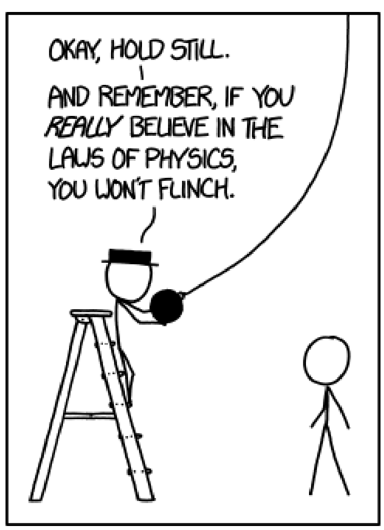
PHYS 4410 Final Exam

Tuesday, May 4, 2021

Instructions: *You will have ample time to complete this exam. Take a deep breath and relax! Read each question carefully, and let me know if anything is unclear. Partial credit may be awarded, so you are encouraged to clearly and legibly show your work for each problem. Extra paper is available at the front of the room if you need it. Write your name on every extra sheet you use, and clearly label what problem you are working on. Staple this to the back of your exam when you turn it in. You may use any information contained within this exam, as well as a calculator.*
Good luck!

Name: ____________________________________



Potentially useful information

Unit analysis

Power	Prefix	Name
10^{12}	T	tera
10^{9}	G	giga
10^{6}	M	mega
10^{3}	k	kilo
10^{0}	—	—
10^{-3}	m	milli
10^{-6}	μ	micro
10^{-9}	n	nano

Constants

Name	Symbol	Value
Boltzmann Constant	k_B	1.381×10^{-23} J/K
Red. Planck Constant	$\hbar$	1.055×10^{-34} J·s
Electron mass	m_e	9.11×10^{-31} kg
Electron charge	e	1.6×10^{-19} C
Avogadro's Number	N_A	6.022×10^{23}
Newton's Constant	G	$6.673 \times 10^{-11} \mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2$

Useful equations

Mono-atomic Ideal Gas

Multiplicity $\Omega = \frac{1}{\hbar^{3N}} \frac{V^N \pi^{\frac{3N}{2}} (2mU)^{\frac{3N}{2}}}{N! \cdot \Gamma(\frac{3N}{2})}$

Entropy $S = Nk_b \left[\ln \left(\frac{V}{N} \left(\frac{4\pi mU}{3N\hbar^2} \right)^{3/2} \right) + \frac{5}{2} \right]$

Einstein Solids

Multiplicity $\Omega = \frac{(q+N-1)!}{q!(N-1)!}$

Hyperbolic Trig Functions

$\cosh x$ $\frac{e^x + e^{-x}}{2}$

$\sinh x$ $\frac{e^x - e^{-x}}{2}$

Unit Conversions

1 atm $= 1.013 \times 10^5$ N/m^2

T [° C] =T[K] - 273.15

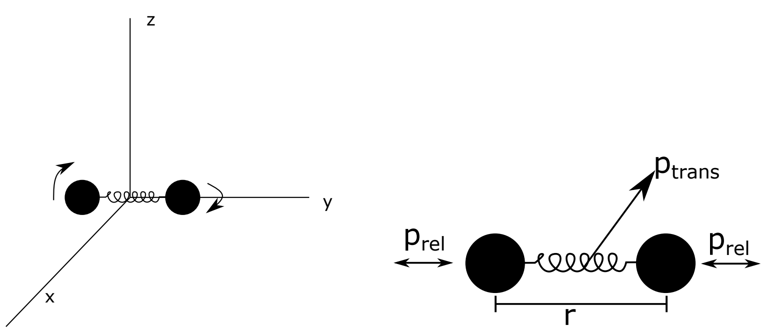
1 cal =4.186 J

1. An ideal gas consists of N many diatomic molecules. Because the interatomic bond resembles a spring, we model each molecule as two equal-mass balls (of mass m) connected by a spring.

   The motion of each molecule consists of:

   - Momentum of the center of mass (translational momentum) in 3 dimensions: $\vec{p}_{trans} = p_x\hat{x}+p_y\hat{y}+p_z\hat{z}$
   - Vibrational momentum relative to the center of mass in one dimension p_{rel}. Note that the two masses do not vibrate independently, but instead can be treated as a single particle with mass $\mu = m_1 m_2/(m_1+m_2) = \frac{1}{2}m$ with momentum p_{rel}.
   - Associated with the vibrational motion is the stretching and squeezing of the interatomic spring. Assume the spring has a spring constant k_s (not to be confused with Boltzmann's constant!) and an equilibrium length r_0.
   - Finally, the molecule is able to rotate in two directions with angular speeds ω_x and ω_z (the molecule is symmetric about the y axis, so rotations about this axis don't contribute to the molecule's energy.) The moment of inertia about the x and z axis is the same; just call it I.

   This motion is sketched in the figure below:



   (a) In terms of the variables defined above, what is the total energy (the Hamiltonian) of each molecule?

   (b) If the gas is in equilibrium at some temperature T, what is the total (average) internal energy of the gas (in terms of Boltzmann's constant k_B and the temperature T)?

   (c) If the temperature is 300 K, and the molecule is a nitrogen N_2 molecule (each atom is a nitrogen atom with a mass $m = 14$ amu $\approx 2.3\times10^{-26}$ kg), what is the "root mean square" speed $v_{rms} = \sqrt{\langle v^2\rangle}$?

2. A system of N distinguishable spin-1/2 particles at temperature T is subject to a constant, uniform magnetic field pointing in the $+z$ direction $\vec{B} = B\hat{z}$. Because the particles are spin-1/2, the dipole moment in the z direction can take only 2 values: spin "up" ($\vec{\mu} = \mu\hat{z}$) or spin "down" ($\vec{\mu} = -\mu\hat{z}$). Recall that the magnetic energy of such a particle is: $U = -\vec{\mu} \cdot \vec{B}$.

$$\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow \ldots$$

$$\vec{B} \quad \uparrow$$

(a) What is the energy of this system at temperature T?

(b) What is the constant volume heat capacity C_V?

(c) What is the magnetization $M = N\langle\mu\rangle$?

3. An Einstein solid in equilibrium at temperature T consists of N oscillators with q many energy "units" of $\hbar\omega$ distributed among them (so that the total energy is $U = q\hbar\omega$). In the "low energy limit" ($q << N$), the multiplicity of this system is given by[1]:

$$\Omega \approx \left(\frac{eN}{q}\right)^q$$

   (a) What is the internal energy of this system (in terms of temperature, don't just write $q\hbar\omega$)

   (b) What is the constant-volume heat capacity C_V?

[1] Actually, this limit is not valid classically, since at low temperature we need to take into account quantum mechanics, but you may blissfully ignore all of that here

4. Consider an ideal gas in a 2D "flatland" Universe. In this case, instead of the gas occupying a volume V, it occupies an area A. Instead of exerting a pressure $P =$ force/area with units of N/m^2, the gas exerts a "linear pressure" $\Pi =$ force/length with units of N/m. The gas still exchanges heat SdT with its surroundings, but the work done on the gas is $-\Pi dA$ rather than $-PdV$. The infinitesimal change in energy of the gas is then:

$$dU = TdS - \Pi dA + \mu dN$$

For this 2D gas, the multiplicity is given by:

$$\Omega(U, A, N) = \frac{1}{N!}\left(\frac{A}{\hbar^2}\right)^N \frac{\pi^N}{N!}(2mU)^N$$

(a) (15 points) Derive the energy vs temperature relationship for this gas (in 3D, it is $U = \frac{3}{2}NkT$, what is it in 2D?)

(b) (15 points) Derive the 2D ideal gas law (in 3D, it is $PV = NkT$, what is the 2D equivalent?)

5. **This is an optional problem. If you attempt it, it can replace your worst scoring problem, if it is to your benefit**
Consider a *relativistic* gas of electrons inside a cube of side L at $T = 0$. The momenta of the electrons are quantized:

$$p_x = \frac{\hbar\pi}{L} n_x$$
$$p_y = \frac{\hbar\pi}{L} n_y$$
$$p_z = \frac{\hbar\pi}{L} n_z$$

Since the electrons energies are well above their rest mass, the energy is given by pc rather than $\frac{p^2}{2m}$.

(a) What is the energy of the highest occupied energy state in the gas (the fermi energy ε_F)?

(b) What is the total internal energy of the gas? (You may write your answer in terms of ε_F if you wish)

Question	Points	Score
1	0	
2	0	
3	0	
4	30	
5	0	
Total:	30	