1 Problem 1

$f(x) = ||\mathbf{Ax} - \mathbf{b}||_2^2$

This is just the ordinary least squares OLS problem
$= (\mathbf{Ax} - \mathbf{b})'(\mathbf{Ax} - \mathbf{b})$
$= (\mathbf{x'A'} - \mathbf{b'})(\mathbf{Ax} - \mathbf{b})$
$= \mathbf{x'A'Ax} - \mathbf{x'A'b} - \mathbf{b'Ax} + \mathbf{b'b}$
$= \mathbf{x'A'Ax} - \mathbf{2x'A'b} + \mathbf{b'b}$

$\nabla f(x) = \mathbf{2A'Ax} - \mathbf{2A'b}$
$\nabla^2 f(x) = \mathbf{2A'A}$

Reminders:
$\mathbf{Av} = \lambda\mathbf{v}$
$\mathbf{v}$ is an eigenvector if it satisfies this equation
λ is the eigenvalue corresponding to $\mathbf{v}$
Eigenvectors are the vectors that linear transformation $\mathbf{A}$ merely elongates or shrinks (as opposed to rotating)
Eigenvalue is the amount that they elongate or shrink
To find, solve $det(\mathbf{A} - \lambda\mathbf{I}) = 0$ (characteristic equation)
Eigendecomposition: $\mathbf{A} = \mathbf{Q\Lambda Q^{-1}}$
$\mathbf{Q}$ is a $n \times n$ matrix whose i'th column is eigenvector q_i of $\mathbf{A}$
Λ is a diagonal matrix for which $\Lambda_{ii} = \lambda_i$
If eigenvalues of Hessian are all positive, then $\mathbf{A}$ is positive semi-definite (multivariate equivalent of convex)
If all negative, then $\mathbf{A}$ is negative-definite (multivariate equivalent of concave)
Matrix $\mathbf{A}$ is positive definite IFF $\mathbf{A} = \mathbf{R'R}$, where $\mathbf{R}$ is a matrix with independent columns (full rank)

So, in our case, assuming $\mathbf{A}$ is of full rank, since it has a Hessian of $\mathbf{AA'}$, it is positive definite and so all eigenvalues are positive.

The necessary and sufficient condition for x^* to be a minimizer of $f(x)$ is that the gradient is 0 at x^*

f has a unique minimizer if A is positive definite

2 Problem 2

By the law of total probability, $f_X(x) = \sum_{k=1}^{K} f_{X|Y=k}(x|k)f_Y(k)$

$P(Y = k|X = x) = \frac{P(X=x|Y=k)P(Y=k)}{P(X=x|Y=k)P(Y=k)+P(X=x|Y\neq k)P(Y\neq k)} = \frac{f_{X|Y=k}(x)P(Y=k)}{f_X(x)}$