Consider a gas of many particles at constent energy E

How many microstates?

- Before we proceed, some new tools to help w/ our analysis

For the QM oscillator, the state of each oscillator is:

- finite (countable, integer factor of $\hbar\omega$) discrete
- Specified by a single number

$$E_{osc} = q\hbar\omega$$

For a gas of particles, the state of each particle is specified by 6 numbers

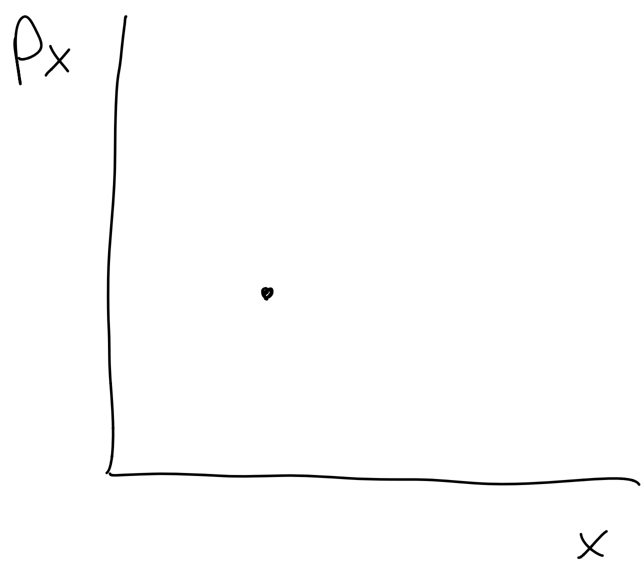
$$(x, y, z, p_x, p_y, p_z)$$

In general they are each continuous

We can specify the state of the system by plotting it in a higher dim. phase space

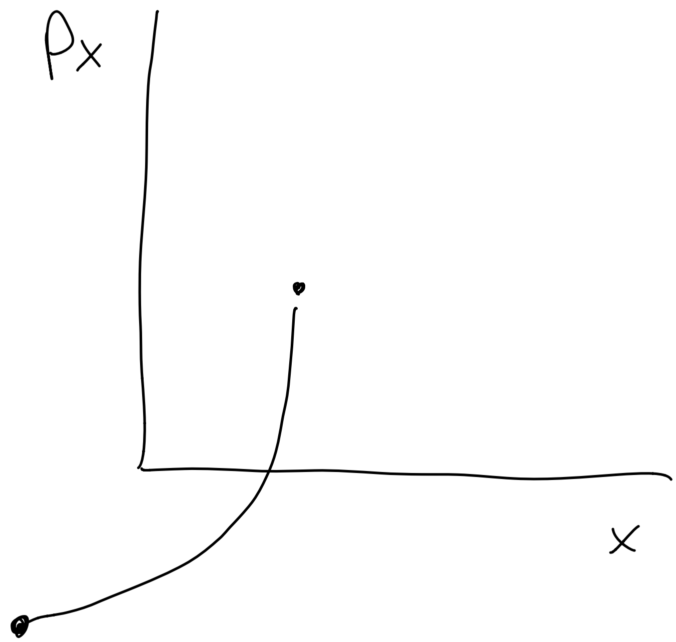
Consider a single particle in 1 dimension

state specified by (x, p_x)



$$p = p_{xi} - mgt$$

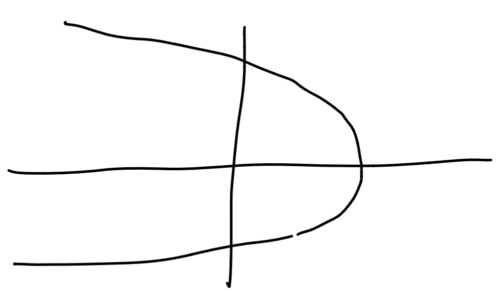
$$x = x_i + \frac{p_{xi}}{m} t - \frac{1}{2} g t^2$$



Possible states of system lie on the curve

$$\frac{1}{2m} p_x^2 + mgx = E$$

$$p_x = \pm\sqrt{2m(E - mgx)}$$



Ex: Single particle

mass on the end of a spring

$$H = \frac{p_x^2}{2m} + \frac{1}{2} k x^2$$

Possible values of (x, p_x)

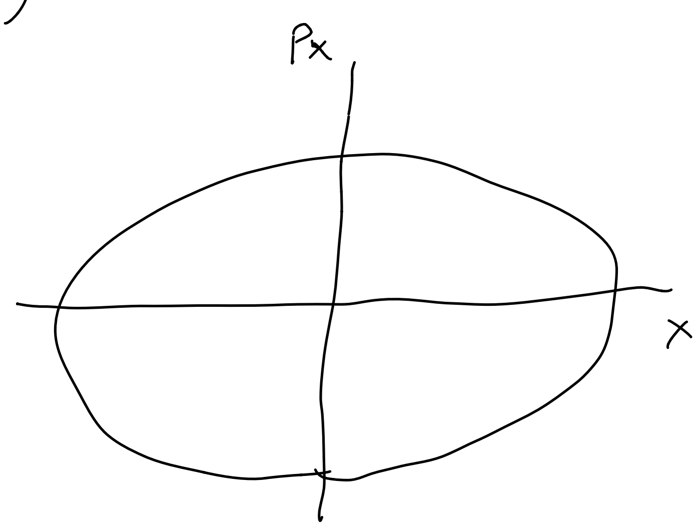
lie on the curve

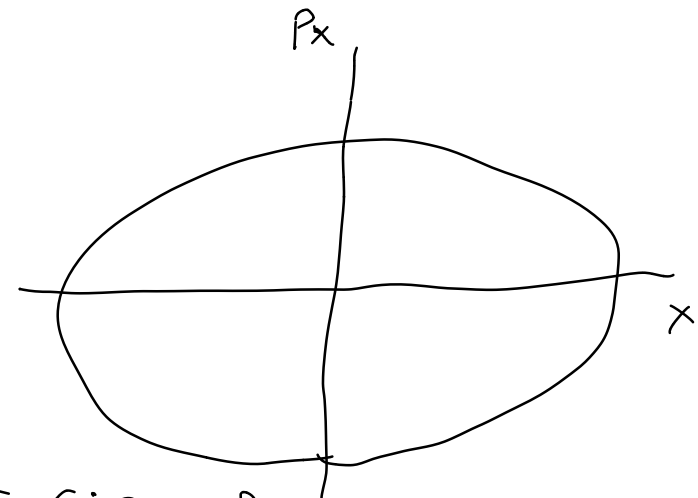
$$\frac{p_x^2}{2m} + \frac{1}{2} k x^2 = E$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$a = \sqrt{\frac{2E}{k}}$$

$$b = \sqrt{2mE}$$





# of states $\propto$ circumference of ellipse

In 3D, we need 6 numbers

x, y, z, p_x, p_y, p_z

The state of the system is a point in 6D space

$$H(x,y,z,p_x,p_y,p_z) = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}k\left(x^2+y^2+z^2\right)$$

$$\frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}k\left(x^2+y^2+z^2\right) = E$$

Our Hamiltonian Eqn forms a

"hyper surface" in a 6D space

# of microstates $\propto$ area of hypersurface

we'll talk about how to find the area in a second

- Consider the general case

N-many particles, in 3 dimensions

6N numbers to specify the state

$x_1, y_1, z_1, p_{1x}, p_{1y}, p_{1z}$

$x_2, y_2, z_2, p_{2x}, p_{2y}, p_{2z}$

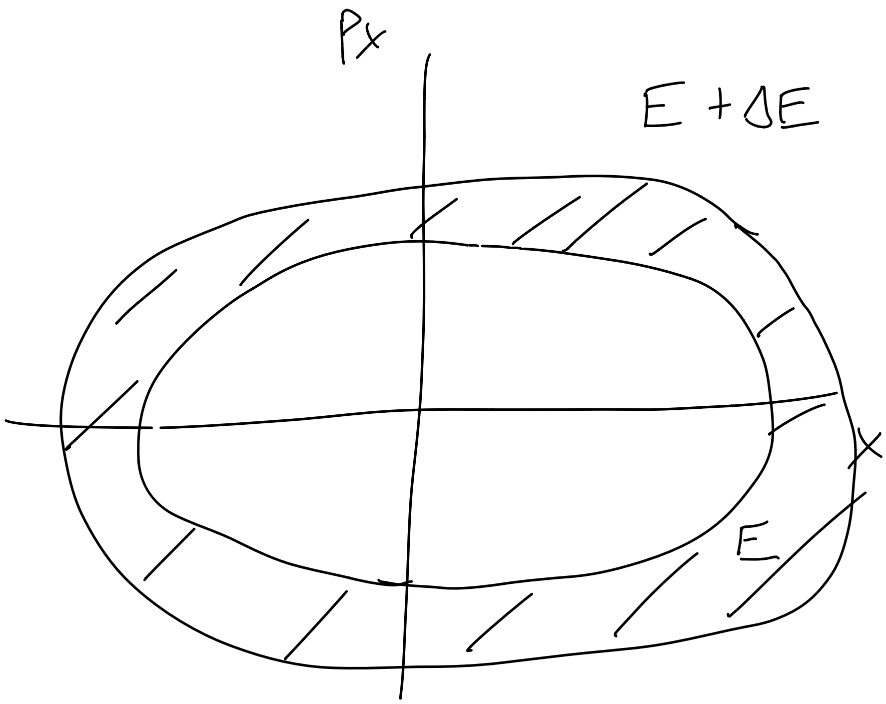
$\ldots$

$$H = \left(\sum_{i=1}^{N} \frac{1}{2m} \left(p_{x_1}^2 + p_{y_1}^2 + p_{z_1}^2 \right) \right) + U(x_1, y_1, z_1, x_2, y_2, z_2, \ldots)$$

To find the area

Consider the small volume in between

E & $E + \Delta E$



Volume in phase space $\longrightarrow w$

$$\Delta w = \int_{E < H < E+\Delta E} dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 \ldots dp_{x_1} dp_{y_1} dp_{z_1} dp_{x_2} dp_{y_2} \ldots$$

$$\Delta w = \int_{E \leq H(r_i, p_i) \leq E+\Delta E} d^{3N} r \, d^{3N} p$$

Total volume bounded by E:

$$w = \int_{H(r_i, p_i) \leq E} d^{3N} r \, d^{3N} p$$

$$\Delta w = w(E+\Delta E) - w(E) = \left(\frac{\partial w}{\partial E} \right) \Delta E$$

If the area is σ

$$\Delta w = \sigma(E)\,\Delta E$$

$$\left(dV = 4\pi r^2 dr \right)$$

so $\sigma(E) = \dfrac{\partial w}{\partial E}$

area = derivative of volume

$$V = \frac{4}{3}\pi r^3$$

$$\frac{\partial V}{\partial r} = 4\pi r^2 = A$$

1) Calculate w

2) Find $\sigma = \dfrac{\partial w}{\partial E}$

3) Then $\Omega \propto \sigma$

Ex: The ideal gas

Potential $= 0$

$$H = \sum_{i=1}^{N} \frac{1}{2m}\left(p_{x_i}^2 + p_{y_i}^2 + p_{z_i}^2\right)$$

$$w = \int_{H<E} d^{3N}r \, d^{3N}p$$

H doesnt depend on r

$$w = \underbrace{\int d^{3N}r}_{} \int_{H<E} d^{3N}p$$

$$\underbrace{\int dx_1 dy_1 dz_1}_{V} \underbrace{\int dx_2 dy_2 dz_2}_{V} \underbrace{\int dx_3 dy_3 dz_3}_{V} \dots$$

$$w = V^N \int_{H<E} d^{3N}p$$

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 + \dots p_{N-1}^2 + p_N^2 = 2m\underline{E}$$

$$\int_{H<E} d^{3N} p \rightarrow \text{Volume of hypersphere of radius } \sqrt{2mE}$$

In d dimensions

$$V_d(R) = \frac{\pi^{d/2}}{\frac{d}{2}\,\Gamma\left(\frac{d}{2}\right)} R^d$$

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$$

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$\Gamma(5) = 4\,\Gamma(4)$$

$$\Gamma(4) = 3\,\Gamma(3)$$

$$\Gamma(3) = 2\,\Gamma(2)$$

$$\Gamma(2) = \Gamma(1)$$

$$\Gamma(5) = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$$

$$\Gamma(z) = (z-1)!$$

$$V_d(R) = \frac{\pi^{d/2}}{\frac{d}{2}\,\Gamma\left(\frac{d}{2}\right)} R^d$$

if $d = 3$

$$V_3(R) = \frac{\pi^{3/2}}{\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)} R^3$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$V_3(R) = \frac{\pi^{3/2}}{\frac{3}{2}\,\frac{\pi^{1/2}}{2}} R^3$$

$$V_3(R) = \frac{4}{3}\pi R^3$$

$$V_2(R) = \pi R^2$$

$$V_d(R) = \frac{\pi^{d/2}}{\frac{d}{2}\,\Gamma\left(\frac{d}{2}\right)} R^d$$

$$d = 3N$$

$$R = (2mE)^{1/2}$$

$$V_d\left(\sqrt{2mE}\right) = \frac{\pi^{\frac{3N}{2}}}{\frac{3N}{2}\,\Gamma\left(\frac{3N}{2}\right)}(2mE)^{3N/2}$$

$$w(E) = V^N \frac{\pi^{\frac{3N}{2}}}{\frac{3N}{2}\,\Gamma\left(\frac{3N}{2}\right)}(2mE)^{3N/2}$$

$$\sigma(E) = \frac{\partial w}{\partial E} = \frac{V^N \pi^{\frac{3N}{2}}}{\frac{3N}{2}\Gamma\left(\frac{3N}{2}\right)}\left(\frac{3N}{2}\right)(2mE)^{\frac{3N}{2}-1}(2m)$$

$$(2mE)^{\frac{3N}{2}-1}(2m) = (2m)^{\frac{3N}{2}} E^{\frac{3N}{2}-1}$$

$$\sigma(E) = \frac{V^N \pi^{\frac{3N}{2}}(2m)^{\frac{3N}{2}} E^{\frac{3N}{2}-1}}{\Gamma\left(\frac{3N}{2}\right)}$$

$$E^{\frac{3N}{2}-1} = E^{\frac{3N}{2}} E^{-1} \approx E^{\frac{3N}{2}}$$

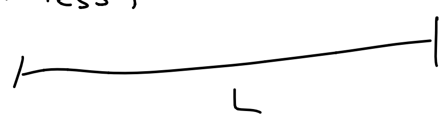
$$\sigma(E) = \frac{V^N \pi^{\frac{3N}{2}} (2mE)^{\frac{3N}{2}}}{\Gamma\left(\frac{3N}{2}\right)}$$

$$\Omega \propto \sigma(E)$$

$$[\sigma] = [L]^{3N} [p]^{3N}$$

We are counting phase space volume

Ω is a number (dim-less)



$$V = \int dx\,dy\,dz \qquad \# = L/\Delta x$$

$$\Omega = \frac{1}{\Delta x^3} \int dx\,dy\,dz \qquad \# = P/\Delta p$$

Split phase space into cells of "volume" $(\Delta x \Delta p)^{3N}$

$$\Omega = \frac{1}{(\Delta x \Delta p)^{3N}} \frac{V^N \pi^{\frac{3N}{2}} (2mE)^{\frac{3N}{2}}}{\Gamma\left(\frac{3N}{2}\right)}$$

Since only the relative amount of Ω matters, the proportionality constant doesn't really matter

The quantity $\Delta x \Delta p$ has an absolute minimum

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Let's pick Δx & Δp so that

$$\Delta x \Delta p = \hbar$$

$$\Omega = \frac{1}{(\hbar)^{3N}} \frac{V^N \pi^{\frac{3N}{2}} (2mE)^{\frac{3N}{2}}}{\Gamma\left(\frac{3N}{2}\right)}$$

if the particles are indistinguishable, then we've over-counted

- we can swap 2 particles w/o changing the state of the system

- number of duplicates

$$\boxed{\Omega = \frac{1}{(\hbar)^{3N}} \frac{V^N \pi^{\frac{3N}{2}} (2mE)^{\frac{3N}{2}}}{N!\,\Gamma\left(\frac{3N}{2}\right)}}$$

$$\Omega \sim V^N U^{3N/2}$$