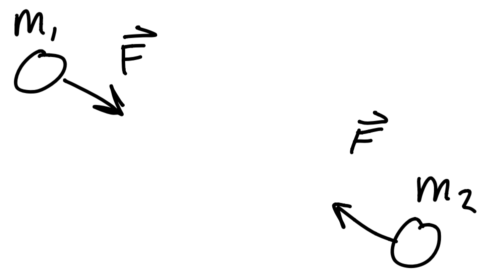
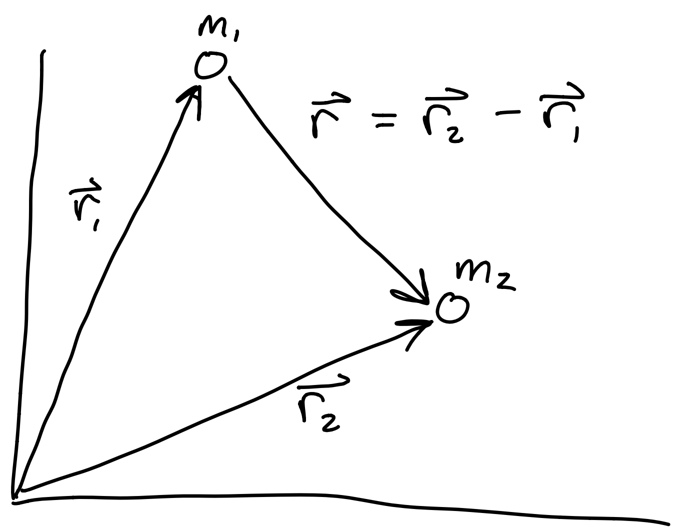
Central Force Problem



$$\vec{F} = -\vec{\nabla}\, U(r)$$

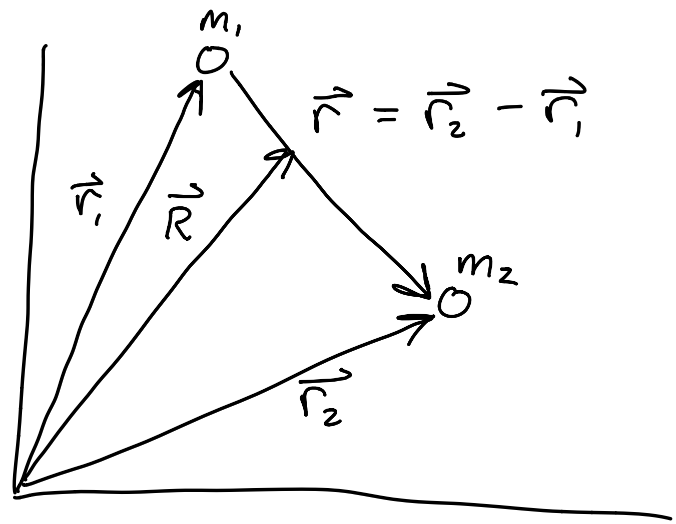
Can convert to 1-D



Potential $U(|\vec{r}|) = U(r)$

$$\mathcal{L} = T - U$$

$$T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$$



Change coordinates: $\vec{r}_1, \vec{r}_2 \longrightarrow \vec{R}, \vec{r}$

$\vec{R} = $ C.O.M

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}, \qquad \vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\vec{r}_2 = \vec{r} + \vec{r}_1$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 (\vec{r} + \vec{r}_1)}{m_1 + m_2}$$

$$\vec{R} = \frac{(m_1 + m_2) \vec{r}_1 + m_2 \vec{r}}{m_1 + m_2}$$

$$(m_1 + m_2) \vec{r}_1 = (m_1 + m_2) \vec{R} - m_2 \vec{r}$$

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \Rightarrow \vec{r}_1 = \vec{r}_2 - \vec{r}$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{R} = \frac{m_1(\vec{r}_2 - \vec{r}) + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{R} = \frac{(m_1 + m_2)\vec{r}_2 - m_1 \vec{r}}{m_1 + m_2}$$

$$(m_1 + m_2)\vec{R} = (m_1 + m_2)\vec{r}_2 - m_1 \vec{r}$$

$$\vec{r}_2 = \vec{R} + \frac{m_1}{m_1 + m_2}\vec{r}$$

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1 + m_2}\vec{r} = \vec{R} - \frac{m_2}{M}\vec{r}$$

$$\vec{r}_2 = \vec{R} + \frac{m_1}{m_1 + m_2}\vec{r} = \vec{R} + \frac{m_1}{M}\vec{r}$$

$$L = T - U(|\vec{r}|) = T - U$$

$$T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$$

$$\vec{r}_1 = \vec{R} - \frac{m_2}{M}\vec{r} \qquad \vec{r}_2 = \vec{R} + \frac{m_1}{M}\vec{r}$$

$$\dot{\vec{r}}_1 = \dot{\vec{R}} - \frac{m_2}{M}\dot{\vec{r}} \qquad \dot{\vec{r}}_2 = \dot{\vec{R}} - \frac{m_2}{M}\dot{\vec{r}}$$

$$\dot{r}_1^{\,2} = \left(\dot{\vec{R}} - \frac{m_2}{M}\dot{\vec{r}}\right) \cdot \left(\dot{\vec{R}} - \frac{m_2}{M}\dot{\vec{r}}\right)$$

$$\boxed{\dot{r}_1^{\,2} = \dot{R}^2 + \left(\frac{m_2}{M}\right)^2 \dot{r}^2 - 2\frac{m_2}{M}\dot{\vec{R}}\cdot\dot{\vec{r}}}$$

$$\dot{r}_2^{\,2} = \left(\dot{\vec{R}} + \frac{m_1}{M}\dot{\vec{r}}\right) \cdot \left(\dot{\vec{R}} + \frac{m_1}{M}\dot{\vec{r}}\right)$$

$$\boxed{\dot{r}_2^{\,2} = \dot{R}^2 + \left(\frac{m_1}{M}\right)^2 \dot{r}^2 + 2\frac{m_1}{M}\dot{\vec{R}}\cdot\dot{\vec{r}}}$$

$$T = \frac{1}{2}m_1\left[\dot{R}^2 + \left(\frac{m_2}{M}\right)^2\dot{r}^2 - 2\frac{m_2}{M}\dot{\vec{R}}\cdot\dot{\vec{r}}\right]$$

$$+ \frac{1}{2}m_2\left[\dot{R}^2 + \left(\frac{m_1}{M}\right)^2\dot{r}^2 + 2\frac{m_1}{M}\dot{\vec{R}}\cdot\dot{\vec{r}}\right]$$

$$T = \frac{1}{2}m_1\dot{R}^2 + \frac{1}{2}m_1\left(\frac{m_2}{M}\right)^2\dot{r}^2$$

$$+ \frac{1}{2}m_2\dot{R}^2 + \frac{1}{2}m_2\left(\frac{m_1}{M}\right)^2\dot{r}^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{R}^2 + \frac{1}{2}\frac{m_1 m_2}{M}\left(\frac{m_2+m_1}{M}\right)\dot{r}^2$$

$$T = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \frac{m_1 m_2}{M} \dot{r}^2$$

$$\mu := \frac{m_1 m_2}{M} \quad \text{reduced mass}$$

$$T = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2$$

T of two different particles!

$$\mathcal{L} = T - U$$

$$= \frac{1}{2} M\left(\dot{x}_R^2 + \dot{y}_R^2 + \dot{z}_R^2\right) + \frac{1}{2} \mu\left(\dot{x}_r^2 + \dot{y}_r^2 + \dot{z}_r^2\right)$$

$$- U(r)$$

$$\frac{\partial \mathcal{L}}{\partial x_R} = 0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_R}\right) = M \ddot{x}_R$$

$$M \ddot{x}_R = M \ddot{y}_R = M \ddot{z}_R = 0$$

$$\dot{\vec{R}} = \text{constant}$$

Choose reference frame where

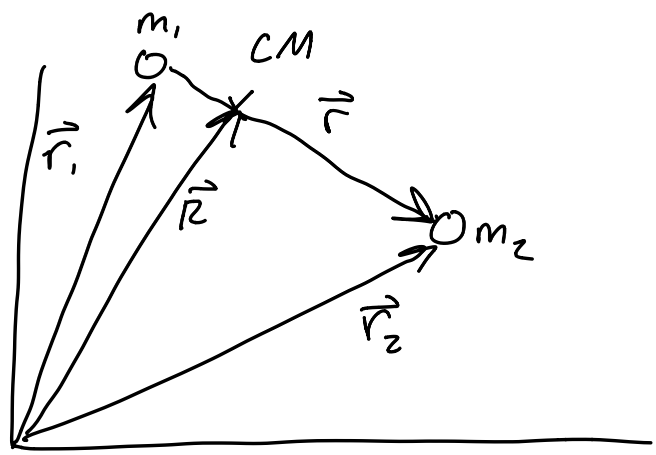
$\vec{R} = 0$

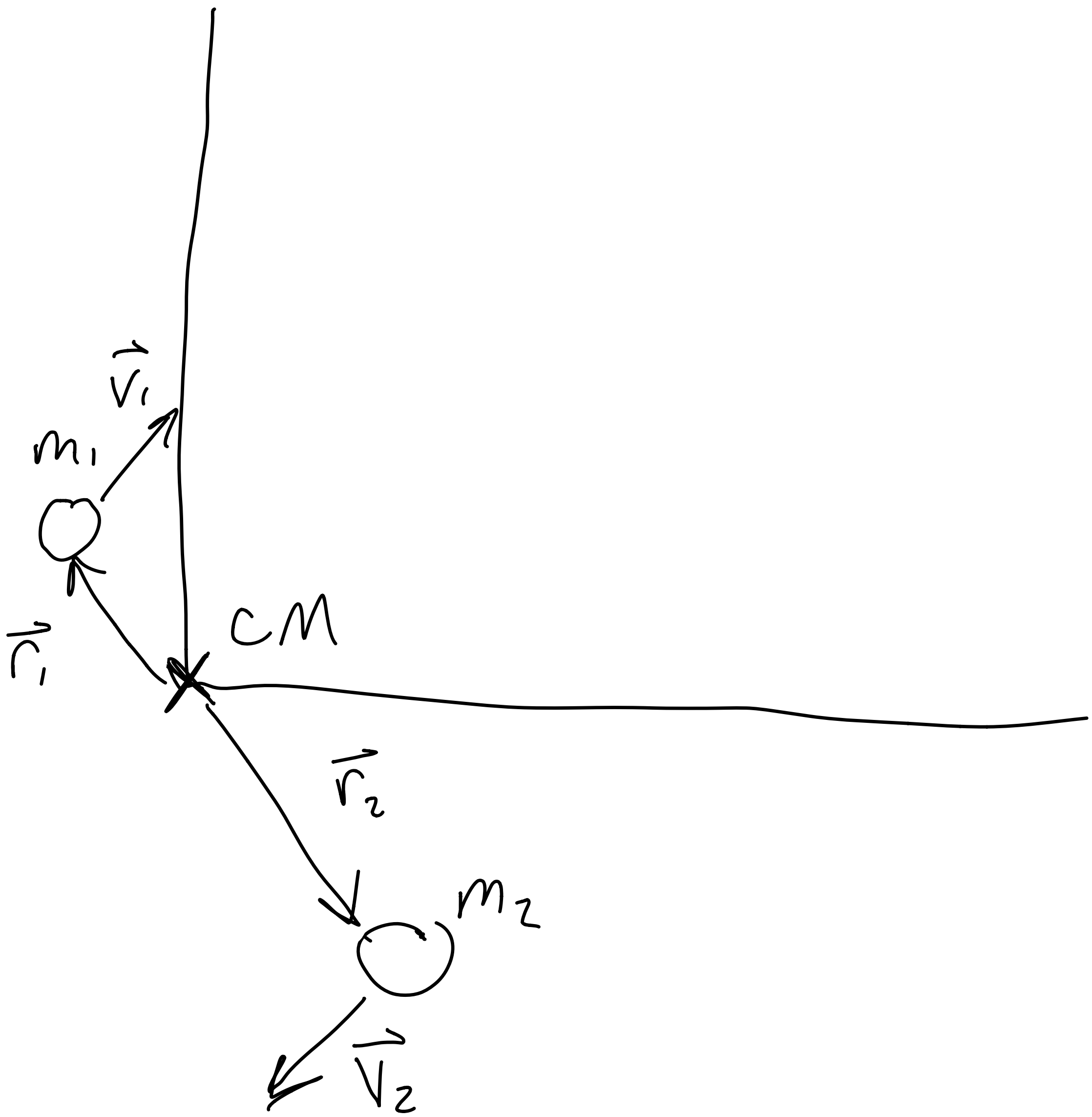
$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}$$

$$\dot{R} = \frac{m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2}{M} = 0$$

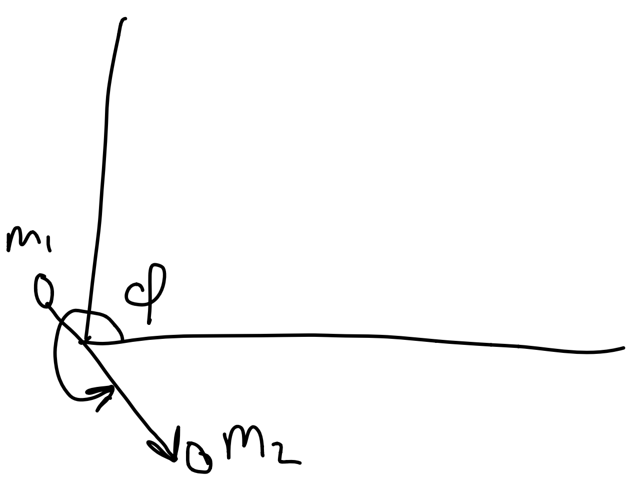
$$m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 = 0$$

Total momentum $= 0$





$\vec{v}_1$
m_1
$\vec{r}_1$
CM
$\vec{r}_2$
m_2
$\vec{V}_2$



m_1
φ
m_2

$$\vec{r} = r\hat{r} + \phi\hat{\phi}$$

$$\vec{r} = r\cos\phi\,\hat{x} + r\sin\phi\,\hat{y}$$

$$\dot{\vec{r}} = (\dot{r}\cos\phi - r\dot{\phi}\sin\phi)\hat{x}$$
$$+ (\dot{r}\sin\phi + r\dot{\phi}\cos\phi)\hat{y}$$

$$\dot{r}^2 = r_x^2 + r_y^2$$
$$= \dot{r}^2\cos^2\phi + r^2\dot{\phi}^2\sin^2\phi - 2r\dot{r}\dot{\phi}\cos\phi\sin\phi$$
$$+ \dot{r}^2\sin^2\phi + r^2\dot{\phi}^2\cos^2\phi + 2r\dot{r}\dot{\phi}\sin\phi\cos\phi$$
$$= \dot{r}^2 + r^2\dot{\phi}^2$$

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2)$$

$$\dot{R} = 0$$

$$T = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2)$$

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - \mathcal{U}(r)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)$$

$$0 = \frac{d}{dt}\left(\mu r^2 \dot{\phi}\right)$$

$$\begin{aligned} \mu r^2 \dot{\phi} &= \mu r (r\dot{\phi}) \\ &= r(\mu v_\phi) \\ &= \left[\vec{r} \times \mu \vec{v}\right]_z \end{aligned}$$

Angular momentum:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\boxed{l = \mu r^2 \dot{\phi}}$$

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

$$\frac{\partial \mathcal{L}}{\partial r} = \mu r \dot{\phi}^2 - \frac{\partial}{\partial r} U(r)$$

$$= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \mu \ddot{r}$$

$$\mu \ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial}{\partial r} U(r)$$

$r\dot{\phi}^2$: $v_\phi = r\dot{\phi}$

$$r\dot{\phi}^2 = \frac{v_\phi^2}{r} = \text{centrifugal}$$