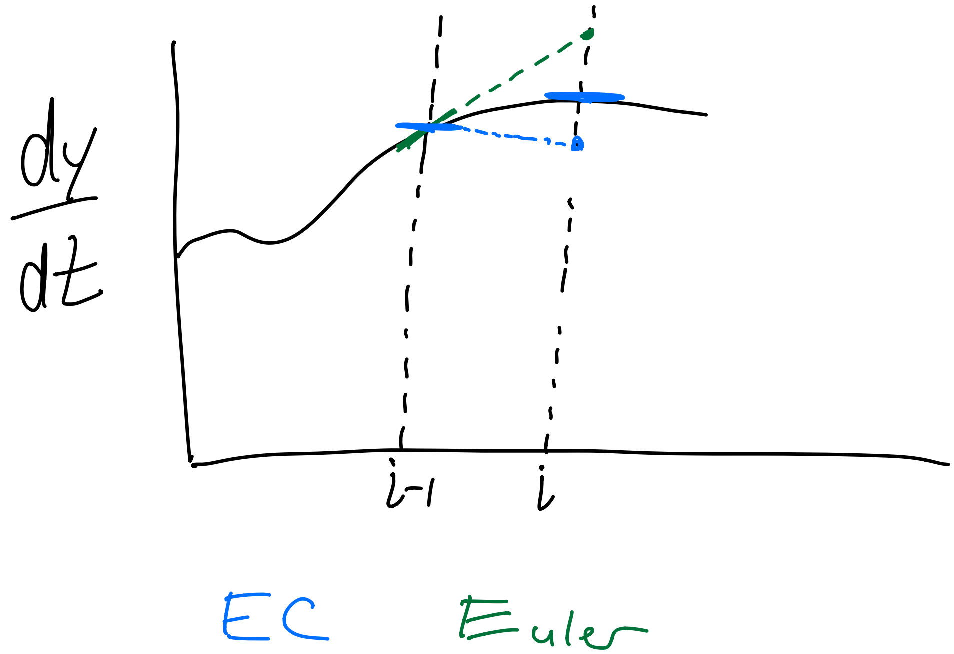
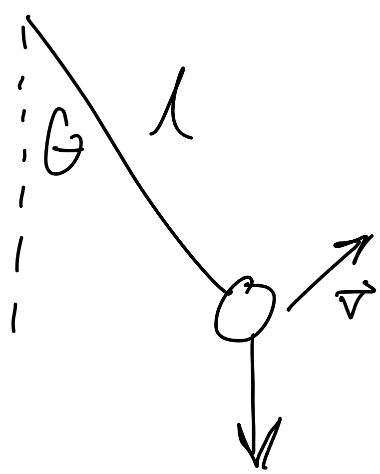
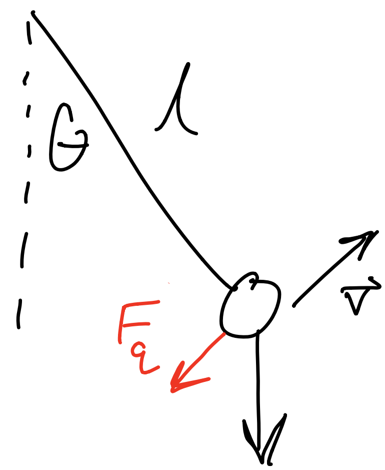
- Outline
- On Wed: we considered an ideal pendulum and solved it with a modified version of Euler





$$\tau_g = -mgl\sin\theta\,\hat{z}$$

restoring force ($\theta \to 0$)



$$F_q = -q\frac{d\theta}{dt}$$

Resistive Force

opposes increasing θ

F_q always along dir of motion

$$\tau_q = -q\,l\,\frac{d\theta}{dt}\,\hat{z}$$

$$I \frac{d^2\theta}{dt^2} = -q\,l \frac{d\theta}{dt}$$

$$\frac{d^2\theta}{dt^2} = \frac{-q\,l}{m l^2} \frac{d\theta}{dt}$$

$$\frac{d^2\theta}{dt^2} = \frac{-q}{m l} \frac{d\theta}{dt}$$

$$t = t_0 \bar{t} = \sqrt{\frac{l}{g}}\, \bar{t}$$

$$\frac{1}{t_0^2} \frac{d^2\theta}{dt^2} = \frac{-q}{m l} \frac{1}{t_0} \frac{d\theta}{d\bar{t}}$$

$$\frac{d^2\theta}{dt^2} = \frac{-q}{ml} t_0 \frac{d\theta}{dt}$$

$$= \underbrace{\frac{-q}{ml}\sqrt{\frac{l}{g}}}_{?} \frac{d\theta}{dt}$$

$$\frac{q}{ml}\sqrt{\frac{l}{g}} = \frac{q}{m}\frac{1}{\sqrt{gl}}$$

$$= \frac{q}{m}\sqrt{\frac{g}{g^2 l}}$$

$$= \frac{q}{mg}\sqrt{\frac{g}{l}} = \frac{q\Omega_0}{mg}$$

$$\boxed{Q = \frac{q\Omega_0}{mg}}$$

$\frac{q}{\text{time}}$ = Force

mg = Force

Q = ratio of damping force to grav force

Q = 0 no damping

Q = 1 ~ balanced

Q > 1 heavy damping

$$\frac{d^2\Theta}{dt^2} = -\Theta - Q\frac{d\Theta}{dt}$$

Can solve analytically

Eqn of the form

$$\frac{d^2\Theta}{dt^2} + Q\frac{d\Theta}{dt} + \Theta = 0$$

2ND ORDER

$$\Theta = C_1\Theta_1 + C_2\Theta_2$$

Try $\Theta_1 = e^{rt}$

$$\frac{d\theta_1}{dt} = re^{rt}$$

$$\frac{d^2\theta_1}{dt} = r^2e^{rt}$$

$$\frac{d^2\theta_1}{dt^2} + Q\frac{d\theta_1}{dt} + \theta_1 = 0$$

$$r^2e^{rt} + Qre^{rt} + e^{rt} = 0$$

$$e^{rt}(r^2+Qr+1) = 0$$

$$r^2+Qr+1 = 0$$

Quadratic Formula

$$ax^2+bx+c=0$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\frac{-Q \pm \sqrt{Q^2-4}}{2} = r$$

have two solutions

$$e^{rt} = e^{\frac{-Q \pm \sqrt{Q^2-4}}{2} t}$$

$$= e^{-\frac{Q}{2}t} e^{\frac{\pm\sqrt{Q^2-4}}{2} t}$$

$\downarrow$ Simplify

$$\frac{\sqrt{Q^2-4}}{2} = \sqrt{\frac{1}{4}\left(Q^2-4\right)}$$

$$= \sqrt{\frac{Q^2}{4} - 1}$$

$$= \pm i \underbrace{\sqrt{1 - \left(\frac{Q}{2}\right)^2}}_{\Omega_q}$$

$$\Omega_q = \sqrt{1 - \left(\frac{Q}{2}\right)^2} \quad (\text{units of } \Omega_0)$$

$$\Theta = e^{-\frac{1}{2}Qt}\left[c_1 e^{i\Omega_q t} + c_2 e^{-i\Omega_q t}\right]$$

Recall

$$e^{\pm i\Omega_q t} = \cos(\Omega_q t) \pm \sin(\Omega_q t)$$

Cases:

$Q \rightarrow 0$

$\Omega_q \rightarrow 1$ Natural Freq

$$\Theta = c_1 e^{it} + c_2 e^{-it}$$

$$= c_1(\cos(t) + \sin(t))$$

$$+ c_2(\cos(t) - \sin(t))$$

$$= (c_1 + c_2)\cos(t) + (c_1 - c_2)\sin(t)$$

ORIGINAL SOLUTION ✓

$0 < Q < 2$

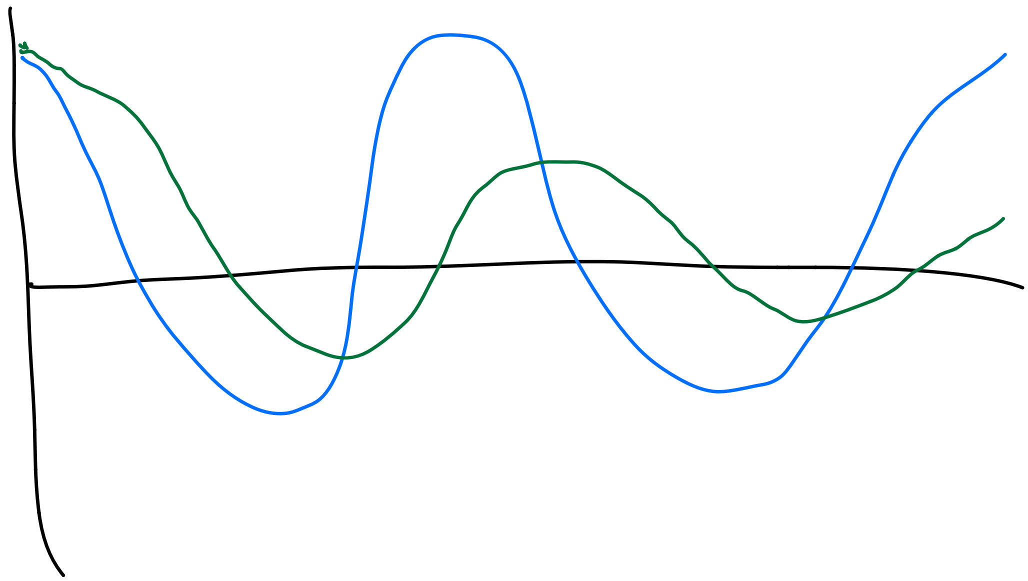
Ω_q is Real

$$e^{-\frac{1}{2}Qt}\left[c_1 e^{i\Omega_q t} + c_2 e^{-i\Omega_q t}\right]$$

$$e^{-\frac{1}{2}Qt}\left[A\sin(\Omega_q t) + B\cos(\Omega_q t)\right]$$

$$\Omega_q = \sqrt{1 - \left(\frac{Q}{2}\right)^2}$$

slower oscillations
than Ω_0
slowly die off
to 0



$Q > 2$

$$Q > 2$$

$$1 - \left(\frac{Q}{2}\right)^2 < 0$$

$$\Omega_q = \sqrt{1 - \left(\frac{Q}{2}\right)^2}$$

$$= \sqrt{-1\left(\left(\frac{Q}{2}\right)^2 - 1\right)}$$

$$= i\sqrt{\left(\frac{Q}{2}\right)^2 - 1}$$

$$e^{-\frac{1}{2}Qt}\left[e^{\pm i\Omega_q t}\right]$$

$$= e^{-\frac{1}{2}Qt}\left[c_1 e^{-\sqrt{\left(\frac{Q}{2}\right)^2 - 1}\,t} + c_2 e^{\sqrt{\left(\frac{Q}{2}\right)^2 - 1}\,t}\right]$$

No oscillation

Decay

$Q=2, \quad \Omega_e=0$

$r^2+2r+1=0$

$(r+1)^2=0$

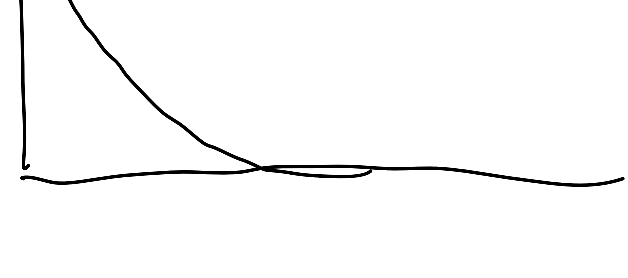
$r=-1$

only one solution

$\Theta_1=c_1e^{-t}$

try $\Theta_2=te^{-t}$

$\Theta=(c_1+tc_2)e^{-t}$



$Q < 2$: underdamped

Decaying oscillations

w/ freq $\sqrt{1-\left(\frac{Q}{2}\right)^2}$

$Q > 2$: overdamped

oscillations completely damped out

$Q = 2$: critically damped

No oscillations

quickly approach eq