Last time:

The Schrödinger Eqn

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi + V\Psi$$

$$\Psi(x,t) = \psi(x)\, e^{-\frac{i}{\hbar} E t}$$

$\psi(x)$ is a solution to:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = E\psi$$

An example:

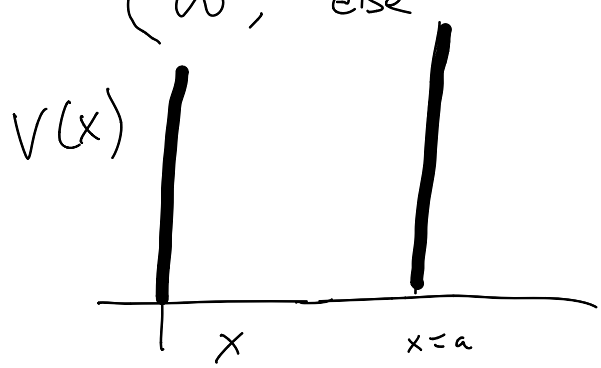
The "infinite square well"

$$V = \begin{cases} 0, & 0 \le x \le a \\ \infty, & \text{else} \end{cases}$$

at $t = 0$

$$\Psi_0 = \Psi_0(x, 0)$$

Same function



Solve for $V=0$

with BC's

$\psi(0) = \psi(a) = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

$V = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$k^2 \equiv \frac{2mE}{\hbar^2}, \quad (E > 0)$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

$$\psi(x) = A\sin kx + B\cos kx$$

Apply BC's

$$\psi(0) = B = 0$$

$$\psi(x) = A\sin(kx)$$

$$\psi(a) = 0 \Rightarrow A\sin(ka) = 0$$

$$ka = \pm\pi, \pm 2\pi, \pm 3\pi, \ \ldots \ n\pi$$

$$k_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots$$

$$k^2 = \frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{a^2}$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

The particle can have only certain energies!

Energy Quantization

$$\psi(x) = A\sin\left(\frac{n\pi}{a}x\right)$$

What is A?

$$\int_{-\infty}^{\infty} |\Psi|^2 dx = 1$$

$$\int \Psi^* \Psi \, dx = 1$$

$$\Psi(x,t) = \psi(x) e^{-\frac{i}{\hbar} Et}$$

$$\Psi^*\Psi = \psi^*(x) e^{\frac{i}{\hbar}Et} \psi(x) e^{-\frac{i}{\hbar}Et} = \psi^*(x)\psi(x)$$

$$\psi(x) = A\sin\left(\frac{n\pi}{a}x\right)$$

$$\psi^*(x) = A^*\left(\sin\left(\frac{n\pi}{a}x\right)\right)^*$$

what is $(\sin(\theta))^*$?

Euler:

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta \qquad \left(e^{i\pi} = -1,\ e^{i\pi}+1=0\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$(\sin\theta)^* = \frac{e^{-i\theta} - e^{i\theta}}{-2i} = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$$

$$\psi^*(x) = A^* \sin\left(\frac{n\pi}{a}x\right)$$

Choose A to be real: $A^* = A$

$$\psi^*(x) = A\sin\left(\frac{n\pi}{a}x\right) = \psi(x)$$

$$\int_{-\infty}^{\infty} \psi(x)^2 dx = 1$$

$$\int_0^a A^2 \sin^2\left(\frac{n\pi}{a}x\right) dx = 1$$

$$\int_0^a \sin^2\left(\frac{n\pi}{a}x\right) dx = \frac{1}{A^2}$$

$$\int \sin^2\theta = ?$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\sin^2\theta = -\frac{1}{4}\left(e^{2i\theta} + e^{-2i\theta} - 2\right)$$

$$= \frac{1}{2}\left[1 - \left(\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}\left(\cos(2\theta) + i\sin(2\theta) + \cos(-2\theta) - i\sin(2\theta)\right)\right]$$

$$= \frac{1}{2}\left[1 - \cos 2\theta\right]$$

$$\int_0^a \sin^2\left(\frac{n\pi}{a}x\right)dx = \frac{1}{2}\int_0^a \left(1 - \cos\left(\frac{2n\pi}{a}x\right)\right)dx = \frac{1}{A^2}$$

$$= \frac{1}{2}\left[x - \frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}x\right)\right]\Bigg|_0^a$$

$$= \frac{a}{2} = \frac{1}{A^2}$$

$$A = \sqrt{\frac{2}{a}}$$

$$\psi(x) = \sqrt{\frac{2}{a}}\ \sin\left(\frac{n\pi}{a}x\right)$$

$$\Psi(x,t) = \sqrt{\frac{2}{a}}\ \sin\left(\frac{n\pi}{a}x\right)e^{-\frac{i}{\hbar}E_n t}, \quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

Initial Condition:

Know $\Psi_0(x,0)$, want $\Psi(x,t)$

$$\Psi(x,t) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar}E_n t}$$

$$\Psi(x,0) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) = \Psi_0(x,0)$$

only works if $\Psi_0 \propto \sin\left(\frac{n\pi}{a}x\right)$

General Solution

$$\Psi(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar}E_n t}$$

$$\Psi(x,0) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}x\right)$$

Fourier Series

with right C_n, $\Psi(x,0)$ can match ~~any~~ function

Pick c_n so that

$$\Psi(x,0) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right) = \Psi_0(x)$$

"Fourier's trick"

multiply by $\sin\left(\frac{m\pi}{a}x\right)$ and $\int_0^a$

$$\sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx = \int_0^a \Psi_0(x)\sin\left(\frac{m\pi}{a}x\right)dx$$

$$\int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right) = \begin{cases} 0, & m \neq n \\ \frac{a}{2}, & m = n \end{cases}$$

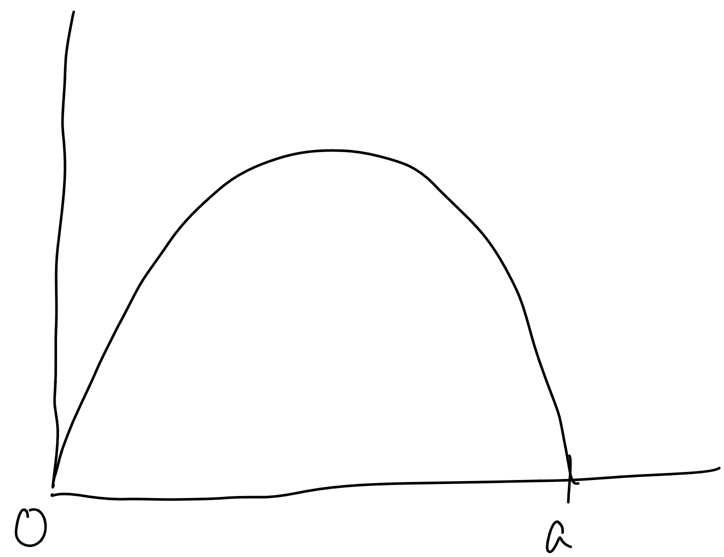
$$\sqrt{\frac{2}{a}}\, c_m \frac{a}{2} = \int_0^a \Psi_0(x)\sin\left(\frac{m\pi}{a}x\right)$$

$$c_m = \sqrt{\frac{2}{a}} \int_0^a \Psi_0(x)\sin\left(\frac{m\pi}{a}x\right)dx$$

$$\Psi(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar}E_n t}$$

Example:

$\Psi_0(x,0)$



$$\Psi_0(x,0) \propto x(a-x)$$

$$\Psi_0(x,0) = Ax(a-x)$$

What is A?

$$\int_{-\infty}^{\infty} A^2 \Psi_0^2 \, dx = 1$$

$$A^2 \int_0^a x^2(a-x^2)\,dx = A^2 \frac{a^5}{30}$$

$$A = \sqrt{\frac{30}{a^5}}$$

$$\Psi_0 = \sqrt{\frac{30}{a^5}}\, x(a-x)$$

Now just find c_n

$$c_m = \sqrt{\frac{2}{a}} \int_0^a \Psi_0(x) \sin\left(\frac{m\pi}{a}x\right) dx$$

$$c_n = \sqrt{\frac{2}{a}} \sqrt{\frac{30}{a^5}} \int_0^a x(a-x) \sin\left(\frac{n\pi}{a}x\right) dx$$

$$= \frac{2\sqrt{15}}{a^3}\left[\int_0^a ax\sin\left(\frac{n\pi}{a}x\right)dx - \int_0^a x^2\sin\left(\frac{n\pi}{a}x\right)dx\right]$$

$$\int x\sin\left(\frac{n\pi}{a}x\right)dx =$$

$$\int u\,dv = uv - \int v\,du$$

$$u = x$$
$$dv = \sin\left(\frac{n\pi}{a}x\right)$$

$$\int_0^a x\sin\left(\frac{n\pi}{a}x\right)dx = -\frac{a}{n\pi}x\cos\left(\frac{n\pi}{a}x\right)\Big|_0^a - \frac{a}{n\pi}\int_0^a \cos\left(\frac{n\pi}{a}x\right)dx$$

$$\int_0^a x^2\sin\left(\frac{n\pi}{a}x\right)dx = \frac{-a}{n\pi}x^2\cos\left(\frac{n\pi}{a}x\right)\Big|_0^a - \frac{2a}{n\pi}\int_0^a x\cos\left(\frac{n\pi}{a}x\right)dx$$

$$C_n = \frac{4\sqrt{15}}{\pi^3 n^3}\left[1 - \cos(n\pi)\right]$$

$$C_n = \begin{cases} 0, & n \text{ is even} \\ \dfrac{8\sqrt{15}}{\pi^3 n^3}, & n \text{ odd} \end{cases}$$

$$\Psi(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i}{\hbar}E_n t}$$

$$\Psi(x,t) = \sqrt{\frac{2}{a}}\, \frac{8\sqrt{15}}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{a}x\right) e^{-iE_n t}$$

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\Psi(x,t) = \sqrt{\frac{2}{a}}\, \frac{8\sqrt{15}}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi}{a}x\right) e^{-\frac{i n^2\pi^2\hbar}{2ma^2}}$$