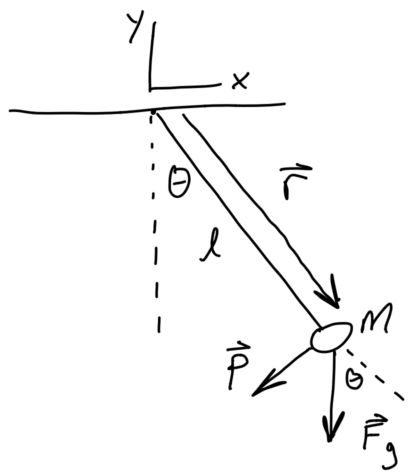
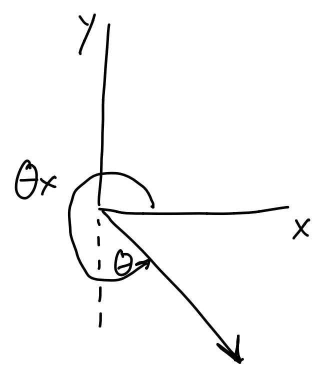
Example: a pendulum



$$\vec{\tau} = \vec{r} \times \vec{F}$$

$\vec{r} = ?$



$$\theta_x = \frac{3\pi}{2} + \theta$$

$$\theta_y = \pi + \theta$$

$$\vec{r} = \ell \left\langle \cos\left(\frac{3\pi}{2} + \theta\right), \cos(\pi + \theta) \right\rangle$$

$$\vec{r} = \ell \langle \sin\theta, -\cos\theta \rangle$$

$$\vec{F} = \langle 0, -mg \rangle$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$= \left(l \sin\theta\, \hat{x} - l\cos\theta\, \hat{y}\right) \times -mg\,\hat{y}$$

$$\vec{\tau} = -mgl\sin\theta\,\hat{z}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

$$\vec{L} = I\vec{\omega}\ , \quad I = \underbrace{I_{cm} + ml^2}_{\text{Parallel axis thm}} = ml^2$$

$$\vec{L} = ml^2\vec{\omega}$$

$$\frac{d\vec{L}}{dt} = ml^2\frac{d\vec{\omega}}{dt}$$

$$ml^2\frac{d\vec{\omega}}{dt} = -mgl\sin\theta\,\hat{z}$$

$$\frac{d\omega}{dt} = -\frac{g}{l}\sin\theta$$

$$\omega = \frac{d\theta}{dt}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$$

Need a computer for this!

(radians)

if $|\theta|$ is very small, then $\sin\theta \approx \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta$$

Compare to:

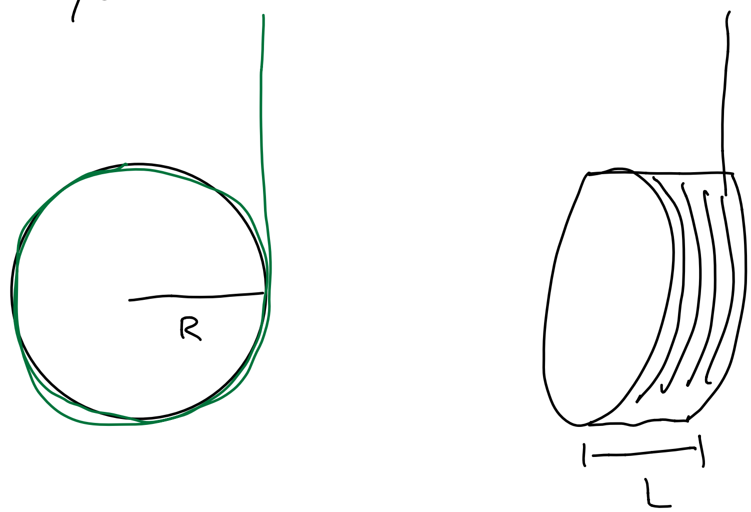
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,x\right)$$

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{\ell}}\,t\right)$$

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

Ex: A yo-yo



How fast is the yo-yo spinning when it is unspooled?

(60 cm)

Idea:

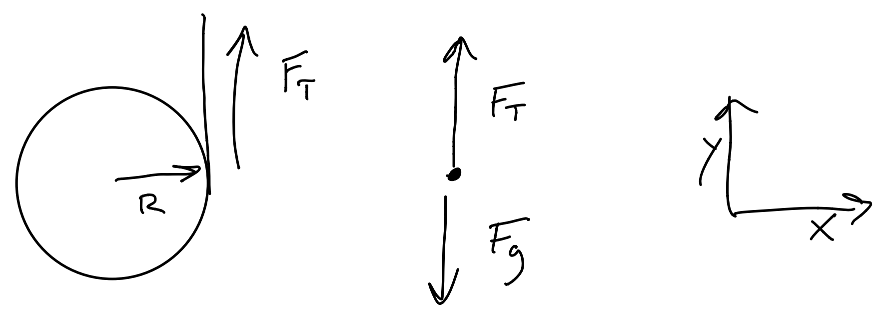
Gravity exerts a torque on the yo-yo which increases $\vec{L}$

Find $\vec{\tau}$

Find Δt (time to unspool)

then $\vec{L} = \vec{\tau}\,\Delta t$

What force exerts the torque?
String tension



Momentum principle:

$$\frac{dp_y}{dt} = F_T - F_g = F_T - mg$$

$$\frac{dp_y}{dt} = -ma = F_T - mg$$

$$F_T = mg - ma$$

Angular momentum principle

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{r} = R\,\hat{x}$$

$$\vec{F} = F_T\,\hat{y}$$

$$\vec{\tau} = R F_T\,\hat{z}$$

$$\vec{\tau} = \frac{d}{dt}\vec{L} = \frac{d}{dt}\left(I\vec{\omega}\right)$$

$$= I\,\frac{d\omega}{dt}\,\hat{z}$$

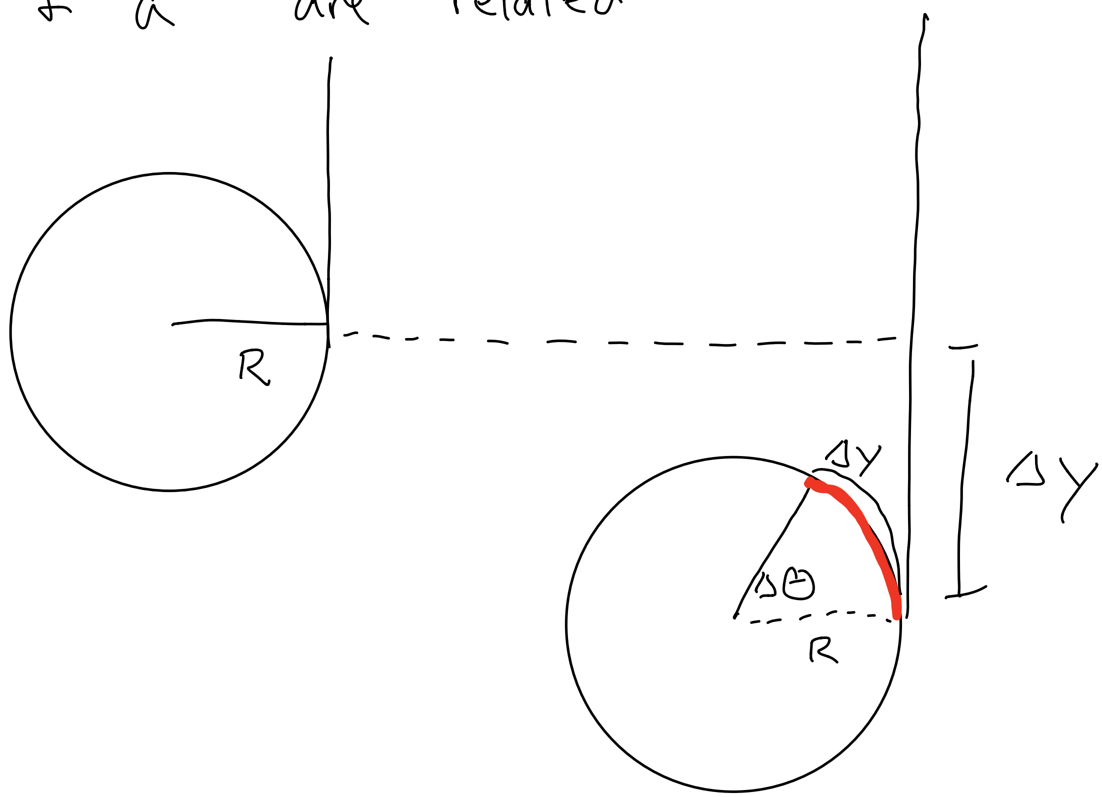
$$I\,\frac{d\omega}{dt} \equiv I\alpha = R F_T$$

$$F_T = \frac{I}{R}\alpha$$

$F_T = mg - ma$

$F_T = \frac{I}{R}\alpha$

α + a are related



$R\Delta\theta = \Delta y$

$R\Delta\theta = v\Delta t$

$R\frac{\Delta\theta}{\Delta t} = v$

$v = R\omega$

$\frac{dv}{dt} = R\frac{d\omega}{dt} = R\alpha$

$\boxed{a = R\alpha}$

$$F_T = mg - ma$$

$$F_T = \frac{I}{R}\alpha$$

$$a = R\alpha$$

so:

$$F_T = \frac{I}{R}\left(\frac{a}{R}\right) = \frac{I}{R^2}a$$

$$F_T = mg - ma = \frac{I}{R^2}a$$

$$a = \frac{mgR^2}{I + mR^2}$$

$$a = g\left(\frac{1}{1 + \frac{I}{mR^2}}\right)$$

constant accel

$$a \leq g, \quad a = g \text{ if } I = 0$$

How fast is the yo-yo spinning after 60 cm of string unspool?

$$\frac{dL}{dt} = \tau$$

$$I\frac{d\omega}{dt} = \tau$$

$$\frac{d\omega}{dt} = \frac{\tau}{I} \qquad \left(\frac{dv}{dt} = \frac{F}{m}\right)$$

$$\omega = \omega_i + \frac{\tau}{I}t \qquad \left(v = v_i + \frac{F}{m}t\right)$$

$$\theta = \theta_i + \omega_i t + \frac{1}{2}\frac{\tau}{I}t^2 \quad \left(y = y_i + v_i t + \frac{1}{2}\frac{F}{m}t^2\right)$$

$$\Delta\theta = \omega_i t + \frac{1}{2}\frac{\tau}{I}t^2$$

$$\omega_i = 0$$

$$\Delta\theta = \frac{1}{2}\frac{\tau}{I}t^2$$

$$R\Delta\theta = \Delta y \implies \Delta\theta = \frac{\Delta y}{R}$$

$$\Delta\theta = \frac{\Delta y}{R}$$

$$\tau = R F_T = R \frac{I}{R^2} a = \frac{I}{R} a \; ; \quad a = g\left(\frac{1}{1 + \frac{I}{mR^2}}\right)$$

$$\frac{\Delta y}{R} = \frac{1}{2} \frac{a}{R} t^2$$

$$\Delta y = \frac{1}{2} a t^2 \quad (!)$$

$$t^2 = \frac{2\Delta y}{a}$$

$$t = \sqrt{\frac{2\Delta y}{a}}$$

$$\omega = \omega_i + \frac{\tau}{I} t$$

$$\omega = \frac{a}{R}\left(\frac{2\Delta y}{a}\right)^{\frac{1}{2}}$$

$$\omega = \left(2a \frac{\Delta y}{R^2}\right)^{\frac{1}{2}}$$

$$a = g\left(\frac{1}{1 + \frac{I}{mR^2}}\right)$$

$$\omega = \sqrt{g\left(\frac{1}{1 + \frac{I}{mR^2}}\right) \cdot \frac{2\Delta y}{R^2}}$$

$$I = \frac{1}{2} mR^2$$

$$\omega = \sqrt{\frac{4}{3} \frac{g \Delta y}{R^2}}$$