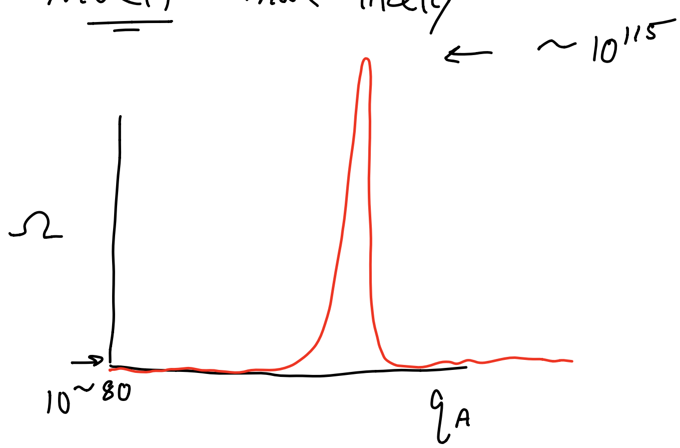
Last lecture

- Considered energy exchange between two systems

- Macrostates with ~ equally shared energy are MUCH more likely



- Explains heat flow from hotter to cooler

- Wrote a program with

  $N, q \sim 100$

- In real systems, $N, q \sim 10^{20-30}$

Since $$\Omega(N,q) = \frac{(N+q-1)!}{q!\,N!}$$

We cant calculate Ω for normal size systems

$$\left(\text{what is } 10^{23}!\right)?$$

To get an idea do 2 things

1) make some approximations

$$10^{23} + 20 \approx 10^{23}$$

$$10^{(10^{23})} \cdot 10^{23} = 10^{10^{23}+23} \approx 10^{23}$$

2) Work in logspace

$$\text{if } \Omega \sim 10^{10^{20}}, \quad \ln(\Omega) \sim 10^{20}$$

OK,

Let's say we have a single $\underline{E}$. solid

w/ $N \sim 10^{23}$, $q >> N$

(high temperature)

What is the functional form of Ω?

$$\Omega(N,q) = \frac{(N+q-1)!}{q!\,(N-1)!}$$

$$(N-1)! = \frac{N!}{N}$$

$$(N+q-1)! = \frac{(N+q)!}{N+q}$$

$$\Omega(N,q) = \frac{(N+q)!\;N}{q!\,N!\,(N+q)}$$

$$\frac{N}{N+q} = \frac{1}{1+\frac{q}{N}} \sim \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$$

$$N! \sim 10^{23}!$$

$$\frac{N}{N+q} \sim 1$$

$$\Omega(N,q) \approx \frac{(N+q)!}{q!\,N!}$$

Now move to logspace

$$\ln(\Omega) \approx \ln\left((N+q)!\right) - \ln(q!) - \ln(N!)$$

Another approximation

$$\ln(n!) \approx n\ln(n) - n$$
$$= n\left(\ln(n) - 1\right)$$

$$\ln(\Omega) \approx (N+q)\ln(N+q) - (N+q)$$
$$- (q\ln q - q)$$
$$- (N\ln N - N)$$

$$= (N+q)\ln(N+q) - N - q$$
$$- q\ln q + q - N\ln N + N$$

$$= (N+q)\ln(N+q) - q\ln q - N\ln N$$

if $q >> N$

$$\ln(N+q) = \ln\left(q\left(1+\frac{N}{q}\right)\right) = \ln q + \ln\left(1+\frac{N}{q}\right)$$

$$\frac{N}{q} << 1$$

$$f(1+\epsilon) = f(1) + f'(1)\epsilon + \frac{1}{2} f''(1)\epsilon^2 + \dots$$

$$\ln(1+\epsilon) \approx \ln(1) + \epsilon$$

$$\approx \epsilon$$

$$\ln\left(1+\frac{N}{q}\right) \approx \frac{N}{q}$$

$$\ln(q+N) \approx \ln q + \frac{N}{q}$$

$$\ln(\Omega) \approx (N+q)\left(\ln q + \frac{N}{q}\right) - q\ln q - N\ln N$$

$$= N\ln q + \frac{N^2}{q} + q\ln q + N - q\ln q - N\ln N$$

$$= N\ln q - N\ln N + N + \frac{N^2}{q}$$

$$\ln\Omega \cong N\ln\frac{q}{N} + N + \frac{N^2}{q}$$

$$N \ln \frac{q}{N} \sim N \quad (10^{20-30})$$

$$N \sim N$$

$$\frac{N^2}{q} \sim N \times 10^{-2-3}$$

$$\Omega \approx e^{N \ln \frac{q}{N} + N}$$

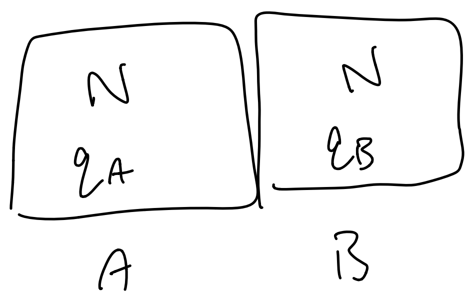
$$\approx e^{\ln\left(\frac{q}{N}\right)^N + N}$$

$$\approx \left(\frac{q}{N}\right)^N e^N$$

$$\Omega \approx \left(\frac{qe}{N}\right)^N \qquad \text{Exponential}$$

What about interacting systems?

Just to get an idea



$$q_{tot} = q_A + q_B := q$$

$$q >> N$$

$$\Omega = \Omega_A \Omega_B$$

$$\Omega \cong \left(\frac{q_A e}{N}\right)^N \left(\frac{q_B e}{N}\right)^N = \left(\frac{e}{N}\right)^{2N} (q_A q_B)^N$$

$$\Omega = \left(\frac{e}{N}\right)^{2N} \left(q_A (q - q_A)\right)^N$$

$$\Omega = \left(\frac{e}{N}\right)^{2N} \left(q q_A - q_A^2\right)^N$$

$$\frac{\partial \Omega}{\partial q_A} = N \left(\frac{e}{N}\right)^{2N} \left(q q_A - q_A^2\right) \left(q - 2 q_A\right) = 0$$

$$q_A = \underbrace{\frac{1}{2} q}_{\text{max}}, \underbrace{0, q}_{\text{min}}$$

$$q_A (q - q_A) \longrightarrow \frac{1}{2} q \left(q - \frac{1}{2} q\right)$$

$$= \frac{1}{2} q \frac{1}{2} q = \frac{1}{4} q^2$$

$$\Omega_{max} = \left(\frac{e}{N}\right)^{2N} \left(\frac{q}{2}\right)^{2N}$$

Consider points very near equilibrium:

$$q_A = \frac{q}{2} + x$$

$$q_B = \frac{q}{2} - x$$

$$|x| << q$$

$$\Omega = \left(\frac{e}{N}\right)^{2N} (q_A q_B)^N \longrightarrow \left(\frac{e}{N}\right)^{2N} (q_A q_B)^N$$

$$q_A q_B = \left(\frac{q}{2} + x\right)\left(\frac{q}{2} - x\right)$$

$$= \left(\frac{q}{2}\right)^2 - x^2$$

$$\Omega = \left(\frac{e}{N}\right)^{2N} \left[\left(\frac{q}{2}\right)^2 - x^2\right]^N$$

what is $\left[\left(\frac{q}{2}\right)^2 - x^2\right]^N$?

go to logspace

$$\ln\left[\left(\frac{q}{2}\right)^2 - x^2\right]^N = N\ln\left[\left(\frac{q}{2}\right)^2 - x^2\right]$$

$$= N\ln\left[\left(\frac{q}{2}\right)^2\left(1 - \left(\frac{2x}{q}\right)^2\right)\right]$$

$$= N\left[\ln\left(\frac{q}{2}\right)^2 + \ln\left(1 - \left(\frac{2x}{q}\right)^2\right)\right]$$

$$\ln\left(1 - \left(\frac{2x}{q}\right)^2\right) \approx -\left(\frac{2x}{q}\right)^2$$

$$\ln\left[\left(\frac{q}{2}\right)^2 - x^2\right]^N \approx N\left[\ln\left(\frac{q}{2}\right)^2 - \left(\frac{2x}{q}\right)^2\right]$$

$$\left[\left(\frac{q}{2}\right)^2 - x^2\right]^N \approx e^{N\ln\left(\frac{q}{2}\right)^2} e^{-N\left(\frac{2x}{q}\right)^2}$$

$$\Omega \simeq \left(\frac{e}{N}\right)^{2N} e^{N\ln\left(\frac{q}{2}\right)^2} e^{-N\left(\frac{2x}{q}\right)^2}$$

$$\Omega \simeq \left(\frac{e}{N}\right)^{2N}\left(\frac{q}{2}\right)^{2N} e^{-N\left(\frac{2x}{q}\right)^2}$$

$$\Omega \simeq \Omega_{max}\, e^{-\frac{4N}{q^2}x^2}$$

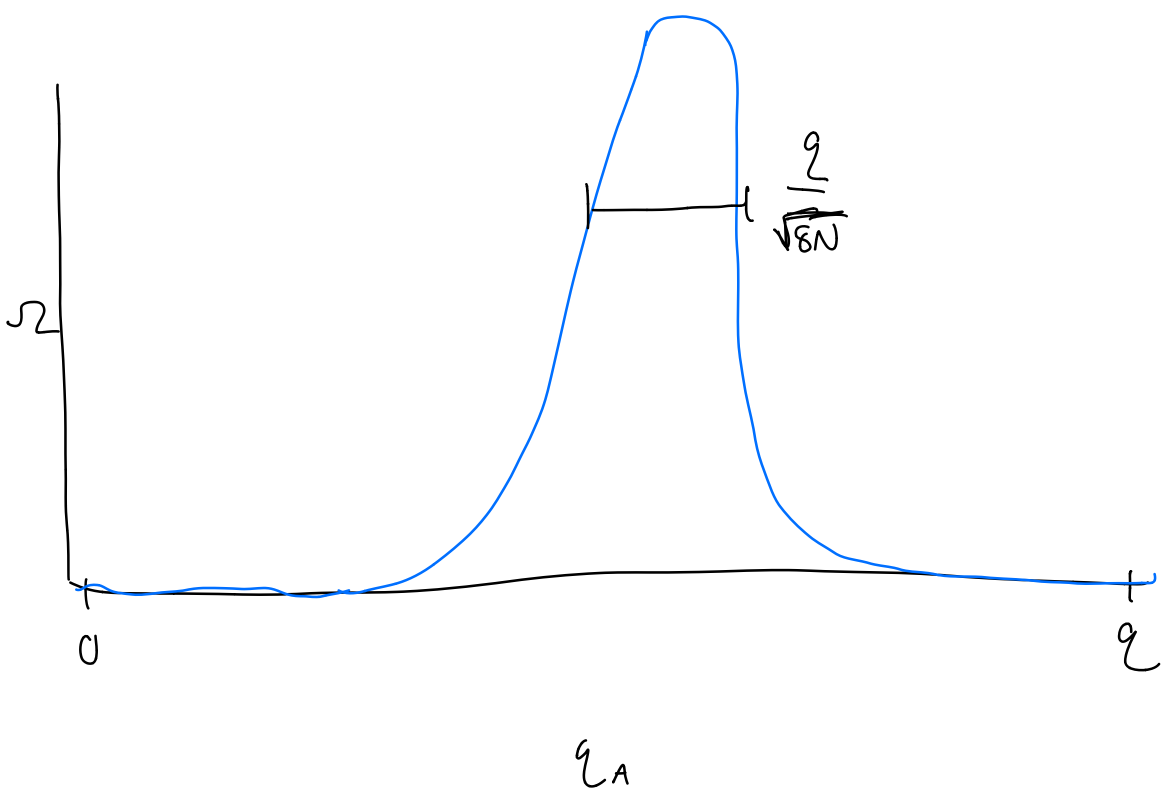
Gaussian: $y(x) = A e^{-\frac{x^2}{2\sigma^2}}$

$$\frac{1}{2\sigma^2} = \frac{4N}{q^2}$$

$$\frac{1}{\sigma^2} = \frac{8N}{q^2}$$

$$\sigma^2 = \frac{q^2}{8N}$$

$$\sigma = \pm \frac{q}{\sqrt{8N}}$$



$\sigma = \frac{q}{\sqrt{8N}}$ is large in the absolute sense, but <u>tiny</u> compared to q

if $N = 10^{23}$, $\sigma = 10^{-12} \cdot q$

- If I draw the curve to scale, the x-axis ranges $\sim 10^{5}$ km
- Cannot distinguish between these macrostates

10 sig figs!