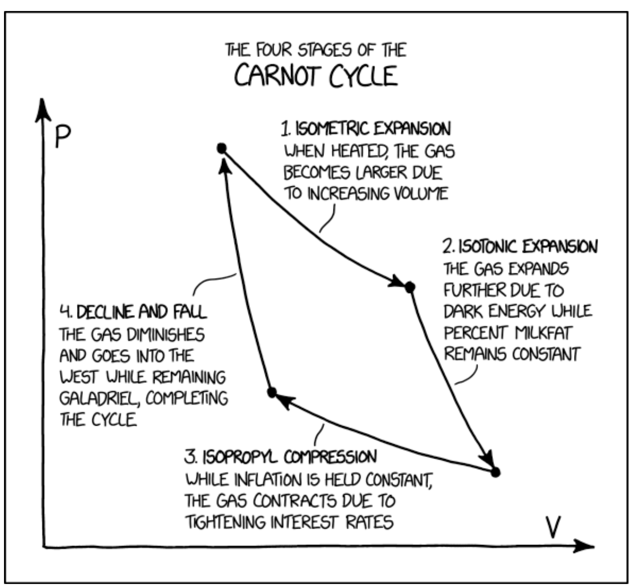
# PHYS 4410 Exam II

Monday, April 19, 2021

***Instructions**: You will have ample time to complete this exam. Take a deep breath and relax! Read each question carefully, and let me know if anything is unclear. Partial credit may be awarded, so you are encouraged to clearly and legibly show your work for each problem. Extra paper is available at the front of the room if you need it. Write your name on every extra sheet you use, and clearly label what problem you are working on. Staple this to the back of your exam when you turn it in. You may use any information contained within this exam, as well as a calculator.*
*Good luck!*

**Name:** ______________________________________



# Potentially useful information

Unit analysis

| Power | Prefix | Name |
|---|---|---|
| $10^{12}$ | T | tera |
| $10^{9}$ | G | giga |
| $10^{6}$ | M | mega |
| $10^{3}$ | k | kilo |
| $10^{0}$ | — | — |
| $10^{-3}$ | m | milli |
| $10^{-6}$ | $\mu$ | micro |
| $10^{-9}$ | n | nano |

Constants

| Name | Symbol | Value |
|---|---|---|
| Boltzmann Constant | $k_B$ | $1.381 \times 10^{-23}$ J/K |
| Red. Planck Constant | $\hbar$ | $1.055 \times 10^{-34}$ J·s |
| Electron mass | $m_e$ | $9.11 \times 10^{-31}$ kg |
| Electron charge | $e$ | $1.6 \times 10^{-19}$ C |
| Avogadro's Number | $N_A$ | $6.022 \times 10^{23}$ |
| Newton's Constant | $G$ | $6.673 \times 10^{-11} \mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2$ |

Useful equations

**Mono-atomic Ideal Gas**

Multiplicity $\Omega = \frac{1}{\hbar^{3N}} \frac{V^N \pi^{\frac{3N}{2}} (2mU)^{\frac{3N}{2}}}{N! \cdot \Gamma(\frac{3N}{2})}$

Entropy $S = Nk_b \left[ \ln \left( \frac{V}{N} \left( \frac{4\pi mU}{3N\hbar^2} \right)^{3/2} \right) + \frac{5}{2} \right]$

**Einstein Solids**

Multiplicity $\Omega = \frac{(q+N-1)!}{q!(N-1)!}$

$\Omega(q >> N)$ $\Omega \approx \left( \frac{eq}{N} \right)^N$

**Unit Conversions**

1 atm $= 1.013 \times 10^5$ N/m$^2$

T [° C] =T[K] - 273.15

1 cal =4.186 J

1. (30 points) A tiny paramagnet consists of 10 electrons with two possible spin states (either parallel or anti-parallel to the magnetic field). The system is at room temperature (300 K). Initially, the magnetic field is very weak, so that 5 of the 10 electrons are aligned with the field and 5 are anti aligned. For what value of the magnetic field will the system spontaneously transition to a state where 80% of the electron magnetic moments are aligned with the field? The magnetic moment of an electron is given by: $\mu_B = \frac{e\hbar}{2m_e}$.

$$U = -N_\uparrow \mu B + N_\downarrow \mu B = (N_\downarrow - N_\uparrow)\mu B$$

$$\Omega = \frac{N!}{N_\uparrow! N_\downarrow!}$$

initially: $U = 0$

$$\Omega = \frac{10!}{5!\,5!} = 252$$

final: $U = (2-8)\mu B = -6\mu B$

$$\Omega = \frac{10!}{8!\,2!} = 45$$

$$\Delta U = -6\mu B$$

$$\frac{\Delta S}{k} = \ln(45) - \ln(252)$$

$$\Delta S = k \ln\left(\frac{45}{252}\right)$$

$$\Delta F = \Delta U - T\Delta S$$

$$\Delta F = -6\mu B - (300K)(k)\ln\left(\frac{45}{252}\right)$$

$\Delta F = 0$:

$$6\mu B = (300K)(k_b)\ln\left(\frac{252}{45}\right)$$

$$B = \frac{(5K)(k_b)\ln\left(\frac{252}{45}\right)}{\mu}$$

$$\mu = \frac{e\hbar}{2m_e} = 9.26\times10^{-24}\left(\frac{C\ J\cdot s}{kg}\right) \swarrow \frac{J}{T}$$

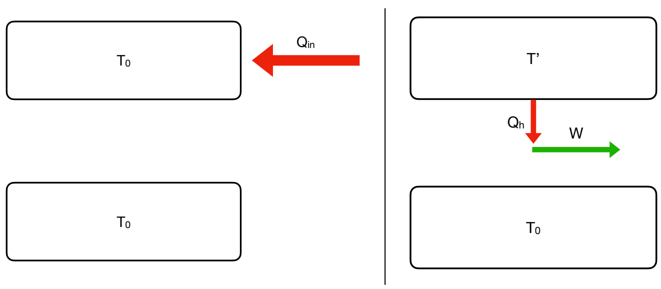
$$\boxed{B = 12.83\ T}$$

2. When a moving car hits the brakes and comes to rest, its kinetic energy is transferred (through friction with the brake pads) to thermal energy within the brakes[1]. A friend of yours has proposed a design that can extract all of the thermal energy from the brakes and convert it back into kinetic energy when the car accelerates, bringing the car up to the same speed it was before hitting the brakes.

   Here's how it works:

   1. The brake consists of two solid blocks, initially at the same temperature $T_0$. Both blocks have the same heat capacity (call it $C_V$)
   2. When the car brakes, the kinetic energy enters the top block in the form of heat ($Q_{in}$) which raises its temperature (left image)
   3. The temperature difference between the top and bottom block causes heat $Q_h$ to flow out of the top block. The heat flows until both blocks are again at the same temperature. 100% of this heat is then converted into work, which is able to power the car.

   Show your friend that this perpetual motion machine is in fact not possible.




   (a) (5 points) After an amount of heat $Q_{in}$ enters the top block, what is its new temperature?

   (b) (5 points) What is the change in entropy of the system as depicted in the right panel of the figure ($\Delta S$ of the top block as an amount of heat $Q_h$ leaves the block and the block returns to its original temperature $T_0$.) Explain why this ensures the mechanism shown in the figure is impossible (assuming the work can be done without creating any new entropy).

   (c) (10 points) Assume that instead of all of $Q_h$ being converted into work, some of it is instead converted into "waste heat" $Q_c$, which flows into the bottom block and raises its temperature, thus $W = Q_h - Q_c$. This heat continues to flow until the temperatures of the top and bottom block are equal. What is the final temperature of the blocks? Show that if $Q_c > 0$, then $Q_h$ must be strictly less than $Q_{in}$.

   (d) (10 points) Then show that, in order to satisfy the laws of thermodynamics, $Q_c$ must be greater than 0, hence $W < Q_h < Q_{in}$; i.e., some of the input energy $Q_{in}$ is necessarily "lost" and cannot be recovered to power the car.

[1] Some of the kinetic energy is also lost to the road and the air, but ignore that for this problem

a) $Q_{in} = C_v \Delta T = C_v \Delta T$

$$\Delta T = \frac{Q_{in}}{C_v}$$

$$T' = T_0 + \frac{Q_{in}}{C_v}$$

b) $dS = \frac{Q}{T} = \frac{C_v\,dT}{T}$

$$\Delta S = \int_{T_i}^{T_f} \frac{C_v}{T}\,dT = C_v \ln\left(\frac{T_f}{T_i}\right)$$

$$T_i = T_0 + \frac{Q_{in}}{C_v}$$

$$T_f = T_0$$

$$\Delta S = C_v \ln\left(\frac{T_0}{T_0 + \frac{Q_{in}}{C_v}}\right)$$

$T_0 + \frac{Q_{in}}{C_v} > T_0$, so $\Delta S < 0$

Not possible!

c) Top: $T_i = T_0 + \frac{Q_{in}}{C_v}$

$$-Q_h = C_v \Delta T = C_v\left(T_f - T_0 - \frac{Q_{in}}{C_v}\right)$$

$$\frac{-Q_h}{C_v} = T_f - T_0 - \frac{Q_{in}}{C_v}$$

$$T_f = T_0 + \frac{Q_{in} - Q_h}{C_v}$$

Bot: $T_i = T_0$

$$Q_c = C_v(T_f - T_0)$$

$$\frac{Q_c}{C_v} = T_f - T_0$$

$$T_f = T_0 + \frac{Q_c}{C_v}$$

$$T_0 + \frac{Q_c}{C_v} = T_0 + \frac{Q_{in} - Q_h}{C_v}$$

$$\boxed{Q_c = Q_{in} - Q_h}$$

if $Q_c = 0$, $Q_h = Q_{in}$

if $Q_c > 0$, $Q_h < Q_{in}$

d) $\Delta S = \Delta S_{top} + \Delta S_{bot}$

$$\Delta S = C_v \ln\left(\frac{T_0 + \frac{Q_{in} - Q_h}{C_v}}{T_0 + \frac{Q_{in}}{C_v}}\right)$$

$$+ C_v \ln\left(\frac{T_0 + \frac{Q_c}{C_v}}{T_0}\right)$$

First term is $< 0$

Second term $\underline{\underline{\text{must}}}$ be $> 0$

$$\frac{T_0 + \frac{Q_c}{C_v}}{T_0} > 1 \implies Q_c > 0$$

3. Consider an ideal gas in a 2D "flatland" Universe. In this case, instead of the gas occupying a volume $V$, it occupies an area $A$. Instead of exerting a pressure $P =$ force/area with units of $N/m^2$, the gas exerts a "linear pressure" $\Pi =$ force/length with units of $N/m$. The gas still exchanges heat $SdT$ with its surroundings, but the work done on the gas is $-\Pi dA$ rather than $-PdV$. The infinitesimal change in energy of the gas is then:

$$dU = \cancel{S}dT - \Pi dA + \mu dN$$

TdS

For this 2D gas, the multiplicity is given by:

$$\Omega(U, A, N) = \frac{1}{N!}\left(\frac{A}{\hbar^2}\right)^N \frac{\pi^N}{N!}(2mU)^N$$

(a) (15 points) Derive the energy vs temperature relationship for this gas (in 3D, it is $U = \frac{3}{2}NkT$, what is it in 2D?)

(b) (15 points) Derive the 2D ideal gas law (in 3D, it is $PV = NkT$, what is the 2D equivalent?)

a) $\Omega = \frac{1}{N!}\left(\frac{A}{\hbar^2}\right)^N \frac{\pi^N}{N!}(2mU)^N$

$$S = k\ln\Omega$$

$$= k\left[N\ln A + N\ln\pi + N\ln(2mU) - 2\ln(N!) - N\ln\hbar^2\right]$$

$$dU = TdS - \pi dA + \mu dN$$

$$dS = \frac{1}{T}dU + \frac{\pi}{T}dA - \frac{\mu}{T}dN$$

so:

$$\left(\frac{\partial S}{\partial U}\right)_{A,N} = \frac{1}{T}$$

$$\left(\frac{\partial S}{\partial U}\right)_{A,N} = \frac{Nk}{U} = \frac{1}{T}$$

$$\boxed{U = NkT}$$

b) $\frac{\Pi}{T} = \left(\frac{\partial S}{\partial A}\right)_{u,N} = \frac{Nk}{A}$

$$\frac{\Pi}{T} = \frac{Nk}{A}$$

$$\boxed{\Pi A = NkT}$$

| Question | Points | Score |
|---|---|---|
| 1 | 30 | |
| 2 | 30 | |
| 3 | 30 | |
| Total: | 90 | |