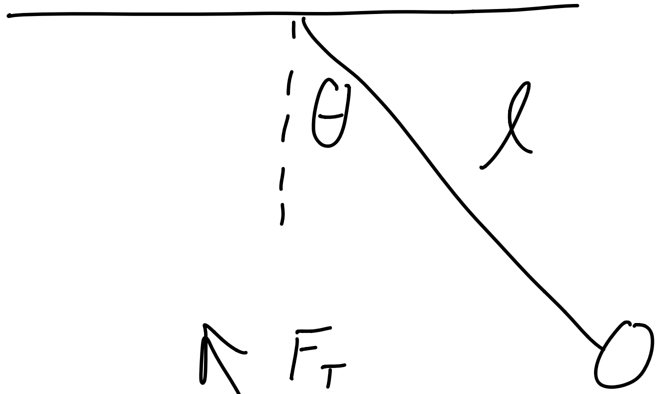
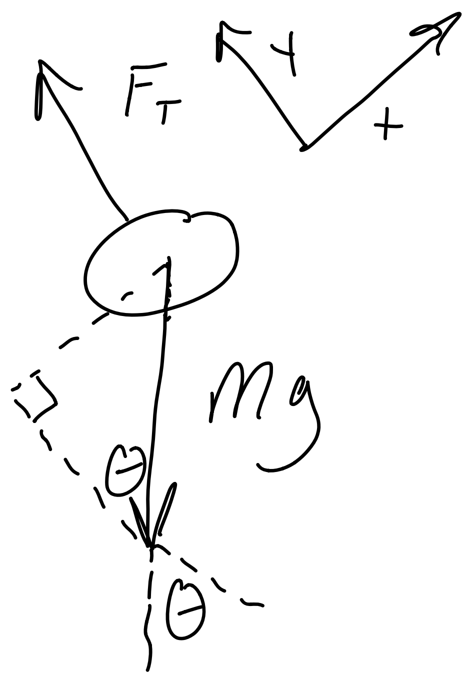
- Outline:
- Apply numerical methods and mechanics learned last chapter to simple harmonic motion
- Present pendulum and obtain equation of motion:
    - Do this two separate ways:
    - Force and Torque considerations
    - Solve analytically
- Solve numerically
    - Write out Euler steps
    - Show python code
    - Amplitude growing out of control!!
- Energy conservation
    - Derive energy in normal units
    - Plot energy vs time
    - Derive energy growth
    - Why didn't we notice this with projectile motion?
        - It was there, but more subtle
        - Derive (quickly) and plot
- How to conserve energy?
    - Euler-Cromer method
        - Implicit (use current system velocity to evaluate current position)
        - Energy deviations cancel out and remains stable
    - Show plot and animation with Euler Cromer

1)



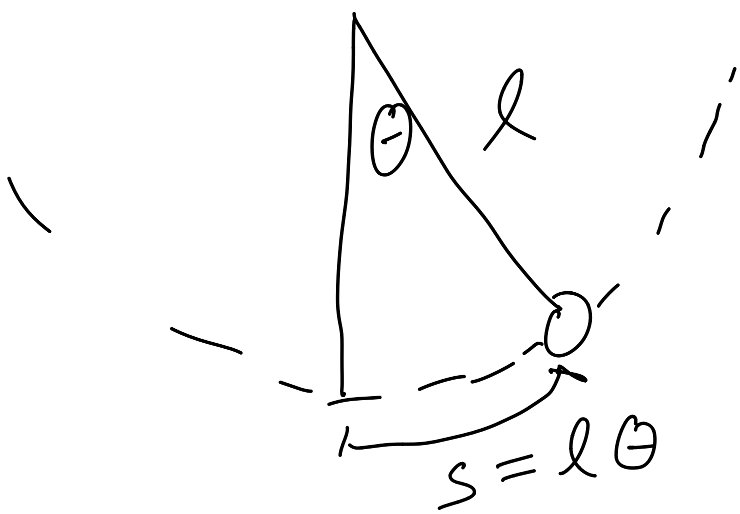
Want forces along motion & tension



$$F_x = -mg\sin\theta \qquad F_y = \underbrace{F_T - mg\cos\theta}_{=0}$$

F = ma

motion is constrained to circle of radius l

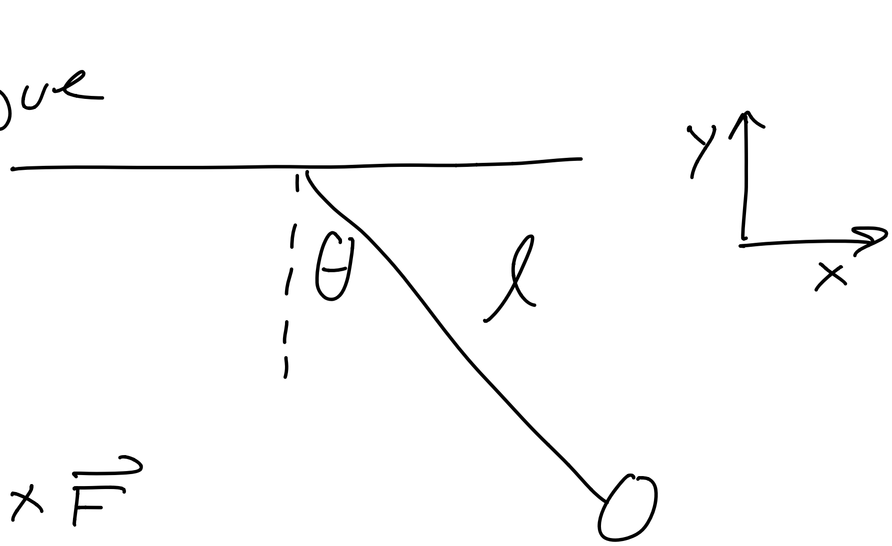


$$m\frac{d^2}{dt^2} s = F_x$$

$$m\frac{d^2}{dt^2}(l\theta) = -mg\sin\theta$$

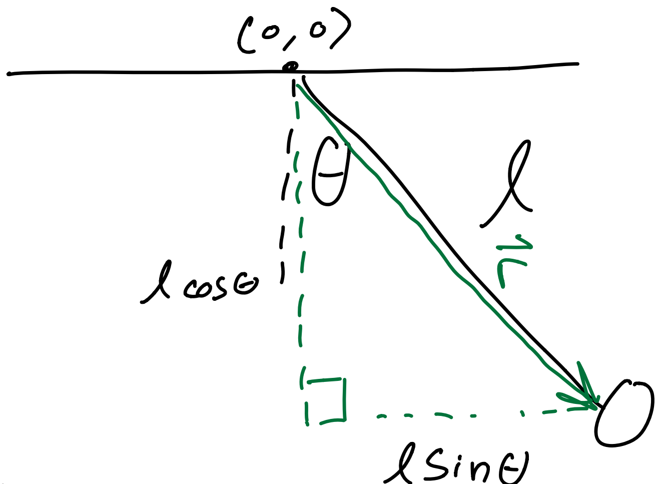
$$\boxed{\frac{d^2}{dt^2}\theta = -\frac{g}{l}\sin\theta}$$

2) Torque



$$\vec{\tau} = \vec{r} \times \vec{F}$$

$\vec{r}$: vector from pivot to force



$$\vec{r} = \ell\sin\theta\,\hat{x} - \ell\cos\theta\,\hat{y}$$

$$\vec{F} = -mg\,\hat{y}$$

(next page, cross prod)

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$= \left(l \sin\theta \hat{x} - l\cos\theta \hat{y}\right) \times -mg\hat{y}$$

$$\hat{y} \times \hat{y} = 0$$

$$\vec{\tau} = -mgl\sin\theta \hat{x} \times \hat{y}$$

$$\vec{\tau} = -mgl\sin\theta \hat{z}$$

$$F = ma \longrightarrow \tau = I\alpha$$

I = moment of inertia $= ml^2$

$$\alpha = \frac{d^2}{dt^2}\theta$$

$$\tau = ml^2 \frac{d^2}{dt^2}\theta = -mgl\sin\theta$$

$$\boxed{\frac{d^2\theta}{dt^2} = \frac{-g}{l}\sin\theta}$$

$$\frac{d^2\theta}{dt^2} = \frac{-g}{l}\sin\theta$$

$$\sin(\theta) = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \cdots$$

$|\theta| << 1$, $\sin\theta \simeq \theta$

$$\frac{d^2\theta}{dt^2} = \frac{-g}{l}\theta$$

CAN SOLVE
ANALYTICALLY

Want $\theta(t)$ such that

$$\frac{d^2}{dt^2}\theta(t) \propto -\theta(t)$$

Try: $\theta(t) = c_1 \sin(\Omega t) + c_2 \cos(\Omega t)$

$$\frac{d}{dt}\Theta = \Omega c_1 \cos(\Omega t) - \Omega c_2 \sin(\Omega t)$$

$$\frac{d^2\Theta}{dt^2} = -\Omega^2 c_1 \sin(\Omega t) - \Omega^2 c_2 \cos(\Omega t)$$

$$= -\Omega^2 \Theta \quad \checkmark$$

$$\Omega^2 = \frac{g}{l} \qquad \Omega = \sqrt{\frac{g}{l}}$$

$$T = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{l}{g}}$$

Period of pendulum

Normalize

$$\frac{d^2\Theta}{dt^2} = \frac{-g}{l}\Theta = -\Omega^2\Theta$$

Θ is already dimensionless

No $\bar{\Theta}$ $\quad (\Theta_0 = 1)$

$$\bar{t} = \frac{t}{t_0}$$

$$\frac{1}{t_0^2}\frac{d^2\Theta}{d\bar{t}^2} = \frac{-g}{l}\Theta = -\Omega^2\Theta$$

$$t_0^2 = \frac{l}{g} = \Omega^2 = \frac{T}{2\pi}$$

$$\bar{t} = 2\pi \longrightarrow 1T$$

$$\bar{t} = 2\pi n = nT$$

$$\frac{d^2\Theta}{d\bar{t}^2} = -\Theta$$

Euler

$$\omega \equiv \frac{d\theta}{dt}$$

$$\frac{d\bar{\omega}}{d\bar{t}} = -\theta \qquad [\omega] = \frac{1}{T}$$

$$\bar{\omega} = t_0 \omega$$

$$= \frac{\omega}{\Omega}$$

$$\frac{d\theta}{d\bar{t}} = \omega$$

Drop the bars ...

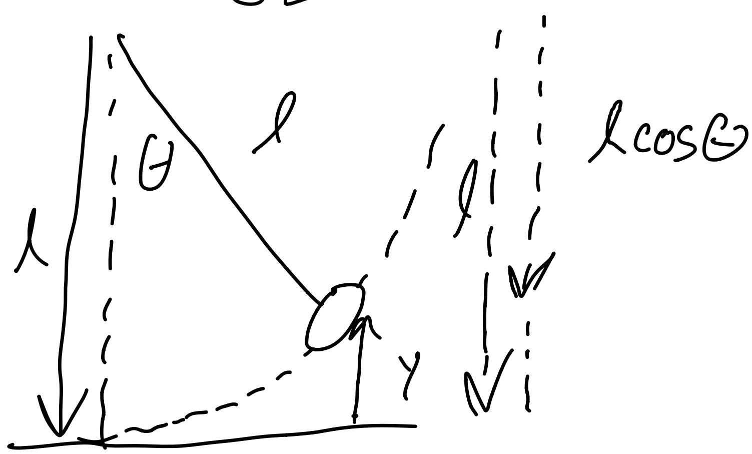
$$\omega_i = \omega_{i-1} - \theta \Delta t$$

$$\theta_i = \theta_{i-1} + \omega_{i-1} \Delta t$$

Now let's code it

???

Should have $\frac{dE}{dt} = 0$



$E = T + U$

$T = \frac{1}{2}mv^2$; $v = \frac{ds}{dt} = l\frac{d\theta}{dt}$

$T = \frac{1}{2}ml^2\omega^2$

$U = mgy$; $y = l - l\cos\theta$
$= l(1-\cos\theta)$

$U = mgl(1-\cos\theta)$

$$E = \frac{1}{2} m l^2 \omega^2 + mgl(1-\cos\theta)$$

$$\bar{E} = \frac{E}{E_0} = 1 + \frac{1}{2}\bar{\omega}^2 - \cos\theta$$

SHow Jupyter

$$E_i = \frac{1}{2}\omega_i^2 + 1 - \cos\theta_i$$

$$|\theta| << 1$$

$$E_i \approx \frac{1}{2}\omega_i^2 + \frac{1}{2}\theta_i^2$$

$$\omega_i = \omega_{i-1} - \theta_{i-1}\Delta t$$

$$\theta_i = \theta_{i-1} + \omega_{i-1}\Delta t$$

$$\omega_i^2 = \omega_{i-1}^2 + \theta_{i-1}^2\Delta t^2 - 2\omega_{i-1}\theta_{i-1}\Delta t$$

$$\theta_i^2 = \theta_{i-1}^2 + \omega_{i-1}^2\Delta t^2 + 2\omega_{i-1}\theta_{i-1}\Delta t$$

$$\omega_i^2 + \theta_i^2 = \omega_{i-1}^2 + \theta_{i-1}^2 + (\omega_{i-1}^2 + \theta_{i-1}^2)\Delta t^2$$

$$E_i = \frac{1}{2}(\omega_i^2 + \theta_i^2)$$

$$E_i = \underbrace{\frac{1}{2}\left(\omega_{i-1}^2 + \theta_{i-1}^2\right)}_{=E_{i-1}} + \frac{1}{2}\left(\omega_{i-1}^2 + \theta_{i-1}^2\right)\Delta t^2$$

$$E_i = E_{i-1} + E_{i-1}\Delta t^2$$

Energy always increasing!

$$E_i - E_{i-1} = E_{i-1}\Delta t^2$$

$$\frac{\Delta E}{\Delta N} = E\Delta t^2$$

$$E \sim e^{N\Delta t^2} = e^{\Delta t t}$$

- But wait, didn't we just use this method to investigate projectile motion???
- Show the plot
- We still gain energy, but at a lower rate (~dt * t)
- Lack of energy conservation is continuously compounded in SHM
- (Velocity depends on position, position depends on velocity)
- Solution quickly becomes unstable
- How do we fix this?

Modified Euler:

Euler-Cromer

$$\omega_i = \omega_{i-1} - \Theta_i \Delta t$$

$$\Theta_i = \Theta_{i-1} + \omega_i \Delta t$$

use updated velocity

Now:

$$E_i - E_{i-1} = \left(E_{i-1} + \omega_i^2 - \omega_{i-1}^2\right)\Delta t^2$$

$$+ \Theta_{i-1}(\omega_i - \omega_{i-1})\Delta t$$

= oscillating "+" term

+

oscillating "−" term

if $\omega_i > \omega_{i-1}$

$\omega_i - \omega_{i-1} > 0$

then $\Theta < 0$