Consider an ideal gas of electrons (fermions)

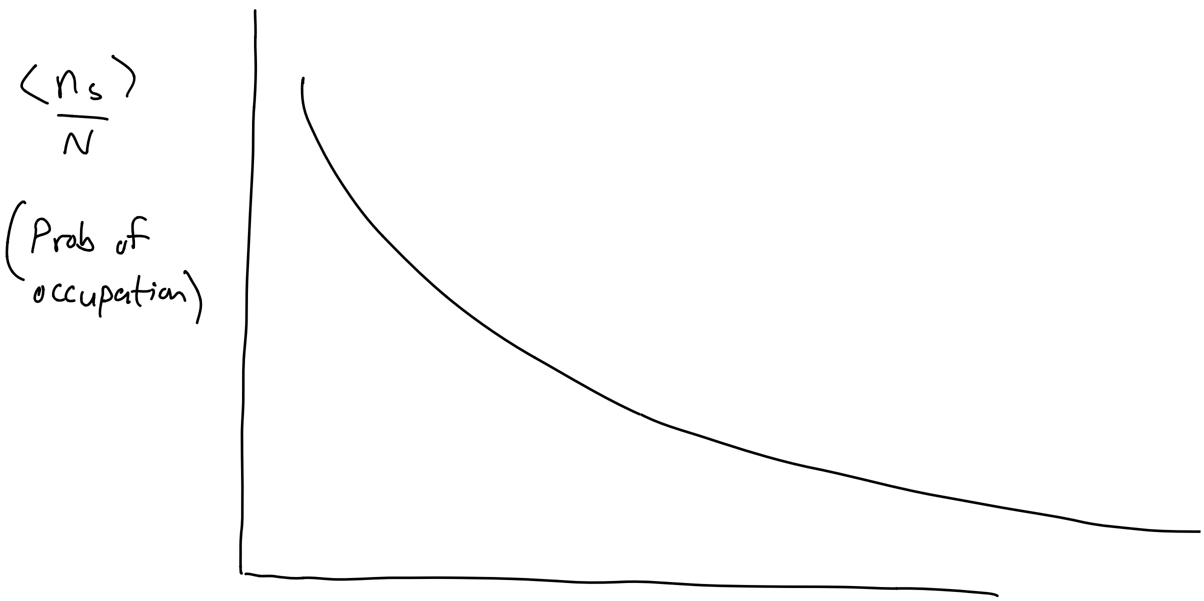
Quantum states given by $n_x, n_y, n_z$

$$\epsilon_s = \frac{\hbar^2 \pi^2}{L^2} (n_x^2 + n_y^2 + n_z^2)$$

| $s$ | $\epsilon_s$ |
|---|---|
| $\langle 1,1,1 \rangle$ | 3 |
| $\langle 2,1,1 \rangle$ | 6 |
| $\langle 1,2,1 \rangle$ | 6 |
| $\langle 1,1,2 \rangle$ | 6 |
| $\langle 3,1,1 \rangle$ | 11 |
| $\vdots$ | $\vdots$ |

If I add an electron to the gas, which state will it occupy?

At "normal" temperatures




- Particles don't necessarily fill the lowest available energy state

- Why? Entropy

$$\Delta F = \Delta U - T\Delta S$$

Particles will spontaneously jump to higher energies as long as $\Delta F < 0$ $(\Delta U < T\Delta S)$

The gas wants to minimize energy while also maximizing entropy

This means there are some empty states between the lowest & highest energy electron

| $S$ | $\epsilon_s$ |
| --- | --- |
| $\langle 1, 1, 1 \rangle$ | 3 |
| $\langle 2, 1, 1 \rangle$ | 6 |
| $\langle 1, 2, 1 \rangle$ | 6 |
| $\langle 1, 1, 2 \rangle$ | 6 |
| $\langle 3, 1, 1 \rangle$ | 11 |
| $\vdots$ | $\vdots$ |

Bring in first electron, it goes to ground state

Second electron skips these & goes here instead

How energetic is the highest energy electron? Can't say for sure.

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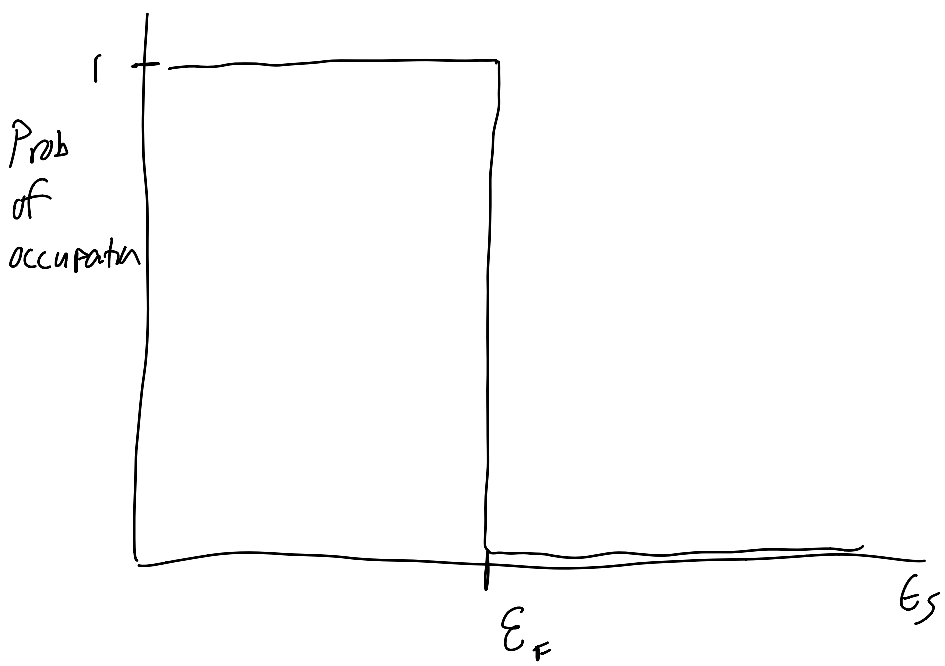
Compare this with the $T = 0$ scenario




States with $\epsilon_s < \varepsilon_F$ are certain to be occupied

States w/ $\epsilon_s > \varepsilon_F$ are certainly empty

(no unoccupied states between ground
+ $\varepsilon_F$)

Particles will spontaneously jump to higher energies as long as $\Delta F < 0$ $(\Delta U < T\Delta S)$

$$T \to 0$$

$$\Delta F < 0 \Rightarrow \Delta U < 0$$

electrons will only move to states with lower energy; these are all occupied

Since all states below $\varepsilon_F$ are filled,

counting states $\longleftrightarrow$ counting particles

| $s$ | $e_s$ | $n_s$ |
|---|---|---|
| $\langle 1,1,1\rangle$ | 3 | 2 $(\uparrow\downarrow)$ |
| $\langle 2,1,1\rangle$ | 6 | 2 |
| $\langle 1,2,1\rangle$ | 6 | 2 |
| $\langle 1,1,2\rangle$ | 6 | 2 |
| $\langle 3,1,1\rangle$ | 11 | 2 |
| $\vdots$ | $\vdots$ | |
| | $\varepsilon_F$ | 2 |
| | $\vdots$ | 0 |
| | | 0 |
| | | 0 |

The possible combinations of $n_x, n_y, n_z$

which result in energy $\epsilon_s$

lie on the sphere with radius

$$\left(\frac{2mL^2}{\hbar^2\pi^2}\epsilon_s\right)^{1/2}$$

$$\epsilon_s = \frac{\hbar^2\pi^2}{L^2}\left(n_x^2+n_y^2+n_z^2\right)$$

$$\Downarrow$$

$$n_x^2+n_y^2+n_z^2 = \frac{2mL^2}{\hbar^2\pi^2}\epsilon_s$$

# of states with energy $\epsilon_s$

$$= 2\cdot\sum_{n_x}\sum_{n_y}\sum_{n_z;\ n_x^2+n_y^2+n_z^2=\frac{2mL^2}{\hbar^2\pi^2}\epsilon_s}(1)$$

# of states with energy below $\epsilon_s$:

$$= 2 \cdot \sum_{n_x} \sum_{n_y} \sum_{n_z \,;\, n_x^2 + n_y^2 + n_z^2 \leq \frac{2mL^2}{\hbar^2 \pi^2} \epsilon_s} (1)$$

$$= 2 \cdot \frac{1}{\Delta n^3} \int_{n_x^2 + n_y^2 + n_z^2 < \frac{2mL^2}{\hbar^2 \pi^2} \epsilon_s} dn_x \, dn_y \, dn_z$$

$$\frac{1}{\Delta n^3} = \frac{1}{1^3} = 1$$

$$dn_x \, dn_y \, dn_z \rightarrow n^2 \sin\theta \, dn \, d\theta \, d\phi$$

$$= 2 \cdot \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \int_0^{\frac{\pi}{2}} d\phi \int_0^{\sqrt{\frac{2mL^2}{\hbar^2 \pi^2} \epsilon_s}} n^2 \, dn$$

$$= 2 \cdot \frac{\pi}{2} \cdot \frac{1}{3} \left( \frac{2mL^2}{\hbar^2 \pi^2} \epsilon_s \right)^{3/2}$$

$$n(\epsilon < \epsilon_s) = 2 \cdot \frac{\pi}{6} \cdot \left( \frac{2mL^2}{\hbar^2 \pi^2} \epsilon_s \right)^{3/2}$$

at $\epsilon_s = \varepsilon_F$, $n(\epsilon < \varepsilon_F) = N$

$$N = \frac{\pi}{3} \cdot \left( \frac{2mL^2}{\hbar^2 \pi^2} \varepsilon_F \right)^{3/2}$$

$$\boxed{\varepsilon_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3}} \; ; \; V = L^3$$

Energy of highest energy electron

@ $T = 0$

What is $U$?

$$U = 2 \cdot \sum_{n_x} \sum_{n_y} \sum_{n_z ;\ n_x^2+n_y^2+n_z^2 \le \frac{2mL^2}{\hbar^2\pi^2}\epsilon_s} \epsilon(n_x, n_y, n_z)$$

$$= 2\int \frac{\hbar^2\pi^2}{L^2}(n_x^2+n_y^2+n_z^2)\, dn_x\, dn_y\, dn_z$$

$$\downarrow$$

$$= 2\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} d\phi \int_0^{\left(\frac{2mL^2}{\hbar^2\pi^2}\right)^{\frac{1}{2}}} \frac{\hbar^2\pi^2}{L^2} n^2 \cdot n^2 dn$$

$$= 2 \cdot 1 \cdot \frac{\pi}{2} \cdot \frac{\hbar^2\pi^2}{L^2}\left(\frac{1}{5}\right)\left(\frac{2mL^2}{\hbar^2\pi^2}\right)^{\frac{5}{2}}$$

$$\boxed{U = \frac{3}{5}N\epsilon_F}$$

Electrons in a metal @ room temp:

(Cu)

$\varepsilon_F \approx 7\ eV$

$kT \approx 3 \times 10^{-2}\ eV$

Since the energy of the gas $\left(\frac{3}{5}\varepsilon_F\right)$

$U >> kT$ (the thermal energy)

we can approximate that $T \approx 0$

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- So our gas gets a large amount of internal energy, even at $T=0$

- The gas also exerts pressure

$$dU = T\,dS - P\,dV + \mu\,dN \Rightarrow P = -\frac{\partial U}{\partial V}$$

$$U = \frac{3}{5}\varepsilon_F = \left(\frac{3}{5}\right)\frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$P = \frac{2}{3}\frac{U}{V}$$

Example: Neutron star

A gas of $N$ neutrons at $T \approx 0$

$$U_{\text{Kinetic}} = \frac{3}{5} N \varepsilon_F$$

$$U_K = \frac{3}{5} \cdot N \cdot \frac{\hbar^2}{2m_n} \left( \frac{3\pi^2 N}{V} \right)^{2/3}$$

$$M = N m_n \Rightarrow N = \frac{M}{m_n}, \quad V = \frac{4}{3}\pi R^3$$

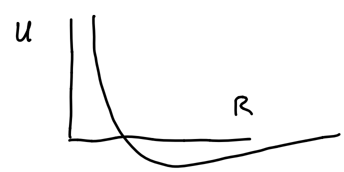
$$U_K = \frac{3}{10} \frac{\hbar^2 \pi^2}{m_n} \left( \frac{M}{m_n} \right)^{5/3} \left( \frac{9}{4\pi^2 R^3} \right)^{2/3}$$

$$U_{grav} = -\frac{3}{5} \frac{GM^2}{R}$$

$$U_{tot} = U_K + U_{grav} =$$

$$\frac{3}{10} \frac{\hbar^2 \pi^2}{m_n} \left( \frac{M}{m_n} \right)^{5/3} \left( \frac{9}{4\pi^2 R^3} \right)^{2/3} - \frac{3}{5} \frac{GM^2}{R}$$

$$\approx \frac{a}{R^2} - \frac{b}{R}$$



Equilibrium: $\frac{dU_{tot}}{dR} = 0$ Solve for $R$

$M \approx 1 M_\odot$

$m_n = 1.67 \times 10^{-27}$ kg

$R \approx 12$ km

$$\rho = \frac{M_\odot}{\frac{4}{3}\pi R^3} \approx 3 \times 10^{17} \frac{kg}{m^3}$$

$$\frac{6 \times 10^{24} \text{ kg}}{R^3} = 3 \times 10^{17}$$

1 spoonfull $\approx$ Mt. Everest

$\varepsilon_F \approx 6 \times 10^7$ eV

$$T_F = \frac{\varepsilon_F}{k} \approx 7 \times 10^{11} \text{ K}$$

$T_{sun} \approx 6\,000$ K