- Differential Equations
    - Why we end up with these a lot in physics
    - Ordinary differential equations

ODEs

- $\frac{dy}{dt} = f(t)$

  Ex: $\frac{dx}{dt} = -gt$

  $\frac{d^2x}{dt^2} = a$

- $\frac{dy}{dt} = f(t, y)$

  Ex: $\frac{dN}{dt} = -\frac{1}{\tau} N$

  $\frac{d^2x}{dt^2} = \frac{GM}{x^2}$

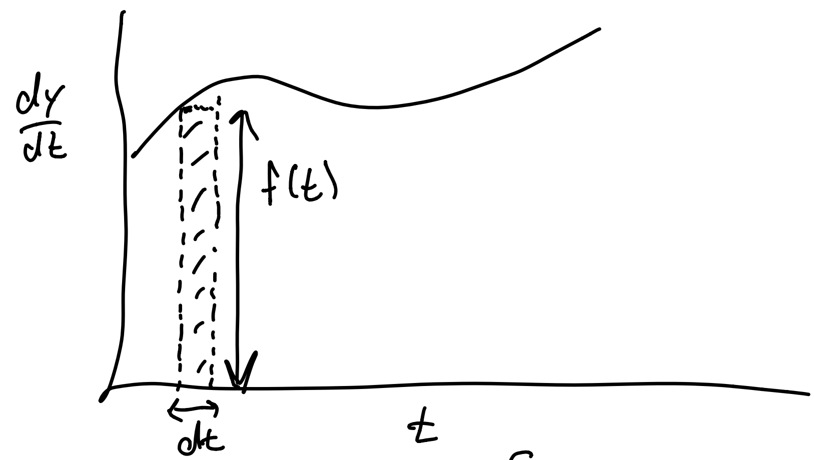
- We want to be able to predict and analyze the motion and behavior of these sorts of systems
    - Not just their derivatives
- How do we solve these?
    - The first type is often very simple
    - Can integrate directly
    - Even if f(t) is very complicated, we can calculate it everywhere and numerically integrate

$$\frac{dx}{dt} = -gt \longrightarrow dx = -gt\,dt$$

$$\int dx = \int -gt\,dt$$

$$x(t) = -\frac{1}{2}gt^2 + C$$

- If f(t) is complicated, we can just do a Reimann sum or some better method (jupyter notebook)



$$A = f(t)dt \qquad \int f(t)dt \approx \sum_{i=0}^{i=N} f(t_i)\Delta t$$

- But what if dy/dt = f(t,y)?
  - We can't just plot the derivate and do a Reimann sum because we don't *know* the derivative everywhere
  - The derivative depends on the function itself, which is changing according to the derivative!
  - Usually, we know the value of y at some starting time t, usually t=0
  - Initial value problem

$$\frac{dy}{dt} = f(t, y)$$

$$\frac{dy}{dt} \longrightarrow y \longrightarrow \frac{dy}{dt}$$

sometimes can still integrate directly . . .

What if we know $y(t_0) = y_0$?

Then $\frac{dy}{dt} = f(t_0, y_0)$

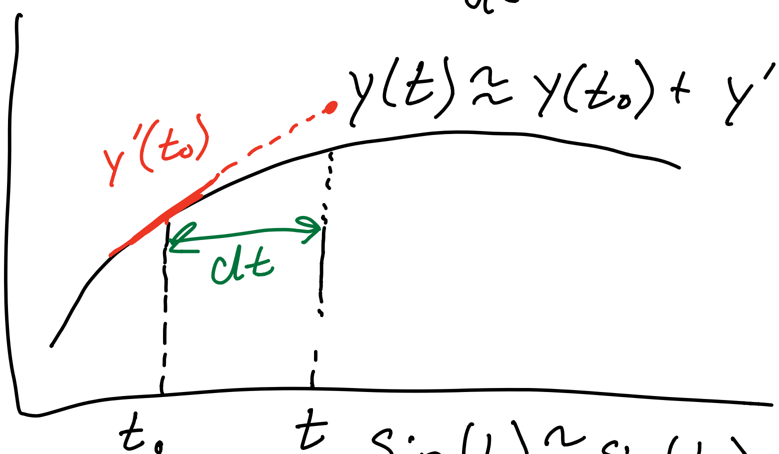
Taylor's Thm:

$$y(t) = y(t_0) + \frac{dy}{dt}(t_0,y_0)\cdot(t-t_0)$$

$$+ \frac{1}{2}\frac{d^2y}{dt^2}(t_0,y_0)\cdot(t-t_0)^2$$

$$+ \frac{1}{6}\frac{d^3y}{dt^3}(t_0,y_0)\cdot(t-t_0)^3 + \dots$$

$y(t) \approx y(t_0) + y'(t_0)\Delta t$



$\sin(t) \approx \sin(t_0) + \cos(t_0)\Delta t - \frac{1}{2}\sin(t_0)\Delta t^2$

- Now "know" $y(t)$ + $f(t, y(t))$
  - can find $y(t+\Delta t)$

- Starting at a known initial value, we can repeat Taylor's approximation indefinitely until we reach the desired stop time
- Often, we don't know the second derivative. If dt is small enough, we can ignore it

Code

- Discretize time in small steps Δt

$$t_n = t_0 + n\Delta t$$

$$y_n = y(t_n)$$

Start: $y(t_0) = y_0$

$$y_1 \approx y_0 + \frac{dy_0}{dt}\Delta t$$

$$y_2 \approx y_1 + \frac{dy_1}{dt}\Delta t$$

$$y_n \approx y_{n-1} + \frac{dy_{n-1}}{dt}\Delta t$$

Ex: $\frac{dy}{dt} = \frac{-1}{\tau} y$

$$y(0) = 5$$

$$\frac{dy}{dt}(0) = \frac{-1}{\tau} 5$$

$$y_1 = y_0 + \frac{dy_0}{dt}\Delta t$$

$$= 5 - \frac{1}{\tau} 5 \Delta t = \# = y_1$$

$$y_2 = y_1 + \frac{dy_1}{dt}\Delta t, \ldots$$

- This is only an approximation!
  - What are some errors associated with it?
    - Truncation error
    - Round off error

$$y_n = y_{n-1} + \frac{dy_{n-1}}{dt}\Delta t \cancel{+ \frac{1}{2}\frac{d^2y_{n-1}}{dt^2}\Delta t^2} + \ldots$$

Approximation will be off by a term $\sim \Delta t^2$

- truncation error $= \mathcal{O}(\Delta t^2)$

$y_n - y_{n-1} \sim \Delta t^2$

- Error at n^{th} pt
  - Each step we incur an error of $\sim \Delta t^2$
  - At point t, we have taken $\sim \frac{1}{\Delta t}$ steps

$$n = \frac{t - t_0}{\Delta t}$$

Each step: $E \sim \Delta t^2$

$$E_n \sim \Delta t^2 \cdot n = \Delta t^2 \cdot \frac{1}{\Delta t} = \Delta t$$

$E_n \sim \Delta t$ Want $E_n \sim 10^{-6}$

need $\Delta t \sim 10^{-6}$

- Show Jupyter lab global error demo
- So do we want to make delta t as small as possible?

- Smaller $\Delta t \Rightarrow$ smaller truncation error

BUT

more computing power required

- on a computer, Δt cannot be arbitrarily small
- smallest pos. #

(machine eps)

$$1 + \epsilon == 1$$

Any float operation is only accurate to $\mathcal{O}(\epsilon)$

Floating Pt Error $\sim \epsilon$

$$E_n \sim \frac{\epsilon}{\Delta t}$$

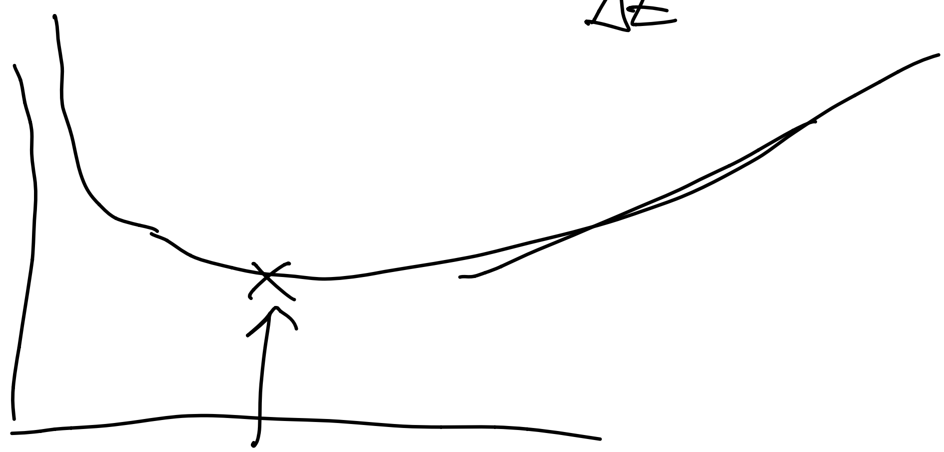
$$E_n = E_n^{trunc} + E_n^{FP}$$

total

$$= \frac{\epsilon}{\Delta t} + \Delta t$$

$\Delta t >> \epsilon$, $E_n \sim \Delta t$

$\Delta t \sim \epsilon$, $E_n \sim \frac{\epsilon}{\Delta t}$



$\Delta t \sim \epsilon^{\frac{1}{2}}$

$\sim 10^{-8}$

Must balance this w/ available resources

Small in comparisan to what?