Lagrangian Mechanics

- Consider an single particle moving with velocity v and subject to a *conservative* force

Conservative Force

Net Work along closed path is 0

$$\oint \vec{F} \cdot d\vec{r} = 0$$

Ex: Gravity

Equivalently:

$$\vec{\nabla} \times \vec{F} = 0$$

Curl of $\vec{F}$ is 0.

Recall:

$$\vec{\nabla} \times (\vec{\nabla}\phi) = 0$$

so there exists some scalar field such that $\vec{F} = \vec{\nabla}\phi$

$\phi = -\mathcal{U}$, $\mathcal{U}$ = Potential Energy

$$\boxed{\vec{F} = -\vec{\nabla}\mathcal{U}, \text{ if } \vec{F} \text{ is conservative}}$$

Why the minus sign?

$$E = T + \mathcal{U} = \text{constant}$$

$$\Delta E = 0 = \Delta T + \Delta \mathcal{U}, \quad \Delta T = W \quad \text{(work)}$$

$$0 = W + \Delta \mathcal{U}$$

$$0 = \int_a^b \vec{F} \cdot d\vec{r} + \Delta \mathcal{U}$$

$$\int_a^b \vec{F} \cdot d\vec{r} = -\Delta \mathcal{U}$$

$$\vec{F} = -\vec{\nabla}\mathcal{U}$$

$\vec{F}$ points toward <u>decreasing</u> potential

- For a conservative system, then:

$$KE = T = \frac{1}{2} m v^2$$

$$dT = \vec{F} \cdot d\vec{r} = \vec{F} \cdot \vec{v}\, dt$$

$$= \frac{d\vec{p}}{dt} \cdot \vec{v}\, dt = d\vec{p} \cdot \vec{v}$$

$$= m\, d\vec{v} \cdot \vec{v}$$

$$d\vec{v} \cdot \vec{v} = \frac{1}{2} d(\vec{v} \cdot \vec{v})$$

$$dT = \frac{1}{2} m\, d(\vec{v} \cdot \vec{v})$$

$$dT = \frac{1}{2} m\, d(v^2)$$

$$T = \frac{1}{2} m v^2$$

$$U = U(\vec{r})$$

$$E = T + U, \quad \frac{dE}{dt} = 0$$

- Consider instead the quantity T - U
  - Why would we do this???
  - Stay with me

$$\mathcal{L} = T - U$$

$$= \frac{1}{2} m v^2 - U(\vec{r})$$

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = F_x \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x \qquad (2)$$

$$F_x = \frac{d}{dt} p_x = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

$$F_x = \frac{\partial \mathcal{L}}{\partial x} \implies \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

Euler-Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)$$

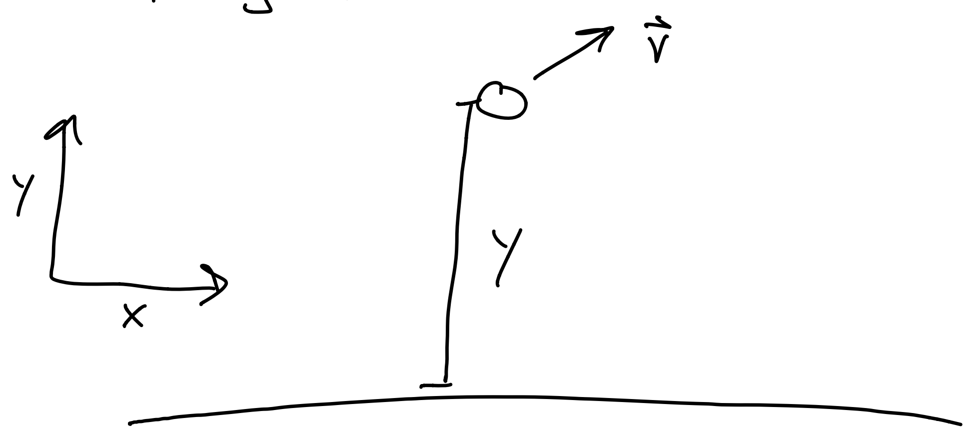
$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right)$$

$$\mathcal{L} = T - U$$

$\mathcal{L}$ = "Lagrangian"

Simple Example

Projectile Motion



$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$U = mgy$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

$$0 = \frac{d}{dt}(m\dot{x}) \longrightarrow m\ddot{x} = 0 \quad \checkmark$$

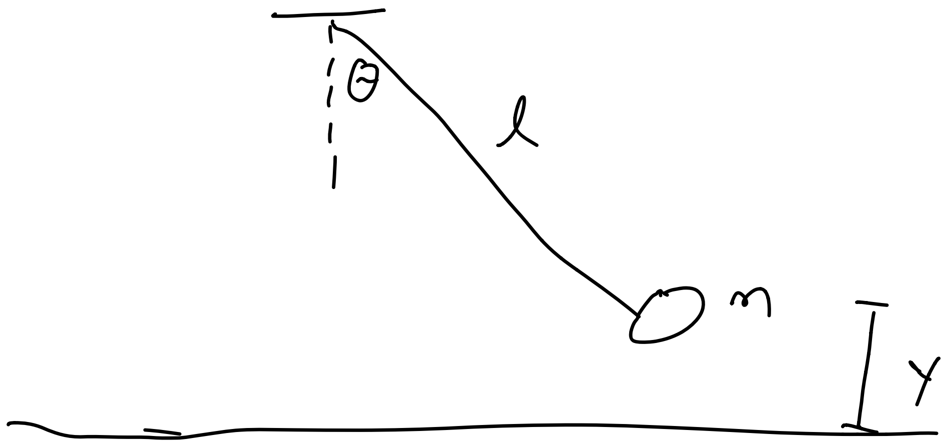
$$\ddot{x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)$$

$$-mg = \frac{d}{dt}(m\dot{y})$$

$$m\ddot{y} = -mg \longrightarrow \ddot{y} = -g \quad \checkmark$$

Independent of coordinate system



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{\vec{r}} \cdot \dot{\vec{r}})$$

$$\vec{r} = l \sin\theta \, \hat{x} - l \cos\theta \, \hat{y}$$

$$\dot{\vec{r}} = l \dot{\theta} \cos\theta \, \hat{x} + l \dot{\theta} \sin\theta \, \hat{y}$$

$$\dot{\vec{r}} \cdot \dot{\vec{r}} = l^2 \dot{\theta}^2 \cos^2\theta + l^2 \dot{\theta}^2 \sin^2\theta$$

$$= l^2 \dot{\theta}^2$$

$$T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$U = mgy$$

$$y = -l\cos\theta - l = -l(\cos\theta - 1)$$

$$y = l(1 - \cos\theta)$$

$$U = mgl(1-\cos\theta)$$

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

(1) $$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right)$$

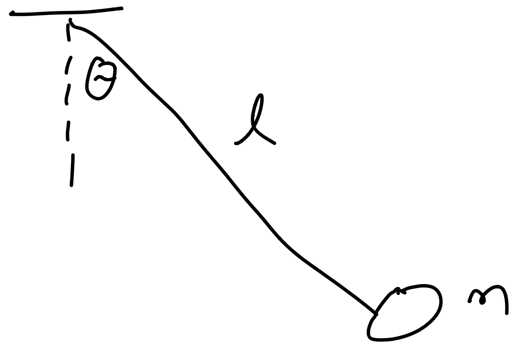
$$mgl\frac{\partial(\cos\theta)}{\partial\theta} = \frac{d}{dt}\left(ml^2\dot{\theta}\right)$$

$$-mgl\sin\theta = ml^2\ddot{\theta}$$

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

Could also use x + y
coords

$-l-y$



$$U = mg(-l-y)$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\mathcal{L} = T - U$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

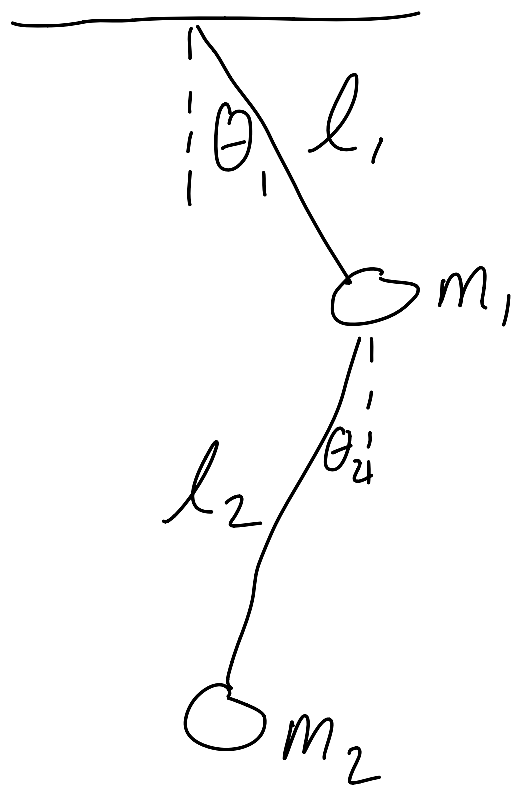
$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)$$

x & y are not independent

$y = \sqrt{\ell^2 - x^2}$

Why use Lagrangian?

Sometimes much easier!



$$\vec{r}_1 = l_1 \sin\theta_1 \hat{x} - l_1 \cos\theta_1 \hat{y}$$

$$\vec{r}_2 = \left(l_1 \sin\theta_1 + l_2 \sin\theta_2\right)\hat{x} - \left(l_1 \cos\theta_1 + l_2 \cos\theta_2\right)\hat{y}$$

$$\mathcal{L} = T - U$$

$$T = T_1 + T_2$$

$$T_1 = \frac{1}{2} m_1 \dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1$$

$$\dot{\vec{r}}_1 = l_1 \dot{\theta}_1 \cos\theta \, \hat{x} + l_1 \dot{\theta}_1 \sin\theta_1 \hat{y}$$

$$\dot{r}_1^2 = l_1^2 \dot{\theta}_1^2$$

$$T_1 = \frac{1}{2} m l_1^2 \dot{\theta}^2$$

$$T_2 = \frac{1}{2} m_2 \dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2$$

$$\dot{\vec{r}}_2 = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2 \, \hat{x}$$

$$- \left(-l_1 \dot{\theta}_1 \sin\theta_1 - l_2 \dot{\theta}_2 \sin\theta_2 \right) \hat{y}$$

$$\dot{\vec{r}}_2 = \left(l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2\right)\hat{x} + \left(l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2\right)\hat{y}$$

$$\dot{r}_2^2 = l_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + l_2^2 \dot{\theta}_2^2 \cos^2\theta_2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 + l_1^2 \dot{\theta}_1^2 \sin^2\theta_1 + l_2^2 \dot{\theta}_2^2 \sin^2\theta_2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2$$

$$\dot{r}_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \left(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2\right)$$

$$\dot{r}_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

$$U = m_1 g \vec{r}_1 \cdot \hat{y}$$
$$+ m_2 g \vec{r}_2 \cdot \hat{y}$$

$$U = -m_1 g l_1 \cos\theta_1$$
$$- m_2 g \left[l_1 \cos\theta_1 + l_2 \cos\theta_2 \right]$$

$$U = -(m_1 + m_2) g l_1 \cos\theta_1$$
$$- m_2 g l_2 \cos\theta_2$$

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

E.L. Eqns

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$
$$-(m_1 + m_2) g l_1 \sin\theta_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1$$
$$+ m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt} \quad \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1$$

$$+ m_2 l_1 l_2 \Big[\ddot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$- \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)$$

$$+ \dot{\theta}_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$= m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1$$

$$+ m_2 l_1 l_2 \ddot{\theta} \cos(\theta_1 - \theta_2)$$

$$+ m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_2 - \dot{\theta}_1) \sin(\theta_1 - \theta_2)$$