Stochastic Gradient Descent

STAT 672 Project

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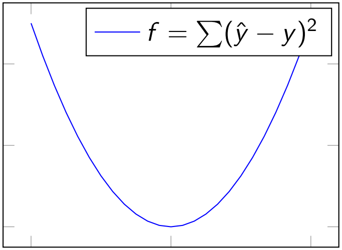
Optimization is everywhere, and sometimes is easy

Many statistical procedures involve minimizing or maximizing some function applied to data

In **parametric** statistics, we often make assumptions that make this optimization "nice"

- Example: in OLS, we do not need to try different values of $\hat{\beta}$ to see which minimizes the loss function, we (typically) can just evaluate $(X'X)^{-1}X'Y$



Other times, optimization is not so easy

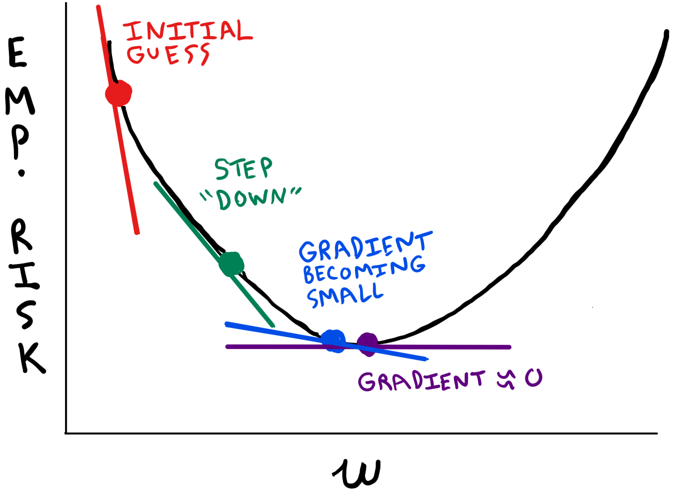
Suppose that we have a typical supervised classification problem

- Non-parametric: no assumptions about distribution of data
- Feature vector $\mathbf{X}_i$, label Y_i
- Want to find best prediction function f_w^* from class $\mathcal{F}$
- Optimization: pick weights $\mathbf{w}$ that minimize empirical risk according to some convex loss function $L(f_w(x), y)$

Our lack of assumptions requires a different approach to optimization

- Cannot analytically identify stationary point
- Need to numerically search for it

Gradient descent is an iterative search procedure



A more formal explanation

$\mathbf{w}_{t+1} = \mathbf{w}_t - \gamma \frac{1}{n} \sum_{i=1}^{n} \nabla_w L(f_w(\mathbf{X}_i), y_i)$

Stop if gradient $\leq \epsilon$

Step size γ can vary over time

Theoretical guarantees on speed of convergence ($\rho :=$ size of error):

- Version presented here: $-\log \rho \sim t$
- More optimized version: $-\log \log \rho \sim t$

But, big difference between:

- Speed $:=$ number of iterations
- Speed $:=$ time (clock on wall)

Batch gradient descent is computationally expensive

In "plain" (batch) gradient descent, for every step, we have to evaluate the gradient at every observation

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \gamma \frac{1}{n} \sum_{i=1}^{n} \nabla_w L(f_w(\mathbf{X}_i), y_i)$$

This becomes computationally intractable as n grows large

Knowing that the quality of our approximation gets linearly or quadratically better with t is not comforting if each t takes days to run

Stochastic gradient descent (SGD) takes less time

For each step, gradient is computed for a **single** randomly chosen observation *i*:

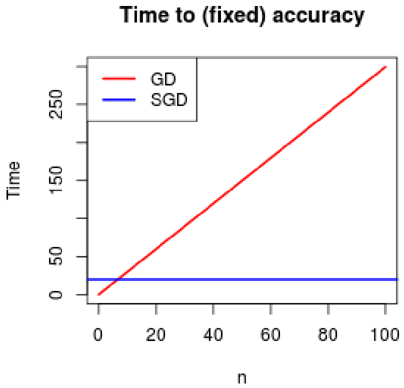
$$\mathbf{w}_{t+1} = \mathbf{w}_t - \gamma \nabla_w L(f_w(\mathbf{X}_i), y_i)$$

This simplification makes approximation much "noisier", and hence SGD requires more iterations

But, each iteration is faster and so SGD can reach a predefined level of risk or error in less time

SGD is particularly useful when n is large and computation time is important

	GD	**SGD**
Time to accuracy ρ	$n \log \frac{1}{\rho}$	$\frac{1}{\rho}$



SGD is widely used in industry

If a Silicon Valley press release uses any of the following phrases...

- "Neural networks"
- "Machine learning"
- "AI"

...SGD probably is involved. Example: Google's **AlphaGo** program.

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