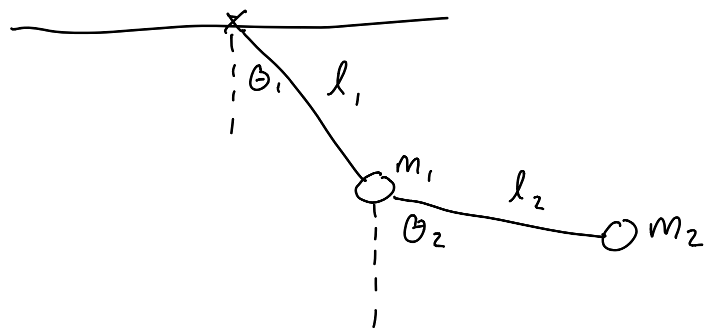
The double pendulum



$$\vec{r}_1 = l_1 \sin\theta_1 \hat{x} - l_1 \cos\theta_1 \hat{y}$$

$$\vec{r}_2 = \vec{r}_1 + l_2 \sin\theta_2 \hat{x} - l_2 \cos\theta_2 \hat{y}$$

$$\vec{r}_2 = (l_1 \sin\theta_1 + l_2 \sin\theta_2)\hat{x} - (l_1\cos\theta_1 + l_2\cos\theta_2)\hat{y}$$

$$\mathcal{L} = T - U$$

$$T = T_1 + T_2$$

$$T_1 = \frac{1}{2} m_1 v^2, \quad v^2 = \dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1$$

$$\dot{\vec{r}}_1 = l_1 \dot{\theta}_1 \cos\theta_1 \hat{x} + l_1 \dot{\theta}_1 \sin\theta_1 \hat{y}$$

$$\dot{\vec{r}}_1 = l_1 \dot{\theta}_1 \left[\cos\theta_1 \hat{x} + \sin\theta_1 \hat{y}\right]$$

$$\dot{r}_1^2 = l_1^2 \dot{\theta}_1^2$$

$$T_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$$

$$\vec{r}_2 = (l_1 \sin\theta_1 + l_2 \sin\theta_2)\hat{x} - (l_1 \cos\theta_1 + l_2 \cos\theta_2)\hat{y}$$

$$\dot{\vec{r}}_2 = (l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2 \cos\theta_2)\hat{x} - (-l_1\dot{\theta}_1 \sin\theta_1 - l_2\dot{\theta}_2 \sin\theta_2)\hat{y}$$

$$\dot{\vec{r}}_2 = (l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2 \cos\theta_2)\hat{x} + (l_1\dot{\theta}_1 \sin\theta_1 + l_2\dot{\theta}_2 \sin\theta_2)\hat{y}$$

$$\dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2 = r_x^2 + r_y^2$$

$$= l_1^2\dot{\theta}_1^2\cos^2\theta_1 + l_2^2\dot{\theta}_2^2\cos^2\theta_2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2$$

$$+ l_1^2\dot{\theta}_1^2\sin^2\theta_1 + l_2^2\dot{\theta}_2^2\sin^2\theta_2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2$$

$$= l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$

$$\dot{r}_2^2 = l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$$

$$T_2 = \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$$

$$T = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}^2$$

$$+ m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$$

$$T = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 +$$

$$\frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)$$

$$U = U_1 + U_2 = m_1 g y_1 + m_2 g y_2$$

$$U_1 = m_1 g y_1 = m_1 g \vec{r}_1 \cdot \hat{y} = -m_1 g l_1 \cos\theta_1$$

$$U_2 = m_2 g y_2 = m_2 g \vec{r}_2 \cdot \hat{y} = -m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$U = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

NORMALIZE

$$E = \frac{1}{2} m_1 l_1^2 \omega_1^2 + \frac{1}{2} m_2 l_1^2 \omega_1^2 + \frac{1}{2} m_2 l_2^2 \omega_2^2 + m_2 l_1 l_2 \omega_1 \omega_2 - (m_1 + m_2) g l_1 - m_2 g l_2$$

$$\bar{E} = \frac{E}{E_0}, \quad \bar{\omega}_{1/2} = \frac{\omega_{1/2}}{\omega_0}$$

$$\bar{E} = \frac{\frac{1}{2} m_1 l_1^2 \bar{\omega}_1^2 \omega_0^2}{E_0} + \frac{\frac{1}{2} m_2 l_1^2 \bar{\omega}_1^2 \omega_0^2}{E_0} + \frac{\frac{1}{2} m_2 l_2^2 \bar{\omega}_2^2 \omega_0^2}{E_0} + \frac{m_2 l_1 l_2 \bar{\omega}_1 \bar{\omega}_2 \omega_0^2}{E_0} - (m_1 + m_2) g l_1 / E_0 - m_2 g l_2 / E_0$$

NORMALIZE (CONT)

$$m_1 l_1^2 = I_1 \;,\; m_2 l_2^2 = I_2$$

$$\omega_0^2 = \frac{E_0}{I_1 + I_2}$$

$$\bar{E} = \frac{1}{2}\frac{I_1 \bar{\omega}_1^2}{I_1 + I_2} + \frac{\frac{1}{2} I_1 \frac{m_2}{m_1} \bar{\omega}_1^2}{I_1 + I_2}$$

$$+ \frac{1}{2}\frac{I_2 \bar{\omega}_2^2}{I_1 + I_2} + \frac{I_2 \frac{l_1}{l_2} \bar{\omega}_1 \bar{\omega}_2}{I_1 + I_2}$$

$$-\frac{(m_1 g l_1)}{E_0} - \frac{m_2 g l_1}{E_0} - \frac{m_2 g l_2}{E_0}$$

$$i_{1/2} = \frac{I_{1/2}}{I_1 + I_2} \qquad \mu = \frac{m_2}{m_1} \qquad \lambda = \frac{l_2}{l_1}$$

$$t = \frac{1}{2} i_1 \bar{\omega}_1^2 (1 + \mu) + \frac{1}{2} i_2 \bar{\omega}_2^2 + \frac{1}{\lambda} i_2 \bar{\omega}_1 \bar{\omega}_2$$

$$\bar{E} = t - \frac{m_1 g l_1}{E_0} - \frac{m_2 g l_1}{E_0} - \frac{m_2 g l_2}{E_0}$$

$$E_0 = (m_1 l_1 + m_2 l_2) g$$

$$\frac{m_1 l_1}{m_1 l_1 + m_2 l_2} = \frac{m_1 l_1}{m_1 l_1 + \mu \lambda m_1} = \frac{1}{1 + \mu \lambda}$$

$$\frac{m_2 l_1}{m_1 l_1 + m_2 l_2} = \frac{m_2 \frac{l_2}{\lambda}}{\frac{m_2 l_2}{\mu \lambda} + m_2 l_2} = \frac{\frac{1}{\lambda}}{\frac{1}{\mu \lambda} + 1}$$

$$= \frac{\mu}{1 + \mu \lambda}$$

$$\frac{m_2 l_2}{m_1 l_1 + m_2 l_2} = \frac{m_2 l_2}{\frac{m_2 l_2}{\mu \lambda} + 1} = \frac{\mu \lambda}{1 + \mu \lambda}$$

$$u = \frac{-1}{1 + \mu \lambda} \left[(1 + \mu) \cos \theta_1 + \mu \lambda \cos \theta_2 \right]$$

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$$t = \frac{1}{2} i_1 \dot{\theta}_1^2 (1+\mu) + \frac{1}{2} i_2 \dot{\theta}_2^2 + \frac{1}{\lambda} i_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$u = \frac{-1}{1+\mu\lambda} \left[(1+\mu)\cos\theta_1 + \mu\lambda\cos\theta_2 \right]$$

$$i_1 = \frac{I_1}{I_1 + I_2} \qquad \mu = \frac{m_2}{m_1} \qquad \lambda = \frac{l_2}{l_1}$$

$$\mathcal{L} = t - u$$

$$= \frac{1}{2} i_1 \dot{\theta}_1^2 (1+\mu) + \frac{1}{2} i_2 \dot{\theta}_2^2 + \frac{1}{\lambda} i_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$+ \frac{1}{1+\mu\lambda} \left[(1+\mu)\cos\theta_1 + \mu\lambda\cos\theta_2 \right]$$

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 +$$
$$\frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$

$$U = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

$$\mathcal{L} = T - U$$
$$\mathcal{L} = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right)$$
$$+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

Euler Lagrange

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \right) \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} \right) \qquad (2)$$

(1) $$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$
$$-(m_1 + m_2) g l_1 \sin\theta_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1$$
$$+ m_2 l_1 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right]$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$- m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$

(1)
$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$
$$= -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1$$

in $(\dot{\theta}_1 - \dot{\theta}_2)$ term $\dot{\theta}_1$ cancels

$\div$ by l_1

$$(m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2)$$
$$+ (m_1+m_2) g \sin\theta_1 = 0 \qquad (a)$$

$$\mathcal{L} = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1-\theta_2)\right)$$
$$+ (m_1+m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

(2) $$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - m_2 g l_2 \sin\theta_2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \left[\ddot{\theta}_1 \cos(\theta_1-\theta_2) - \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2)\right]$$

$$= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2)$$
$$- m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1-\theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

$(\dot{\theta}_1 - \dot{\theta}_2)$, $\dot{\theta}_2$ cancels

$\div$ by l_2

$\downarrow$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0 \qquad (b)$$

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin\theta_1 = 0 \qquad (a)$$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0 \qquad (b)$$