Mathematics

Contents

Continuous

Functions

Functions

- We have seen different functions at A-level. For example:
  Polynomials
  Exponential functions
  Logarithmic functions
  Trigonometric functions
  Inverse trigonometric functions
- But what exactly is a function and how do we define a function?

Define a Function

- A proper definition of a function can be derived from a mapping process, i.e.:
- For numbers belonging to the (number group) D to be mapped to another (number group) R, a function *f* can be defined as:
- $y = f(x)$
- if any given number within D is mapped to definite values of y within R. D is termed the Domain of the function, whereas R is called the Range.

Function Properties

- Functions are sometimes bounded in Domain or Range:
  - $y = \sin x$
  - $y = \ln x$
- Functions, or a certain region within a function, can be monotonically increasing or monotonically decreasing
  - $y = x^2$
  - $y = e^x$

Function Properties – Cont'd

- Functions may be even or odd around a certain point
  - $y = x^2$
  - $y = \sin x$
- Some functions are cyclic (or periodic)
  - $y = \tan x$
- Functions may have their explicit or implicit inverse functions
  - $y = x^3$

Exercise

- There are three pairs of functions listed below. Are the two functions within each pair identical?

1. $f(x) = \ln x^2$
   - $f(x) = 2\ln x$
2. $f(x) = x$
   - $f(x) = \sqrt{x^2}$
3. $f(x) = \sqrt[3]{x^4 - x^3}$
   - $f(x) = x\sqrt[3]{x-1}$

Exercise

- Try to prove the following:
  - The sum of two even functions is an even function, while the sum of two odd functions is an odd function.
  - The product of two even functions is an even function, while the product of two odd functions is an even function. The product of an even function with an odd function is an odd function.

Exercise

- Define the following hyperbolic functions:
  - $\sinh x = \frac{e^x - e^{-x}}{2}$
  - $\cosh x = \frac{e^x + e^{-x}}{2}$
  - $\tanh x = \frac{\sinh x}{\cosh x}$
- What are the domain and range for each hyperbolic function? Are any of these functions even or odd?
- Plot these functions

Exercise

- For those definitions, try to prove the following equalities:
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
- $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
- $\cosh(x+y) = \cosh x \ \cosh y + \sinh x \ \sinh y$
- $\cosh(x-y) = \cosh x \ \cosh y \ - \sinh x \ \sinh y$

Exercise

- Suppose f is a non-zero function that satisfies $f(x+y)=f(x)f(y)$. First prove that $f(0)=1$. Then prove that $f(nx)=f(x)n$ for any $n\in N$.
- What happens is the “non- zero” condition is not given?

Limits

Function limits

- In general, for any function $y = f(x)$, if a constant A exists so that for any given positive ϵ, we can always find a positive δ so that for values of x within the range $0 < |x - x_0| < \delta$, we have the corresponding $f(x)$ satisfying the following inequality:
- $|f(x) - A| < \epsilon$
- then constant A is termed the limit of the function $y = f(x)$ when x tends to x_0. We can use the following shorthand notation to express this limit relationship:
- $\lim_{x \to x_0} f(x) = A$

Exercise

- $\lim_{x \to x_0} C = C$
- $\lim_{x \to x_0} x = x_0$
- $\lim_{x \to x_0} \frac{1}{x-1} = \infty$

Derivations of Limit

1. If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = 0$,
   then $\lim_{x\to x_0}[f(x)+g(x)] = 0$, $\lim_{x\to x_0} f(x)g(x) = 0$ and $\lim_{x\to x_0} Cf(x) = 0$
2. If $|f(x)| < C$ and $\lim_{x\to x_0} g(x) = 0$,
   then $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \infty$ and $\lim_{x\to x_0} f(x)g(x) = 0$
3. If $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B$,
   then $\lim_{x\to x_0}[f(x) \pm g(x))] = A \pm B$ and $\lim_{x\to x_0} f(x)g(x) = A \cdot B$
4. $\lim_{x\to x_0}[Cf(x)] = C\lim_{x\to x_0} f(x)$

Exercise

- Prove the following function limits.
- $\lim_{x \to \infty} \frac{1+x^3}{2x^3} = \frac{1}{2}$
- $\lim_{x \to \infty} \frac{3x^3+4x^2+2}{7x^3+5x^2-3} = \frac{3}{7}$
- $\lim_{x \to \infty} \frac{3x^2-2x-1}{2x^3-x^2+5} = 0$
- $\lim_{x \to \infty} \frac{x^2+1}{x-1} = x$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{\tan x}{x} = 1$
- $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

More Derivations

- If $\delta \rightarrow 0$, and $\varepsilon \rightarrow 0$, and both δ and ε are of the same order, then we have the following:

1. $C_1\delta + C_2\epsilon \rightarrow 0$, where C_1 and C_2 are arbitrary finite constants
2. $\frac{\delta^M}{\epsilon^N} \rightarrow \begin{cases} \infty \text{ if } M < N \\ 0 \text{ if } M > N \end{cases}$
3. $\frac{x^M}{x^N} \rightarrow \begin{cases} \infty \text{ if } M < N \\ 0 \text{ if } M > N \end{cases} \text{ (as } x \rightarrow 0)$

More Limits

- There are two important limits that need memorizing:
- $\lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^n = e$
- $\lim_{t \to 0} (1 + t)^{\frac{1}{t}} = e$

Exercise

- Try to prove:
- $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$

Graph Sketching

Graph Sketching

- Sketch the following functions:
- $y = \frac{x^2+5x+1}{x^2}$
- $y = e^{\sin x}$

Graph Sketching

- Question: What's the sequence of steps?

Graph Sketching

1. Domain & Range
2. Analyse the function:
   a) Can the graph be simplified into a more recognisable and easier to sketch function?
   b) Can you split the function into two functions and then add/subtract/multiply them?
   c) Odd, even or periodic?
   d) Asymptotes?
3. Intersection points
4. Turning Points and their nature
5. Monotonicity
6. Special Points

Homework

- $y = 2x^3 - 15x^2 + 36x - 25$
- $y = \frac{x^2-4x+4}{x^2+x-6}$
- $y = 3x^5 - 5x^3$
- $y = \frac{x^2}{x-1}$
- $y = \frac{x^3}{x-1}$
- $y = \frac{x^3-4x+1}{x-3}$
- $y = \frac{x+1}{x-1}$
- $y = \frac{x^3-4x+1}{x^2-2x-1}$
- $y = \sqrt{1-x^2}$
- $y = \frac{1}{\sqrt{1-x^2}}$
- $y = \frac{(x^2+1)}{(x^2-9)}$
- $x^2 + y^2 = 6y$
- $y = \frac{x^2+4x-1}{3x^2+2}$
- $(2y-2)^2 + (2x-2)^2 = 2$
- $y = x^x$

Homework

- $y = sinx + cosx$
- $y = sinx + sin2x$
- $y = xsinx$
- $y = e^x \sin x$
- $y = \cos(\cos x)$
- $y = \sin\frac{1}{x}$
- $y = \sin e^x$
- $y = \cos(\sin \mathrm{x})$
- $y = \ln(\sin x)$
- $y = \cos(x^2)$
- $y = (\cos x)^2$
- $y = \frac{\sin x}{x}$
- $y = \frac{x}{\sin x}$
- $y = \frac{\ln x}{x}$
- $y = \frac{\cos x}{x+\frac{\pi}{2}}$
- $y = e^{e^x}$

Calculus - Differentiation

Taylor's Expansion (Taylor's Series)

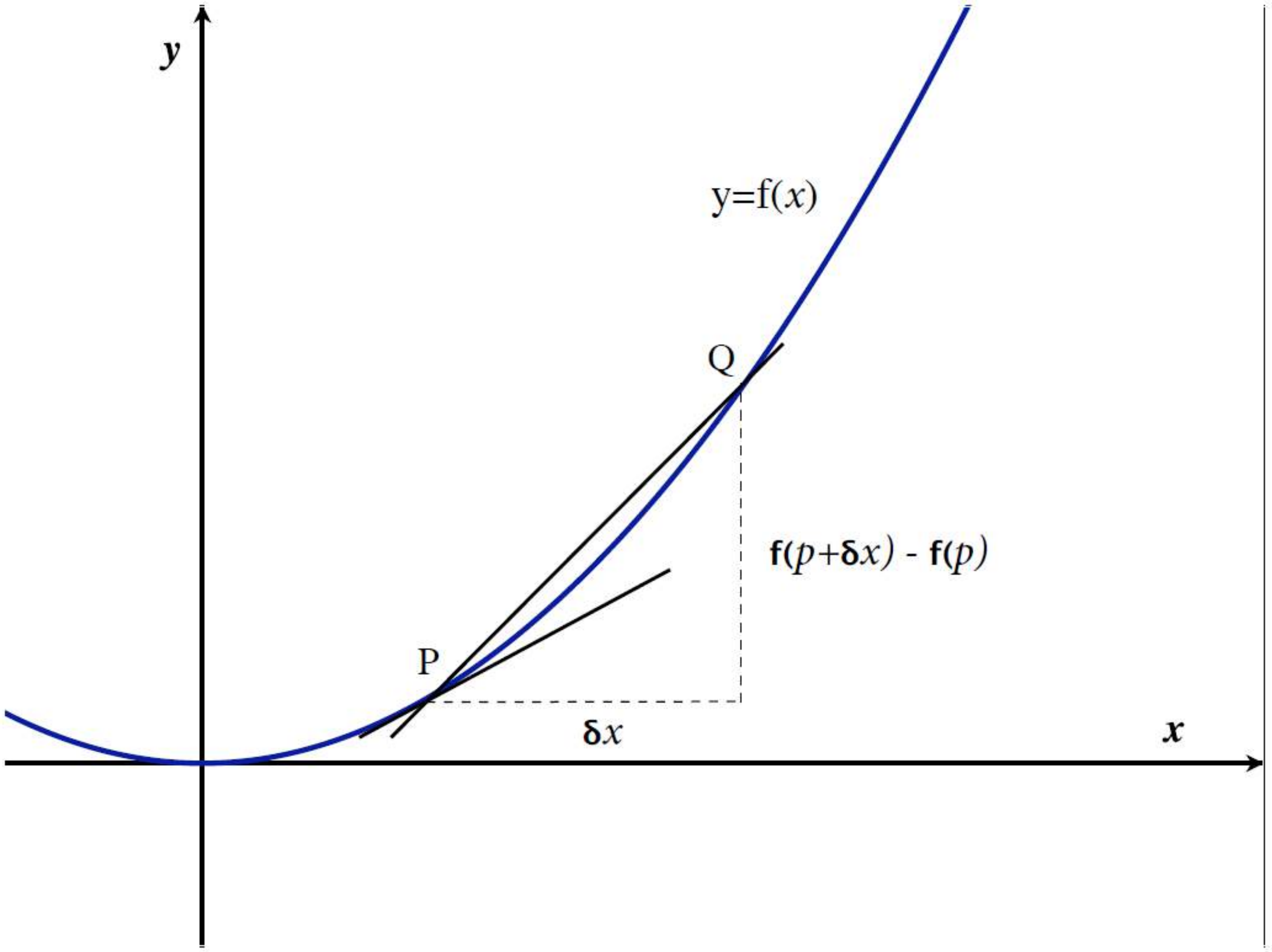
- $f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

Taylor's Series of Some Functions

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$
- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ for $|x| < 1$

Differentiation

- Fundamental of Differentiation: (The First Principle)
- $\mathbf{y}' = \frac{\mathbf{dy}}{\mathbf{dx}} = \frac{df(x)}{dx} = f'(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta)-f(x)}{\Delta}$



y
y=f(x)
Q
f(p+δx) - f(p)
P
δx
x

Exercise

- Use the first principle to obtain the derivatives of the following functions
- $y = x^n$
- $y = sin\, x$
- $y = a^x$
- $y = log_a\, x$

Important Rules

- $y = u(x) + v(x), \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ (Addition rule)
- $y = u(x)\, v(x), \frac{dy}{dx} = \frac{du}{dx} v + u \frac{dv}{dx}$ (Multiplication rule)
- $y = \frac{u(x)}{v(x)}, \frac{dy}{dx} = \frac{\frac{du}{dx} v - u \frac{dv}{dx}}{v^2}$ (Division rule)

Proof

Exercise

- Find the First Derivatives of the Following Functions
- $y = tan\,x$
- $y = sec\,x = \frac{1}{\cos x}$

Chain Rule

- $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Proof

Exercise

- Find the First Derivatives of the Following Functions
- $y = e^{x^3}$
- $y = ln\, sin\, x$
- $y = ln\, cos(e^x)$
- $y = e^{sin\frac{1}{x}}$

Differentiating Inverse Functions

- $x = f(y), \frac{dy}{dx} = ?$

Proof

- From definition:
- $\delta y = f^{-1}(x + \delta x) - f^{-1}(x)$
- $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta y \to 0} \frac{1}{\frac{\delta x}{\delta y}} = \frac{1}{f'(y)}$

Exercise

- Find the First Derivatives of the Following Functions
- $x = tan\, y$
- $x = a^y$

(1) $(C)' = 0$,

(2) $(x^{\mu})' = \mu x^{\mu-1}$,

(3) $(\sin x)' = \cos x$,

(4) $(\cos x)' = -\sin x$,

(5) $(\tan x)' = \sec^2 x$,

(6) $(\cot x)' = -\csc^2 x$,

(7) $(\sec x)' = \sec x \tan x$,

(8) $(\csc x)' = -\csc x \cot x$,

(9) $(a^x)' = a^x \ln a$,

(10) $(e^x)' = e^x$,

(11) $(\log_a x)' = \dfrac{1}{x \ln a}$,

(12) $(\ln x)' = \dfrac{1}{x}$,

(13) $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$,

(14) $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$,

(15) $(\arctan x)' = \dfrac{1}{1+x^2}$,

(16) $(\operatorname{arccot} x)' = -\dfrac{1}{1+x^2}$.

17) $(\sinh x)' = \cosh x$

18) $(\cosh x)' = \sinh x$

19) $(\tanh x)' = \operatorname{sech}^2 x$

Differentiating Implicit Functions

- $e^y + xy - e = 0, \frac{dy}{dx} = ?$

Exercise

- Find the value of $\frac{dy}{dx}$ for the function $y^5 + 2y - x - 3x^7 = 0$ at $x = 0$.
- Find the equation of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at $\left(2, \frac{3}{2}\sqrt{3}\right)$

Another Technique:

- Substitution
- $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$
- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2} \cdot \frac{1}{\frac{dx}{dt}} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$

Exercise

- Find $\frac{d^2y}{dx^2}$ for the function defined by: $\begin{cases} x = a\,(t - \sin t) \\ y = a\,(1 - \cos t) \end{cases}$

The Logarithmic Method in Differentiation

- $y = a^x$

- $\ln y = \ln f(x)$
- $d\frac{(\ln y)}{dx} = \frac{d \ln[f(x)]}{dx}$
- $\frac{1}{y}\frac{dy}{dx} = \frac{d \ln[f(x)]}{dx}$
- $\frac{dy}{dx} = y \cdot \frac{d \ln[f(x)]}{dx}$

Exercise

- Find the First Derivatives of the Following Functions
- $y = x^x$
- $y = x^{\sin x}$
- $y = x^{x^x}$
- $y = \frac{\sqrt{x+2}\,(3-x)^4}{(x+1)^5}$

L'Hôpital's rule (L'Hospital's rule)

- $f(x) = \frac{g(x)}{h(x)}$
- When $g(x_0) = h(x_0) = 0 \; or \; \infty$ and $g(x)$ and $h(x_0)$ are differentiable around x_0 and $h'(x_0) \neq 0$
- $f(x_0) = \frac{g(x_0)}{h(x_0)} = \frac{g'(x_0)}{h'(x_0)}$

- $\lim_{x \to 0} \sin(x) = x$