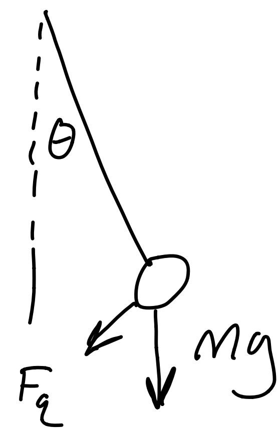


$$F_q = -q\frac{d\theta}{dt}$$

$$\frac{d^2\theta}{dt^2} = -\theta - Q\frac{d\theta}{dt}$$

$$Q = \frac{q\Omega_0}{mg}$$

Soln

$$\Theta = e^{-\frac{1}{2}Qt}\left[c_1 e^{i\Omega_q t} + c_2 e^{-i\Omega_q t}\right]$$

$$\Omega_q = \sqrt{1 - \left(\frac{Q}{2}\right)^2}$$

$$0 < Q < 2$$

$$\Theta = \underbrace{e^{-\frac{1}{2}Qt}}_{\text{Exp Decay}}\underbrace{\left[c_1 e^{i\Omega_q t} + c_2 e^{-i\Omega_q t}\right]}_{\text{oscillations with } \Omega_q}$$

Ω_q is Real

$$Q > 2$$

$$\Theta = \underbrace{e^{-\frac{1}{2}Qt}}_{\text{Decay}}\underbrace{\left[c_1 e^{i\Omega_q t} + c_2 e^{-i\Omega_q t}\right]}_{\text{Decay}}$$

Ω_q is complex

No oscillations $\Theta(t \to \infty) = 0$

What about $Q = 2$?

We assumed $\Theta = e^{rt}$

Characteristic Eqn

$$r^2 + Qr + 1 = 0$$

Solve w/ Quadratic Eqn

$$Q = 2$$

$$r^2 + 2r + 1 = (r+1)^2$$

$r = -1$ only solution $\Theta_1 = e^{-rt}$

Need to try something else

Try $\Theta_2 = te^{-rt}$

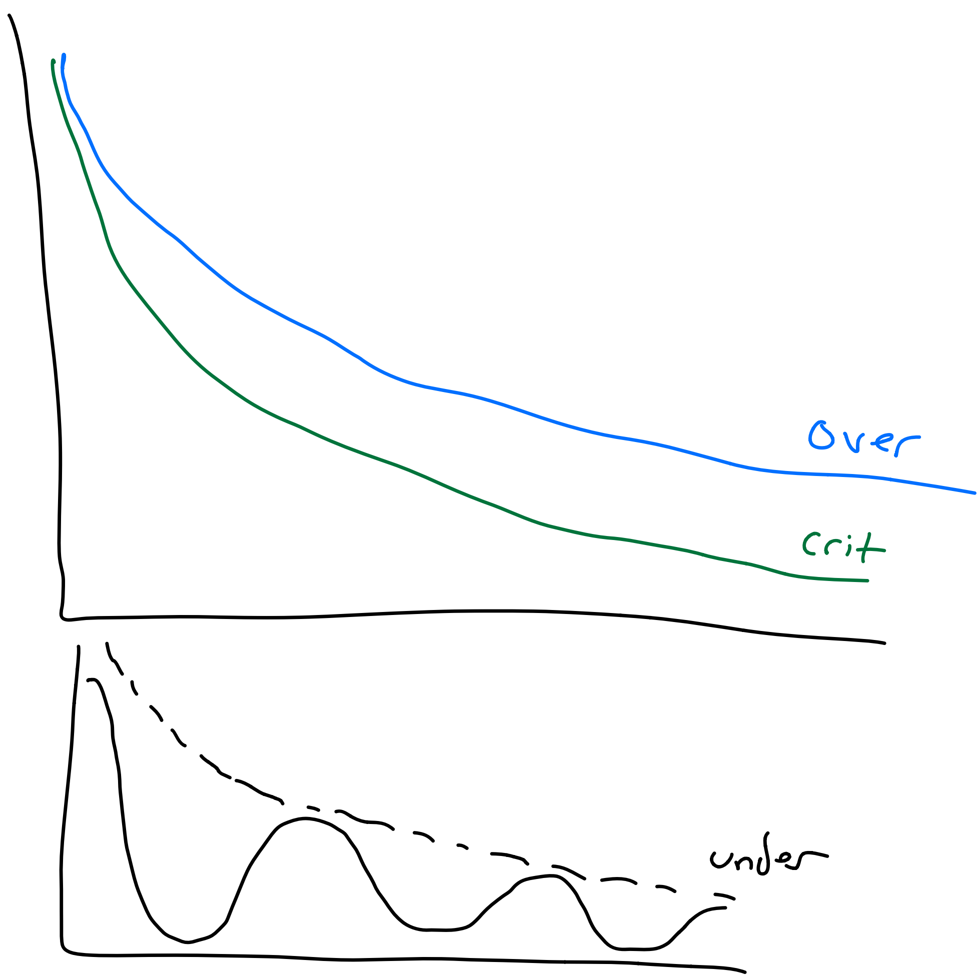
$$\Theta = c_1 e^{-rt} + c_2 t e^{-rt}$$

$$= e^{-rt}(c_1 + c_2 t)$$

$$\Theta = e^{-rt}(c_1 + c_2 t)$$

Critical Damping

No oscillations, approach equillibrium



Critical vs overdamping

Both approach Equilibrium
$(\Theta = 0, \omega = 0)$

- Critical damping is fastest possible

- Just enough damping to kill off oscillations

- Not so much damping to slow down approach to equilibrium

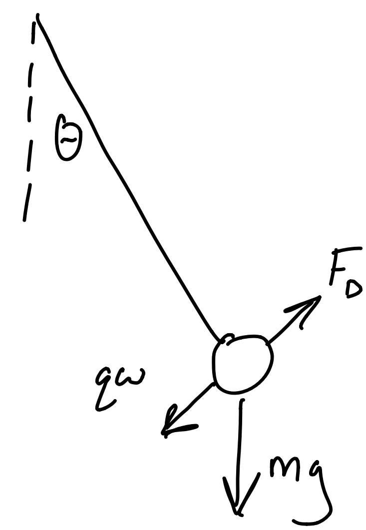
Code

$$\omega_i = \omega_{i-1} - \left(\theta_{i-1} + Q\omega_{i-1}\right)\Delta t$$

$$\theta_i = \theta_{i-1} + \omega_i \Delta t$$

Jupyter

Add Driving Term



Add oscillating driving force
+/- direction of motion

$$F_D = F_0 \cos(\Omega_d t)$$

$$\frac{d^2\theta}{dt^2} = -\theta - Q\frac{d\theta}{dt} + A\cos(\Omega_d t)$$

$$A = F_0/mg$$

$$\Theta = \Theta_H + \Theta_P$$

$$\frac{d^2\Theta_H}{dt^2} + Q\frac{d\Theta_H}{dt} + \Theta_H = 0$$

Want Θ_P such that

$$\frac{d^2\Theta_P}{dt^2} + Q\frac{d\Theta_P}{dt} + \Theta_P = A\cos(\Omega_d t)$$

Guess:

$$\Theta_P = a\cos(\Omega_d t) + b\sin(\Omega_d t)$$

$$\frac{d\Theta_P}{dt} = -a\Omega_d\sin(\Omega_d t) + b\Omega_d\cos(\Omega_d t)$$

$$\frac{d^2\Theta_P}{dt} = -a\Omega_d^2\cos(\Omega_d t) - b\Omega_d^2\sin(\Omega_d t)$$

$$\frac{d^2\Theta_p}{dt^2} + Q\frac{d\Theta_p}{dt} + \Theta_p = A\cos(\Omega_d t)$$

$$\frac{d\Theta_p}{dt} = -a\Omega_d \sin(\Omega_d t) + b\Omega_d \cos(\Omega_d t)$$

$$\frac{d^2\Theta_p}{dt} = -a\Omega_d^2 \cos(\Omega_d t) - b\Omega_d^2 \sin(\Omega_d t)$$

$$-a\Omega_d^2\cos(\Omega_d t) - b\Omega_D^2 \sin(\Omega_D t)$$

$$- Qa\Omega_D \sin(\Omega_D t) + Qb\Omega_D \cos(\Omega_D t)$$

$$+ a\cos(\Omega_D t) + b\sin(\Omega_D t) = A\cos(\Omega_D t)$$

$$(-a\Omega_D^2 + Qb\Omega_D + a)\cos(\Omega_D t)$$

$$+ (-b\Omega_D^2 - Qa\Omega_D + b)\sin(\Omega_D t) = A \cos\Omega_D t$$

$$-a\Omega_D^2 + Qb\Omega_D + a = A \qquad 1)$$

$$-b\Omega_D^2 - Qa\Omega_D + b = 0 \qquad 2)$$

solve for a & b

$$b(1-\Omega_D^2) = Qa\Omega_D \qquad 2)$$

$$b = \frac{Q\,a\Omega_0}{1-\Omega_D^2}$$

$b \rightarrow (1)$

$$-a\Omega_D^2 + Q\Omega_D\left(\frac{Qa\Omega_D}{1-\Omega_D^2}\right) + a = A$$

$$a\left[1-\Omega_D^2 + \frac{Q^2\Omega_D^2}{1-\Omega_D^2}\right] = A$$

$$a = \frac{A(1-\Omega_D^2)}{(1-\Omega_D^2)^2 + Q^2\Omega_D^2} \; ; \; b = \frac{Q\Omega_0}{(1-\Omega_D^2)^2 + Q^2\Omega_D^2}$$

General

$$\Theta = e^{-\frac{1}{2}Qt}\left[C_1 e^{i\Omega_q t} + C_2 e^{-i\Omega_q t}\right]$$

$$+ \frac{A(1-\Omega_D^2)}{(1-\Omega_D^2)^2+Q^2\Omega_D^2}\cos(\Omega_D t)$$

$$+ \frac{Q\Omega_D}{(1-\Omega_D^2)^2+Q^2\Omega_D^2}\sin(\Omega_D t)$$

Resonance

$$\Omega_D = 1$$

cos term cancels

$$\frac{Q}{Q^2}\sin(\Omega_D t) \rightarrow \frac{1}{Q}\sin(\Omega_D t)$$