Last lecture

Derived Poisson's Eqn

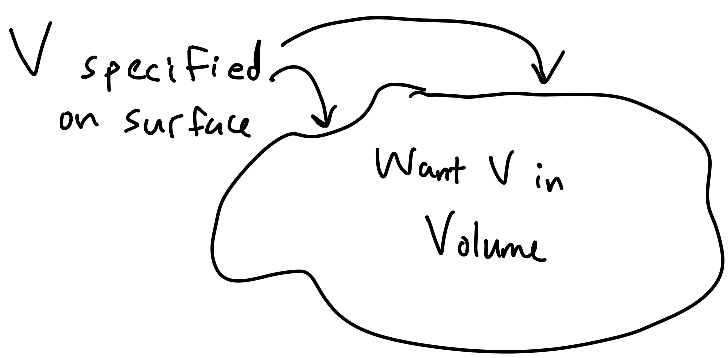
$$\nabla^2 V = \frac{-\rho}{\epsilon_0}$$

Today:

Consider Laplace's Eqn:

$$\nabla^2 V = 0$$

ρ is not 0 everywhere



Uniqueness: The solution to $\nabla^2 V = 0$ in a volume $\mathcal{V}$ is uniquely determined if V is specified on the surface

If we can find a solution that

a) has the correct value at the boundary

&

b) satisfies $\nabla^2 V = 0$

then it has to be correct

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

IN 1 D:

$$\frac{\partial^2 V}{\partial x^2} = 0 \implies \frac{d^2 V}{dx^2} = 0$$

Integrate twice

$$\frac{dV}{dx} = C_1$$

$$V = C_1 x + C_2$$

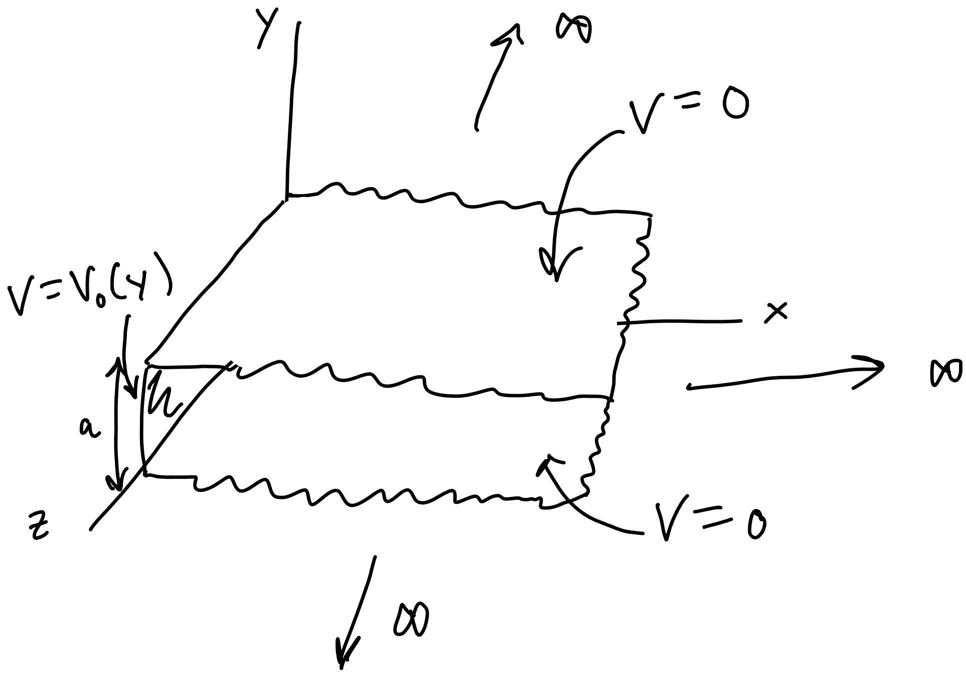
In 2D:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

2ND ORDER PDE

USE Separation of variables

Example



$V = V(x,y)$

Boundaries

$V(0,y) = V_0(y)$

$V(x,0) = V(x,a) = 0$

$V(x \to \infty, y) = 0$

$\rho = 0$, so $\nabla^2 V = 0$

Separation of variables

Assume $V(x,y) = X(x)\,Y(y)$

I know this seems ridiculous, but watch what happens

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} V + \frac{\partial^2 V}{\partial y^2}$$

$$= Y(y)\frac{\partial^2}{\partial x^2} X(x) + X(x)\frac{\partial^2}{\partial y^2} Y(y) = 0$$

Now, divide by $X(x)\,Y(y)$

$$\underbrace{\frac{1}{X(x)}\frac{\partial^2}{\partial x^2} X(x)}_{f(x)} + \underbrace{\frac{1}{Y(y)}\frac{\partial^2}{\partial y^2} Y(y)}_{g(y)} = 0$$

$$f(x) + g(y) = 0$$

$\Rightarrow$ $f(x)$, $g(y)$ are constants

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = C_1$$

$$\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = C_2$$

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = C_1$$

$$\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} = C_2$$

$$C_1 + C_2 = 0$$

$$C_1 = k^2 , \quad C_2 = -k^2$$

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = k^2$$

$$\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} = -k^2$$

$$\frac{d^2 X(x)}{dx^2} = k^2 X(x)$$

$$a y'' + b y' + c y = 0$$

$$y = e^{\lambda x}$$

$$a \lambda^2 e^{\lambda x} + b \lambda e^{\lambda x} + c e^{\lambda x} = 0$$

$$a \lambda^2 + b \lambda + c = 0$$

$$\lambda^2 - k^2 = 0 \Rightarrow \lambda = \pm k$$

$$X(x) = Ae^{kx} + Be^{-kx}$$

$$Y(y) = C\sin(ky) + D\cos(ky)$$

$$V(x,y) = X(x)Y(y) = \left(Ae^{kx} + Be^{-kx}\right)\left(C\sin(ky) + D\cos(ky)\right)$$

Boundary Conditions:

$$V(x \to \infty) = 0 \Rightarrow A = 0$$

$$V(x,y) = Be^{-kx}\left(C\sin(ky) + D\cos(ky)\right)$$

$$V(x,0) = 0 \Rightarrow D = 0$$

$$V(x,y) = Be^{-kx}\,C\sin(ky)$$

$$V(x,y) = Ce^{-kx}\sin(ky)$$

$$V(x,a) = 0 \Rightarrow \sin(ka) = 0$$

$$ka = n\pi$$

$$k = \frac{n\pi}{a}, \quad n = 1,2,3\ldots$$

$$V(x,y) = Ce^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi}{a} y\right)$$

$$V(0,y) = V_0(y)$$

$$C \sin\left(\frac{n\pi}{a} y\right) \stackrel{?}{=} V_0(y) \; ?$$

- Solution doesn't work unless $V_0(y)$ looks like $\sin\left(\frac{n\pi}{a} y\right)$

- What is n?

Laplace's Eqn is Linear

$$\nabla^2(V_1+V_2) = \nabla^2 V_1 + \nabla^2 V_2$$

if V_1, V_2, V_3 are solutions,

so is $V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 \cdots$

$$\nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \ldots$$

$$= \alpha_1 (0) + \alpha_2 (0) + \ldots$$

Write our general solution as a sum over n

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

Still satisfies boundary conditions

What about $V(0,y)$?

$$V(0,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

Can we choose C_n to make this true?

Fourier sine series. We <u>can</u> but how?

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

Use a trick

multiply by $\sin\left(\frac{n'\pi y}{a}\right)$

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right) = V_0(y)\sin\left(\frac{n'\pi y}{a}\right)$$

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \begin{cases} 0, & n' \neq n \\ \frac{a}{2}, & n' = n \end{cases}$$

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right) dy$$

$$n = n'$$

$$C_n \frac{a}{2} = \int_0^a V_0(y)\sin\left(\frac{n\pi}{a}y\right) dy$$

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi}{a}y\right) dy$$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

Suppose $V_0(y) = V_0$ (const)

Then:

$$C_n = \frac{2}{a} V_0 \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= -\frac{2V_0}{n\pi}\left(\cos(n\pi) - 1\right)$$

$$= \frac{2V_0}{n\pi}\left(1 - \cos(n\pi)\right)$$

$$C_n = \begin{cases} 0, & n \text{ even} \\ \frac{4V_0}{n\pi}, & n \text{ odd} \end{cases}$$

So

$$V(x,y) = \sum_{n=1,3,5} \frac{4V_0}{n\pi} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$V(x,y) = 2V_0 \arctan\left[\frac{\sin(\pi y/a)}{\sin(\pi x/a)}\right]$$