What is a kernel?

To classify, we need some measure of similarity. Let's define some function k that takes two arguments x and x' and returns a real number expressing their similarity:

$$k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

As an example, take the dot product.

$$k(\mathbf{x}, \mathbf{x}') = \langle \mathbf{x}, \mathbf{x}' \rangle = \sum x_i x_i'$$

Geometrically, the dot product expresses the cosine of the angle between $\mathbf{x}$ and $\mathbf{x}'$. The dot product also expresses the *length* of a vector, i.e. $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. Lastly, the *distance* between two vectors is computed as the length of the difference vector. So, being able to compute dot products amounts to being able to carry out all mathematical operations that can be characterized in terms of angles, lengths, and distances (cough cough, linear algebra and analytic geometry).

BUT, we cannot take for granted that our data actually lives in a dot product space. For example, it might be text strings. So, we need some transformation that takes possibly non-vectorial data x and maps it to the dot product space $\mathcal{H}$ (which need not be $\mathbb{R}^n$).

$$\Phi : \mathcal{X} \to \mathcal{H}$$

$$x \mapsto \mathbf{x} := \Phi(x)$$

We can cleverly pick our kernel function so that we do not actually need to carry out this mapping.

The "kernel trick"

Suppose that we have the feature map Φ that takes two-dimensional inputs and maps them to a three-dimensional space.

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^3$$

Specifically:

$$\Phi : [x_1, x_2] \to \left[x_1^2, x_2^2, \sqrt{2} x_1 x_2\right] \tag{1}$$

Now suppose that we will perform standard soft-margin SVM on the transformed data. In its dual form:

$$\max_{\alpha \in \mathbb{R}^n} \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_i \alpha_j Y_i Y_j \langle \Phi(X_i), \Phi(X_j) \rangle$$

Let's examine that inner product bit on the right in more detail.

$$\begin{aligned}
\langle \Phi(X_i), \Phi(X_j) \rangle &= \langle (X_{i1}^2, X_{i2}^2, \sqrt{2} X_{i1} X_{j2}), (X_{j1}^2, X_{j2}^2, \sqrt{2} X_{i1} X_{j2}) \rangle \\
&= X_{i1}^2 X_{j1}^2 + 2 X_{i1} X_{i_2} X_{j1} X_{j2} + X_{i2}^2 X_{j2}^2 \\
&= (X_{i1} X_{j1} + X_{j1} X_{j2})^2 \\
&= \langle X_i, X_j \rangle^2 \\
&:= k(x, x')
\end{aligned}$$

(i.e. we are defining our kernel function to be the inner product, squared). Now we rewrite our SVM:

$$\max_{\alpha \in \mathbb{R}^n} \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_i \alpha_j Y_i Y_j k(X_i, X_j)$$

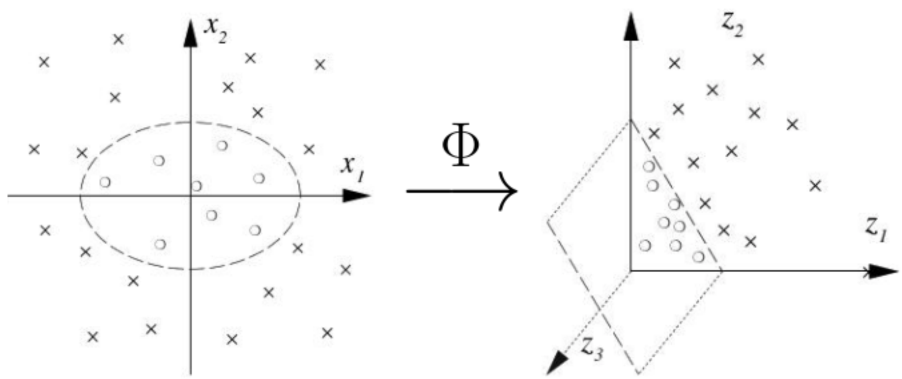
Note that we are operating directly on the untransformed X, i.e. without having to conduct the transformation Φ. We have defined k such that $k(x, x') = \langle \Phi(x), \Phi(x') \rangle$. This is the **kernel trick**. It works for any kernel function k that is symmetric and positive semi-definite. Some common such kernels are:

- Linear: $k(x, x') = \langle x, x' \rangle$
- Polynomial: $k(x, x') = \langle x, x' \rangle^\gamma$
- Gaussian (RBF): $k(x, x') = \exp(-\frac{||x-x'||_2^2}{2\sigma^2})$. This is the de facto standard for machine learning.

Why do we care?

The kernel trick, and kernel methods in general, have some nice value propositions.

The first is **applying linear methods to non-linear problems**. Choosing a kernel that lets us map between dimensions can allow us to apply linear methods when we otherwise wouldn't be able to. Consider the below classification problem. Clearly, the best decision boundary is an ellipse, which is non-linear, so we cannot apply something like SVM. But if we map $\mathbb{R}^2$ to $\mathbb{R}^3$ using (1), we can train a SVM to separate the classes.



Similarly, we can use kernels to apply machine learning methods to non-vectorial data such as text strings or k-grams or labelled graphs.

The second is **computational efficiency**. Because of the kernel trick (and some other stuff covered later), we can explore enormous features spaces for classification purposes without actually computing such spaces. Suppose we have $d = 100$ and want to consider up to 3rd degree interactions. This already leads to $d = 171,700$—not trivial to compute! Yet through the kernel trick, we do not actually need to conduct the laborious mapping Φ, we can operate directly on the input data through the kernel.

Reproducing kernel Hilbert spaces and the representer theorem
HOLY SHIT THIS IS ABSTRACT
As previously covered, a Hilbert space basically is one where we have the dot product.
Reproducing property:

$$\langle k(x, \bullet), k(x', \bullet) \rangle_{\mathcal{H}_k} := k(x, x')$$