$$\vec{r} = r\hat{r} + \phi\hat{\phi}$$

$$\vec{r} = r\cos\phi\,\hat{x} + r\sin\phi\,\hat{y}$$

$$\dot{\vec{r}} = \left(\dot{r}\cos\phi - r\dot{\phi}\sin\phi\right)\hat{x}$$

$$+\left(\dot{r}\sin\phi + r\dot{\phi}\cos\phi\right)\hat{y}$$

$$\dot{r}^2 = r_x^2 + r_y^2$$

$$= \dot{r}^2\cos^2\phi + r^2\dot{\phi}^2\sin^2\phi - 2r\dot{r}\dot{\phi}\cos\phi\sin\phi$$

$$+ \dot{r}^2\sin^2\phi + r^2\dot{\phi}^2\cos^2\phi + 2r\dot{r}\dot{\phi}\sin\phi\cos\phi$$

$$= \dot{r}^2 + r^2\dot{\phi}^2$$

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$$

$$\dot{R} = 0$$

$$T = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$$

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \mathcal{U}(r)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)$$

$$0 = \frac{d}{dt}\left(\mu r^2 \dot{\phi}\right)$$

$$\begin{aligned} \mu r^2 \dot{\phi} &= \mu r (r\dot{\phi}) \\ &= r(\mu v_\phi) \\ &= [\vec{r} \times \mu \vec{v}]_z \end{aligned}$$

Angular momentum:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\boxed{l = \mu r^2 \dot{\phi}}$$

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

$$\frac{\partial \mathcal{L}}{\partial r} = \mu r \dot{\phi}^2 - \frac{\partial}{\partial r} U(r)$$

$$= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \mu \ddot{r}$$

$$\mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial}{\partial r} U(r)$$

$r\dot{\phi}^2$: $v_\phi = r\dot{\phi}$

$$r\dot{\phi}^2 = \frac{v_\phi^2}{r} = \text{centrifugal}$$

$$\mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial}{\partial r} U(r) \quad (1)$$

$$l = \mu r^2 \dot{\phi}$$

$$\dot{\phi}^2 = \left(\frac{l}{\mu}\right)^2 \frac{1}{r^4}$$

$$\mu \ddot{r} = \mu r \left(\frac{l}{\mu}\right)^2 \frac{1}{r^4} - \frac{\partial}{\partial r} U(r)$$

$$\ddot{r} = \left(\frac{l}{\mu}\right)^2 \frac{1}{r^3} - \frac{1}{\mu} \frac{\partial}{\partial r} U(r)$$

$$U = -W$$

$$W = \int_\infty^r -\frac{G m_1 m_2}{r'^2} \hat{r}' \cdot d\vec{r}'$$

$$= \frac{G m_1 m_2}{r}$$

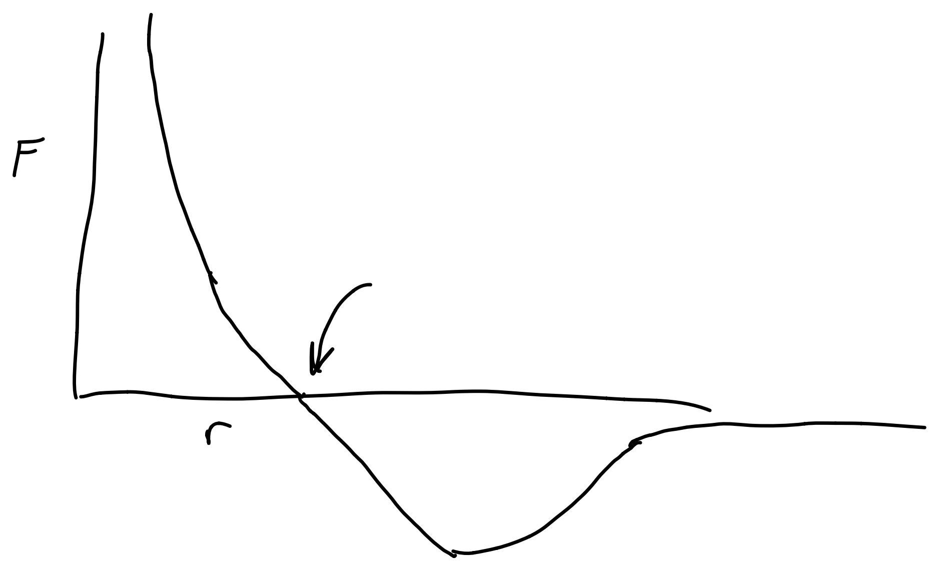
$$U = -\frac{G m_1 m_2}{r}$$

$$\ddot{r} = \left(\frac{l}{\mu}\right)^2 \frac{1}{r^3} - \frac{1}{\mu}\frac{\partial}{\partial r}\left(-\frac{Gm_1m_2}{r}\right)$$

$$\ddot{r} = \left(\frac{l}{\mu}\right)^2 \frac{1}{r^3} - \frac{1}{\mu}\frac{Gm_1m_2}{r^2}$$

$$\mu = \frac{m_1m_2}{M} \Rightarrow m_1m_2 = \mu M$$

$$\ddot{r} = \left(\frac{l}{\mu}\right)^2 \frac{1}{r^3} - \frac{GM}{r^2}$$



$$\ddot{r} = 0 = \left(\frac{l}{\mu}\right)^2 \frac{1}{r_0^3} - \frac{GM}{r_0^2}$$

$$0 = \left(\frac{l}{\mu}\right)^2 - GMr_0 \Rightarrow r_0 = \frac{1}{GM}\left(\frac{l}{\mu}\right)^2$$

$$\boxed{\begin{gathered} r_0 = \frac{1}{GM}\left(\frac{l}{\mu}\right)^2 \\ \text{Equilibrium Orbit} \end{gathered}}$$

$$\ddot{r}(r) = \left(\frac{l}{\mu}\right)^2 \frac{1}{r^3} - \frac{GM}{r^2}$$

$$\ddot{r}(r_0+\Delta r) = \left(\frac{l}{\mu}\right)^2 \frac{1}{(r+\Delta r)^3} - \frac{GM}{(r+\Delta r)^2}$$

Let $\Delta r << r$

$$(r_0+\Delta r)^{-\alpha} = r_0^{-\alpha}\left(1+\frac{\Delta r}{r_0}\right)^{-\alpha}$$

$$\left(1+\frac{\Delta r}{r_0}\right)^{-\alpha} \approx 1 - \alpha\frac{\Delta r}{r_0}$$

$$(r_0+\Delta r)^{-\alpha} \approx \left(1-\alpha\frac{\Delta r}{r_0}\right) r_0^{-\alpha}$$

$$\ddot{r}(r_0+\Delta r) \approx \left(\frac{l}{\mu}\right)^2 \left(1-3\frac{\Delta r}{r_0}\right)\frac{1}{r_0^3} - GM\left(1-2\frac{\Delta r}{r_0}\right)\frac{1}{r_0^2}$$

$$\ddot{r} \approx \underbrace{\left(\frac{l}{\mu}\right)^2 \frac{1}{r_0^3} - \frac{GM}{r_0^2}}_{=0} - 3\left(\frac{l}{\mu}\right)^2 \frac{\Delta r}{r_0^4} + \frac{2GM}{r_0^3}\Delta r$$

$$\ddot{r} \approx 2GM\frac{\Delta r}{r_0^3} - 3\left(\frac{l}{\mu}\right)^2 \frac{\Delta r}{r_0^4}$$

$$\ddot{r} \approx \left[\frac{2GM}{r_0^3} - 3\left(\frac{l}{\mu}\right)^2 \frac{1}{r_0^4}\right]\Delta r$$

$$r_0 = \frac{1}{GM}\left(\frac{l}{\mu}\right)^2$$

$$\frac{2GM}{r_0^3} = 2GM(GM)^3\left(\frac{\mu}{l}\right)^6 = 2(GM)^4\left(\frac{\mu}{l}\right)^6$$

$$3\left(\frac{l}{\mu}\right)^2\frac{1}{r_0^4} = 3\left(\frac{l}{\mu}\right)^2 (GM)^4\left(\frac{\mu}{l}\right)^8 = 3(GM)^4\left(\frac{\mu}{l}\right)^6$$

$$\ddot{r} \approx -(GM)^4\left(\frac{\mu}{l}\right)^6 \Delta r$$

oscillation with $\omega^2 = (GM)^4\left(\frac{\mu}{l}\right)^6$

$$\omega^2 = (GM)^4 \left(\frac{\mu}{\ell}\right)^6 = \frac{GM}{r_0^3}$$

$$\boxed{\omega^2 = \frac{GM}{r_0^3}}$$