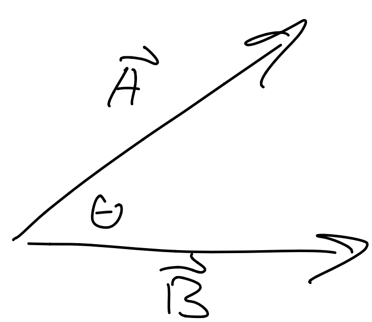
$$|\vec{L}| = |\vec{r}||\vec{p}| \sin\theta$$

$\vec{L}$ is product of $\vec{r}$ & $\vec{p}$

How do we multiply two vectors?

We've seen one way:

Dot product



$$\vec{A} = \langle A_x, A_y, A_z \rangle$$

$$\vec{B} = \langle B_x, B_x, B_z \rangle$$

$$\vec{A} \cdot \vec{B} = A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z$$

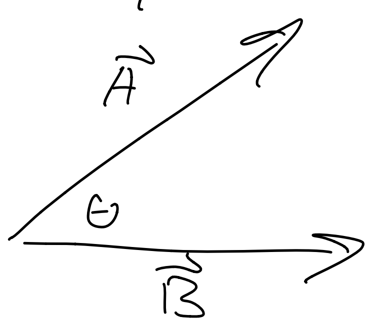
$$= |\vec{A}||\vec{B}| \cos \theta$$

$$W = \vec{F} \cdot \Delta \vec{r}$$

Dot Product is a scalar

There is another way to multiply vectors:

the cross product



$$\vec{A} = \langle A_x, A_y, A_z \rangle$$
$$\vec{B} = \langle B_x, B_x, B_z \rangle$$

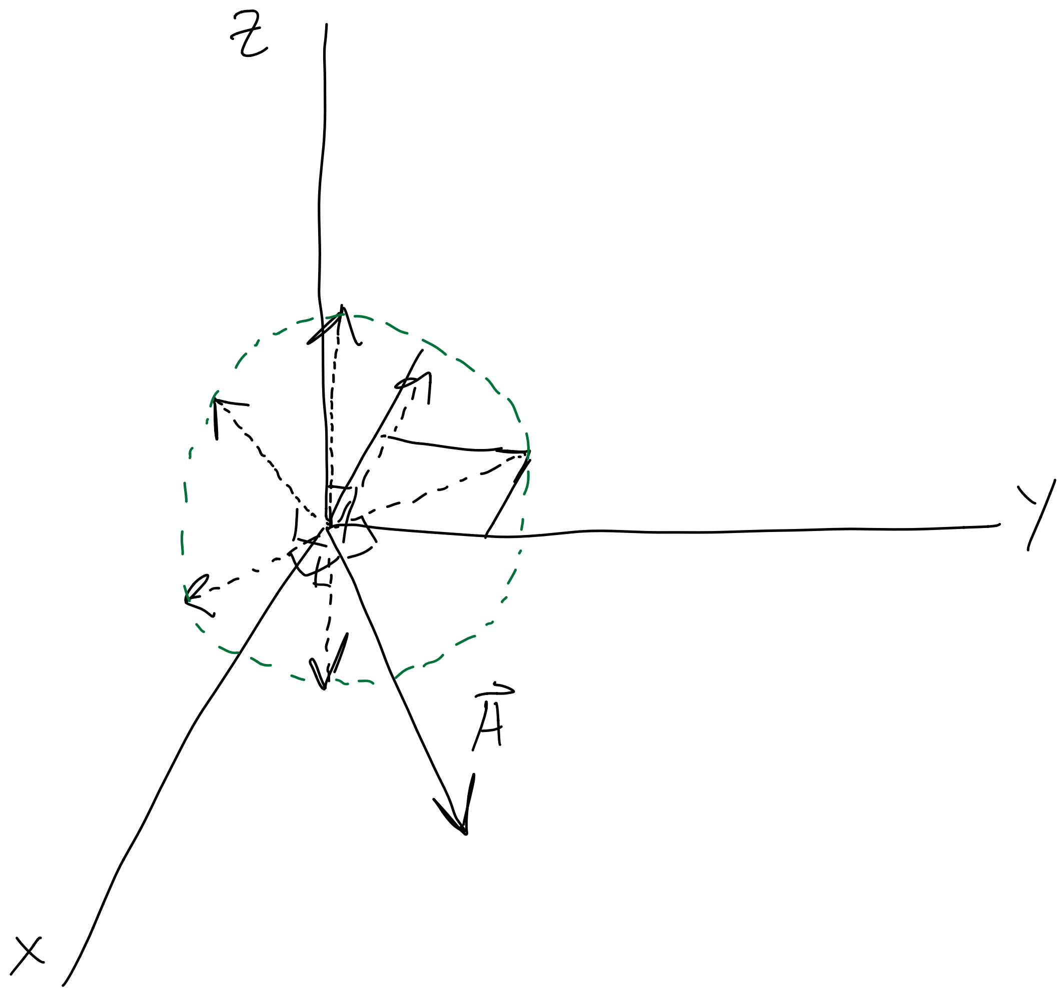
$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$$

Unlike the dot product, the cross product results in a vector!

vector $\cdot$ vector = scalar

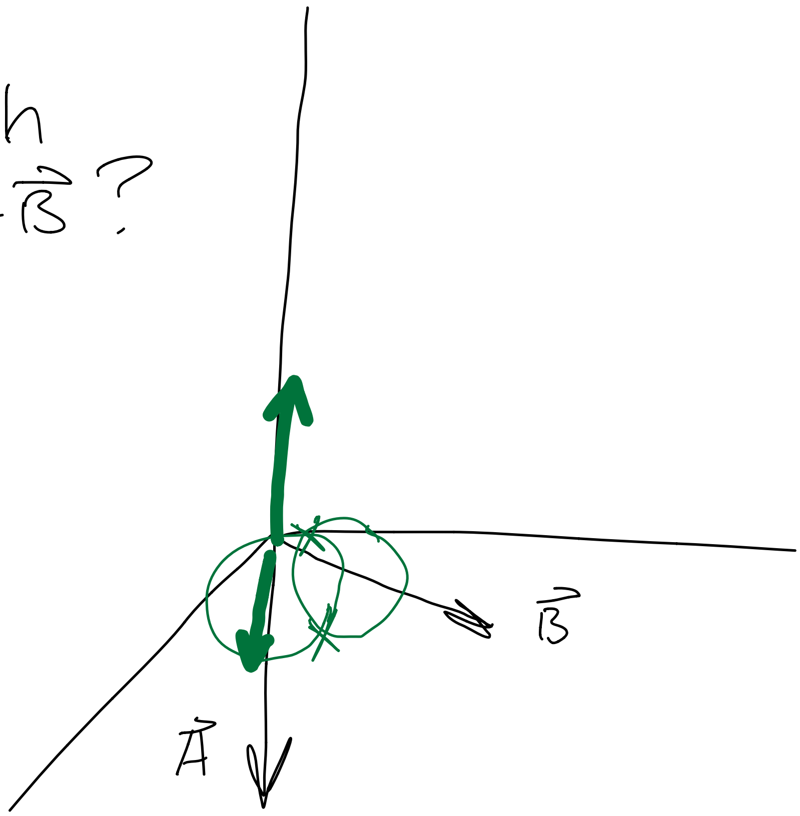
vector $\times$ vector = vector

The direction? Perp to <u>both</u> $\vec{A}$ + $\vec{B}$



Infinite $\perp$ vectors in a circle

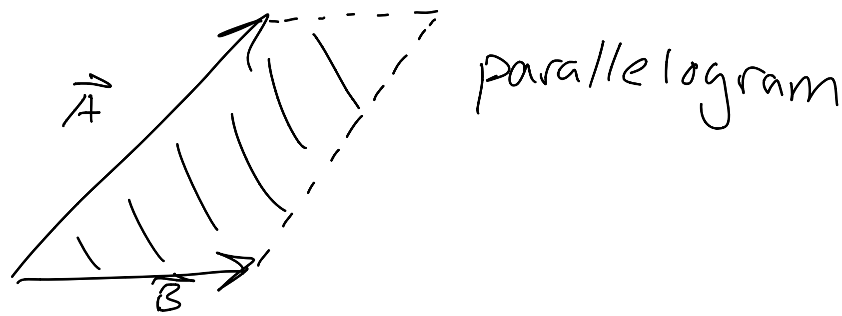
$\perp$ to both $\vec{A}$ + $\vec{B}$?



Two vectors that are perp to
<u>both</u> vectors

Another way:

Any two vectors define the edges of a flat surface



In this room:

The $\hat{x}$ + $\hat{z}$ vectors define the surface of the floor

$\hat{x}$ + $\hat{y}$ define this back wall

$\hat{z}$ + $\hat{y}$ define the L + R walls

Consider the 2D surface
defined by 2 vectors

Now imagine a vector that is
perpendicular to this surface
(it points directly away)

There are two (up + down)

- So the cross product has
magnitude

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$$

And direction

- directly away from the plane formed
by $\vec{A} + \vec{B}$

The direction is ambiguous

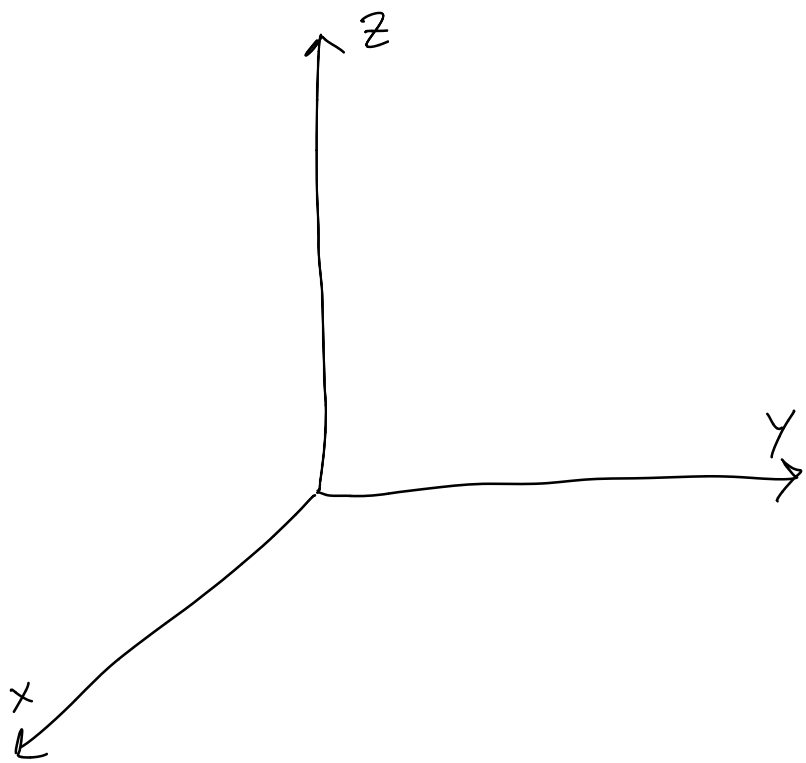
How do we decide?

Use RHR

- Lay your R hand along vector $\vec{A}$

- Curl fingers in direction of $\vec{B}$

- Thumb now points in direction of $\vec{A} \times \vec{B}$



Reminder:

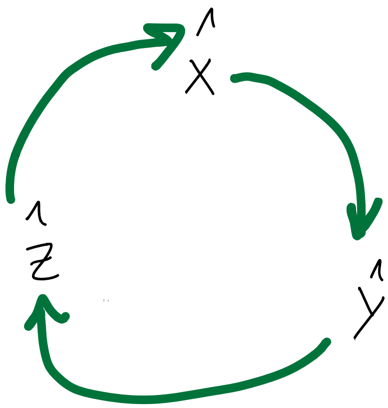
$\hat{x} = \langle 1, 0, 0 \rangle$
$\hat{y} = \langle 0, 1, 0 \rangle$
$\hat{z} = \langle 0, 0, 1 \rangle$

What is: $\hat{x} \times \hat{x}$? (0)

$\hat{x} \times \hat{y}$? $(\hat{z})$

$\hat{y} \times \hat{x}$? $(-\hat{z})$

$\hat{y} \times \hat{z} = \hat{x}$



$$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$$

$\hat{z} \times \hat{x}$: $= \pm \hat{y}$ which one?

Start @ z + move to x

moved with the arrow, so pos

$$\hat{y} \times \hat{x} = -\hat{z}$$

$$\vec{A} = (A_x, A_y)$$

$$\vec{B} = \langle B_x, B_y \rangle$$

$$\vec{A} \times \vec{B} = \left(A_x \hat{x} + A_y \hat{y}\right) \times \left(B_x \hat{x} + B_y \hat{y}\right)$$

$$= A_x \hat{x} \times B_x \hat{x} + A_x \hat{x} + B_y \hat{y}$$

$$+ A_y \hat{y} \times B_x \hat{x} + A_y \hat{y} \times B_y \hat{y}$$

$$= A_x B_x (\hat{x} \times \hat{x}) + A_x B_y (\hat{x} \times \hat{y})$$

$$+ A_y B_x (\hat{y} \times \hat{x}) + A_y B_y (\hat{y} \times \hat{y})$$

$$= 0 + A_x B_y \hat{z} - A_y B_x \hat{z} + 0$$

$$= (A_x B_y - A_y B_x) \hat{z}$$

$\vec{A} = \langle A_x, A_y, A_z \rangle$

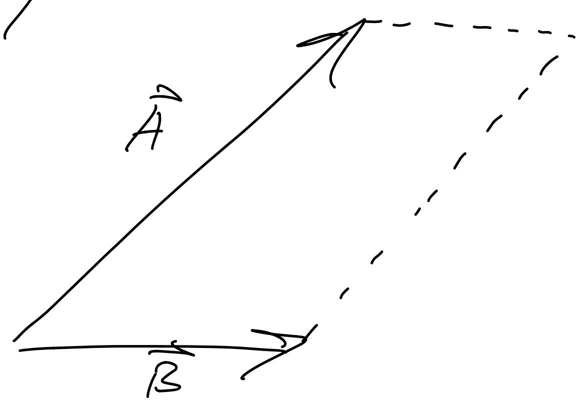
$\vec{B} = \langle B_x, B_x, B_z \rangle$

$\vec{A} \times \vec{B} =$

$\langle A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x \rangle$

$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$

Geometry:



Area of parallelogram $= |\vec{A} \times \vec{B}|$