- Brief review of last class main topics
    - Review Euler method
    - Re-derive error (both delta t and delta t squared)
- What does it mean to have a small time step?
    - If I'm doing a radioactive decay problem, do I still need a time step of 10^-3?
    - Small time step *relative to the system*
    - Show jupyter example
- In fact, when numerically solving systems it often behooves to work with a system a units defined relative to the system itself

$$f(t) = f(t_0) + f'(t_0)\Delta t + \frac{1}{2} f''(t_0)\Delta t^2$$

Keep using units of τ

$$t_{stop} = 5 \cdot \tau, \quad dt = 0.1\,\tau$$

Let's just make a variable change!

$$\bar{t} = \frac{t}{\tau} \qquad t \to \bar{t}\,\tau$$

$$\frac{dN}{dt} = -\frac{1}{\tau} N$$

$$\frac{dN}{dt} \to \frac{dN}{d(\bar{t}\tau)} = \frac{1}{\tau}\frac{dN}{d\bar{t}} = -\frac{1}{\tau} N$$

$$\frac{dN}{d\bar{t}} = -N$$

Problem is timescale independent!

Q: At what time is N 25% of N_0?

- Code solution together in jupyter
- Often times, it's more convenient to just work in terms of fraction of initial value.
- We can make this problem completely general if we do this.

$$\frac{dN}{dt} = -N$$

$$\bar{N} = \frac{N}{N(0)} \Rightarrow N = N_0 \bar{N}$$

$$\frac{d}{dt}(N_0 \bar{N}) = -N_0 \bar{N}$$

$$\cancel{N_0} \frac{d}{dt} \bar{N} = -\cancel{N_0} \bar{N}$$

$$\frac{d}{dt} \bar{N} = -\bar{N}$$

- Radioactive decay ~ 50 orders of mag
- RC circuit ~ μs
- Reaction rates ~ s

...

just solved them all!

- It's standard practice to work with dimensionless equations like this
    - Plus it means we don't need to keep track of units!
- How to make an equation dimensionless:

dynamic variable = y

dependent var = t

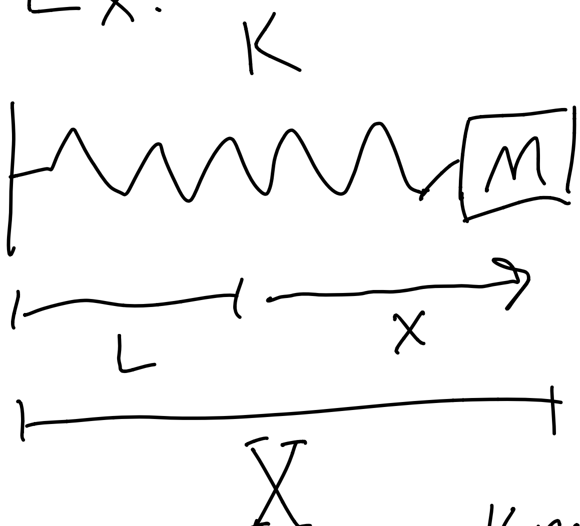
$$\bar{y} = \frac{y}{y_0} \; ; \quad \bar{t} = \frac{t}{t_0}$$

can then choose y_0 & t_0

$$\frac{d}{dt} \rightarrow \frac{1}{t_0}\frac{d}{d\bar{t}}$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\frac{d}{dt} = \frac{1}{t_0^2}\frac{d^2}{d\bar{t}^2}$$

Ex:



$$x = X - L$$

know $X(0)$, $V(0)$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = \frac{-k}{m}x$$

$$\bar{x} = \frac{x}{x_0} \; ; \; \bar{t} = \frac{t}{t_0}$$

Do we normalize k & m?

NO

$$\frac{d^2}{dt^2} \rightarrow \frac{1}{t_0}\frac{d}{d\bar{t}}\left(\frac{1}{t_0}\frac{d}{d\bar{t}}\right)$$

$$= \frac{1}{t_0}\frac{d^2}{d\bar{t}^2}$$

$$\frac{1}{t_0^2}\frac{d^2}{d\bar{t}^2}(x_0\bar{x}) = \frac{-k}{m}x_0\bar{x}$$

$$\frac{\cancel{x_0}}{t_0^2}\frac{d^2}{d\bar{t}^2}\bar{x} = \frac{-k}{m}\cancel{x_0}\bar{x}$$

$$\frac{d^2}{d\bar{t}^2}\bar{x} = t_0^2\left(\frac{-k}{m}\right)\bar{x}$$

We can choose t_0!

dimensions of k/m?

$$[k] = \frac{F}{L} = \frac{MA}{L} = \frac{ML}{LT^2} = \frac{M}{T^2}$$

$$\left[\frac{k}{m}\right] = \frac{1}{M}\frac{M}{T^2} = \frac{1}{T^2}$$

$$t_0^2\left(\frac{k}{m}\right) = 1$$

$$t_0 = \sqrt{\frac{m}{k}} = \omega$$

What ab X_0?

use initial condition!

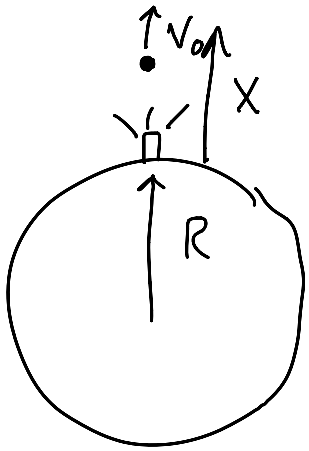
$$\bar{X} = \frac{X}{X(0)}$$

$\bar{X}$ = frac of max amplitude

$$\frac{d^2}{d\bar{t}^2}\bar{X} = -\bar{X} \quad \checkmark$$

units of v? $v = v_0\bar{v} = \frac{x_0}{t_0}\bar{v} = \frac{X(0)}{\sqrt{\frac{m}{k}}}$

Ex:



$x(0) = 0$

$v(0) = v_i$

$$m\frac{d^2x}{dt^2} = -\frac{GMm}{(x+R)^2}$$

$$\cancel{m}\frac{d^2x}{dt^2} = -\frac{GM\cancel{m}}{(x+R)^2} \qquad GM \to gR^2$$

$$g = \frac{GM}{R^2}$$

$$\frac{d^2x}{dt^2} = \frac{-gR^2}{(x+R)^2}$$

$$\bar{x} = \frac{x}{x_0}$$

$$\bar{t} = \frac{t}{t_0}$$

$$\frac{x_0}{t_0^2}\frac{d^2\bar{x}}{d\bar{t}^2} = \frac{-gR^2}{(x_0\bar{x}+R)^2}$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} = \frac{t_0^2}{x_0}\frac{(-gR^2)}{(x_0\bar{x}+R)^2}$$

$$[gR] = \frac{L}{T^2}\cdot L$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} = \frac{-gR^2t_0^2}{x_0R^2\left(\frac{x_0\bar{x}}{R}+1\right)^2}$$

$$x_0 = R$$

$$\frac{d^2x}{d\bar{t}^2} = \frac{-gt_0^2}{R(\bar{x}+1)^2}$$

$$\left[\frac{g}{R}\right] = \frac{1}{T^2} \qquad \frac{K}{T^2}\ \frac{1}{K}$$

Choices

Cannot normalize

x_0, t_0, v_0 independently

pick 2, get 3rd

$$v = \frac{dx}{dt} = \frac{x_0}{t_0}\frac{d\bar{x}}{d\bar{t}}$$

$$\frac{d\bar{x}}{d\bar{t}} = \frac{t_0}{x_0} v$$

could choose $t_0^2 = \frac{R}{g}$

then

$$\frac{d\bar{x}}{d\bar{t}} = \sqrt{\frac{R}{g}} \cdot \frac{1}{R} v = \frac{v}{\sqrt{gR}}$$

$\underline{\underline{OR}}$ $\quad v_0 = v_i$

$$\frac{d\bar{x}}{d\bar{t}} = \frac{v}{v_i}$$

$$v_i = \frac{x_0}{t_0} = \frac{R}{t_0} \rightarrow t_0 = \frac{R}{v_i}$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} = \frac{-g t_0^2}{R(\bar{x}+1)^2}$$

$$t_0 = \frac{R}{v_i}$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} = -\frac{g}{R}\left(\frac{R}{v_i}\right)^2 \frac{1}{(\bar{x}+1)^2}$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} = -\frac{gR}{v_i^2}\frac{1}{(\bar{x}+1)^2}$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} = -\frac{\beta}{(\bar{x}+1)^2} ; \beta = \frac{gR}{v_i^2}$$

$$\bar{x}(0) = 0 \;,\; \bar{v}(0) = 1$$