$$\mathcal{U} = G' m_1 m_2 r^2$$

From notes:

$$\ddot{r} = \left(\frac{\ell}{\mu}\right)^2 \frac{1}{r^3} - \frac{1}{\mu} \frac{\partial}{\partial r} \mathcal{U}(r)$$

$$\ddot{r} = \left(\frac{\ell}{\mu}\right)^2 \frac{1}{r^3} - \frac{2}{\mu} G' m_1 m_2 r$$

$$\frac{m_1 m_2}{\mu} = \frac{m_1 m_2}{m_1 m_2 / M} = M$$

a) $$\ddot{r} = \left(\frac{\ell}{\mu}\right)^2 \frac{1}{r^3} - 2G' M r$$

b) $$0 = \left(\frac{\ell}{\mu}\right)^2 \frac{1}{r_0^3} - 2G' M r_0$$

$$0 = \left(\frac{\ell}{\mu}\right)^2 - 2G' M r_0^4$$

$$r_0^4 = \frac{1}{2G'M}\left(\frac{\ell}{\mu}\right)^2, \quad r_0 = \left[\frac{1}{2G'M}\left(\frac{\ell}{\mu}\right)^2\right]^{1/4}$$

Stability Analysis

$$(\ddot{r_o+\Delta r}) = \left(\frac{l}{\mu}\right)^2 \frac{1}{(r_o+\Delta r)^3} - 2G'M\,(r_o+\Delta r)$$

$$(r_o+\Delta r)^{-3} = r_o^{-3}\left(1+\frac{\Delta r}{r_o}\right)^{-3} \approx r_o^{-3}\left(1-3\frac{\Delta r}{r_o}\right)$$

$$\Delta\ddot{r} = \left(\frac{l}{\mu}\right)^2 \frac{1}{r_o^3}\left(1-3\frac{\Delta r}{r_o}\right) - 2G'M\,(r_o+\Delta r)$$

$$= \underbrace{\left(\frac{l}{\mu}\right)^2 \frac{1}{r_o^3} - 2G'Mr_o}_{=0} - 3\left(\frac{l}{\mu}\right)^2 \frac{\Delta r}{r_o^4} - 2G'M\Delta r$$

$$\Delta\ddot{r} = -\left[3\left(\frac{l}{\mu}\right)^2 \frac{1}{r_o^4} + 2G'M\right]\Delta r$$

$$= -\left[3\left(\frac{l}{\mu}\right)^2 2G'M\left(\frac{\mu}{l}\right)^2 + G'M\right]\Delta r$$

$$= -\left[6G'M + G'M\right]\Delta r$$

$$\Delta\ddot{r} = -7G'M$$

$$\omega^2 = 7G'M$$

Normalize

$$\ddot{r} = \left(\frac{\ell}{\mu}\right)^2 \frac{1}{r^3} - 2G'Mr$$

$$\frac{d^2 r}{dt^2} = \left(\frac{\ell}{\mu}\right)^2 \frac{1}{r^3} - 2G'Mr$$

$$r = r_0 \bar{r}$$
$$t = t_0 \bar{t}$$

$$\frac{r_0}{t_0^2} \frac{d^2 \bar{r}}{d\bar{t}^2} = \left(\frac{\ell}{\mu}\right)^2 \frac{1}{r_0^3 \bar{r}^3} - 2G'M r_0 \bar{r}$$

$$\frac{d^2 \bar{r}}{d\bar{t}^2} = \left(\frac{\ell}{\mu}\right)^2 \frac{t_0^2}{r_0^4} \frac{1}{\bar{r}^3} - t_0^2 2G'M\bar{r}$$

$$\frac{r_0^4}{t_0^2} \stackrel{?}{=} \left(\frac{\ell}{\mu}\right)^2$$

$$[\ell] = [p \times r] = [MVL] = \frac{ML^2}{T}$$

$$[\mu] = M$$

$$\left[\frac{\ell}{\mu}\right] = \frac{L^2}{T} \quad , \quad \left[\left(\frac{\ell}{\mu}\right)^2\right] = \frac{L^4}{T^2}$$

$$\frac{r_0^4}{t_0^2} = \left(\frac{\ell}{\mu}\right)^2 \checkmark$$

$[G'M]$

$$F = -G' m_1 m_2 r$$

$$[F] = \frac{ML}{T^2} = [G'] M^2 L$$

$$[G'] = \frac{1}{MT^2}$$

$$[G'M] = \frac{1}{T^2}$$

Choose

$$t_0^2 = \frac{1}{G'M}$$

$$\frac{r_0^4}{t_0^2} = \left(\frac{l}{\mu}\right)^2$$

$$r_0^4 = \frac{1}{G'M}\left(\frac{l}{\mu}\right)^2$$

$$\boxed{\frac{d^2\bar{r}}{d\bar{t}^2} = \frac{1}{\bar{r}^3} - 2\bar{r}}$$

$1 - 2r^4 = 0$

$r^4 = \frac{1}{2}$

$l \downarrow$

$$l = \mu r^2 \dot{\phi}$$

$$\frac{l}{\mu} = r_0^2 \bar{r}^2 \frac{1}{t_0} \dot{\phi}$$

$$\left(\frac{l}{\mu}\right)\left(\frac{t_0}{r_0^2}\right) = \bar{r}^2 \dot{\phi}$$

$$\frac{r_0^4}{t_0^2} = \left(\frac{l}{\mu}\right)^2 \Rightarrow \frac{r_0^2}{t_0} = \frac{l}{\mu} \Rightarrow \frac{t_0}{r_0^2} = \frac{\mu}{l}$$

$$\bar{r}^2 \dot{\phi} = 1$$

$$\boxed{\dot{\phi} = \frac{1}{\bar{r}^2}}$$

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\phi}^2 + U(r)$$

$$l = \mu r^2 \dot{\phi}$$

$$\dot{\phi} = \frac{l}{\mu r^2}$$

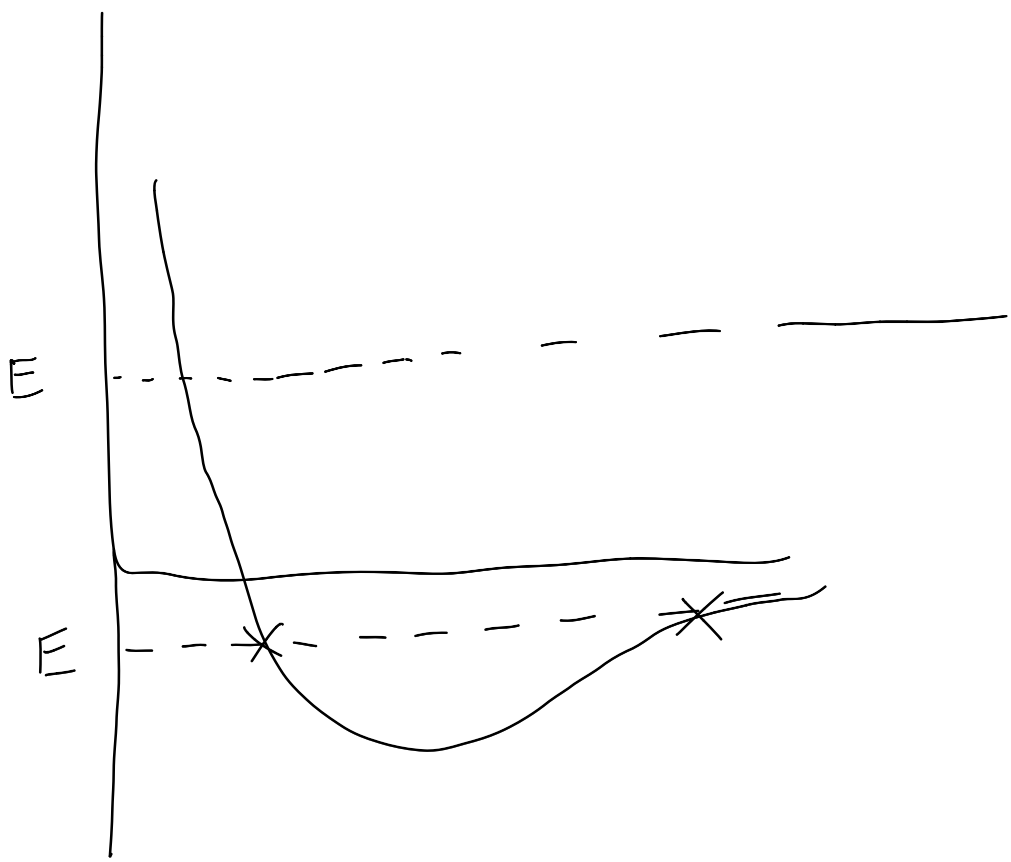
$$\dot{\phi}^2 = \frac{l^2}{\mu^2 r^4} = \left(\frac{l}{\mu}\right)^2 \frac{1}{r^4}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\left(\frac{l}{\mu}\right)^2\frac{1}{r^4} + U(r)$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$$

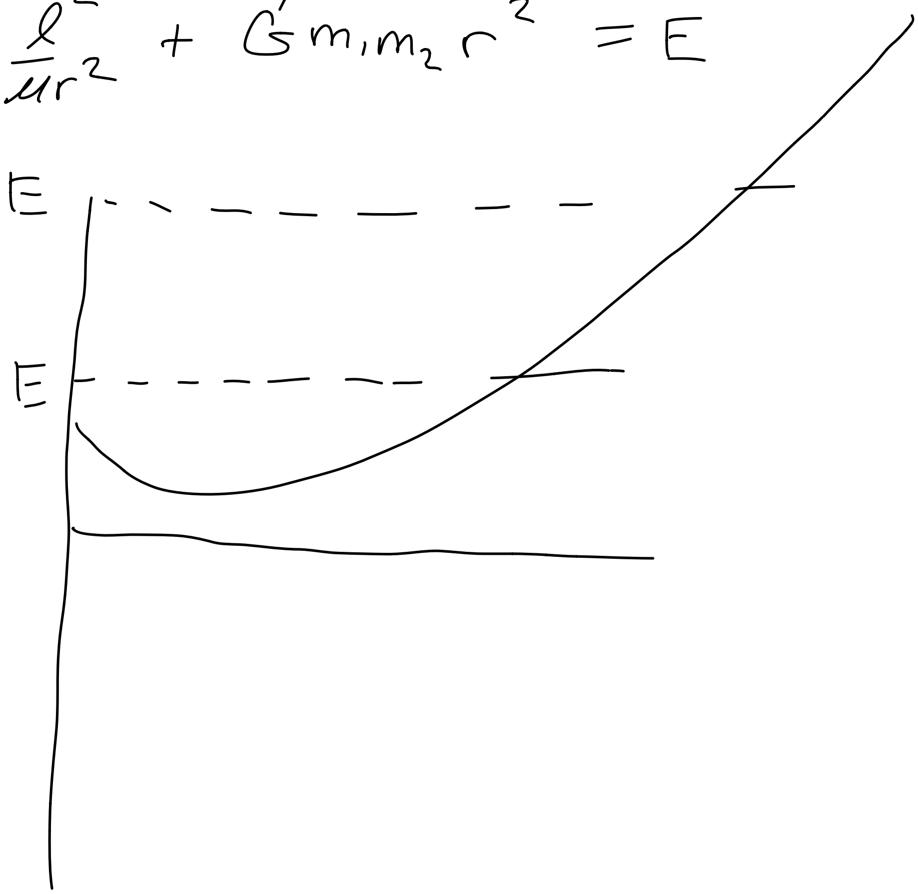
"Normal gravity"

$$\frac{1}{2}\frac{l^2}{\mu r^2} - \frac{G m_1 m_2}{r} = E$$



Spring Gravity

$$\frac{1}{2}\frac{\ell^2}{\mu r^2} + G' m_1 m_2 r^2 = E$$



$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

$$t_0^2 = \frac{1}{G'M}$$

$$\frac{r_0^4}{t_0^2} = \left(\frac{\ell}{\mu}\right)^2$$

$$r_0^4 = \frac{1}{G'M}\left(\frac{\ell}{\mu}\right)^2$$

$$E = \frac{1}{2}\mu \frac{r_0^2}{t_0^2}\dot{\bar{r}} + \frac{1}{2}\frac{\ell^2}{\mu r_0^2}\frac{1}{\bar{r}^2} + G'M\mu r_0^2 \bar{r}^2$$

$$r_0^2 = \frac{1}{\sqrt{G'M}}\frac{\ell}{\mu}$$

$$E = \frac{1}{2}\mu \frac{1}{\sqrt{G'M}}\frac{\ell}{\mu} G'M \dot{\bar{r}}$$

$$+ \frac{1}{2}\frac{\ell^2}{\mu}\sqrt{G'M}\frac{\mu}{\ell}\frac{1}{\bar{r}^2} + G'M\mu\frac{1}{\sqrt{G'M}}\frac{\ell}{\mu}\bar{r}^2$$

$$E = \frac{1}{2} \ell (G'M)^{1/2} \dot{\bar{r}}$$

$$+ \frac{1}{2} \ell (G'M)^{1/2} \frac{1}{\bar{r}^2}$$

$$+ \ell (GM') \bar{r}^2$$

$$\bar{E} = \frac{1}{2} \dot{\bar{r}} + \frac{1}{2} \frac{1}{\bar{r}^2} + \bar{r}^2$$