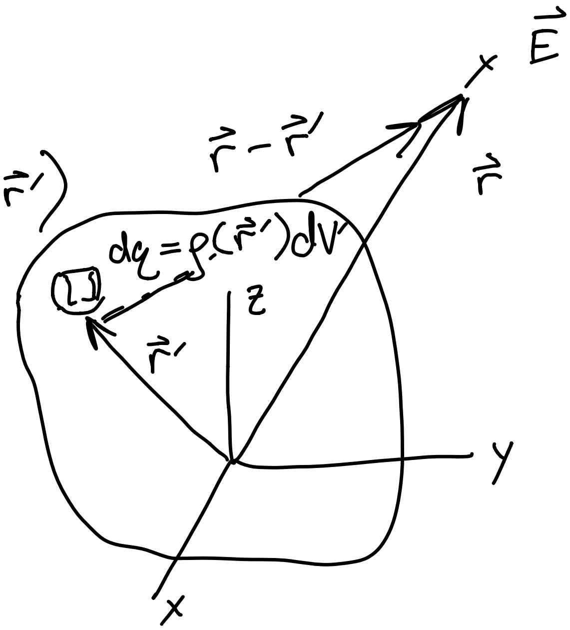
Principle Question of Electrostatics:

What is the field $\vec{E}$ of a distribution of charges?

1) $\vec{E}_{pt} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$

2) $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')\, dV'}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}')$$



Cartesian

$\vec{r} = x, y, z$

$\vec{r}' = x', y', z'$

$dV' = dx'dy'dz'$

etc ...

This integral is very hard!

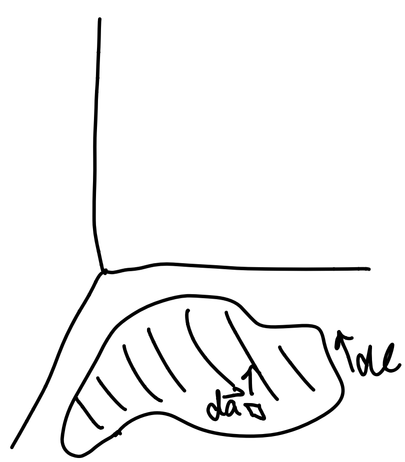
Exploit some properties of $\vec{E}$

$\vec{E}$ (electrostatics) is <u>conservative</u>

$$\oint \vec{E} \cdot d\vec{l} = 0$$

Stokes Thm

$$\oint \vec{E} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a}$$



$\Rightarrow \vec{\nabla} \times \vec{E} = 0$

Scalar function $A(x,y,z)$

$$\vec{\nabla} A = \frac{\partial}{\partial x} A \hat{x} + \frac{\partial}{\partial y} A \hat{y} + \frac{\partial}{\partial z} A \hat{z}$$

$$\vec{\nabla} \times \vec{\nabla} A$$

$$\left(\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) \times \left(\frac{\partial A}{\partial x} \hat{x} + \frac{\partial A}{\partial x} \hat{y} + \frac{\partial A}{\partial z} \hat{z} \right)$$

$$\frac{\partial}{\partial x} \frac{\partial A}{\partial y} \hat{z} - \frac{\partial}{\partial x} \frac{\partial A}{\partial z} \hat{y}$$

$$- \frac{\partial}{\partial y} \frac{\partial A}{\partial x} \hat{z} + \frac{\partial}{\partial y} \frac{\partial A}{\partial z} \hat{x}$$

$$+ \frac{\partial}{\partial z} \frac{\partial A}{\partial x} \hat{y} - \frac{\partial}{\partial z} \frac{\partial A}{\partial y} \hat{x} = \vec{\nabla} \times \vec{\nabla} A$$

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} (A) = \frac{\partial}{\partial y} \frac{\partial}{\partial x} (A)$$

$$\frac{\partial}{\partial x} \cancel{\frac{\partial A}{\partial y}} \hat{z} - \frac{\partial}{\partial x} \cancel{\frac{\partial A}{\partial z}} \hat{y}$$

$$- \frac{\partial}{\partial y} \cancel{\frac{\partial A}{\partial x}} \hat{z} + \frac{\partial}{\partial y} \cancel{\frac{\partial A}{\partial z}} \hat{x}$$

$$+ \frac{\partial}{\partial z} \cancel{\frac{\partial A}{\partial x}} \hat{y} - \frac{\partial}{\partial z} \cancel{\frac{\partial A}{\partial y}} \hat{x} = \vec{\nabla} \times \vec{\nabla} A = 0$$

So: $\vec{\nabla} \times \vec{E} = 0 = \vec{\nabla} \times \vec{\nabla}(\text{scalar})$

Electric Potential V

$$\vec{\nabla} \times \vec{E} = 0 = \vec{\nabla} \times \vec{\nabla}(-V)$$

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

$$\int_a^b -\vec{\nabla} V \cdot d\vec{\ell} = -\left(V(\vec{b}) - V(\vec{a})\right)$$
$$= -\Delta V = \int_a^b \vec{E} \cdot d\vec{\ell}$$

$$V \text{ point charge} = -\int_\infty^r \vec{E}_{pt} \cdot d\vec{\ell}$$

so $V(\vec{r}) = \dfrac{1}{4\pi\epsilon_0} \dfrac{q}{|\vec{r} - \vec{r}_{chg}|}$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, dV$$

- Even this integral is often too tough.

- We don't often know the charge distribution ahead of time.

- Question

  What is the divergence of $\vec{E}$?

- Gauss' Law:

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$Q_{enc} = \int_V \rho \, dV$$

$$\oint \vec{E} \cdot d\vec{a} = \int_V \frac{\rho}{\epsilon_0} dV$$

Divergence Thm

$$\oint \vec{E} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{E} \, dV$$

$$\int_V \vec{\nabla}\cdot\vec{E}\,dV = \int \frac{\rho}{\epsilon_0}\,dV$$

so:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{E} = -\vec{\nabla}V$$

$$-\vec{\nabla}\cdot\vec{\nabla}V = \frac{\rho}{\epsilon_0}$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}}$$

Poisson's Eqn

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$