What is $\langle n_s \rangle$?

- Expected / Avg # of particles in single-particle state "s"

$$0 \leq \langle n_s \rangle \leq N$$

How are the N particles distributed among the possible states?

Back in chapter 6, we would have written:

$$\langle n_s \rangle = N \cdot P(s)$$

# of particles in system

Prob that particle is in state "s"

$$\langle n_s \rangle = \frac{N e^{-\beta \epsilon_s}}{Z_1}$$

$$\mu = \frac{\partial F}{\partial N} = -\beta \frac{\partial}{\partial N} \ln\left(\frac{1}{N!} Z_1^N\right) \approx -\frac{1}{\beta} \ln \frac{Z_1}{N}$$

$$\frac{Z_1}{N} = e^{-\mu\beta}$$

$$\langle n_s \rangle = e^{-\beta(\epsilon_s - \mu)}$$

$$\langle n_s \rangle = e^{\frac{-(\epsilon_s - \mu)}{kT}} \qquad \text{MB}$$

$$\langle n_s \rangle = \frac{1}{e^{\frac{\epsilon_s - \mu}{kT}} + 1} \qquad \text{FD}$$

$$\langle n_s \rangle = \frac{1}{e^{\frac{\epsilon_s - \mu}{kT}} - 1} \qquad \text{BE}$$

Show plot $\frac{1}{e^{\frac{\epsilon_s - \mu}{kT}} \pm 1} \approx e^{-\frac{(\epsilon_s - \mu)}{kT}}$

if $\frac{\epsilon_s - \mu}{kT} \gg 1$,

Note: μ is determined by: $\mu \ll kT \rightarrow$ MB stats

$$\sum_s \langle n_s \rangle = N$$

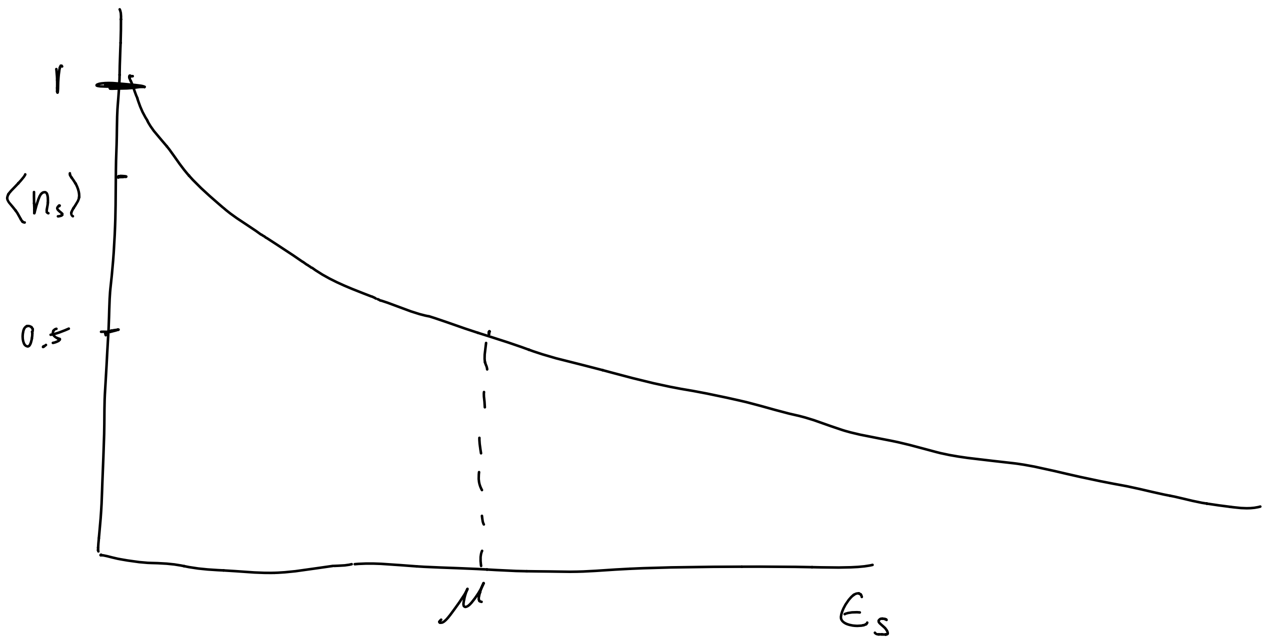
μ acts as a normalization factor on the sum/integral

μ: Energy required to add a particle to the system

ϵ_s: Energy required for that particle to occupy state s

Physical implications

Fermi-Dirac



For $\epsilon_s << \mu$

$$\langle n_s \rangle = \frac{1}{e^{\frac{\epsilon_s - \mu}{kT}} + 1} \approx \frac{1}{e^{-\frac{\mu}{kT}} + 1}$$

$$< 1$$

For $\epsilon_s >> \mu$

$$\langle n_s \rangle \longrightarrow 0$$

$$\epsilon_s = \mu, \quad \langle n_s \rangle = \frac{1}{2}$$

if $kT \to 0$

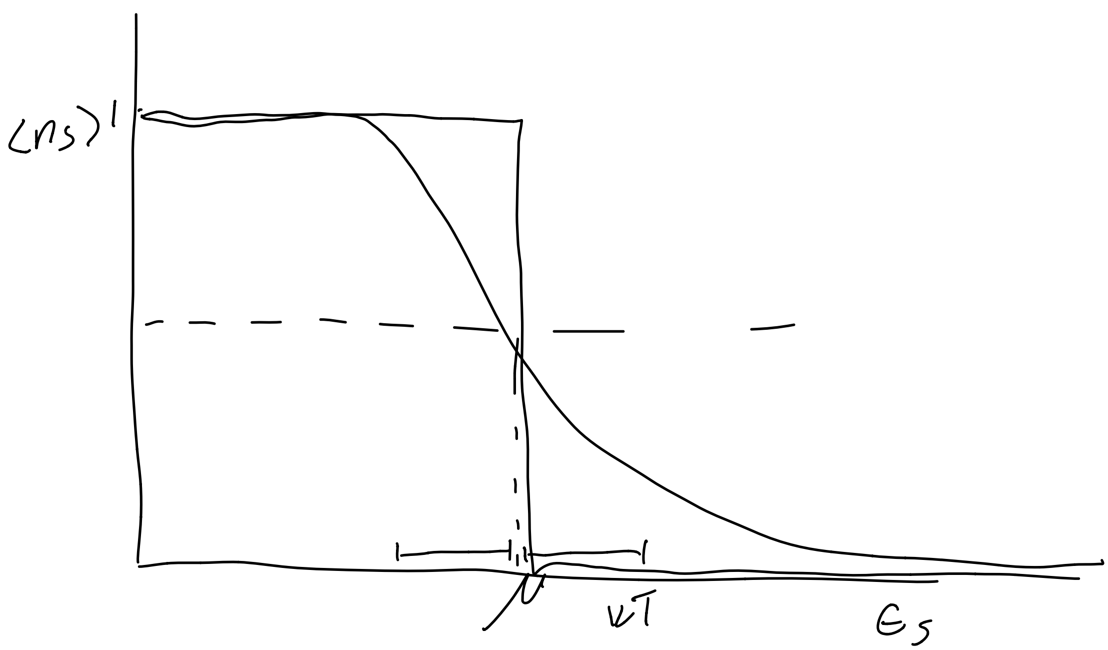
$\epsilon_s << \mu$

$$e^{\frac{\epsilon_s - \mu}{kT}} \longrightarrow 0 \implies \langle n_s \rangle \approx 1$$

$\epsilon_s >> \mu$

$$e^{\frac{\epsilon_s - \mu}{kT}} \longrightarrow \infty \implies \langle n_s \rangle \approx 0$$

$\epsilon_s = \mu$, $\langle n_s \rangle = \frac{1}{2}$



At $T = 0$

$\langle n_s \rangle$ is a step function

$$\langle n_s \rangle = \begin{cases} 1, & \epsilon_s < \mu \\ 0, & \epsilon_s > \mu \end{cases}$$

Consider an ideal gas of electrons (fermions) at $T = 0$

- Classically, $U = \frac{3}{2} NkT = 0 \Rightarrow \langle v^2 \rangle = 0$
  no motion

- In actuallity: No electrons can share a state
  Place first particle in lowest energy state, second in next lowest . . .

Net result: All states with energies below μ are occupied; all with energy above are empty

μ is an energy

In this context $(T=0)$ μ is the energy of the most energetic particle in the gas

(the energy of the last occupied state)

- Called the Fermi Energy: ε_F

$$\varepsilon_F = \mu(N, T=0)$$

$\uparrow \varepsilon = \varepsilon_F$

- Value of ε_F depends on # of particles

(more particles, higher ε_F)

Find ε_F + then find U + P for the gas

Gas of N particles in a box of Volume V $(L \times L \times L)$

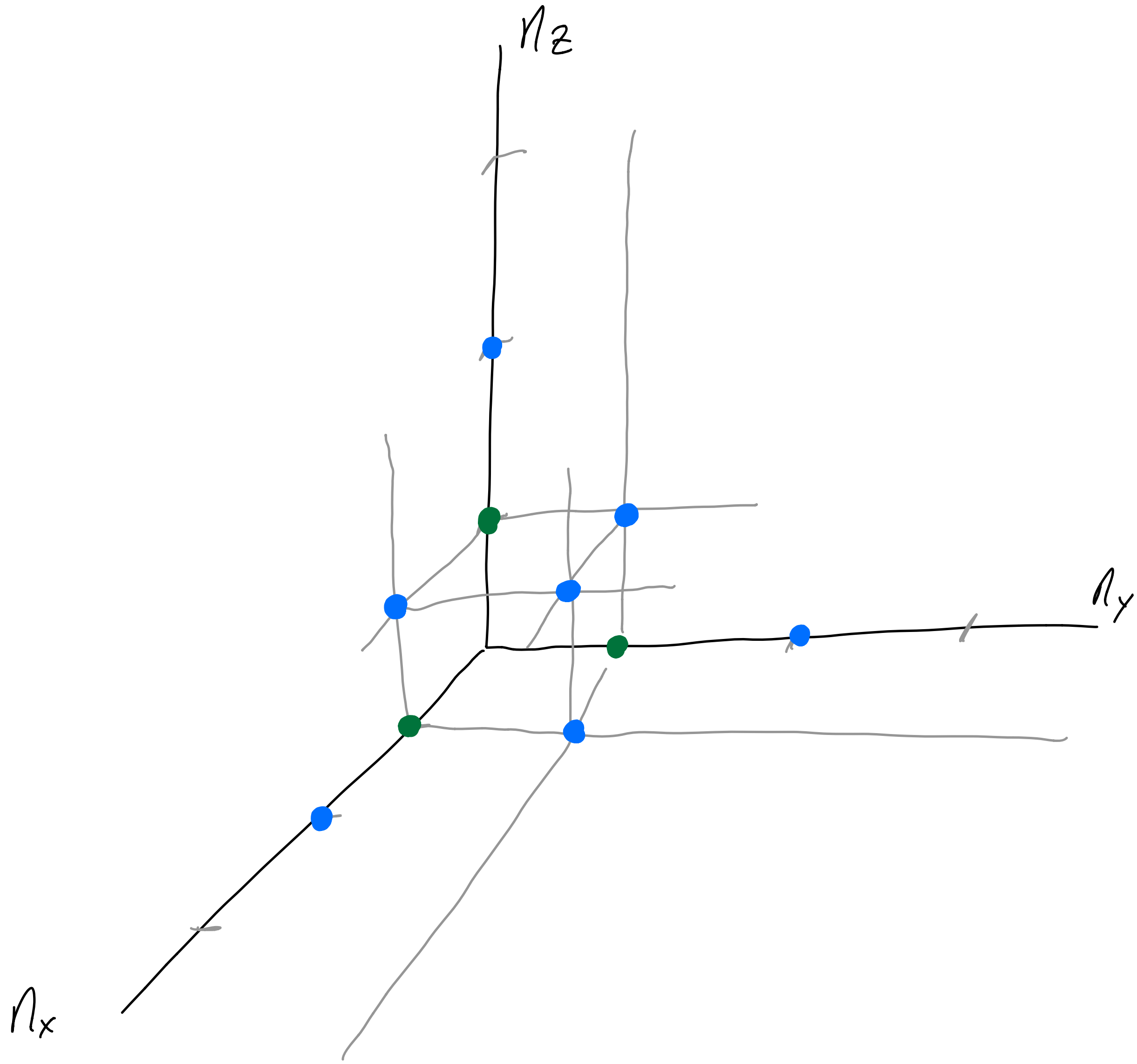
$$\psi = \sqrt{\frac{8}{V}} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right)$$

$$p_x = \frac{\hbar \pi n_x}{L}$$

$$E = \frac{p^2}{2m} = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

$$E = \frac{\hbar^2 \pi^2}{L^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

$$\epsilon_s = \epsilon_{n_x n_y n_z} = \frac{\hbar^2 \pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$



$$\epsilon_s = \frac{\hbar^2 \pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

$$n_x^2 + n_y^2 + n_z^2 = \frac{2mL^2}{\hbar^2\pi^2}\epsilon_s$$

The states corresponding to each energy level lie on the sphere of radius $\left(\frac{2mL^2}{\hbar^2\pi^2}\epsilon_s\right)^{1/2}$

If there are no empty states ($T=0$)

Then:

Total # of particles below ϵ_s is just $2 \times$ # of states below ϵ_s

# of states below ϵ_s is just the volume of the sphere of radius

$$R = \left(\frac{2mL^2}{\hbar^2 \pi^2} \epsilon_s \right)^{1/2}$$

Actually, since n_x, n_y, n_z are all positive, $\frac{1}{8}$ the volume

$$\# = 2 \cdot \frac{1}{8} V$$

$$= 2 \cdot \frac{1}{8} \cdot \frac{4}{3} \pi R^3$$

$$= 2 \cdot \frac{1}{8} \cdot \frac{4}{3} \pi \left(\frac{2mL^2}{\hbar^2 \pi^2} \epsilon_s \right)^{3/2}$$

$$\# = \frac{1}{3}\pi\left(\frac{2m}{\hbar^2\pi^2}\right)^{3/2} L^3 \, \epsilon_s^{3/2}$$

$$L^3 = V$$

$$\# = \frac{\pi}{3}\left(\frac{2m}{\hbar^2\pi^2}\right)^{3/2} V \, \epsilon_s^{3/2}$$

if $\epsilon_s = \varepsilon_F$, then $\# = N$

$$N = \frac{\pi}{3}\left(\frac{2m}{\hbar^2\pi^2}\right)^{3/2} V \, \varepsilon_F^{3/2}$$

$$\varepsilon_F^{3/2} = \frac{3N}{\pi V}\left(\frac{\hbar^2\pi^2}{2m}\right)^{3/2}$$

$$\boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}}$$