CHAPTER 3

Potentials

3.1 ■ LAPLACE'S EQUATION

3.1.1 ■ Introduction

The primary task of electrostatics is to find the electric field of a given stationary charge distribution. In principle, this purpose is accomplished by Coulomb's law, in the form of Eq. 2.8:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\scriptstyle\mathcal{R}}}}{\mathcal{R}^2}\rho(\mathbf{r}')\,d\tau'. \tag{3.1}$$

Unfortunately, integrals of this type can be difficult to calculate for any but the simplest charge configurations. Occasionally we can get around this by exploiting symmetry and using Gauss's law, but ordinarily the best strategy is first to calculate the *potential,* V, which is given by the somewhat more tractable Eq. 2.29:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathcal{R}}\rho(\mathbf{r}')\,d\tau'. \tag{3.2}$$

Still, even *this* integral is often too tough to handle analytically. Moreover, in problems involving conductors ρ itself may not be known in advance; since charge is free to move around, the only thing we control directly is the *total* charge (or perhaps the potential) of each conductor.

In such cases, it is fruitful to recast the problem in differential form, using Poisson's equation (2.24),

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \tag{3.3}$$

which, together with appropriate boundary conditions, is equivalent to Eq. 3.2. Very often, in fact, we are interested in finding the potential in a region where $\rho = 0$. (If $\rho = 0$ *everywhere,* of course, then $V = 0$, and there is nothing further to say—that's not what I mean. There may be plenty of charge *elsewhere,* but we're confining our attention to places where there is no charge.) In this case, Poisson's equation reduces to Laplace's equation:

$$\nabla^2 V = 0, \tag{3.4}$$

or, written out in Cartesian coordinates,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{3.5}$$

This formula is so fundamental to the subject that one might almost say electrostatics *is* the study of Laplace's equation. At the same time, it is a ubiquitous equation, appearing in such diverse branches of physics as gravitation and magnetism, the theory of heat, and the study of soap bubbles. In mathematics, it plays a major role in analytic function theory. To get a feel for Laplace's equation and its solutions (which are called **harmonic functions**), we shall begin with the one- and two-dimensional versions, which are easier to picture, and illustrate all the essential properties of the three-dimensional case.

3.1.2 ■ Laplace's Equation in One Dimension

Suppose V depends on only one variable, x. Then Laplace's equation becomes

$$\frac{d^2 V}{dx^2} = 0.$$

The general solution is

$$V(x) = mx + b, \tag{3.6}$$

the equation for a straight line. It contains two undetermined constants (m and b), as is appropriate for a second-order (ordinary) differential equation. They are fixed, in any particular case, by the boundary conditions of that problem. For instance, it might be specified that $V = 4$ at $x = 1$, and $V = 0$ at $x = 5$. In that case, $m = -1$ and $b = 5$, so $V = -x + 5$ (see Fig. 3.1).

I want to call your attention to two features of this result; they may seem silly and obvious in one dimension, where I can write down the general solution explicitly, but the analogs in two and three dimensions are powerful and by no means obvious:

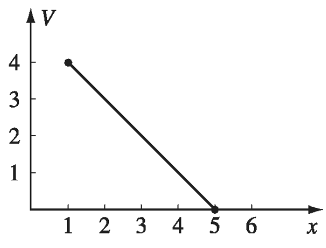

FIGURE 3.1

1. $V(x)$ is the *average* of $V(x+a)$ and $V(x-a)$, for any a:

$$V(x) = \frac{1}{2}[V(x+a) + V(x-a)].$$

Laplace's equation is a kind of averaging instruction; it tells you to assign to the point x the average of the values to the left and to the right of x. Solutions to Laplace's equation are, in this sense, *as boring as they could possibly be,* and yet fit the end points properly.

2. Laplace's equation tolerates *no local maxima or minima;* extreme values of V must occur at the end points. Actually, this is a consequence of (1), for if there *were* a local maximum, V would be greater at that point than on either side, and therefore could not be the average. (Ordinarily, you expect the second derivative to be negative at a maximum and positive at a minimum. Since Laplace's equation requires, on the contrary, that the second derivative is zero, it seems reasonable that solutions should exhibit no extrema. However, this is not a *proof,* since there exist functions that have maxima and minima at points where the second derivative vanishes: x^4, for example, has such a minimum at the point $x = 0$.)

3.1.3 ■ Laplace's Equation in Two Dimensions

If V depends on two variables, Laplace's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

This is no longer an *ordinary* differential equation (that is, one involving ordinary derivatives only); it is a *partial* differential equation. As a consequence, some of the simple rules you may be familiar with do not apply. For instance, the general solution to this equation doesn't contain just two arbitrary constants—or, for that matter, *any* finite number—despite the fact that it's a second-order equation. Indeed, one cannot write down a "general solution" (at least, not in a closed form like Eq. 3.6). Nevertheless, it is possible to deduce certain properties common to all solutions.

It may help to have a physical example in mind. Picture a thin rubber sheet (or a soap film) stretched over some support. For definiteness, suppose you take a cardboard box, cut a wavy line all the way around, and remove the top part (Fig. 3.2). Now glue a tightly stretched rubber membrane over the box, so that it fits like a drum head (it won't be a *flat* drumhead, of course, unless you chose to cut the edges off straight). Now, if you lay out coordinates (x, y) on the bottom of the box, the height $V(x, y)$ of the sheet above the point (x, y) will satisfy Laplace's

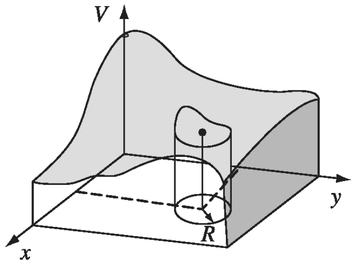

FIGURE 3.2

equation.[1] (The one-dimensional analog would be a rubber band stretched between two points. Of course, it would form a straight line.)

Harmonic functions in two dimensions have the same properties we noted in one dimension:

1. The value of V at a point (x, y) is the average of those *around* the point. More precisely, if you draw a circle of any radius R about the point (x, y), the average value of V on the circle is equal to the value at the center:

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V\, dl.$$

   (This, incidentally, suggests the **method of relaxation**, on which computer solutions to Laplace's equation are based: Starting with specified values for V at the boundary, and reasonable guesses for V on a grid of interior points, the first pass reassigns to each point the average of its nearest neighbors. The second pass repeats the process, using the corrected values, and so on. After a few iterations, the numbers begin to settle down, so that subsequent passes produce negligible changes, and a numerical solution to Laplace's equation, with the given boundary values, has been achieved.)[2]

2. V has no local maxima or minima; all extrema occur at the boundaries. (As before, this follows from (1).) Again, Laplace's equation picks the most featureless function possible, consistent with the boundary conditions: no hills, no valleys, just the smoothest conceivable surface. For instance, if you put a ping-pong ball on the stretched rubber sheet of Fig. 3.2, it will

[1] Actually, the equation satisfied by a rubber sheet is

$$\frac{\partial}{\partial x}\left(g\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(g\frac{\partial V}{\partial y}\right) = 0, \quad \text{where } g = \left[1 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right]^{-1/2};$$

it reduces (approximately) to Laplace's equation as long as the surface does not deviate too radically from a plane.

[2] See, for example, E. M. Purcell, *Electricity and Magnetism,* 2nd ed. (New York: McGraw-Hill, 1985), problem 3.30.

roll over to one side and fall off—it will not find a "pocket" somewhere to settle into, for Laplace's equation allows no such dents in the surface. From a geometrical point of view, just as a straight line is the shortest distance between two points, so a harmonic function in two dimensions minimizes the surface area spanning the given boundary line.

3.1.4 ■ Laplace's Equation in Three Dimensions

In three dimensions I can neither provide you with an explicit solution (as in one dimension) nor offer a suggestive physical example to guide your intuition (as I did in two dimensions). Nevertheless, the same two properties remain true, and this time I will sketch a proof.[3]

1. The value of V at point $\mathbf{r}$ is the average value of V over a spherical surface of radius R centered at $\mathbf{r}$:

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V\, da.$$

2. As a consequence, V can have no local maxima or minima; the extreme values of V must occur at the boundaries. (For if V had a local maximum at $\mathbf{r}$, then by the very nature of maximum I could draw a sphere around $\mathbf{r}$ over which all values of V—and *a fortiori* the average—would be less than at $\mathbf{r}$.)

Proof. Let's begin by calculating the average potential over a spherical surface of radius R due to a *single* point charge q located outside the sphere. We may as well center the sphere at the origin and choose coordinates so that q lies on the z-axis (Fig. 3.3). The potential at a point on the surface is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}},$$

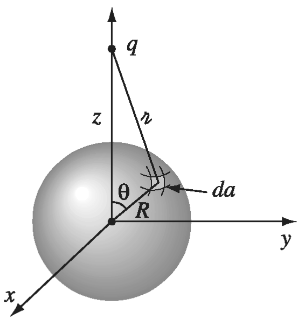

FIGURE 3.3

[3]For a proof that does not rely on Coulomb's law (only on Laplace's equation), see Prob. 3.37.

where

$$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta,$$

so

$$\begin{aligned}
V_{\text{ave}} &= \frac{1}{4\pi R^2}\frac{q}{4\pi\epsilon_0}\int [z^2 + R^2 - 2zR\cos\theta]^{-1/2} R^2 \sin\theta\, d\theta\, d\phi \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}\sqrt{z^2 + R^2 - 2zR\cos\theta}\,\bigg|_0^{\pi} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}[(z+R) - (z-R)] = \frac{1}{4\pi\epsilon_0}\frac{q}{z}.
\end{aligned}$$

But this is precisely the potential due to q at the *center* of the sphere! By the superposition principle, the same goes for any *collection* of charges outside the sphere: their average potential over the sphere is equal to the net potential they produce at the center. □

Problem 3.1 Find the average potential over a spherical surface of radius R due to a point charge q located *inside* (same as above, in other words, only with $z < R$). (In this case, of course, Laplace's equation does not hold within the sphere.) Show that, in general,

$$V_{\text{ave}} = V_{\text{center}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R},$$

where V_{center} is the potential at the center due to all the *external* charges, and Q_{enc} is the total enclosed charge.

Problem 3.2 In one sentence, justify **Earnshaw's Theorem:** *A charged particle cannot be held in a stable equilibrium by electrostatic forces alone.* As an example, consider the cubical arrangement of fixed charges in Fig. 3.4. It *looks,* off hand, as though a positive charge at the center would be suspended in midair, since it is repelled away from each corner. Where is the leak in this "electrostatic bottle"? [To harness nuclear fusion as a practical energy source it is necessary to heat a plasma (soup of charged particles) to fantastic temperatures—so hot that contact would vaporize any ordinary pot. Earnshaw's theorem says that electrostatic containment is also out of the question. Fortunately, it *is* possible to confine a hot plasma magnetically.]

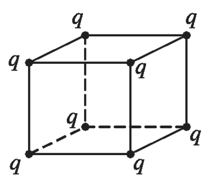

FIGURE 3.4

Problem 3.3 Find the general solution to Laplace's equation in spherical coordinates, for the case where V depends only on r. Do the same for cylindrical coordinates, assuming V depends only on s.

Problem 3.4

(a) Show that the average electric *field* over a spherical surface, due to charges outside the sphere, is the same as the field at the center.

(b) What is the average due to charges *inside* the sphere?

3.1.5 ■ Boundary Conditions and Uniqueness Theorems

Laplace's equation does not by itself determine V; in addition, suitable boundary conditions must be supplied. This raises a delicate question: What are appropriate boundary conditions, sufficient to determine the answer and yet not so strong as to generate inconsistencies? The one-dimensional case is easy, for here the general solution $V = mx + b$ contains two arbitrary constants, and we therefore require two boundary conditions. We might, for instance, specify the value of the function at each end, or we might give the value of the function and its derivative at one end, or the value at one end and the derivative at the other, and so on. But we cannot get away with *just* the value or *just* the derivative at *one* end—this is insufficient information. Nor would it do to specify the derivatives at both ends—this would either be redundant (if the two are equal) or inconsistent (if they are not).

In two or three dimensions we are confronted by a *partial* differential equation, and it is not so obvious what would constitute acceptable boundary conditions. Is the shape of a taut rubber membrane, for instance, uniquely determined by the frame over which it is stretched, or, like a canning jar lid, can it snap from one stable configuration to another? The answer, as I think your intuition would suggest, is that V *is* uniquely determined by its value at the boundary (canning jars evidently do not obey Laplace's equation). However, other boundary conditions can also be used (see Prob. 3.5). The *proof* that a proposed set of boundary conditions will suffice is usually presented in the form of a **uniqueness theorem.** There are many such theorems for electrostatics, all sharing the same basic format—I'll show you the two most useful ones.[4]

> **First uniqueness theorem:** The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$.

Proof. In Fig. 3.5 I have drawn such a region and its boundary. (There could also be "islands" inside, so long as V is given on all their surfaces; also, the outer

[4] I do not intend to prove the *existence* of solutions here—that's a much more difficult job. In context, the existence is generally clear on physical grounds.

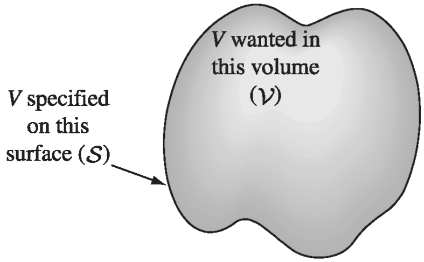

FIGURE 3.5

boundary could be at infinity, where V is ordinarily taken to be zero.) Suppose there were *two* solutions to Laplace's equation:

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0,$$

both of which assume the specified value on the surface. I want to prove that they must be equal. The trick is look at their *difference:*

$$V_3 \equiv V_1 - V_2.$$

This obeys Laplace's equation,

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0,$$

and it takes the value *zero* on all boundaries (since V_1 and V_2 are equal there). But Laplace's equation allows no local maxima or minima—all extrema occur on the boundaries. So the maximum and minimum of V_3 are both zero. Therefore V_3 must be zero everywhere, and hence

$$V_1 = V_2. \qquad \square$$

Example 3.1. Show that the potential is *constant* inside an enclosure completely surrounded by conducting material, provided there is no charge within the enclosure.

Solution

The potential on the cavity wall is some constant, V_0 (that's item (iv), in Sect. 2.5.1), so the potential inside is a function that satisfies Laplace's equation and has the constant value V_0 at the boundary. It doesn't take a genius to think of *one* solution to this problem: $V = V_0$ everywhere. The uniqueness theorem guarantees that this is the *only* solution. (It follows that the *field* inside an empty cavity is zero—the same result we found in Sect. 2.5.2 on rather different grounds.)

The uniqueness theorem is a license to your imagination. It doesn't matter *how* you come by your solution; if (a) it satisfies Laplace's equation and (b) it has the correct value on the boundaries, then it's *right*. You'll see the power of this argument when we come to the method of images.

Incidentally, it is easy to improve on the first uniqueness theorem: I assumed there was no charge inside the region in question, so the potential obeyed Laplace's equation, but we may as well throw in some charge (in which case V obeys Poisson's equation). The argument is the same, only this time

$$\nabla^2 V_1 = -\frac{1}{\epsilon_0}\rho, \qquad \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho,$$

so

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{1}{\epsilon_0}\rho + \frac{1}{\epsilon_0}\rho = 0.$$

Once again the *difference* ($V_3 \equiv V_1 - V_2$) satisfies Laplace's equation and has the value zero on all boundaries, so $V_3 = 0$ and hence $V_1 = V_2$.

Corollary: The potential in a volume $\mathcal{V}$ is uniquely determined if (a) the charge density throughout the region, and (b) the value of V on all boundaries, are specified.

3.1.6 ■ Conductors and the Second Uniqueness Theorem

The *simplest* way to set the boundary conditions for an electrostatic problem is to specify the value of V on all surfaces surrounding the region of interest. And this situation often occurs in practice: In the laboratory, we have conductors connected to batteries, which maintain a given potential, or to **ground**, which is the experimentalist's word for $V = 0$. However, there are other circumstances in which we do not know the *potential* at the boundary, but rather the *charges* on various conducting surfaces. Suppose I put charge Q_a on the first conductor, Q_b on the second, and so on—I'm not telling you how the charge distributes itself over each conducting surface, because as soon as I put it on, it moves around in a way I do not control. And for good measure, let's say there is some specified charge density ρ in the region between the conductors. Is the electric field now uniquely determined? Or are there perhaps a number of different ways the charges could arrange themselves on their respective conductors, each leading to a different field?

Second uniqueness theorem: In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the *total charge* on each conductor is given (Fig. 3.6). (The region as a whole can be bounded by another conductor, or else unbounded.)

Proof. Suppose there are *two* fields satisfying the conditions of the problem. Both obey Gauss's law in differential form in the space between the conductors:

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\epsilon_0}\rho, \qquad \nabla \cdot \mathbf{E}_2 = \frac{1}{\epsilon_0}\rho.$$

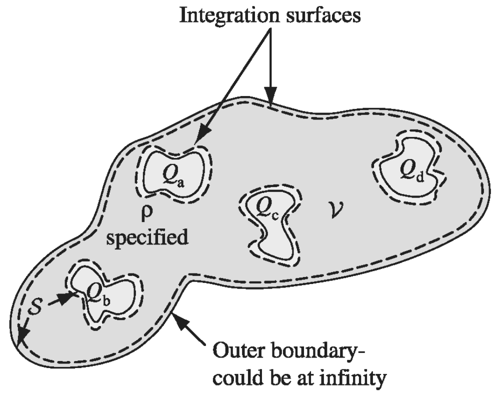

FIGURE 3.6

And both obey Gauss's law in integral form for a Gaussian surface enclosing each conductor:

$$\oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_i, \qquad \oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_i.$$

Likewise, for the outer boundary (whether this is just inside an enclosing conductor or at infinity),

$$\oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}, \qquad \oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{tot}}.$$

As before, we examine the difference

$$\mathbf{E}_3 \equiv \mathbf{E}_1 - \mathbf{E}_2,$$

which obeys

$$\nabla \cdot \mathbf{E}_3 = 0 \tag{3.7}$$

in the region between the conductors, and

$$\oint \mathbf{E}_3 \cdot d\mathbf{a} = 0 \tag{3.8}$$

over each boundary surface.

Now there is one final piece of information we must exploit: Although we do not know how the charge Q_i distributes itself over the ith conductor, we *do* know that each conductor is an equipotential, and hence V_3 is a *constant* (not necessarily the *same* constant) over each conducting surface. (It need not be *zero,* for the potentials V_1 and V_2 may not be equal—all we know for sure is that *both* are *constant* over any given conductor.) Next comes a trick. Invoking product rule number 5 (inside front cover), we find that

$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3(\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla V_3) = -(E_3)^2.$$

Here I have used Eq. 3.7, and $\mathbf{E}_3 = -\nabla V_3$. Integrating this over $\mathcal{V}$, and applying the divergence theorem to the left side:

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{E}_3)\, d\tau = \oint_{\mathcal{S}} V_3 \mathbf{E}_3 \cdot d\mathbf{a} = -\int_{\mathcal{V}} (E_3)^2\, d\tau.$$

The surface integral covers all boundaries of the region in question—the conductors and outer boundary. Now V_3 is a constant over each surface (if the outer boundary is infinity, $V_3 = 0$ there), so it comes outside each integral, and what remains is zero, according to Eq. 3.8. Therefore,

$$\int_{\mathcal{V}} (E_3)^2\, d\tau = 0.$$

But this integrand is never negative; the only way the integral can vanish is if $E_3 = 0$ everywhere. Consequently, $\mathbf{E}_1 = \mathbf{E}_2$, and the theorem is proved. □

This proof was not easy, and there is a real danger that the theorem itself will seem more plausible to you than the proof. In case you think the second uniqueness theorem is "obvious," consider this example of Purcell's: Figure 3.7 shows a simple electrostatic configuration, consisting of four conductors with charges $\pm Q$, situated so that the plusses are near the minuses. It all looks very comfortable. Now, what happens if we join them in pairs, by tiny wires, as indicated in Fig. 3.8? Since the positive charges are very near negative charges (which is where they *like* to be) you might well guess that *nothing* will happen—the configuration looks stable.

Well, that sounds reasonable, but it's wrong. The configuration in Fig. 3.8 is *impossible.* For there are now effectively *two* conductors, and the total charge on each is *zero*. *One* possible way to distribute zero charge over these conductors is to have no accumulation of charge anywhere, and hence zero field

FIGURE 3.7

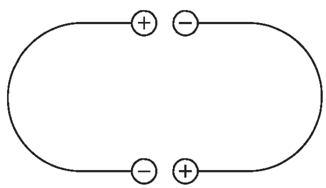

FIGURE 3.8

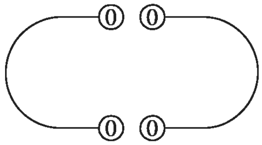

FIGURE 3.9

everywhere (Fig. 3.9). By the second uniqueness theorem, this must be *the* solution: The charge will flow down the tiny wires, canceling itself off.

Problem 3.5 Prove that the field is uniquely determined when the charge density ρ is given and *either* V *or* the normal derivative $\partial V/\partial n$ is specified on each boundary surface. Do not assume the boundaries are conductors, or that V is constant over any given surface.

Problem 3.6 A more elegant proof of the second uniqueness theorem uses Green's identity (Prob. 1.61c), with $T = U = V_3$. Supply the details.

3.2 ■ THE METHOD OF IMAGES

3.2.1 ■ The Classic Image Problem

Suppose a point charge q is held a distance d above an infinite grounded conducting plane (Fig. 3.10). *Question:* What is the potential in the region above the plane? It's not just $(1/4\pi\epsilon_0)q/\mathscr{r}$, for q will induce a certain amount of negative charge on the nearby surface of the conductor; the total potential is due in part to q directly, and in part to this induced charge. But how can we possibly calculate the potential, when we don't know how much charge is induced or how it is distributed?

From a mathematical point of view, our problem is to solve Poisson's equation in the region $z > 0$, with a single point charge q at $(0, 0, d)$, subject to the boundary conditions:

1. $V = 0$ when $z = 0$ (since the conducting plane is grounded), and
2. $V \to 0$ far from the charge $\left(\text{that is, for } x^2 + y^2 + z^2 \gg d^2\right)$.

The first uniqueness theorem (actually, its corollary) guarantees that there is only one function that meets these requirements. If by trick or clever guess we can discover such a function, it's got to be the answer.

Trick: Forget about the actual problem; we're going to study a *completely different* situation. This new configuration consists of *two* point charges, $+q$ at

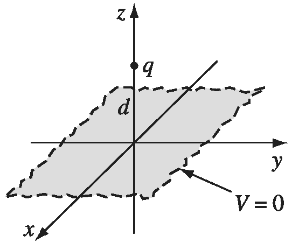

FIGURE 3.10

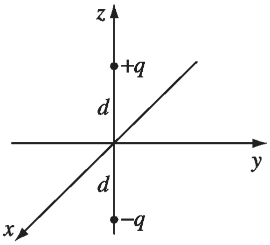

FIGURE 3.11

$(0, 0, d)$ and $-q$ at $(0, 0, -d)$, and *no* conducting plane (Fig. 3.11). For this configuration, I can easily write down the potential:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]. \tag{3.9}$$

(The denominators represent the distances from (x, y, z) to the charges $+q$ and $-q$, respectively.) It follows that

1. $V = 0$ when $z = 0$,
2. $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$,

and the only charge in the region $z > 0$ is the point charge $+q$ at $(0, 0, d)$. But these are precisely the conditions of the original problem! Evidently the second configuration happens to produce exactly the same potential as the first configuration, in the "upper" region $z \geq 0$. (The "lower" region, $z < 0$, is completely different, but who cares? The upper part is all we need.) *Conclusion:* The potential of a point charge above an infinite grounded conductor is given by Eq. 3.9, for $z \geq 0$.

Notice the crucial role played by the uniqueness theorem in this argument: without it, no one would believe this solution, since it was obtained for a completely different charge distribution. But the uniqueness theorem certifies it: If it satisfies Poisson's equation in the region of interest, and assumes the correct value at the boundaries, then it must be right.

3.2.2 ■ Induced Surface Charge

Now that we know the potential, it is a straightforward matter to compute the surface charge σ induced on the conductor. According to Eq. 2.49,

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n},$$

where $\partial V/\partial n$ is the normal derivative of V at the surface. In this case the normal direction is the z direction, so

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}.$$

From Eq. 3.9,

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\left\{\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\},$$

so[5]

$$\sigma(x, y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}. \tag{3.10}$$

As expected, the induced charge is negative (assuming q is positive) and greatest at $x = y = 0$.

While we're at it, let's compute the *total* induced charge

$$Q = \int \sigma\, da.$$

This integral, over the xy plane, could be done in Cartesian coordinates, with $da = dx\, dy$, but it's a little easier to use polar coordinates (r, ϕ), with $r^2 = x^2+y^2$ and $da = r\, dr\, d\phi$. Then

$$\sigma(r) = \frac{-qd}{2\pi(r^2+d^2)^{3/2}},$$

and

$$Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}} r\, dr\, d\phi = \left.\frac{qd}{\sqrt{r^2+d^2}}\right|_0^{\infty} = -q. \tag{3.11}$$

The total charge induced on the plane is $-q$, as (with benefit of hindsight) you can perhaps convince yourself it *had* to be.

3.2.3 ■ Force and Energy

The charge q is attracted toward the plane, because of the negative induced charge. Let's calculate the force of attraction. Since the potential in the vicinity of q is the same as in the analog problem (the one with $+q$ and $-q$ but no conductor), so also is the field and, therefore, the force:

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{\mathbf{z}}. \tag{3.12}$$

[5]For an entirely different derivation of this result, see Prob. 3.38.

Beware: It is easy to get carried away, and assume that *everything* is the same in the two problems. Energy, however, is *not* the same. With the two point charges and no conductor, Eq. 2.42 gives

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}. \tag{3.13}$$

But for a single charge and conducting plane, the energy is *half* of this:

$$W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}. \tag{3.14}$$

Why half? Think of the energy stored in the fields (Eq. 2.45):

$$W = \frac{\epsilon_0}{2}\int E^2\,d\tau.$$

In the first case, both the upper region ($z > 0$) and the lower region ($z < 0$) contribute—and by symmetry they contribute equally. But in the second case, only the upper region contains a nonzero field, and hence the energy is half as great.[6]

Of course, one could also determine the energy by calculating the work required to bring q in from infinity. The force required (to oppose the electrical force in Eq. 3.12) is $(1/4\pi\epsilon_0)(q^2/4z^2)\hat{\mathbf{z}}$, so

$$\begin{aligned} W &= \int_\infty^d \mathbf{F}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_\infty^d \frac{q^2}{4z^2}\,dz \\ &= \frac{1}{4\pi\epsilon_0}\left(-\frac{q^2}{4z}\right)\Bigg|_\infty^d = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}. \end{aligned}$$

As I move q toward the conductor, I do work *only on* q. It is true that induced charge is moving in over the conductor, but this costs me nothing, since the whole conductor is at potential zero. By contrast, if I simultaneously bring in *two* point charges (with no conductor), I do work on *both* of them, and the total is (again) twice as great.

3.2.4 ■ Other Image Problems

The method just described is not limited to a single point charge; *any* stationary charge distribution near a grounded conducting plane can be treated in the same way, by introducing its mirror image—hence the name **method of images.** (Remember that the image charges have the *opposite sign;* this is what guarantees that the xy plane will be at potential zero.) There are also some exotic problems that can be handled in similar fashion; the nicest of these is the following.

[6]For a generalization of this result, see M. M. Taddei, T. N. C. Mendes, and C. Farina, *Eur. J. Phys.* **30**, 965 (2009), and Prob. 3.41b.

Example 3.2. A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R (Fig. 3.12). Find the potential outside the sphere.

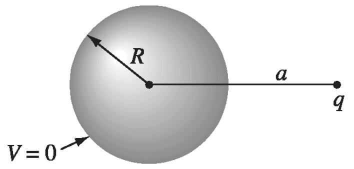

FIGURE 3.12

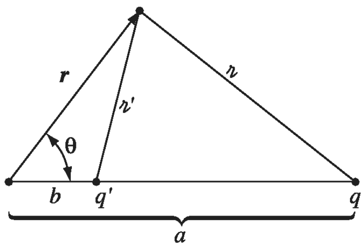

FIGURE 3.13

Solution

Examine the *completely different* configuration, consisting of the point charge q together with another point charge

$$q' = -\frac{R}{a}q, \tag{3.15}$$

placed a distance

$$b = \frac{R^2}{a} \tag{3.16}$$

to the right of the center of the sphere (Fig. 3.13). No conductor, now—just the two point charges. The potential of this configuration is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right), \tag{3.17}$$

where $\mathscr{r}$ and $\mathscr{r}'$ are the distances from q and q', respectively. Now, it happens (see Prob. 3.8) that this potential vanishes at all points on the sphere, and therefore fits the boundary conditions for our original problem, in the exterior region.[7]

Conclusion: Eq. 3.17 is the potential of a point charge near a grounded conducting sphere. (Notice that b is less than R, so the "image" charge q' is safely inside the sphere—*you cannot put image charges in the region where you are calculating V*; that would change ρ, and you'd be solving Poisson's equation with

[7]This solution is due to William Thomson (later Lord Kelvin), who published it in 1848, when he was just 24. It was apparently inspired by a theorem of Apollonius (200 BC) that says the locus of points with a fixed ratio of distances from two given points is a sphere. See J. C. Maxwell, "Treatise on Electricity and Magnetism, Vol. I," Dover, New York, p. 245. I thank Gabriel Karl for this interesting history.

the wrong source.) In particular, the force of attraction between the charge and the sphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}. \tag{3.18}$$

The method of images is delightfully simple ... when it works. But it is as much an art as a science, for you must somehow think up just the right "auxiliary" configuration, and for most shapes this is forbiddingly complicated, if not impossible.

Problem 3.7 Find the force on the charge $+q$ in Fig. 3.14. (The xy plane is a grounded conductor.)

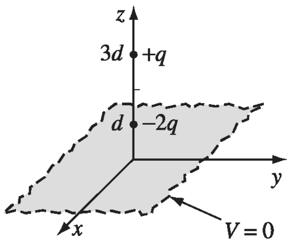

FIGURE 3.14

Problem 3.8

(a) Using the law of cosines, show that Eq. 3.17 can be written as follows:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right], \tag{3.19}$$

where r and θ are the usual spherical polar coordinates, with the z axis along the line through q. In this form, it is obvious that $V = 0$ on the sphere, $r = R$.

(b) Find the induced surface charge on the sphere, as a function of θ. Integrate this to get the total induced charge. (What *should* it be?)

(c) Calculate the energy of this configuration.

Problem 3.9 In Ex. 3.2 we assumed that the conducting sphere was grounded ($V = 0$). But with the addition of a second image charge, the same basic model will handle the case of a sphere at *any* potential V_0 (relative, of course, to infinity). What charge should you use, and where should you put it? Find the force of attraction between a point charge q and a *neutral* conducting sphere.

! **Problem 3.10** A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. (Let's say the wire runs parallel to the x-axis and directly above it, and the conducting plane is the xy plane.)

(a) Find the potential in the region above the plane. [*Hint:* Refer to Prob. 2.52.]

(b) Find the charge density σ induced on the conducting plane.

Problem 3.11 Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q, situated as shown in Fig. 3.15. Set up the image configuration, and calculate the potential in this region. What charges do you need, and where should they be located? What is the force on q? How much work did it take to bring q in from infinity? Suppose the planes met at some angle other than 90°; would you still be able to solve the problem by the method of images? If not, for what particular angles *does* the method work?

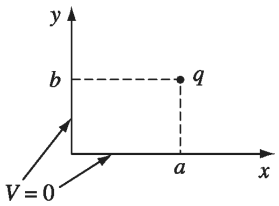

FIGURE 3.15

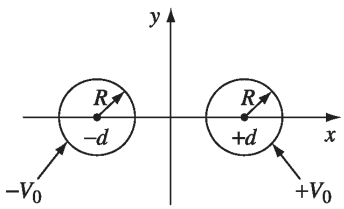

FIGURE 3.16

! **Problem 3.12** Two long, straight copper pipes, each of radius R, are held a distance $2d$ apart. One is at potential V_0, the other at $-V_0$ (Fig. 3.16). Find the potential everywhere. [*Hint:* Exploit the result of Prob. 2.52.]

3.3 ■ SEPARATION OF VARIABLES

In this section we shall attack Laplace's equation directly, using the method of **separation of variables**, which is the physicist's favorite tool for solving partial differential equations. The method is applicable in circumstances where the potential (V) or the charge density (σ) is specified on the boundaries of some region, and we are asked to find the potential in the interior. The basic strategy is very simple: *We look for solutions that are* products *of functions, each of which depends on only* one *of the coordinates*. The algebraic details, however, can be formidable, so I'm going to develop the method through a sequence of examples. We'll start with Cartesian coordinates and then do spherical coordinates (I'll leave the cylindrical case for you to tackle on your own, in Prob. 3.24).

3.3.1 ■ Cartesian Coordinates

Example 3.3. Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$ (Fig. 3.17). The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."

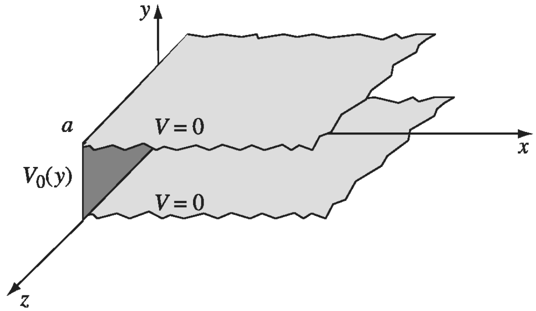

FIGURE 3.17

Solution

The configuration is independent of z, so this is really a *two*-dimensional problem. In mathematical terms, we must solve Laplace's equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \tag{3.20}$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0(y) \text{ when } x = 0, \\ \text{(iv)} & V \to 0 \text{ as } x \to \infty. \end{array}\right\} \tag{3.21}$$

(The latter, although not explicitly stated in the problem, is necessary on physical grounds: as you get farther and farther away from the "hot" strip at $x = 0$, the potential should drop to zero.) Since the potential is specified on all boundaries, the answer is uniquely determined.

The first step is to look for solutions in the form of products:

$$V(x, y) = X(x)Y(y). \tag{3.22}$$

On the face of it, this is an absurd restriction—the overwhelming majority of solutions to Laplace's equation do *not* have such a form. For example, $V(x, y) =$

$(5x + 6y)$ satisfies Eq. 3.20, but you can't express it as the product of a function x times a function y. Obviously, we're only going to get a tiny subset of all possible solutions by this means, and it would be a *miracle* if one of them happened to fit the boundary conditions of our problem ... But hang on, because the solutions we *do* get are very special, and it turns out that by pasting them together we can construct the general solution.

Anyway, putting Eq. 3.22 into Eq. 3.20, we obtain

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0.$$

The next step is to "separate the variables" (that is, collect all the x-dependence into one term and all the y-dependence into another). Typically, this is accomplished by dividing through by V:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0. \tag{3.23}$$

Here the first term depends only on x and the second only on y; in other words, we have an equation of the form

$$f(x) + g(y) = 0. \tag{3.24}$$

Now, there's only one way this could possibly be true: *f and g must both be constant.* For what if $f(x)$ *changed,* as you vary x—then if we held y fixed and fiddled with x, the sum $f(x) + g(y)$ would *change,* in violation of Eq. 3.24, which says it's always zero. (That's a simple but somehow rather elusive argument; don't accept it without due thought, because the whole method rides on it.)

It follows from Eq. 3.23, then, that

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad \text{with} \quad C_1 + C_2 = 0. \tag{3.25}$$

One of these constants is positive, the other negative (or perhaps both are zero). In general, one must investigate all the possibilities; however, in our particular problem we need C_1 positive and C_2 negative, for reasons that will appear in a moment. Thus

$$\frac{d^2X}{dx^2} = k^2X, \qquad \frac{d^2Y}{dy^2} = -k^2Y. \tag{3.26}$$

Notice what has happened: A *partial* differential equation (3.20) has been converted into two *ordinary* differential equations (3.26). The advantage of this is obvious—ordinary differential equations are a lot easier to solve. Indeed:

$$X(x) = Ae^{kx} + Be^{-kx}, \qquad Y(y) = C\sin ky + D\cos ky,$$

so

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky). \tag{3.27}$$

This is the appropriate separable solution to Laplace's equation; it remains to impose the boundary conditions, and see what they tell us about the constants. To begin at the end, condition (iv) requires that A equal zero.[8] Absorbing B into C and D, we are left with

$$V(x, y) = e^{-kx}(C \sin ky + D \cos ky).$$

Condition (i) now demands that D equal zero, so

$$V(x, y) = Ce^{-kx} \sin ky. \tag{3.28}$$

Meanwhile (ii) yields $\sin ka = 0$, from which it follows that

$$k = \frac{n\pi}{a}, \qquad (n = 1, 2, 3, \ldots). \tag{3.29}$$

(At this point you can see why I chose C_1 positive and C_2 negative: If X were sinusoidal, we could never arrange for it to go to zero at infinity, and if Y were exponential we could not make it vanish at both 0 and a. Incidentally, $n = 0$ is no good, for in that case the potential vanishes *everywhere*. And we have already excluded negative n's.)

That's as far as we can go, using separable solutions, and unless $V_0(y)$ just happens to have the form $\sin(n\pi y/a)$ for some integer n, we simply *can't fit* the final boundary condition at $x = 0$. But now comes the crucial step that redeems the method: Separation of variables has given us an *infinite family* of solutions (one for each n), and whereas none of them *by itself* satisfies the final boundary condition, it is possible to combine them in a way that *does*. Laplace's equation is *linear,* in the sense that if V_1, V_2, V_3, ... satisfy it, so does any **linear combination**, $V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \ldots$, where α_1, α_2, ... are arbitrary constants. For

$$\nabla^2 V = \alpha_1 \nabla^2 V_1 + \alpha_2 \nabla^2 V_2 + \ldots = 0\alpha_1 + 0\alpha_2 + \ldots = 0.$$

Exploiting this fact, we can patch together the separable solutions (Eq. 3.28) to construct a much more general solution:

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a). \tag{3.30}$$

This still satisfies three of the boundary conditions; the question is, can we (by astute choice of the coefficients C_n) fit the final boundary condition (iii)?

$$V(0, y) = \sum_{n=1}^{\infty} C_n \sin(n\pi y/a) = V_0(y). \tag{3.31}$$

[8] I'm assuming k is positive, but this involves no loss of generality—negative k gives the same solution (Eq. 3.27), only with the constants shuffled ($A \leftrightarrow B$, $C \to -C$). Occasionally (though not in this example) $k = 0$ must also be included (see Prob. 3.54).

Well, you may recognize this sum—it's a **Fourier sine series**. And Dirichlet's theorem[9] guarantees that virtually *any* function $V_0(y)$—it can even have a finite number of discontinuities—can be expanded in such a series.

But how do we actually *determine* the coefficients C_n, buried as they are in that infinite sum? The device for accomplishing this is so lovely it deserves a name—I call it **Fourier's trick**, though it seems Euler had used essentially the same idea somewhat earlier. Here's how it goes: Multiply Eq. 3.31 by $\sin(n'\pi y/a)$ (where n' is a positive integer), and integrate from 0 to a:

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin(n\pi y/a)\,\sin(n'\pi y/a)\,dy = \int_0^a V_0(y)\sin(n'\pi y/a)\,dy. \tag{3.32}$$

You can work out the integral on the left for yourself; the answer is

$$\int_0^a \sin(n\pi y/a)\,\sin(n'\pi y/a)\,dy = \begin{cases} 0, & \text{if } n' \neq n, \\ \dfrac{a}{2}, & \text{if } n' = n. \end{cases} \tag{3.33}$$

Thus all the terms in the series drop out, save only the one where $n = n'$, and the left side of Eq. 3.32, reduces to $(a/2)C_{n'}$. *Conclusion:*[10]

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy. \tag{3.34}$$

That *does* it: Eq. 3.30 is the solution, with coefficients given by Eq. 3.34. As a concrete example, suppose the strip at $x = 0$ is a metal plate with constant potential V_0 (remember, it's insulated from the grounded plates at $y = 0$ and $y = a$). Then

$$C_n = \frac{2V_0}{a}\int_0^a \sin(n\pi y/a)\,dy = \frac{2V_0}{n\pi}(1 - \cos n\pi) = \begin{cases} 0, & \text{if } n \text{ is even}, \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd}. \end{cases} \tag{3.35}$$

Thus

$$V(x, y) = \frac{4V_0}{\pi}\sum_{n=1,3,5\ldots} \frac{1}{n} e^{-n\pi x/a}\sin(n\pi y/a). \tag{3.36}$$

Figure 3.18 is a plot of this potential; Fig. 3.19 shows how the first few terms in the Fourier series combine to make a better and better approximation to the constant V_0: (a) is $n = 1$ only, (b) includes n up to 5, (c) is the sum of the first 10 terms, and (d) is the sum of the first 100 terms.

[9]Boas, M., *Mathematical Methods in the Physical Sciences,* 2nd ed. (New York: John Wiley, 1983).
[10]For aesthetic reasons I've dropped the prime; Eq. 3.34 holds for $n = 1, 2, 3, \ldots$, and it doesn't matter (obviously) what letter you use for the "dummy" index.

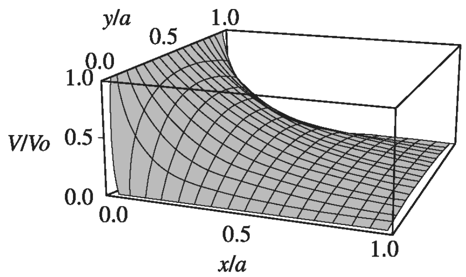

FIGURE 3.18

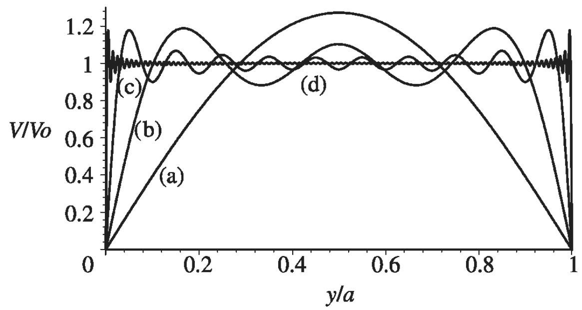

FIGURE 3.19

Incidentally, the infinite series in Eq. 3.36 can be summed explicitly (try your hand at it, if you like); the result is

$$V(x, y) = \frac{2V_0}{\pi} \tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh(\pi x/a)}\right). \tag{3.37}$$

In this form, it is easy to check that Laplace's equation is obeyed and the four boundary conditions (Eq. 3.21) are satisfied.

The success of this method hinged on two extraordinary properties of the separable solutions (Eqs. 3.28 and 3.29): **completeness** and **orthogonality**. A set of functions $f_n(y)$ is said to be **complete** if any other function $f(y)$ can be expressed as a linear combination of them:

$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y). \tag{3.38}$$

The functions $\sin(n\pi y/a)$ are complete on the interval $0 \le y \le a$. It was this fact, guaranteed by Dirichlet's theorem, that assured us Eq. 3.31 could be satisfied, given the proper choice of the coefficients C_n. (The *proof* of completeness, for a particular set of functions, is an extremely difficult business, and I'm afraid physicists tend to *assume* it's true and leave the checking to others.) A set of functions is **orthogonal** if the integral of the product of any two different members of the set is zero:

$$\int_0^a f_n(y) f_{n'}(y)\, dy = 0 \qquad \text{for } n' \neq n. \tag{3.39}$$

The sine functions are orthogonal (Eq. 3.33); this is the property on which Fourier's trick is based, allowing us to kill off all terms but one in the infinite series and thereby solve for the coefficients C_n. (Proof of orthogonality is generally quite simple, either by direct integration or by analysis of the differential equation from which the functions came.)

Example 3.4. Two infinitely-long grounded metal plates, again at $y = 0$ and $y = a$, are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0, as shown in Fig. 3.20 (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe.

Solution

Once again, the configuration is independent of z. Our problem is to solve Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = V_0 \text{ when } x = b, \\ \text{(iv)} & V = V_0 \text{ when } x = -b. \end{array}\right\} \tag{3.40}$$

The argument runs as before, up to Eq. 3.27:

$$V(x, y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky).$$

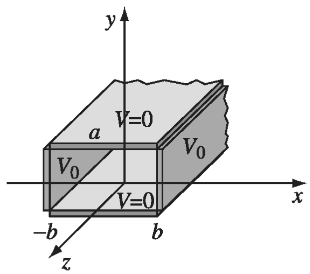

FIGURE 3.20

This time, however, we cannot set $A = 0$; the region in question does not extend to $x = \infty$, so e^{kx} is perfectly acceptable. On the other hand, the situation is *symmetric* with respect to x, so $V(-x, y) = V(x, y)$, and it follows that $A = B$. Using

$$e^{kx} + e^{-kx} = 2\cosh kx,$$

and absorbing $2A$ into C and D, we have

$$V(x, y) = \cosh kx\,(C \sin ky + D \cos ky).$$

Boundary conditions (i) and (ii) require, as before, that $D = 0$ and $k = n\pi/a$, so

$$V(x, y) = C \cosh(n\pi x/a)\, \sin(n\pi y/a). \tag{3.41}$$

Because $V(x, y)$ is even in x, it will automatically meet condition (iv) if it fits (iii). It remains, therefore, to construct the general linear combination,

$$V(x, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/a)\, \sin(n\pi y/a),$$

and pick the coefficients C_n in such a way as to satisfy condition (iii):

$$V(b, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi b/a)\, \sin(n\pi y/a) = V_0.$$

This is the same problem in Fourier analysis that we faced before; I quote the result from Eq. 3.35:

$$C_n \cosh(n\pi b/a) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Conclusion: The potential in this case is given by

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \sin(n\pi y/a). \tag{3.42}$$

This function is shown in Fig. 3.21.

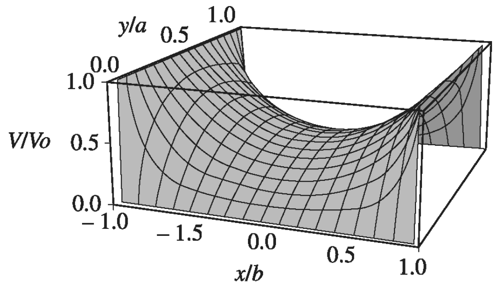

FIGURE 3.21

Example 3.5. An infinitely long rectangular metal pipe (sides a and b) is grounded, but one end, at $x = 0$, is maintained at a specified potential $V_0(y, z)$, as indicated in Fig. 3.22. Find the potential inside the pipe.

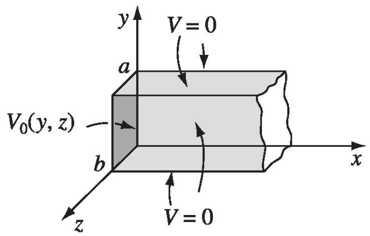

FIGURE 3.22

Solution

This is a genuinely three-dimensional problem,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{3.43}$$

subject to the boundary conditions

$$\left.\begin{array}{ll} \text{(i)} & V = 0 \text{ when } y = 0, \\ \text{(ii)} & V = 0 \text{ when } y = a, \\ \text{(iii)} & V = 0 \text{ when } z = 0, \\ \text{(iv)} & V = 0 \text{ when } z = b, \\ \text{(v)} & V \to 0 \text{ as } x \to \infty, \\ \text{(vi)} & V = V_0(y, z) \text{ when } x = 0. \end{array}\right\} \tag{3.44}$$

As always, we look for solutions that are products:

$$V(x, y, z) = X(x)Y(y)Z(z). \tag{3.45}$$

Putting this into Eq. 3.43, and dividing by V, we find

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

It follows that

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = C_3, \quad \text{with } C_1 + C_2 + C_3 = 0.$$

Our previous experience (Ex. 3.3) suggests that C_1 must be positive, C_2 and C_3 negative. Setting $C_2 = -k^2$ and $C_3 = -l^2$, we have $C_1 = k^2 + l^2$, and hence

$$\frac{d^2X}{dx^2} = (k^2 + l^2)X, \quad \frac{d^2Y}{dy^2} = -k^2Y, \quad \frac{d^2Z}{dz^2} = -l^2Z. \tag{3.46}$$

Once again, separation of variables has turned a *partial* differential equation into *ordinary* differential equations. The solutions are

$$\begin{aligned} X(x) &= Ae^{\sqrt{k^2+l^2}\,x} + Be^{-\sqrt{k^2+l^2}\,x}, \\ Y(y) &= C \sin ky + D \cos ky, \\ Z(z) &= E \sin lz + F \cos lz. \end{aligned}$$

Boundary condition (v) implies $A = 0$, (i) gives $D = 0$, and (iii) yields $F = 0$, whereas (ii) and (iv) require that $k = n\pi/a$ and $l = m\pi/b$, where n and m are positive integers. Combining the remaining constants, we are left with

$$V(x, y, z) = Ce^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a) \sin(m\pi z/b). \tag{3.47}$$

This solution meets all the boundary conditions except (vi). It contains *two* unspecified integers (n and m), and the most general linear combination is a *double* sum:

$$V(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x} \sin(n\pi y/a) \sin(m\pi z/b). \tag{3.48}$$

We hope to fit the remaining boundary condition,

$$V(0, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin(n\pi y/a) \sin(m\pi z/b) = V_0(y, z), \tag{3.49}$$

by appropriate choice of the coefficients $C_{n,m}$. To determine these constants, we multiply by $\sin(n'\pi y/a)\,\sin(m'\pi z/b)$, where n' and m' are arbitrary positive integers, and integrate:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \int_0^a \sin(n\pi y/a)\,\sin(n'\pi y/a)\,dy \int_0^b \sin(m\pi z/b)\,\sin(m'\pi z/b)\,dz$$
$$= \int_0^a \int_0^b V_0(y,z)\sin(n'\pi y/a)\,\sin(m'\pi z/b)\,dy\,dz.$$

Quoting Eq. 3.33, the left side is $(ab/4)C_{n',m'}$, so

$$C_{n,m} = \frac{4}{ab}\int_0^a \int_0^b V_0(y,z)\sin(n\pi y/a)\,\sin(m\pi z/b)\,dy\,dz. \tag{3.50}$$

Equation 3.48, with the coefficients given by Eq. 3.50, is the solution to our problem.

For instance, if the end of the tube is a conductor at *constant* potential V_0,

$$C_{n,m} = \frac{4V_0}{ab}\int_0^a \sin(n\pi y/a)\,dy \int_0^b \sin(m\pi z/b)\,dz$$
$$= \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if } n \text{ and } m \text{ are odd.} \end{cases} \tag{3.51}$$

In this case

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n,m=1,3,5\ldots}^{\infty} \frac{1}{nm} e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin(n\pi y/a)\,\sin(m\pi z/b). \tag{3.52}$$

Notice that the successive terms decrease rapidly; a reasonable approximation would be obtained by keeping only the first few.

Problem 3.13 Find the potential in the infinite slot of Ex. 3.3 if the boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$, is held at a constant potential V_0, and the other, from $y = a/2$ to $y = a$, is at potential $-V_0$.

Problem 3.14 For the infinite slot (Ex. 3.3), determine the charge density $\sigma(y)$ on the strip at $x = 0$, assuming it is a conductor at constant potential V_0.

Problem 3.15 A rectangular pipe, running parallel to the z-axis (from $-\infty$ to $+\infty$), has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specified potential $V_0(y)$.

(a) Develop a general formula for the potential inside the pipe.

(b) Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).

Problem 3.16 A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (Fig. 3.23). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box. [What should the potential at the center $(a/2, a/2, a/2)$ be? Check numerically that your formula is consistent with this value.][11]

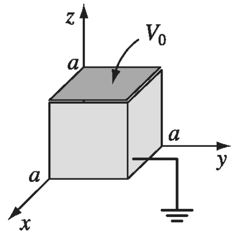

FIGURE 3.23

3.3.2 ■ Spherical Coordinates

In the examples considered so far, Cartesian coordinates were clearly appropriate, since the boundaries were planes. For round objects, spherical coordinates are more natural. In the spherical system, Laplace's equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0. \tag{3.53}$$

I shall assume the problem has **azimuthal symmetry**, so that V *is independent of* ϕ;[12] in that case, Eq. 3.53 reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0. \tag{3.54}$$

As before, we look for solutions that are products:

$$V(r,\theta)=R(r)\Theta(\theta). \tag{3.55}$$

Putting this into Eq. 3.54, and dividing by V,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0. \tag{3.56}$$

[11] This cute test was suggested by J. Castro.

[12] The general case, for ϕ-dependent potentials, is treated in all the graduate texts. See, for instance, J. D. Jackson's *Classical Electrodynamics*, 3rd ed. (New York: John Wiley, 1999), Chapter 3.

Since the first term depends only on r, and the second only on θ, it follows that each must be a constant:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1), \qquad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1). \tag{3.57}$$

Here $l(l+1)$ is just a fancy way of writing the separation constant—you'll see in a minute why this is convenient.

As always, separation of variables has converted a *partial* differential equation (3.54) into *ordinary* differential equations (3.57). The radial equation,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R, \tag{3.58}$$

has the general solution

$$R(r) = Ar^l + \frac{B}{r^{l+1}}, \tag{3.59}$$

as you can easily check; A and B are the two arbitrary constants to be expected in the solution of a second-order differential equation. But the angular equation,

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta, \tag{3.60}$$

is not so simple. The solutions are **Legendre polynomials** in the variable $\cos\theta$:

$$\Theta(\theta) = P_l(\cos\theta). \tag{3.61}$$

$P_l(x)$ is most conveniently defined by the **Rodrigues formula:**

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l. \tag{3.62}$$

The first few Legendre polynomials are listed in Table 3.1.

$P_0(x)$	$=$	1
$P_1(x)$	$=$	x
$P_2(x)$	$=$	$(3x^2-1)/2$
$P_3(x)$	$=$	$(5x^3-3x)/2$
$P_4(x)$	$=$	$(35x^4-30x^2+3)/8$
$P_5(x)$	$=$	$(63x^5-70x^3+15x)/8$

TABLE 3.1 Legendre Polynomials.

Notice that $P_l(x)$ is (as the name suggests) an lth-order *polynomial* in x; it contains only *even* powers, if l is even, and *odd* powers, if l is odd. The factor in front $(1/2^l l!)$ was chosen in order that

$$P_l(1) = 1. \tag{3.63}$$

The Rodrigues formula obviously works only for nonnegative integer values of l. Moreover, it provides us with only *one* solution. But Eq. 3.60 is *second-order*, and it should possess *two* independent solutions, for *every* value of l. It turns out that these "other solutions" blow up at $\theta = 0$ and/or $\theta = \pi$, and are therefore unacceptable on physical grounds.[13] For instance, the second solution for $l = 0$ is

$$\Theta(\theta) = \ln\left(\tan\frac{\theta}{2}\right). \tag{3.64}$$

You might want to check for yourself that this satisfies Eq. 3.60.

In the case of azimuthal symmetry, then, the most general separable solution to Laplace's equation, consistent with minimal physical requirements, is

$$V(r, \theta) = \left(Ar^l + \frac{B}{r^{l+1}}\right) P_l(\cos\theta).$$

(There was no need to include an overall constant in Eq. 3.61 because it can be absorbed into A and B at this stage.) As before, separation of variables yields an infinite set of solutions, one for each l. The *general* solution is the linear combination of separable solutions:

$$\boxed{V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta).} \tag{3.65}$$

The following examples illustrate the power of this important result.

Example 3.6. The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of radius R. Find the potential inside the sphere.

Solution

In this case, $B_l = 0$ for all l—otherwise the potential would blow up at the origin. Thus,

$$V(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta). \tag{3.66}$$

[13] In rare cases where the z axis is excluded, these "other solutions" do have to be considered.

At $r = R$ this must match the specified function $V_0(\theta)$:

$$V(R, \theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta). \tag{3.67}$$

Can this equation be satisfied, for an appropriate choice of coefficients A_l? *Yes:* The Legendre polynomials (like the sines) constitute a complete set of functions, on the interval $-1 \leq x \leq 1$ $(0 \leq \theta \leq \pi)$. How do we determine the constants? Again, by Fourier's trick, for the Legendre polynomials (like the sines) are *orthogonal* functions:[14]

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta = \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1}, & \text{if } l' = l. \end{cases} \tag{3.68}$$

Thus, multiplying Eq. 3.67 by $P_{l'}(\cos\theta)\sin\theta$ and integrating, we have

$$A_{l'} R^{l'} \frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta,$$

or

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta\, d\theta. \tag{3.69}$$

Equation 3.66 is the solution to our problem, with the coefficients given by Eq. 3.69.

It can be difficult to evaluate integrals of the form 3.69 analytically, and in practice it is often easier to solve Eq. 3.67 "by eyeball."[15] For instance, suppose we are told that the potential on the sphere is

$$V_0(\theta) = k \sin^2(\theta/2), \tag{3.70}$$

where k is a constant. Using the half-angle formula, we rewrite this as

$$V_0(\theta) = \frac{k}{2}(1 - \cos\theta) = \frac{k}{2}[P_0(\cos\theta) - P_1(\cos\theta)].$$

[14]M. Boas, *Mathematical Methods in the Physical Sciences,* 2nd ed. (New York: John Wiley, 1983), Section 12.7.

[15]This is certainly true whenever $V_0(\theta)$ can be expressed as a polynomial in $\cos\theta$. The degree of the polynomial tells us the highest l we require, and the leading coefficient determines the corresponding A_l. Subtracting off $A_l R^l P_l(\cos\theta)$ and repeating the process, we systematically work our way down to A_0. Notice that if V_0 is an *even* function of $\cos\theta$, then only even terms will occur in the sum (and likewise for odd functions).

Putting this into Eq. 3.67, we read off immediately that $A_0 = k/2$, $A_1 = -k/(2R)$, and all other A_l's vanish. Therefore,

$$V(r,\theta) = \frac{k}{2}\left[r^0 P_0(\cos\theta) - \frac{r^1}{R}P_1(\cos\theta)\right] = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right). \tag{3.71}$$

Example 3.7. The potential $V_0(\theta)$ is again specified on the surface of a sphere of radius R, but this time we are asked to find the potential *outside*, assuming there is no charge there.

Solution

In this case it's the A_l's that must be zero (or else V would not go to zero at ∞), so

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta). \tag{3.72}$$

At the surface of the sphere, we require that

$$V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta).$$

Multiplying by $P_{l'}(\cos\theta)\sin\theta$ and integrating—exploiting, again, the orthogonality relation 3.68—we have

$$\frac{B_{l'}}{R^{l'+1}} \frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(\cos\theta)\sin\theta\, d\theta,$$

or

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.73}$$

Equation 3.72, with the coefficients given by Eq. 3.73, is the solution to our problem.

Example 3.8. An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. The field will push positive charge to the "northern" surface of the sphere, and—symmetrically—negative charge to the "southern" surface (Fig. 3.24). This induced charge, in turn, distorts the field in the neighborhood of the sphere. Find the potential in the region outside the sphere.

Solution

The sphere is an equipotential—we may as well set it to zero. Then by symmetry the entire xy plane is at potential zero. This time, however, V does *not* go to zero at large z. In fact, far from the sphere the field is $E_0\hat{\mathbf{z}}$, and hence

$$V \to -E_0 z + C.$$

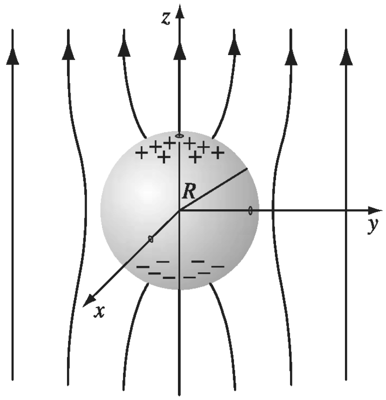

FIGURE 3.24

Since $V = 0$ in the equatorial plane, the constant C must be zero. Accordingly, the boundary conditions for this problem are

$$\left.\begin{array}{lll} \text{(i)} & V = 0 & \text{when } r = R, \\ \text{(ii)} & V \to -E_0 r\cos\theta & \text{for } r \gg R. \end{array}\right\} \tag{3.74}$$

We must fit these boundary conditions with a function of the form 3.65.

The first condition yields

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0,$$

or

$$B_l = -A_l R^{2l+1}, \tag{3.75}$$

so

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta).$$

For $r \gg R$, the second term in parentheses is negligible, and therefore condition (ii) requires that

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta.$$

Evidently only one term is present: $l = 1$. In fact, since $P_1(\cos\theta) = \cos\theta$, we can read off immediately

$$A_1 = -E_0, \quad \text{all other } A_l\text{'s zero.}$$

Conclusion:

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta. \tag{3.76}$$

The first term $(-E_0 r\cos\theta)$ is due to the external field; the contribution attributable to the induced charge is

$$E_0\frac{R^3}{r^2}\cos\theta.$$

If you want to know the induced charge density, it can be calculated in the usual way:

$$\sigma(\theta) = -\epsilon_0\frac{\partial V}{\partial r}\bigg|_{r=R} = \epsilon_0 E_0\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta\bigg|_{r=R} = 3\epsilon_0 E_0\cos\theta. \tag{3.77}$$

As expected, it is positive in the "northern" hemisphere $(0 \le \theta \le \pi/2)$ and negative in the "southern" $(\pi/2 \le \theta \le \pi)$.

Example 3.9. A specified charge density $\sigma_0(\theta)$ is glued over the surface of a spherical shell of radius R. Find the resulting potential inside and outside the sphere.

Solution

You could, of course, do this by direct integration:

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma_0}{\mathscr{r}}\,da,$$

but separation of variables is often easier. For the interior region, we have

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad (r \le R) \tag{3.78}$$

(no B_l terms—they blow up at the origin); in the exterior region

$$V(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) \qquad (r \ge R) \tag{3.79}$$

(no A_l terms—they don't go to zero at infinity). These two functions must be joined together by the appropriate boundary conditions at the surface itself. First, the potential is *continuous* at $r = R$ (Eq. 2.34):

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta). \tag{3.80}$$

It follows that the coefficients of like Legendre polynomials are equal:

$$B_l = A_l R^{2l+1}. \tag{3.81}$$

(To prove that formally, multiply both sides of Eq. 3.80 by $P_{l'}(\cos\theta)\sin\theta$ and integrate from 0 to π, using the orthogonality relation 3.68.) Second, the radial derivative of V suffers a discontinuity at the surface (Eq. 2.36):

$$\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\Bigg|_{r=R} = -\frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.82}$$

Thus

$$-\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_0(\theta),$$

or, using Eq. 3.81,

$$\sum_{l=0}^{\infty}(2l+1) A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\epsilon_0}\sigma_0(\theta). \tag{3.83}$$

From here, the coefficients can be determined using Fourier's trick:

$$A_l = \frac{1}{2\epsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta. \tag{3.84}$$

Equations 3.78 and 3.79 constitute the solution to our problem, with the coefficients given by Eqs. 3.81 and 3.84.

For instance, if

$$\sigma_0(\theta) = k\cos\theta = k P_1(\cos\theta), \tag{3.85}$$

for some constant k, then all the A_l's are zero except for $l = 1$, and

$$A_1 = \frac{k}{2\epsilon_0}\int_0^{\pi} [P_1(\cos\theta)]^2 \sin\theta \, d\theta = \frac{k}{3\epsilon_0}.$$

The potential inside the sphere is therefore

$$V(r,\theta) = \frac{k}{3\epsilon_0} r\cos\theta \qquad (r \le R), \tag{3.86}$$

whereas outside the sphere

$$V(r, \theta) = \frac{kR^3}{3\epsilon_0} \frac{1}{r^2} \cos\theta \qquad (r \geq R). \tag{3.87}$$

In particular, if $\sigma_0(\theta)$ is the induced charge on a metal sphere in an external field $E_0\hat{\mathbf{z}}$, so that $k = 3\epsilon_0 E_0$ (Eq. 3.77), then the potential inside is $E_0 r\cos\theta = E_0 z$, and the field is $-E_0\hat{\mathbf{z}}$—exactly right to cancel off the external field, as of course it *should* be. Outside the sphere the potential due to this surface charge is

$$E_0 \frac{R^3}{r^2} \cos\theta,$$

consistent with our conclusion in Ex. 3.8.

Problem 3.17 Derive $P_3(x)$ from the Rodrigues formula, and check that $P_3(\cos\theta)$ satisfies the angular equation (3.60) for $l = 3$. Check that P_3 and P_1 are orthogonal by explicit integration.

Problem 3.18

(a) Suppose the potential is a *constant* V_0 over the surface of the sphere. Use the results of Ex. 3.6 and Ex. 3.7 to find the potential inside and outside the sphere. (Of course, you know the answers in advance—this is just a consistency check on the method.)

(b) Find the potential inside and outside a spherical shell that carries a uniform surface charge σ_0, using the results of Ex. 3.9.

Problem 3.19 The potential at the surface of a sphere (radius R) is given by

$$V_0 = k\cos 3\theta,$$

where k is a constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume there's no charge inside or outside the sphere.)

Problem 3.20 Suppose the potential $V_0(\theta)$ at the surface of a sphere is specified, and there is no charge inside or outside the sphere. Show that the charge density on the sphere is given by

$$\sigma(\theta) = \frac{\epsilon_0}{2R} \sum_{l=0}^{\infty} (2l+1)^2 C_l P_l(\cos\theta), \tag{3.88}$$

where

$$C_l = \int_0^\pi V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta. \tag{3.89}$$