Project 2

Gravity in Two Spatial Dimensions
Due: Friday, November 13

Overview

We have spent the past several lectures of class investigating the interaction of objects under the influence of a gravitational force $\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$. As it turns out, the $1/r^2$ dependence of this force is deeply connected to the fact that our Universe has 3 spatial dimensions, and thus the field strength is spread over a sphere with an area of $4\pi r^2$. The same reasoning is true of the electrostatic force.

What would gravity look like if we were confined to 2 spatial dimensions instead of 3? This question is the focus of this project.

In this case, the field strength is spread over a *circle* with *perimeter* $2\pi r$, and so the field is proportional to $1/r$ instead of $1/r^2$. Our gravitational force law is then:

$$\vec{F} = -\frac{G_{2\text{D}}m_1m_2}{r}\hat{r} \tag{1}$$

Where $G_{2\text{D}}$ is the 2D equivalent of its 3D cousin G. Since G has dimensions of $[G] = L^3T^{-2}M^{-1}$, $G_{2\text{D}}$ has dimensions of $G_{2\text{D}} = [G]/L = L^2T^{-2}M^{-1}$ (we don't need to specify an actual value for $G_{2\text{D}}$, as it should drop out when we normalize.)

Since the force law (equation 1) is conservative ($\vec{\nabla} \times \vec{F}$=0), the force can be expressed as the gradient of a potential, where

$$U = G_{2\text{D}}m_1m_2 \ln\left(\frac{r}{R_0}\right) \tag{2}$$

Where R_0 is an arbitrary constant of integration.

Questions

2-bodies

1. What is the equilibrium orbit r_0 with this gravitational potential?

2. Is this orbit stable? If so, what is the frequency of small oscillations about r_0?
   Hint: We did this in class for the $1/r^2$ force. Using the force equation derived from the Lagrangian, find the force at a small distance Δr from the equilibrium orbit r_0, and then approximate using the lowest-order terms of the Taylor expansion. The resulting differential equation corresponds either to harmonic oscillation or exponential growth.

3. In the above problem, you should have uncovered either exponential growth or periodic motion. Use the time scale of this motion (1 / angular frequency, or the term in the denominator of the exponential) as your unit for t_0, and use the equilibrium orbit from part 1 for r_0 to normalize your equations.

4. Write a program to numerically solve the 2-body problem. Plot the motion (x vs y for different sets of initial conditions (at least 3)).

5. For the $1/r^2$ gravitational force, the bounded orbits close on themselves (ϕ and r have the same period). Is this true of the $1/r$ force? Plot ϕ vs time together with r vs time for a variety of initial conditions and check whether they have the same period.

6. For the $1/r^2$ force, if the total energy $E \geq 0$, then the motion is unbounded (the objects never return). Is this unbounded motion possible for the $1/r$ force? Explain.

3-bodies

1. Write a program to numerically solve the 3-body problem.

2. Plot the motion for a few (at least 3) sets of initial conditions. Try to make a fun looking plot!

   – If you like, you can use my animation code animate3B.py to produce animations of the motion. This is not required, though.

3. Use your 3-body code to construct a miniature solar system. Start with a massive central object (at least 1000 times more massive than any other object) and two planets in different circular orbits. Plot the result.

4. Now give both planets some extra energy (non circular orbits) and plot the results. Do this several times for increasing values of total energy.

5. Simulate the case of a comet which originates far from the central object and its planet (say ~ 15 times farther out than the planet) with some small initial velocity. Plot the motion of the system. Is it possible to give the comet enough energy for it to escape altogether?

6. In two dimensions, Kepler's Law states that the orbital period τ is directly proportional to the orbital radius r: $\tau \propto r$. Make a plot of two bodies orbiting a massive central object in *resonance* (the inner body completes an integer number n orbits for every one orbit the outer body completes).