Write our general solution as a sum over n

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

Still satisfies boundary conditions

What about $V(0,y)$?

$$V(0,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

Can we choose C_n to make this true?

Fourier sine series. We can but how?

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

Use a trick

multiply by $\sin\left(\frac{n'\pi y}{a}\right)$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right) = V_0(y)\sin\left(\frac{n'\pi y}{a}\right)$$

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \begin{cases} 0, & n' \neq n \\ \frac{a}{2}, & n' = n \end{cases}$$

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right)dy$$

$$n = n'$$

$$C_n \frac{a}{2} = \int_0^a V_0(y)\sin\left(\frac{n\pi}{a}y\right)dy$$

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi}{a}y\right)dy$$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

Suppose $V_0(y) = V_0$ (const)

Then:

$$C_n = \frac{2}{a} V_0 \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{2V_0}{n\pi}\Big(\cos(n\pi) - 1\Big)$$

$$= \frac{2V_0}{n\pi}\Big(1 - \cos(n\pi)\Big)$$

$$C_n = \begin{cases} 0 & , \text{ n even} \\ \frac{4V_0}{n\pi} & , \text{ n odd} \end{cases}$$

So

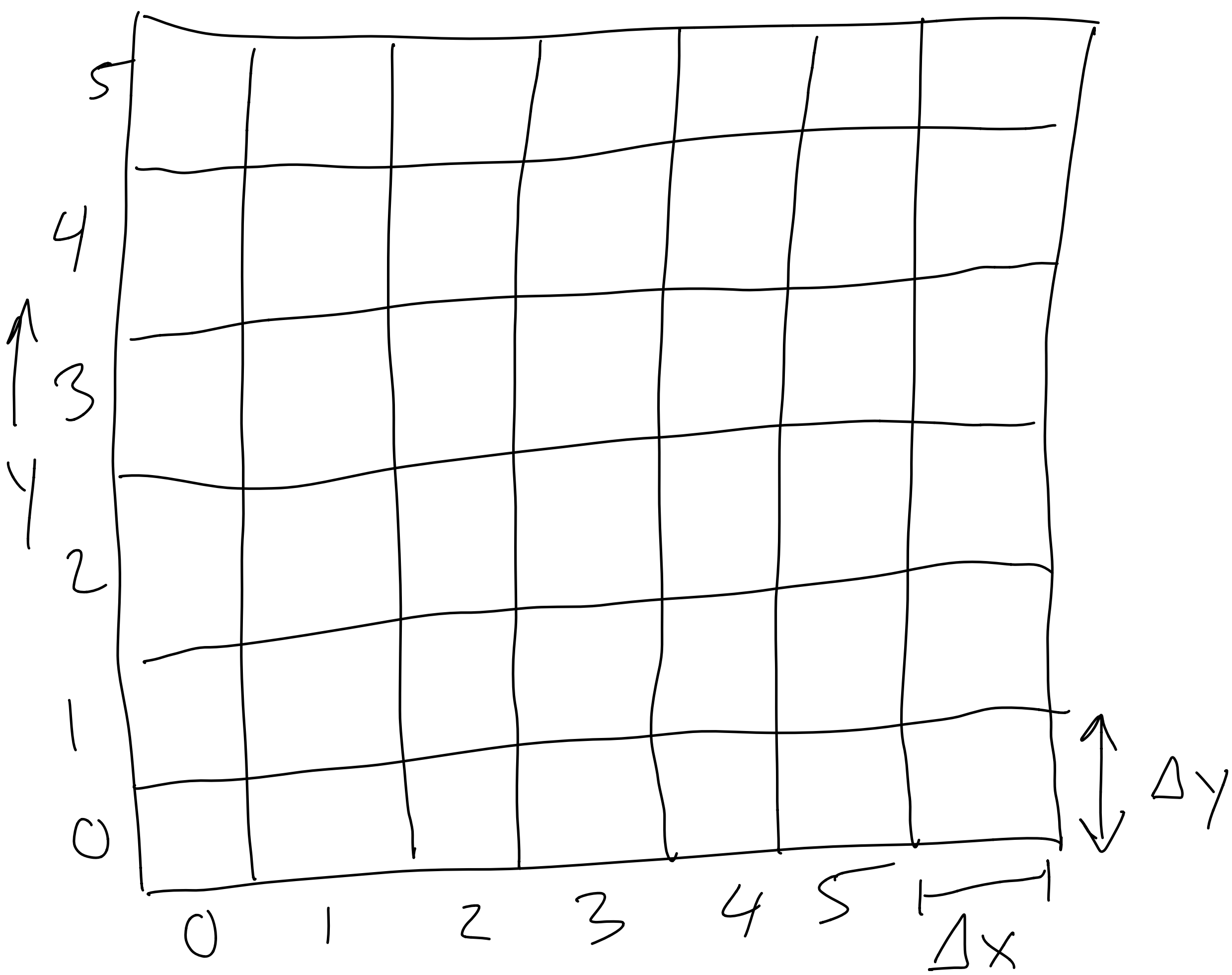
$$V(x,y) = \sum_{n=1,3,5} \frac{4V_0}{n\pi} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$V(x,y) = 2V_0 \arctan\left[\frac{\sin(\pi y/a)}{\sin(\pi x/a)}\right]$$

Show plot

How to solve numerically?

Discretize Space



$x \longrightarrow$

$i, j \longrightarrow x = i\Delta x,\ y = j\Delta y$

$3, 5 \longrightarrow 3\Delta x,\ 5\Delta y$

$$V(i,j,k) \equiv V(i\Delta x, j\Delta y, k\Delta z)$$

Want:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

What is $\dfrac{\partial V}{\partial x}$?

$$\frac{\partial V}{\partial x} = \lim_{h \to 0} \frac{V(x+h, y, z) - V(x, y, z)}{h}$$

$$\frac{\partial V}{\partial x} \approx \frac{V(i\Delta x + \Delta x, j\Delta y, k\Delta z) - V(i\Delta x, j\Delta y, k\Delta z)}{\Delta x}$$

$$\frac{\partial V}{\partial x} \approx \frac{V(i+1,j,k) - V(i,j,k)}{\Delta x}$$

equiv to

$$\frac{\partial V}{\partial x} \approx \frac{V(i,j,k) - V(i-1,j,k)}{\Delta k}$$

Finally

$$\frac{\partial V}{\partial x} \approx \frac{V(i+1,j,k) - V(i-1,j,k)}{2\Delta x}$$

2ND Deriv:

$$\frac{\partial^2 V}{\partial x^2} = \frac{\frac{\partial V}{\partial x}(i+1) - \frac{\partial V}{\partial x}(i)}{\Delta x}$$

$$= \frac{\frac{\partial V}{\partial x}(i) - \frac{\partial V}{\partial x}(i-1)}{\Delta x}$$

etc

Symmetric:

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{\Delta x}\left[\frac{V(i+1,j,k) - V(i,j,k)}{\Delta x} - \left(\frac{V(i,j,k) - V(i-1,j,k)}{\Delta x}\right)\right]$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{V(i+1,j,k) + V(i-1,j,k) - 2V(i,j,k)}{\Delta x^2}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{V(i,j+1,k) + V(i,j-1,k) - 2V(i,j,k)}{\Delta y^2}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{V(i,j,k+1) + V(i,j,k-1) - 2V(i,j,k)}{\Delta y^2}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} = 0$$

Assume $\Delta x = \Delta y = \Delta z$

$$V(i+1,j,k) + V(i-1,j,k) + V(i,j+1,k) + V(i,j-1,k) + V(i,j,k+1) + V(i,j,k-1) - 6V(i,j,k) = 0$$

$$V(i,j,k) = \frac{1}{6}\Big[V(i+1,j,k) + V(i-1,j,k) + V(i,j+1,k) + V(i,j-1,k) + V(i,j,k+1) + V(i,j,k-1) \Big]$$

	$V(i,j+1)$	
$V(i-1,j)$	$V(i,j)$	$V(i+1,j)$
	$V(i,j-1)$	