Logic

Propositional Logic

What is Propositional Logic?

What is a Proposition?

- A proposition is a meaningful declarative sentence that may be true or false in a situation.

Examples

- I am hungry
- Socrates is mortal
- 我在RBC工作
- It is raining and my head is wet
- If I wear a hat and it is raining then my head stays dry

- ~~Do you like Chinese food?~~
- ~~Let's go!~~
- ~~Don't tell me that sad story.~~
- ~~This statement is False.~~

Atomic Propositions

- An atomic proposition is a proposition with no logical connectives in it.

Examples

- I am hungry
- Socrates is mortal
- 我在RBC工作

- ~~It is raining and my head is wet~~
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Symbolic representation

- Atomic propositions denoted by letters/identifiers
- Propositional connectives written in symbols

Symbolic representation Cont'd

- Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite set of propositional variables (atomic propositions).
- Formulas of propositional logic are inductively defined as follows:
  - *true* and *false* are formulas
  - Every propositional variable x_i is a formula
  - If F is a formula, then ¬F is a formula.
  - If F and G are formulas, then $(F \wedge G)$ and $(F \vee G)$ are formulas.

Logical Connectives

not	$\neg P$	Negation
… and …	$P \wedge Q$	Conjunction
… or …	$P \vee Q$	Disjunction
if … then …	$P \rightarrow Q$	Implication
… if and only if …	$P \leftrightarrow Q$	Bi-Implication
	$P \oplus Q$	Exclusive-or

Derived connectives

- $(F_1 \rightarrow F_2) := (\neg F_1 \vee F_2)$
- $(F_1 \leftrightarrow F_2) := (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$
- $(F_1 \oplus F_2) := (F_1 \wedge \neg F_2) \vee (\neg F_1 \wedge F_2)$
- $\bigvee_{i=1}^{n} F_i = F_1 \vee F_2 \vee F_3 \vee \ldots \vee F_n$
- $\bigwedge_{i=1}^{n} F_i = F_1 \wedge F_2 \wedge F_3 \wedge \ldots \wedge F_n$

Boolean Representations

- We can evaluate the (Boolean) value of a logical formula by evaluate each sub-formula.

Formalising natural language

- A device consists of a thermostat, a pump, and a warning light. Suppose we are told the following four facts about the pump:
  - The thermostat or the pump (or both) are broken.
  - If the thermostat is broken then the pump is also broken.
  - If the pump is broken and the warning light is on then the thermostat is not broken.
  - The warning light is on.
- Is it possible for all four to be true at the same time?

Express in a formula

- $F := (t \vee p) \wedge (t \rightarrow p) \wedge ((p \wedge w) \rightarrow \neg t) \wedge w$

Uses of Truth Tables

t	p	w	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

	2		5		1		9	
8			2		3			6
	3			6			7	
		1				6		
5	4						1	9
		2				7		
	9			3			8	
2			8		4			7
	1		9		7		6	

Equivalence

- How do we know that two formulas are equivalent?

Equivalence

- If for all different ways to assign Boolean values to all propositional variables in both formulas, the whole formulas have the same Boolean values.
- In another word, two formulas have the same truth tables.

Boolean Algebra - Axioms

- Idempotence
  - $F \wedge F \equiv F$
  - $F \vee F \equiv F$
- Commutativity
  - $F \wedge G \equiv G \wedge F$
  - $F \vee G \equiv G \vee F$
- Associativity
  - $(F \wedge G) \wedge H \equiv F \wedge (G \wedge H)$
  - $(F \vee G) \vee H \equiv F \vee (G \vee H)$
- Absorption
  - $F \wedge (F \vee G) \equiv F$
  - $F \vee (F \wedge G) \equiv F$
- Distributivity
  - $F \wedge (G \vee H) \equiv (F \wedge G) \vee (F \wedge H)$
  - $F \vee (G \wedge H) \equiv (F \vee G) \wedge (F \vee H)$

Boolean Algebra - Axioms

- Double negation
  - $\neg\neg F \equiv F$
- De Morgan's laws
  - $\neg(F \land G) \equiv (\neg F \lor \neg G)$
  - $\neg(F \lor G) \equiv (\neg F \land \neg G)$
- Complementation
  - $F \lor \neg F \equiv \mathit{true}$
  - $F \land \neg F \equiv \mathit{false}$
- Zero Laws
  - $F \lor \mathit{true} \equiv \mathit{true}$
  - $F \land \mathit{false} \equiv \mathit{false}$
- Identity Laws
  - $F \lor \mathit{false} \equiv F$
  - $F \land \mathit{true} \equiv F$

Exercise

- Prove the following equivalence:
- $(P \vee (Q \vee R) \wedge (R \vee \neg P)) \equiv R \vee (\neg P \wedge Q)$

First-Order Logic

Thinking

- What is the limitation of propositional logic?

Limitation of Propositional Logic

- Can only reason about true or false
- Atomic formulas have no internal structure
- Impossible to express “real” mathematical statements

Example

- Every natural number x is either odd or even.

What is First-Order Logic

First Order Logic (Predicate Logic)

- First Order Logic consists of:
- Objects (Constants): Tony, VA, the United Kingdom, fish and chips…
- Functions: The_father_of (), The_capital_of(), The_most_handsome_teacher_in()
  - functions return objects
- Predicates (Relations): is_Male(), is_a_teacher_of(,), whose_famous_food_is(,), likes(,)
  - Note: Predicates typically correspond to verbs
- Connectives: ¬, ∧, ∨
- Quantifiers:
  - Universal: ∀x: (is_Man(x)) is_Mortal(x))
  - Existential: ∃y: (is_Father(y, fred))

Predicates

- In traditional grammar, a predicate is one of the two main parts of a sentence the other being the subject, which the predicate modifies.
- "John is tall" John acts as the subject, and is tall acts as the predicate.
- The predicate is much like a verb phrase.
- In linguistic semantics a predicate is an expression that can be true of something.

Points to remember

- The main connective for universal quantifier ∀ is implication →
- The main connective for existential quantifier ∃ is and ∧

Properties of Quantifiers

- In universal quantifier, $\forall x \forall y$ is similar to $\forall y \forall x$
- In Existential quantifier, $\exists x \exists y$ is similar to $\exists y \exists x$
- $\exists x \forall y$ is not similar to $\forall y \exists x$

Negation of Quantifiers

- $\neg \forall x \mathrm{P}(x) \Leftrightarrow \exists x \neg \mathrm{P}(x)$
- $\neg \exists x \mathrm{P}(x) \Leftrightarrow \forall x \neg \mathrm{P}(x)$

Examples

- All birds fly

Examples

- Every man respects his parent

Examples

- Some boys play football

Examples

- Not all students like both Mathematics and Science

Examples

- Only one student failed in Mathematics

Types of formal logic

- Propositional logic
  - Propositions are interpreted as true or false
  - Infer truth of new propositions
- First order logic
  - Contains predicates, quantifiers and variables
    - E.g. Philosopher(a) → Scholar(a)
    - $\forall x$, King(x) ∧ Greedy (x) → Evil (x)
  - Variables range over individuals (domain of discourse)

Types of formal logic

- Second order logic
  - Quantify over predicates and over sets of variables
- Other logics
  - Temporal logic
    - Truths and relationships change and depend on time
  - Fuzzy logic
    - Uncertainty, contradictions

Proof

Proofs

- *Direct proof (Introduction)*
- *Proof by contrapositive*
- *Proof by Contradiction*
- *Proof by Mathematical Induction*
- *Proof of Strong Induction*

Direct Proof (Introduction)

- In mathematics and logic, a direct proof is a way of showing the truth or falsehood of a given statement by a straightforward combination of established facts, usually axioms, existing lemmas and theorems, without making any further assumptions.
- $P \to Q$
- $P \to P_1 \to P_2 \to \cdots \to P_n \to Q$

Notice

- Make sure whether you are using implication or bi-implication

Example

- The square of an odd number is odd

Proof by contrapositive

- Assume $\neg Q$ and show $\neg P$
- $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$

Examples

- If x and y are two integers whose product is even, then at least one of the two must be even.
- If x and y are two integers whose product is odd, then both must be odd.
- If a and b are real numbers such that the product a b is an irrational number, then either a or b must be an irrational number

Proof by Contradiction

- Assume P and $\neg Q$, then derive a contradiction

Examples

- Prove that $\sqrt{2}$ is irrational.
- No least positive rational number.
- There are infinitely many prime numbers.
- There are no positive integer solutions to the equation $x^2 - y^2 = 1$

Mathematical Induction

- P(n) is a mathematical statement where a natural number n is involved. If we want to proof P is satisfied for all natural numbers, we need to proof the following two statements:
  - i) P(0) is true
  - ii) P(m+1) is true whenever P(m) is true, i.e. P(m) is true implies that P(m+1) is true.
- Then P(n) is true for all natural numbers n.

Examples

- The sum of the first n natural numbers is $n(n+1)/2$
- Assume an infinite supply of 4- and 5-dollar coins. For any amount n > 11, there is a combination of 4- and 5-dollar coins to form an amount n.

Proof of Strong Induction

- one proves the statement $P(m + 1)$ under the assumption that $P(n)$ holds for **all** natural n less than $m + 1$;

Example of error in the inductive step

- All horses are of the same color

Example of error in induction

- Everyone is bald.
- No one is bald.

Exercises

- Prove that there are no rational number solutions to the equation $x^3 + x + 1 = 0$.
- Prove $4^n - 1$ is divisible by 3.
- Prove that $2 + 2^2 + 2^3 + 2^4 + \ldots + 2^n = 2^{n+1} - 2$ for $n \geq 1$.
- Prove that $4n < 2^n$ for $n \geq 5$.
- Prove that $(1\times2) + (2\times3) + (3\times4) + \ldots + (n)(n+1) = \frac{(n)(n+1)(n+2)}{3}$ for $n \geq 1$.