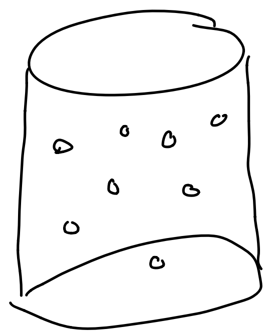
Macrostate VS Microstate

- We have a gas w 10^{20} molecules



How do we describe this system?

- Pressure
- Temperature
- Volume
- Energy (U)
- etc ...

Another way:

m1: $\vec{r} = \langle 3, 7, -1 \rangle$
$\vec{p} = \langle 13, 12, 9 \rangle$

m2: $\vec{r} = \langle -5, 6, 4 \rangle$ $\vec{p} = \langle -1, 0, 13 \rangle$

m3: ...

Macrostate: Specify P, V, T, U

Microstate: List $\vec{r}$ + $\vec{p}$ of every molecule

If we know these, we could then calculate the macroscopic variables

- Can't actually specify a microstate entirely (too many molecules!)
  - Nonetheless, can use it as a starting point for explaining macro-phenomena

If we are in equilibrium, macro variables don't change (P, V, T, U)

- But micro properties do!

This suggests that multiple microscopic configurations can correspond to one single macroscopic state

- Very basic example

system of 2 particles in $\underline{1}$ dimension

Total Energy $U = 5$ $(m=1)$

$$U = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2$$

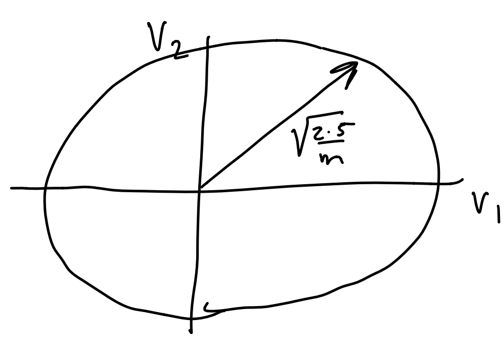
v_1	v_2	U
0	3.2	5
0	-3.2	5
3.2	0	5
-3.2	0	5
2	2.45	5
2.45	2	5
-2.45	-2	5

$\frac{1}{2} m v_2^2 = 5$

$v_2 = \sqrt{\frac{10}{m}} = \sqrt{10}$

$$\frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 = 5$$

$$v_1^2 + v_2^2 = \frac{2(5)}{m}$$



The question we're getting at:
How many micro states correspond to a given macrostate?

Technically, infinitely many (in the above example). But you can see that different Macrostates have different numbers of microstates (what if I increased the energy of the system above?)

As we will soon see, the relation between micro and macro states is very important in statistical mechanics.
As it turns out, macro states which have more associated micro states are more probable (we will see what this means)

In the previous example, each particle had infinitely many momenta it could assume
Let's consider a system where each particle only has two possible states (even this has some physical applications) and go from there

Let's count micro + macro states of flipped coins

Example:

Flipping coins

Flip 3 coins

What are the possible outcomes?

c1	c2	c3
H	H	H
H	H	T
H	T	H
T	H	H
H	T	T
T	T	H
T	H	T
T	T	T

Each row is a microstate

End result (# heads, # tails)
is a macrostate

Possible macrostates:

3H, 2H 1T, 2T 1H, 3T

If you know the microstate,
you know the macrostate

(HHT = 2H, 1T)

Reverse is not true

(1H, 2T) has many possible
microstates

Notation call the # of microstates
associated with a given macrostate

Ω

- What is $\Omega(2 \text{ heads})$?

$$\Omega = 3$$

(HHT, HTH, THH)

$$\Omega(1 \text{ heads}) = 3$$
$$\Omega(2 \text{ heads}) = 3$$
$$\Omega(3 \text{ heads}) = 1$$
$$\Omega(0 \text{ heads}) = 1$$

$$3 + 3 + 1 + 1 = 8$$

We relate this to probability

$$\text{prob}(2 \text{ heads}) = \frac{\Omega(2)}{\Omega(0) + \Omega(1) + \Omega(2) + \Omega(3)}$$

$$\text{prob}(2) = \frac{3}{8}$$

$$\text{prob}(3) = \frac{1}{8}$$ etc...

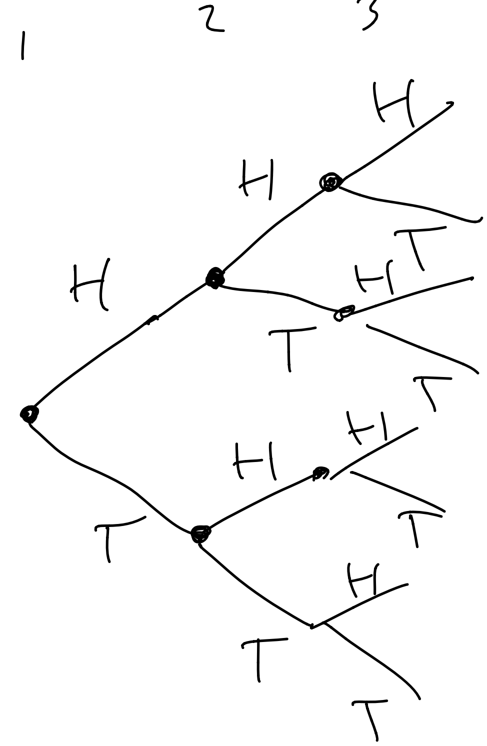
$$\text{prob(macrostate)} = \frac{\#\ \text{microstates}}{\text{all microstates}}$$

Now let's flip 100 coins

There are 101 macrostates

$n_{heads} = 0, 1, 2, \ldots, 100$

can't list all microstates



After n flips, there are 2^n possible outcomes

For 100 coins, $2^{100} \simeq 1.3 \times 10^{30}$ outcomes

$$\Omega(\text{all}) = 2^{100}$$

microstates per macrostate?

Consider

$$\Omega(n=0)$$

Every coin must be tails

only $\underline{1}$ state

$$\Omega(0) = 1 \qquad (\Omega(100) = 1)$$

$$\Omega(1) \ ?$$

1. H T T T T . . .

T H T T T . . . .

T T H T T . . . $\Omega = 100$

each coin could be the single heads.

$\Omega(1) = 100 \qquad (\Omega(99) = 100)$

$\Omega(2)$?

Take each $\Omega(1)$ state & choose one more head

$\underline{H\ T\ T\ T\ \ldots\ T\ T\ T}$

$$\left.\begin{array}{l} H\,H\,T\,T\ \ldots\ T\,T\,T \\ H\,T\,H\,T\ \ldots\ T\,T\,T \\ \quad\ \ldots \\ H\,T\,T\,T\ \ldots\ T\,T\,H \end{array}\right\} 99$$

$T\ \underline{H\ T\ T\ \ldots\ T\ T\ T}$

$$\left.\begin{array}{l} H\,H\,T\,T\ \ldots\ T\,T\,T \\ T\,H\,H\,T\ \ldots\ T\,T\,T \\ T\,H\,T\,H\ \ldots\ T\,T\,T \\ T\,H\,T\,T\ \ldots\ T\,T\,H \end{array}\right\} 99$$

100 microstates for one head, each with 99 outcomes for the second head

$$\Omega(n=2) = 100 \cdot 99$$

But! Some of the states are duplicates!

$$\underline{H\;T\;T\;T\;\ldots\;T\;T\;T}$$

Counted twice!

$$\left.\begin{array}{l} H\,H\,T\,T\;\ldots\;T\,T\,T \\ H\,T\,H\,T\;\ldots\;T\,T\,T \\ \quad\vdots \\ H\,T\,T\,T\;\ldots\;T\,T\,H \end{array}\right\} 99$$

$$\underline{T\;H\;T\;T\;\ldots\;T\;T\;T}$$

$$\left.\begin{array}{l} H\,H\,T\,T\;\ldots\;T\,T\,T \\ T\,H\,H\,T\;\ldots\;T\,T\,T \\ T\,H\,T\,H\;\ldots\;T\,T\,T \\ T\,H\,T\,T\;\ldots\;T\,T\,H \end{array}\right\} 99$$

How many duplicates?

Well, how did this happen...

We considered choosing a first head, & then choosing a second

1) $T \; T \; T \; H_1 \; T \; T \; T \; H_2 \; T \ldots$

$T \; T \; T \; H_2 \; T \; T \; T \; H_1 \; T \ldots$

these are the same!

Each microstate will have one duplicate

↓

$$\Omega(2) = \frac{100 \cdot 99}{2} = 4950$$

$$\Omega(3) = ?$$

100 ways to choose first coin

99 for second

98 for third

$$\Omega(3) = 100 \cdot 99 \cdot 98$$

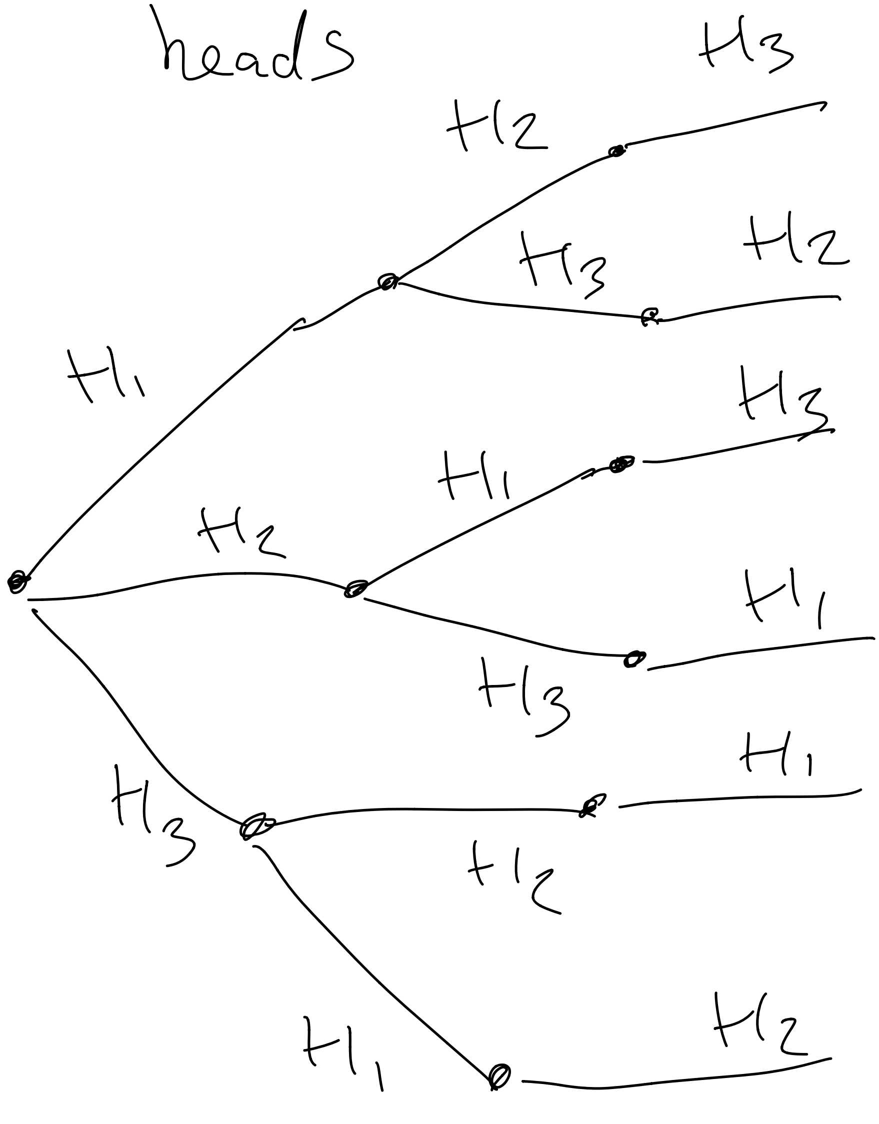
But what about duplicates?

$T\,H_1\,T\,T\,H_2\,T\,H_3\,T\,T$

$H_2 \qquad H_1 \qquad H_3$

$H_3 \qquad H_2 \qquad H_1$, etc...

# of duplicates = # of ways to arrange my 3 heads



$\# = 6 \quad (3 \cdot 2 \cdot 1)$

$$\Omega(3) = \frac{100 \cdot 99 \cdot 98}{3 \cdot 2 \cdot 1} = \boxed{161700}$$

Pattern?

$$\Omega(2) = \frac{100 \cdot 99}{2 \cdot 1}$$

$$\Omega(3) = \frac{100 \cdot 99 \cdot 98}{3 \cdot 2 \cdot 1}$$

$$\Omega(2) = \frac{100!}{(100-2)!} \; \frac{1}{2!}$$

$$= \frac{100 \cdot 99 \cdot \cancel{98} \cdot \cancel{97} \cdot \cancel{96} \cdots}{\cancel{98} \cdot \cancel{97} \cdot \cancel{96} \cdots} \; \frac{1}{2 \cdot 1}$$

$$\Omega(3) = \frac{100!}{(100-3!)} \; \frac{1}{3!}$$

$$= \frac{100 \cdot 99 \cdot 98}{3 \cdot 2 \cdot 1}$$

$$\Omega(n) = \frac{100!}{n!\,(100-n)!}$$

N coins instead of 100

$$\Omega(N,n) = \frac{N!}{n!\,(N-n)!}$$

This is just the # of combinations

N things to choose from, want to pick r.

$$\Omega(N,n) = \binom{N}{n}$$

Applicable to paramagnetic systems.

Will do this later!

Einstein Solids