Paramagnet

- Usually unmagnetized
- Become weakly magnetized in the presence of a magnetic field
- Loses magnetization if $\vec{B}$ is removed

What is the mechanism for magnetization?
- electrons

Quantum mechanics: electrons have their own angular momentum

electron spin $S = 1/2$

This momentum can only ever point one of two directions spin "up" + spin "down"

This means that electrons behave like tiny magnetic dipoles

Our model:

Assume our system consists of N-many dipoles

Each dipole has magnetic moment $\vec{\mu} = \pm \mu \hat{z}$

Neglect intra-dipole interactions

Along a chosen axis, each one can be up or down

up $\uparrow$ $\downarrow$ dwn

$\uparrow \hat{z}$

$\uparrow \ \uparrow \ \downarrow \ \uparrow \ \downarrow \ \downarrow \ \uparrow \ \downarrow \ \cdots$

What happens when we apply an external field along $+\hat{z}$?

$\uparrow \ \uparrow \ \downarrow \ \uparrow \ \downarrow \ \downarrow \ \uparrow \ \downarrow \ \cdots$

$\uparrow\uparrow$
B

Energy

$$U_{dip} = -\vec{\mu} \cdot \vec{B}$$

Non-aligned dipoles will feel a torque toward the direction of $\vec{B}$

They won't <u>all</u> align though

Random thermal motion competes with magnetic torque

What is the relationship between:

Field Strength B

Energy U

Temp T

Magnetization M

Total number of dipoles $= N$

pointing "up" (aligned) $= N_\uparrow$

down $= N_\downarrow$

$$U_{tot} = U = \sum_{i=1}^{N} \vec{\mu}_i \cdot \vec{B} = N_\downarrow \mu B - N_\uparrow \mu B$$

$$= (N_\downarrow - N_\uparrow) \mu B$$

$$N_\uparrow + N_\downarrow = N$$

$$N_\downarrow = N - N_\downarrow$$

$$U = (N - 2N_\uparrow) \mu B$$

$$M = \sum_{1}^{N} \mu_i = (N_\uparrow - N_\downarrow) \mu = -\frac{U}{B}$$

Plan:

$$\Omega \xrightarrow{k \ln \Omega} S \xrightarrow{\frac{\partial S}{\partial U} = \frac{1}{T}} U(T \ldots)$$

Find

Ω

Macrostates specified by $N_\uparrow$

How many ways to have $N_\uparrow$ up out of N total?

"How many ways to have N_h heads of N total"

$$\Omega(N_\uparrow) = \frac{N!}{N_\uparrow!\,(N-N_\uparrow)!}$$

Now find S

$$U = U(N_\uparrow, \ldots)$$

$$S = k \ln \Omega$$

$$\ln \Omega = \ln N! - \ln N_\uparrow! - \ln (N-N_\uparrow)!$$

$$\ln N! \approx N \ln N - N$$

$$\ln\Omega \approx N\ln N - N - (N_\uparrow \ln N_\uparrow - N_\uparrow) - \left[(N-N_\uparrow)\ln(N-N_\uparrow) - (N-N_\uparrow)\right]$$

$$\approx N\ln N - N - N_\uparrow \ln N_\uparrow + N_\uparrow - N\ln(N-N_\uparrow) + N_\uparrow \ln(N-N_\uparrow) + N - N_\uparrow$$

$$\approx N\ln N \cancel{-N} - N_\uparrow \ln N_\uparrow \cancel{+N_\uparrow} - N\ln(N-N_\uparrow) + N_\uparrow \ln(N-N_\uparrow) \cancel{+N} \cancel{-N_\uparrow}$$

$$\boxed{\ln\Omega \approx N\ln N - N_\uparrow \ln N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow)}$$

$$U = (N - 2N_\uparrow)\mu B$$

$$\frac{U}{\mu B} = N - 2N_\uparrow$$

$$2N_\uparrow = N - \frac{U}{\mu B}$$

$$N_\uparrow = \frac{1}{2}\left(N - \frac{U}{\mu B}\right)$$

$$\boxed{\ln\Omega \approx N\ln N - N_\uparrow \ln N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow)}$$

$$N - N_\uparrow = N - \frac{1}{2}N + \frac{1}{2}\frac{U}{\mu B}$$

$$= \frac{1}{2}\left(N + \frac{U}{\mu B}\right)$$

$$\ln\Omega \approx N\ln N - \frac{1}{2}\left(N - \frac{U}{\mu B}\right)\ln\left[\frac{1}{2}\left(N - \frac{U}{\mu B}\right)\right] - \frac{1}{2}\left(N + \frac{U}{\mu B}\right)\ln\left[\frac{1}{2}\left(N + \frac{U}{\mu B}\right)\right]$$

$$\ln\left[\frac{1}{2}\left(N \pm \frac{U}{\mu B}\right)\right] = \ln\frac{1}{2} + \ln N + \ln\left(1 \pm \frac{U}{\mu B N}\right)$$

$$\approx \ln\frac{N}{2} \pm \frac{U}{\mu B N}$$

$$\ln\Omega \approx N\ln N \underbrace{-\frac{1}{2}\left(N-\frac{U}{\mu B}\right)\ln\left[\frac{1}{2}\left(N-\frac{U}{\mu B}\right)\right]}_{-} \underbrace{-\frac{1}{2}\left(N+\frac{U}{\mu B}\right)\ln\left[\frac{1}{2}\left(N+\frac{U}{\mu B}\right)\right]}_{+}$$

$$\left(N \pm \frac{U}{\mu B}\right)\left(\ln\frac{N}{2} \pm \frac{U}{\mu B N}\right)$$

$$N\ln\frac{N}{2} \pm \frac{U}{\mu B} \pm \frac{U}{\mu B}\ln\frac{N}{2} + \frac{1}{N}\left(\frac{U}{\mu B}\right)^2$$

$$\ln\Omega \approx N\ln N - \frac{1}{2}\left[N\ln\frac{N}{2} - \frac{U}{\mu B} - \frac{U}{\mu B}\ln\frac{N}{2} + \frac{1}{N}\left(\frac{U}{\mu B}\right)^2\right]$$
$$-\frac{1}{2}\left[N\ln\frac{N}{2} + \frac{U}{\mu B} + \frac{U}{\mu B}\ln\frac{N}{2} + \frac{1}{N}\left(\frac{U}{\mu B}\right)^2\right]$$

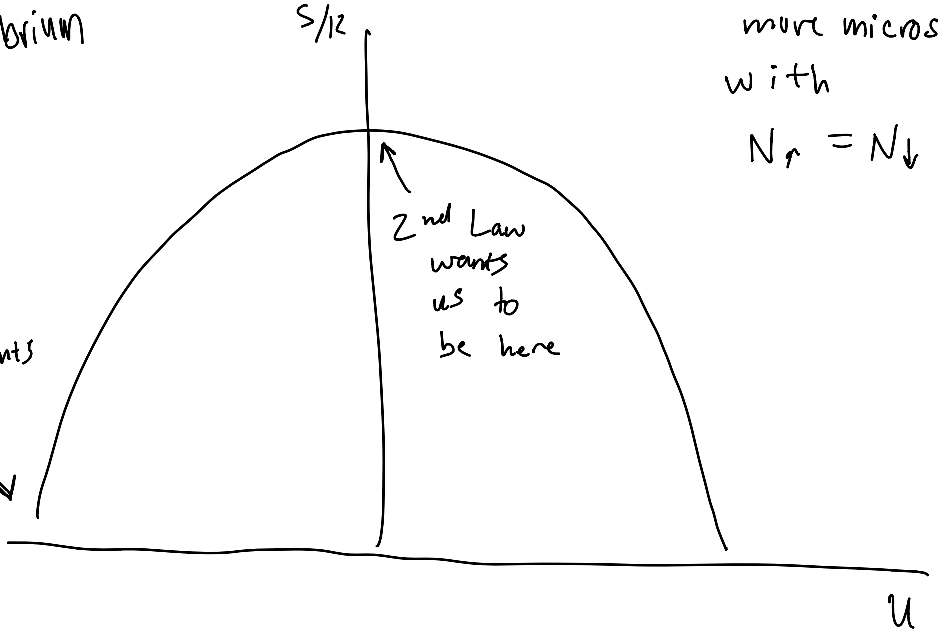
$$\ln\Omega \approx N\ln N - \frac{1}{2}\left[N\ln\frac{N}{2} - \cancel{\frac{U}{\mu B}} - \cancel{\frac{U}{\mu B}\ln\frac{N}{2}} + \frac{1}{N}\left(\frac{U}{\mu B}\right)^2\right]$$
$$-\frac{1}{2}\left[N\ln\frac{N}{2} + \cancel{\frac{U}{\mu B}} + \cancel{\frac{U}{\mu B}\ln\frac{N}{2}} + \frac{1}{N}\left(\frac{U}{\mu B}\right)^2\right]$$

$$\approx N\ln N - N\ln\left(\frac{N}{2}\right) - \frac{1}{N}\left(\frac{U}{\mu B}\right)^2$$
$$\approx N\left(\ln N - \ln\frac{N}{2}\right) - \frac{1}{N}\left(\frac{U}{\mu B}\right)^2 = N\ln 2 - \frac{1}{N}\left(\frac{U}{\mu B}\right)^2$$

Equilibrium

more microstates with $N_\uparrow = N_\downarrow = \frac{N}{2}$

B field wants us to be here



S_{max} @ $U = 0$

$$U = (N - 2N_\uparrow)\mu B$$

$$N_\uparrow = N_\downarrow = \frac{1}{2}N$$

$$\left(\frac{\partial S}{\partial U}\right)_{V,N} = \frac{1}{T}$$

$$\ln\Omega \simeq N\ln N - \frac{1}{2}\left(N - \frac{U}{\mu B}\right)\ln\left[\frac{1}{2}\left(N - \frac{U}{\mu B}\right)\right] - \frac{1}{2}\left(N + \frac{U}{\mu B}\right)\ln\left[\frac{1}{2}\left(N + \frac{U}{\mu B}\right)\right]$$

$$\frac{\partial}{\partial U}\left[\left(N \pm \frac{U}{\mu B}\right)\ln\left[\frac{1}{2}\left(N \pm \frac{U}{\mu B}\right)\right]\right]$$

$$= \left(N \pm \frac{U}{\mu B}\right)\left(\frac{\pm 1/2\mu B}{\frac{1}{2}\left(N \pm \frac{U}{\mu B}\right)}\right) \pm \frac{1}{\mu B}\ln\left[\frac{1}{2}\left(N \pm \frac{U}{\mu B}\right)\right]$$

$$= \pm\frac{1}{\mu B}\left(1 + \ln\left(N \pm \frac{U}{\mu B}\right) + \ln\frac{1}{2}\right)$$

$$\ln\Omega \simeq N\ln N - \frac{1}{2}\left(N - \frac{U}{\mu B}\right)\ln\left[\frac{1}{2}\left(N - \frac{U}{\mu B}\right)\right] - \frac{1}{2}\left(N + \frac{U}{\mu B}\right)\ln\left[\frac{1}{2}\left(N + \frac{U}{\mu B}\right)\right]$$

$$\frac{\partial}{\partial U}\ln\Omega \simeq \frac{1}{2\mu B}\left(1 + \ln\left(N - \frac{U}{\mu B}\right) + \ln\frac{1}{2}\right)$$

$$- \frac{1}{2\mu B}\left(1 + \ln\left(N + \frac{U}{\mu B}\right) + \ln\frac{1}{2}\right)$$

$$\frac{1}{k}\frac{\partial S}{\partial U} \simeq \frac{1}{2\mu B}\ln\left(\frac{N - \frac{U}{\mu B}}{N + \frac{U}{\mu B}}\right)$$

$$\frac{\partial S}{\partial U} = \frac{1}{T} = \frac{k}{2\mu B}\ln\left(\frac{N - \frac{U}{\mu B}}{N + \frac{U}{\mu B}}\right)$$

Solve for U

$$\frac{2\mu B}{kT} = \ln\left(\frac{N - \frac{U}{\mu B}}{N + \frac{U}{\mu B}}\right)$$

$$e^{2\frac{\mu B}{kT}} = \frac{N - \frac{U}{\mu B}}{N + \frac{U}{\mu B}}$$

$$\left(N + \frac{U}{\mu B}\right) e^{\frac{2\mu B}{kT}} = N - \frac{U}{\mu B}$$

$$N e^{\frac{2\mu B}{kT}} + \frac{U}{\mu B} e^{\frac{2\mu B}{kT}} + \frac{U}{\mu B} = N$$

$$\frac{U}{\mu B}\left(e^{\frac{2\mu B}{kT}} + 1\right) = N\left(1 - e^{\frac{2\mu B}{kT}}\right)$$

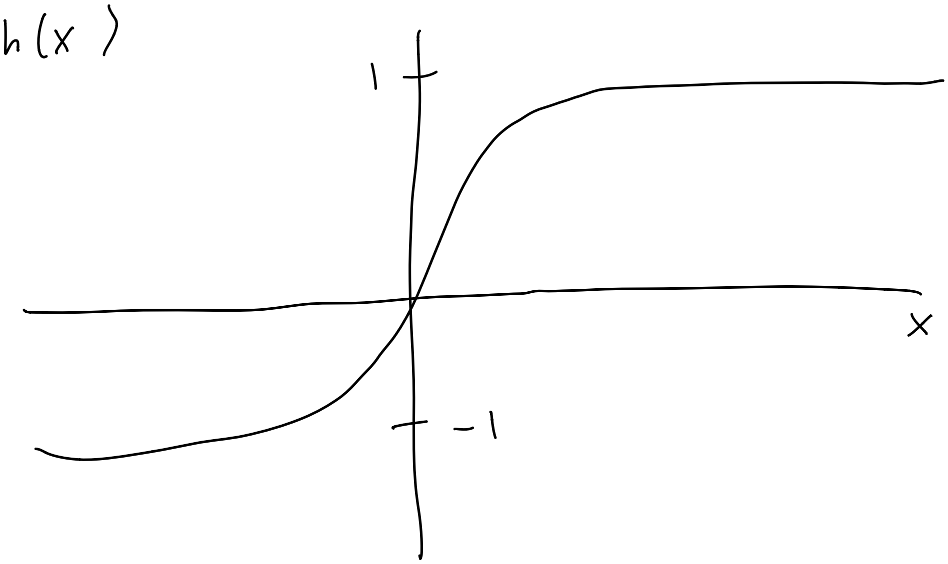
$$U = \mu B N \, \frac{\left(1 - e^{\frac{2\mu B}{kT}}\right)}{\left(1 + e^{\frac{2\mu B}{kT}}\right)}$$

$$= \mu B N \, \frac{e^{\frac{\mu B}{kT}}\left(e^{-\frac{\mu B}{kT}} - e^{\frac{\mu B}{kT}}\right)}{e^{\frac{\mu B}{kT}}\left(e^{-\frac{\mu B}{kT}} + e^{\frac{\mu B}{kT}}\right)}$$

$$= \mu B N \left(\frac{-2\sinh\left(\frac{\mu B}{kT}\right)}{2\cosh\left(\frac{\mu B}{kT}\right)}\right)$$

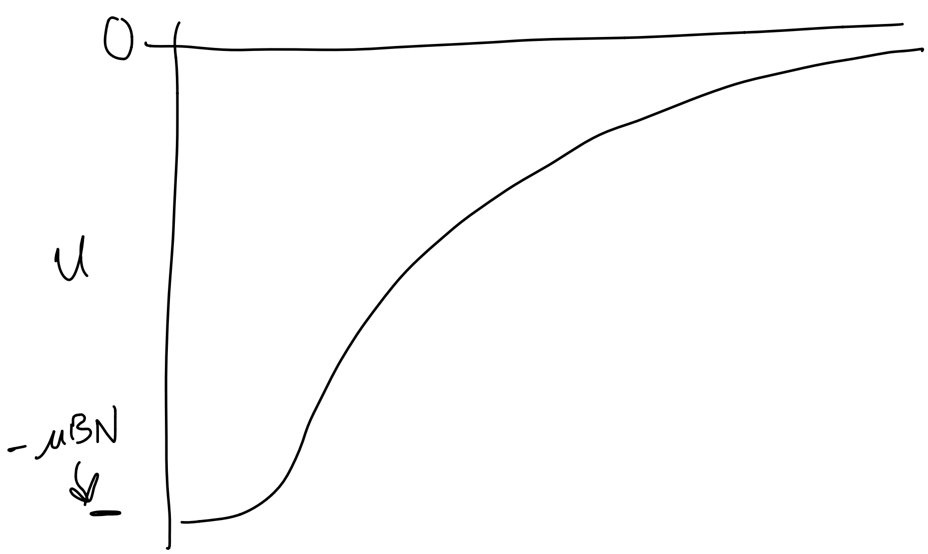
$$U = -\mu B N \tanh\left(\frac{\mu B}{kT}\right)$$

$\tanh(x)$



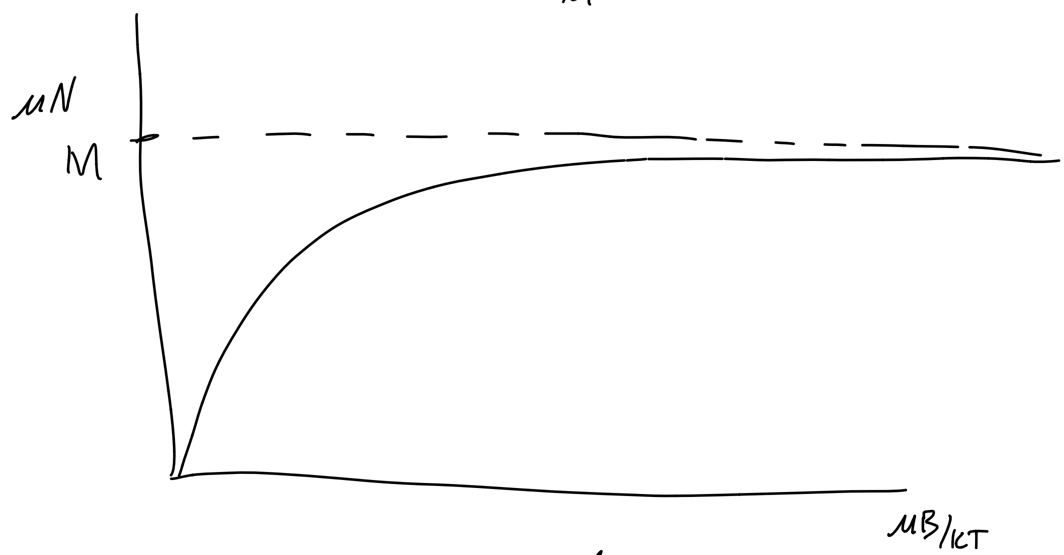
U vs $\frac{\mu B}{kT}$

$kT/\mu B$



$$M = -\frac{U}{B} = \mu N \tanh\left(\frac{\mu B}{kT}\right)$$

M vs $\frac{\mu B}{kT}$

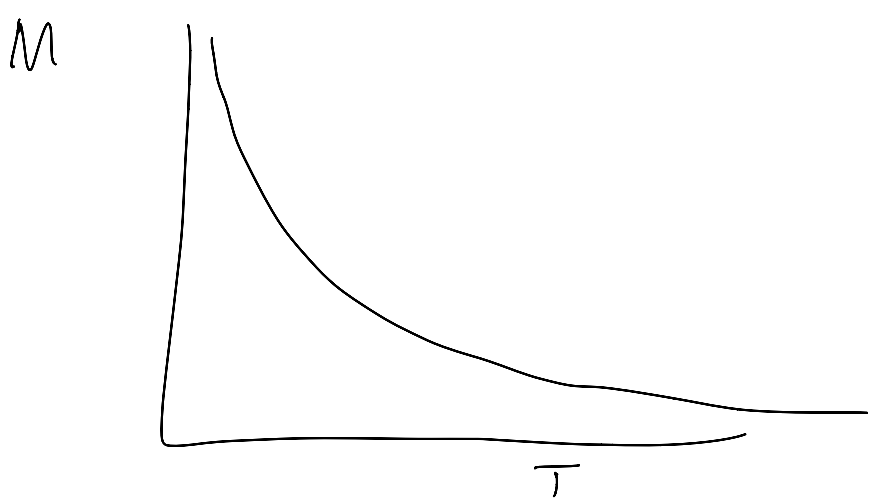


$\mu B << kT$ (high temp or low field)

$M \to 0$

$\mu B >> kT$

$M \to N\mu B$



Negative temperature?

- If energy is positive, $T < 0$
- Negative temperature is hotter than positive temperature!
- Don't worry too much

  Our model is simple

$$U = -\mu B N \tanh\left(\frac{\mu B}{kT}\right)$$

- According to this, if $B = 0$ the $U = 0$
- We have neglected vibrational / translational motion
- Artificial max energy (only 1 way to arrange!)
- Still a useful model

  at high temps: $\mu B / kT \ll 1$

$$\tanh\left(\frac{\mu B}{kT}\right) \approx \frac{\mu B}{kT}$$

$$M = \mu N \tanh\left(\frac{\mu B}{kT}\right) \approx \frac{\mu^2 B}{kT}$$

$M \sim \frac{1}{T}$ agrees w/ experiment

Curié Law (Pierre)

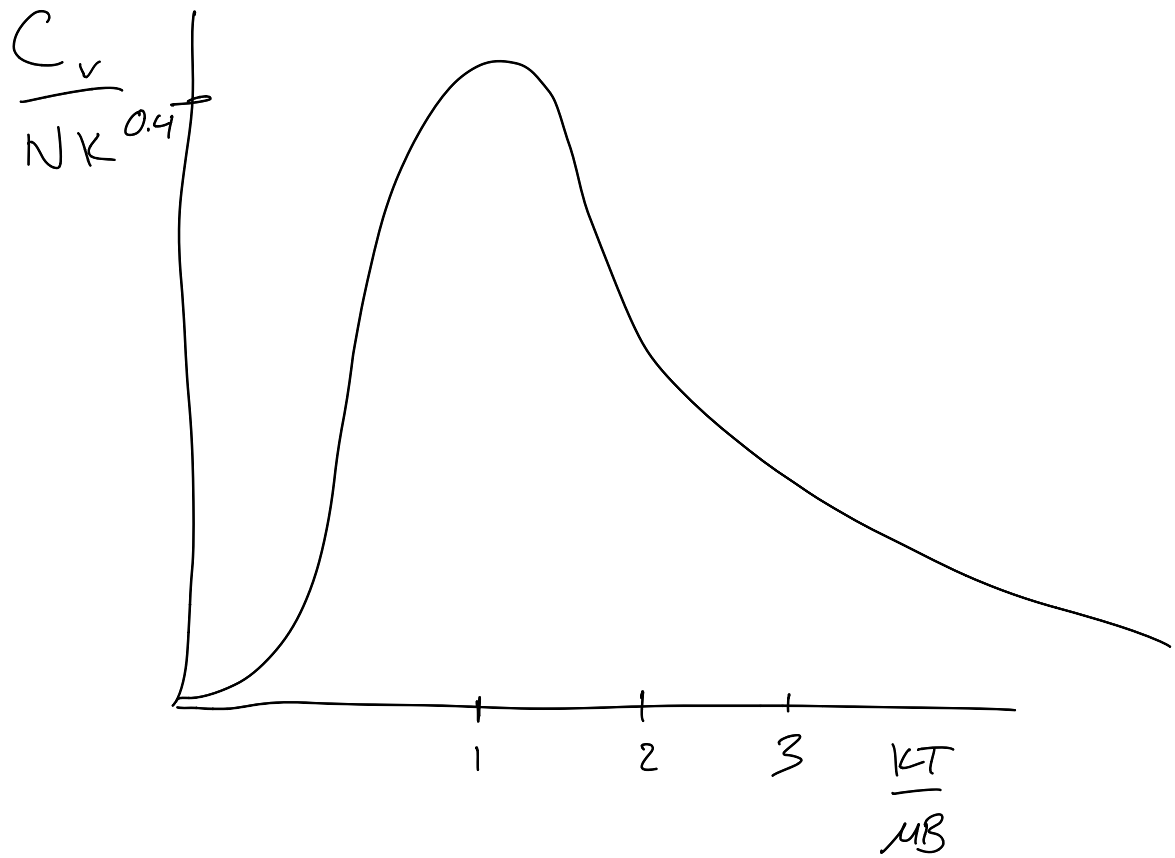
Heat Capacity:

$$C_v = \left(\frac{\partial U}{\partial T}\right)_{N,V}$$

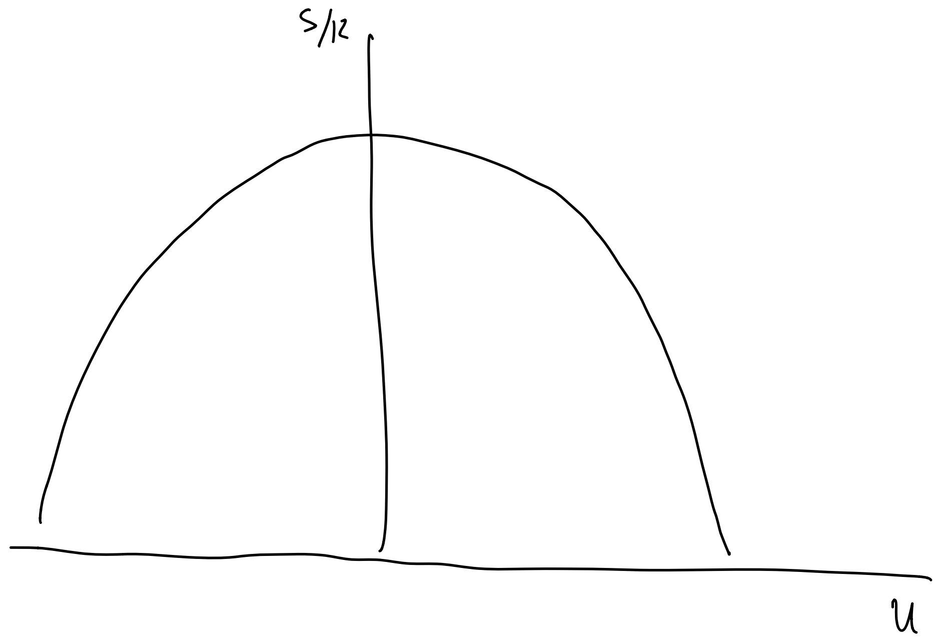
$$= \frac{\partial}{\partial T}\left(-\mu B N \tanh\left(\frac{\mu B}{kT}\right)\right)$$

$$= -\mu B N \left(-\frac{\mu B}{kT^2}\right) \operatorname{sech}^2\left(\frac{\mu B}{kT}\right)$$

$$C_v = \frac{N k \left(\frac{\mu B}{kT}\right)^2}{\cosh^2\left(\frac{\mu B}{kT}\right)}$$



- This is a simple model but it cleanly demonstrates the relationship btwn S, U, & T



Assume we start in the lowest entropy state (prepare our state as a ~0 K isolated system)

Say $N = 100$

$\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \ldots$

Only one way to arrange, low entropy

- Now allow heat to flow into the system

Energy required to flip a single dipole is $\mu B - (-\mu B) = 2\mu B$

- Pump in heat @ $2\mu B / s$ will flip 1 dipole / second

What does this do to the temperature?

Initially, each flipping dipole dramatically increases Ω + S

$\Omega(N_\uparrow = N) = 1$

$\Omega(N_\uparrow = 99) = 100$

$\Omega(N_\uparrow = 98) = 5000$

$\frac{\Delta S}{\Delta U}$ is large, $\frac{1}{T}$ is large, T is small

As more dipoles flip, the entropy increases, but slower

$\Omega(N_\uparrow = 60) \approx 1.4 \times 10^{28}$

$\Omega(N_\uparrow = 59) \approx 2 \times 10^{28}$

$\Omega(N_\uparrow = 58) \approx 2.8 \times 10^{28}$

$\frac{\Delta S}{\Delta U}$ is small $\frac{1}{T}$ is small, T is large

Eventually, we reach $N_\uparrow = N_\downarrow = 50$

Here $\frac{\Delta S}{\Delta U} \approx 0$, adding energy doesn't alter the entropy

Technically, $\frac{1}{T} = 0 \Rightarrow T \sim \infty$

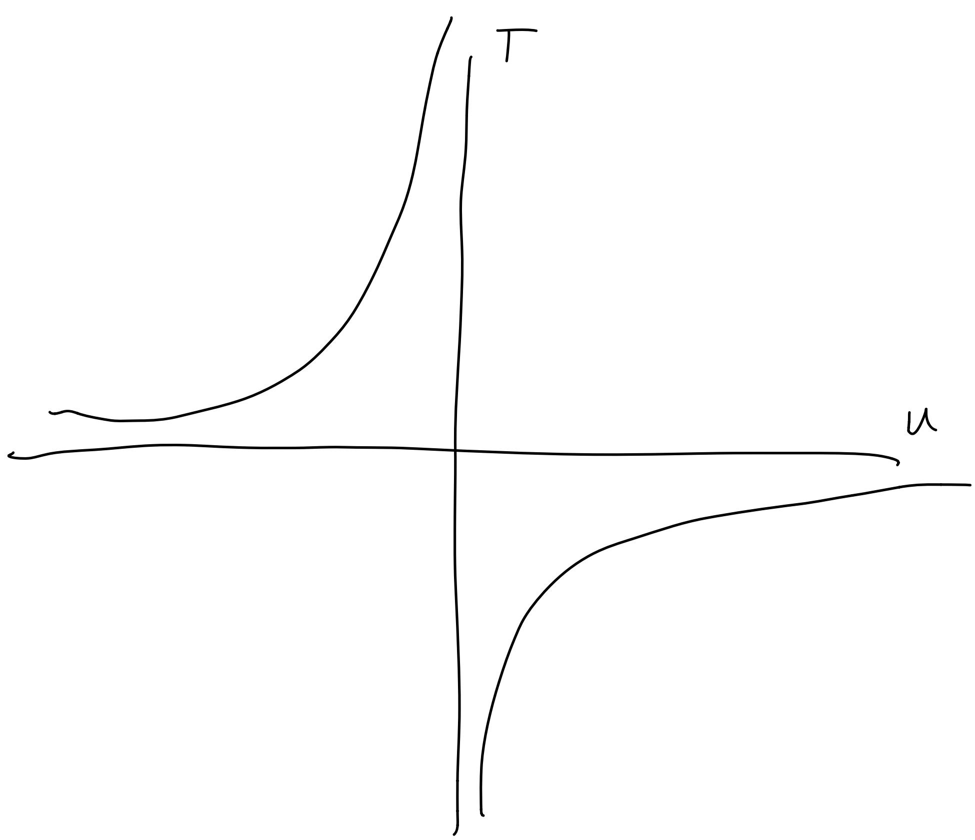
Now what if increase U further?

Now $N_\uparrow > 50$, & Ω starts to decrease

This means $\frac{\Delta S}{\Delta U} < 0$, & T is negative!

Neg temp $\rightarrow$ higher energy

Neg temp is "hotter" than pos temp



What does this mean?

We only considered magnetic energy

- puts an artificial max on U_{tot}
- No outlet for thermal energy

In any real system, dipoles have translational, rotational, vibrational energy

- Initial input energy goes to flipping dipoles against the field
- Additional energy goes into random kinetic energy with no preferred axis

$$\Omega = \Omega_{para} \times \Omega_{solid}$$

$$\frac{S}{k} = \ln \Omega = S_{para} + S_{solid}$$

$$= N \ln 2 - \frac{1}{N}\left(\frac{U}{\mu B}\right)^2 + N \ln U - N \ln(N\hbar\omega) + N$$

$$\frac{S}{Nk} = \ln 2 - \frac{1}{N^2}\left(\frac{u}{\mu B}\right)^2 + \ln U - \ln(N\hbar\omega) + 1$$

$$= \ln\left(\frac{2U}{N\hbar\omega}\right) - \frac{1}{N^2}\left(\frac{u}{\mu B}\right)^2 + 1$$

$$\frac{U}{N\hbar\omega} \sim \xi \sim N$$

$$\sim \ln N - \frac{1}{N^2} + \sim 1$$