# STAT 672: Homework 3

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## SVD and Ridge Regression

Estimated coefficients in ridge regression are given by:

$$\hat{\boldsymbol{\beta}}_{\text{ridge}} = (\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{y}$$

We take the singular value decomposition of the feature matrix:

$$= ((\mathbf{UDV}^T)^T(\mathbf{UDV}^T) + \lambda\mathbf{I})^{-1}(\mathbf{UDV}^T)^T\mathbf{y}$$
$$= (\mathbf{VD}^T\mathbf{U}^T\mathbf{UDV}^T + \lambda\mathbf{I})^{-1}\mathbf{VD}^T\mathbf{U}^T\mathbf{y}$$

We know that $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{D}^T\mathbf{D} = \mathbf{D}^2$ and so can simplify this to:

$$= (\mathbf{VD}^2\mathbf{V}^T + \lambda\mathbf{I})^{-1}\mathbf{VDU}^T\mathbf{y}$$

We substitute in $\mathbf{VV}^T$ for $\mathbf{I}$:

$$= (\mathbf{VD}^2\mathbf{V}^T + \lambda\mathbf{VV}^T)^{-1}\mathbf{VDU}^T\mathbf{y}$$

And factor out $\mathbf{VV}^T$:

$$= \mathbf{V}(\mathbf{D}^2 + \lambda)^{-1}\mathbf{V}^T\mathbf{VDU}^T\mathbf{y}$$

And make use of the fact that $\mathbf{V}$ is orthonormal:

$$= \mathbf{V}(\mathbf{D}^2 + \lambda)^{-1}\mathbf{DU}^T\mathbf{y}$$

Since $\mathbf{D}$ is diagonal, we can rewrite the expression involving it and $\lambda$:

$$\mathbf{D}_\lambda := (\mathbf{D}^2 + \lambda)^{-1}\mathbf{D}$$

$$= \text{diag}\left(\frac{d_1}{d_1^2 + \lambda} \cdots \frac{d_D}{d_D^2 + \lambda}\right)$$

Thus, computation of estimated coefficients in ridge regression via SVD is given by:

$$\hat{\boldsymbol{\beta}}_{\text{ridge}} = \mathbf{VD}_\lambda\mathbf{U}^T\mathbf{y}$$

$$\blacksquare$$

## Efficiency of Computation

There are an inefficient method and an efficient method of re-calculating ridge regression coefficients for a new regularization parameter $\lambda$.

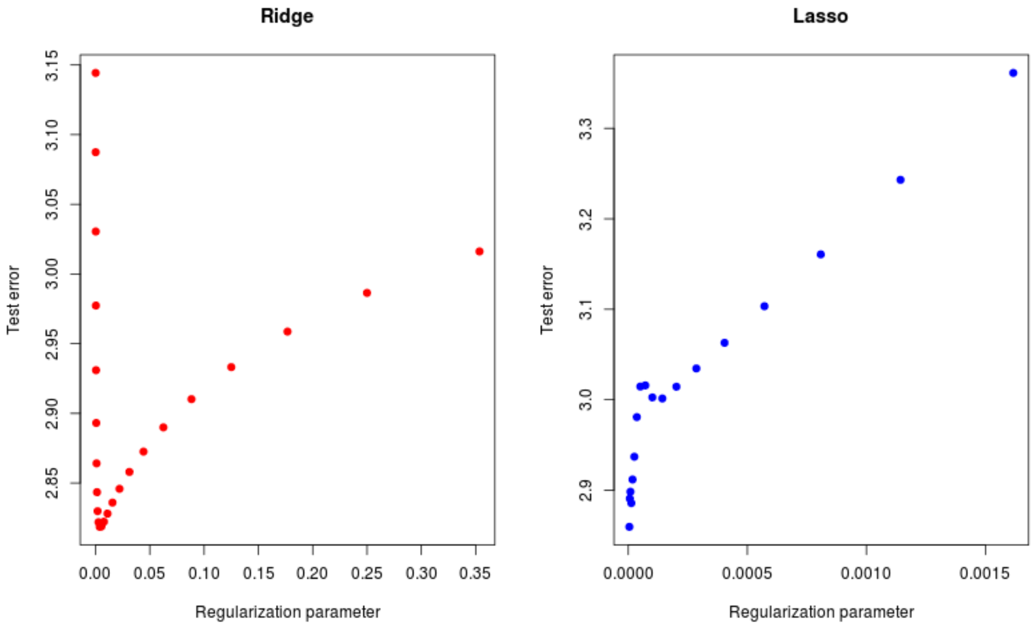
In the inefficient method, we redo the entire singular value decomposition every time we update $\lambda$. SVD has a computational complexity on the order of $O(nd^2)$ (with $n$ corresponding to the number of rows of the feature matrix and $d$ corresponding to the number of columns, i.e. the dimensionality of the data). We then multiply $\mathbf{V}\mathbf{D}_\lambda\mathbf{U}^T\mathbf{y}$. A matrix-matrix product $C = AB$, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, costs $O(2mnp)$ flops. In our case, $\mathbf{U}^T \in \mathbb{R}^{d\times n}$ and $\mathbf{Y} \in \mathbb{R}^{n\times 1}$, and so multiplying the two costs $O(2\times n\times d\times 1) = O(2nd)$ flops and results in a $d \times 1$ matrix. Multiplying $\mathbf{D}_\lambda \in \mathbb{R}^{d\times d}$ and this $d \times 1$ matrix costs $O(2d^2)$ flops (assuming we do not take advantage of the diagonal structure of $\mathbf{D}_\lambda$) and results in a $d \times 1$ matrix. Multiplying $\mathbf{V} \in \mathbb{R}^{d\times d}$ by this $d \times 1$ matrix costs $O(2d^2)$ flops. Adding together all these steps, we have $O(nd^2 + 2nd + 2d^2 + 2d^2)$. Dropping all constant coefficients and only considering the highest-order polynomial, we conclude that the inefficient method costs $O(nd^2)$ flops.

A more efficient method notes that $\mathbf{D}_\lambda$ is the only part of $\mathbf{V}\mathbf{D}_\lambda\mathbf{U}^T\mathbf{y}$ that depends on $\lambda$ ($\mathbf{U}$ and $\mathbf{V}$ depend solely on the feature matrix $\mathbf{X}$, and $\mathbf{y}$ depends only on itself) and so we do not not need to recompute the SVD for every new value of $\lambda$. Assume that we have pre-calculated and cached $\mathbf{U}^T\mathbf{y}$ and $\mathbf{V}$. Modifying the $d$ non-zero values of $\mathbf{D}_\lambda$ to reflect our new value of $\lambda$ costs $O(d)$ flops. Computing new regression coefficients requires multiplying the new $\mathbf{D}_\lambda$ and $\mathbf{U}^T\mathbf{y}$, which costs $O(2d^2)$ flops, and then multiplying that result by $\mathbf{V}$, which also costs $O(2d^2)$ flops. Adding together these two steps, we have $O(d + 2d^2 + 2d^2)$. Dropping all constant coefficients and only considering the highest-order polynomial, we conclude that the efficient method costs $O(d^2)$ flops.

| | **Complexity** |
|---|---|
| **Inefficient method** | $O(nd^2)$ |
| **Efficient method** | $O(d^2)$ |

## Results

Below, I compare the performance of ridge and lasso regression on the homework dataset for different values of $\lambda$. For ridge regression, a regularization parameter value of 0.00390625 achieves the lowest test error (2.81855); for lasso regression, a regularization parameter of $4.46e^{-0.6}$ (i.e. the tested value closest to 0) achieves the lowest test error (2.85948). The ridge method achieves the better minimum test error.


Ridge
Test error
3.15
3.10
3.05
3.00
2.95
2.90
2.85
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
Regularization parameter
Lasso
Test error
3.3
3.2
3.1
3.0
2.9
0.0000
0.0005
0.0010
0.0015
Regularization parameter