We want to find function f from class $\mathcal{F}$, which we call the **hypothesis class**.

We choose a **loss function** to measure how well f does at the learning problem.

Risk or **expected loss** is:

$$R(f) = E(X,Y) \sim_P [L(Y, f(x))]$$

or

$$R(f) = E(X) \sim_P [L(X, f(x))]$$

We denote the optimal risk as r^* and the optimal function as f^*

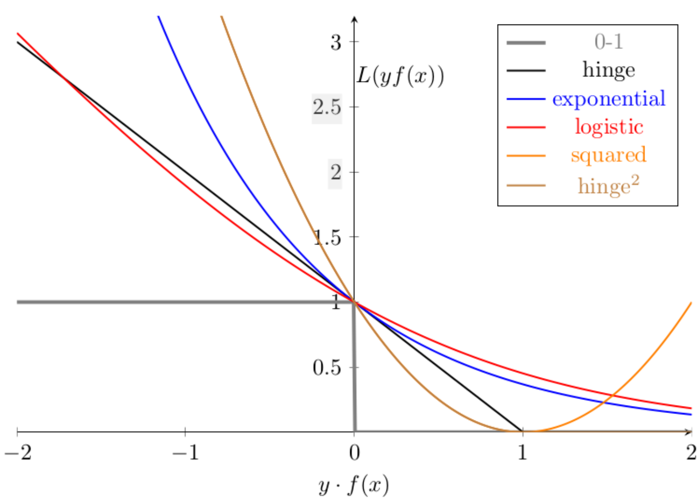
The Bayes classifier, $f^*(x) = argmax_y P(Y = y|X = x)$, is the theoretical optimum (at least for 0-1 loss). But, it isn't a practical scheme because we don't actually know the conditional distribution. Rather, we use it as a benchmark.

We have a host of **margin-based** loss functions (margin is $yf(\mathbf{X})$ for $\mathbf{X} \in \mathbb{R}, y \in -1, 1)$. The margin only takes positive values when the classification is correct. A margin-based loss function is: $L(y, f(x)) = L(yf(x))$. The simple 0-1 loss function can be written as $I(yf(x) < 0$ (we want to minimize this). We are interested in alternate loss functions that have two key properties:

- L is a convex function of the margin
- L upper bounds L_{0-1} (our above simple 0-1 loss function)

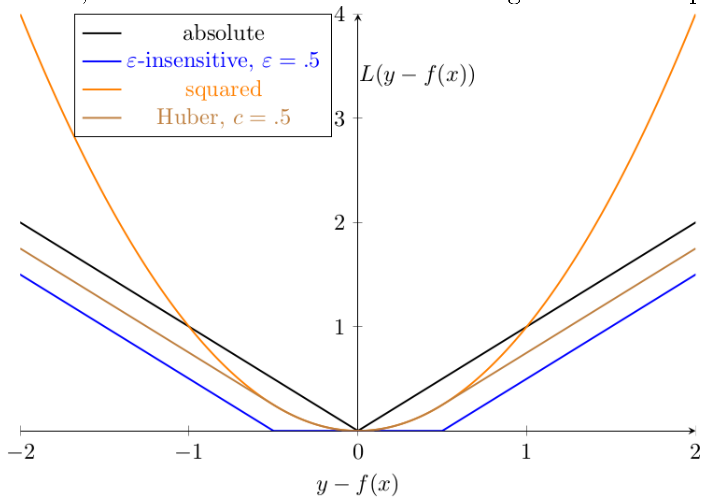
Some candidate functions (for classification) are below. All of them have the same sign as Bayes loss and so are called **classification-calibrated** or **surrogate** loss functions, in that minimizing them is equivalent to minimizing Bayesian loss.

Loss function	**Form**	**Notes**
Hinge	$\max\{0, 1 - yf(x)\}$	Tightest convex upper bound to 0-1; non-smooth
Exponential	$\exp(-yf(x))$	1-Lipschitz (robust)
Logistic	$\log_2(1 + \exp(-yf(x)))$	1-Lipschitz (robust)
Squared	$(1 - yf(x))^2$	Not recommended for class. (weird errors)
Squared hinge	$\max\{0, 1 - yf(x)\}^2$	Only 1x differentiable at 1



Loss function	**Form**
Squared	$(y - f(x))^2$
Absolute	$L_1(y, f(x)) = \|y - f(x)\|$
ϵ-insensitive	$\|y - f(x)\| I(\|y - f(x)\| \geq \epsilon)$
Huber	loooong

Now, let's switch from classification to regression. Some potential loss functions are:



We want minimize risk (expected loss) ($E_{(X,Y)\sim P[L(Y,f(X))]}$) over all possible f. But that's ambitious, so lets replace R by its empirical counterpart:

$$R_{\text{emp}}(f) = \frac{1}{n} \sum_{i=1} nL(Y_i, f(X_i))$$

By LLN, we know that for any **fixed** f, $R_{\text{emp}}(f) \to R(f)$ as $n \to \infty$, but we can't be assured that we have infinite n, nor is f fixed (we are considering lots of f), so this is a weak guarantee. Let $\mathcal{F}$ be the hypothesis class the minimum is taken over:

$$\min_{f \in \mathcal{F}} R_{\text{emp}}(f)$$

Denote empirical risk minimizer as $\hat{f}$. Denote minimizer of risk over the class $\mathcal{F}$ as $\bar{f}$. The difference between the minimum risk $\bar{R}$ over $\mathcal{F}$ and the minimum risk R^* is called **excess risk**. Reminder: $R(f^*)$ is theoretical optimum. Theorem:

$$R(\hat{f}) \leq R(\bar{f}) + 2\sup_{f \in \mathcal{F}} |R_{\text{emp}}(f) - R(f)|$$

$$= R(f^*) + \epsilon(\bar{f}) + 2\sup_{f \in \mathcal{F}} |R_{\text{emp}}(f) - R(f)|$$

$$= \text{intrinsicerror} + \text{approximationerror} + \text{estimationerror}$$

The latter two are bias and variance by another name. They are antagonists: decreasing one means increasing another.

You can fit any series of datapoints with interpolating polynomial of degree $n - 1$. You are guaranteed to minimize empirical risk, but it will generalize poorly to new data. Or, you can have some really general method, but it will have high empirical risk.

A good way to find the optimal balance is **regularization**, covered next class.