# Differentiating Implicit Functions

- $e^y + xy - e = 0, \frac{dy}{dx} = ?$

# Exercise

- Find the value of $\frac{dy}{dx}$ for the function $y^5 + 2y - x - 3x^7 = 0$ at $x = 0$.
- Find the equation of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at $\left(2, \frac{3}{2}\sqrt{3}\right)$

# Another Technique:

- Substitution
- $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$
- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2} \cdot \frac{1}{\frac{dx}{dt}} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$

# Exercise

- Find $\frac{d^2y}{dx^2}$ for the function defined by: $\begin{cases} x = a\,(t - \sin t) \\ y = a\,(1 - \cos t) \end{cases}$

# The Logarithmic Method in Differentiation

- $y = a^x$

- $\ln y = \ln f(x)$
- $d\frac{(\ln y)}{dx} = \frac{d\ln[f(x)]}{dx}$
- $\frac{1}{y}\frac{dy}{dx} = \frac{d\ln[f(x)]}{dx}$
- $\frac{dy}{dx} = y \cdot \frac{d\ln[f(x)]}{dx}$

# Exercise

- Find the First Derivatives of the Following Functions
- $y = x^x$
- $y = x^{\sin x}$
- $y = x^{x^x}$
- $y = \frac{\sqrt{x+2}\,(3-x)^4}{(x+1)^5}$

# L'Hôpital's rule (L'Hospital's rule)

- $f(x) = \frac{g(x)}{h(x)}$
- When $g(x_0) = h(x_0) = 0\ or\ \infty$ and $g(x)$ and $h(x_0)$ are differentiable around $x_0$ and $h'(x_0) \neq 0$
- $f(x_0) = \frac{g(x_0)}{h(x_0)} = \frac{g'(x_0)}{h'(x_0)}$

- $\lim_{x \to 0} \sin(x) = x$

# Integration

# Fundamentals of Indefinite Integration

- Integration is the inverse of differentiation
- If a function F(x) is differentiable and the differentiated product is f(x), we call F(x) the original function of f(x). The written expression is:
- $F(x) = \int f(x)\, dx$

1. $\int k\,dx = kx + C$
2. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int \frac{dx}{1+x^2}dx = \arctan x + C$
5. $\int \frac{dx}{\sqrt{1-x^2}}dx = \arcsin x + C$
6. $\int \cos x\,dx = \sin x + C$
7. $\int \sin x\,dx = -\cos x + C$
8. $\int \frac{dx}{\cos^2 x}dx = \tan x + C$
9. $\int \frac{dx}{\sin^2 x}dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = -\sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int e^x\,dx = e^x + C$
13. $\int a^x\,dx = \frac{a^x}{\ln a} + C$

# Exercise

- Derive the following integration results.
- $\int \sqrt{x}(x^2 - 5)dx$
- $\int \frac{(x-1)^3}{x^2} dx$
- $\int tan^2 \, x \, dx$
- $\int sin^2 \frac{x}{2} dx$
- $\int \frac{2x^4+x^2+3}{x^2+1}$

# Techniques in Indefinite Integration/Substitution

- $\int g(x)dx = \int f[\varphi(x)]\varphi'(x)dx = \left[\int f(u)du\right]_{u=\varphi(x)}$

# Exercise

Derive the following integration results:

1. $\int \frac{x^2}{(x+2)^3} dx$
2. $\int 2xe^{x^2} dx$
3. $\int x\sqrt{1-x^2}\, dx$
4. $\int \frac{1}{a^2+x^2} dx$
5. $\int \frac{1}{\sqrt{a^2-x^2}} dx$
6. $\int \frac{1}{x^2-a^2} dx$
7. $\int \frac{dx}{x(1+2\ ln\ x)}$
8. $\int tan\ x\ dx$
9. $\int sin^3\ x\ dx$
10. $\int sin^2\ x\ cos^5\ x\ dx$
11. $\int sec^6\ x\ dx$
12. $\int tan^5\ x\ sec^3\ x\ dx$
13. $\int csc\ x\ dx$

14. $\int \sec x \, dx$

15. $\int \cos 3x \cos 2x \, dx$

16. $\int \sqrt{a^2 - x^2} dx$

17. $\int \frac{x}{\sqrt{x^2+a^2}} dx$

18. $\int \frac{dx}{\sqrt{x^2-a^2}}$

19. $\int \frac{\sqrt{a^2-x^2}}{x^4} dx$

- By making use of substitution, we can add another 12 formulae to the previous list.

14. $\int \tan x \, dx = -\ln|\cos x| + C$
15. $\int \cot x \, dx = \ln|\sin x| + C$
16. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$
17. $\int \csc x \, dx = \ln|\csc x - \cot x| + C$
18. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$
19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$
20. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C$
21. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$
22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C$
23. $\int \sinh x \, dx = \cosh x + C$
24. $\int \cosh x \, dx = \sinh x + C$

# Exercise

- Derive the following integration results:
- $\int \frac{1}{x^2-2x+2} dx$
- $\int \frac{1}{x^2-2ax+b} dx$
- $\int sin^4 x \, dx$

# Additional Questions:

- $\int_{-\infty}^{\infty} e^{-x^2}\,dx$
- $\int_{0}^{\infty} e^{-\frac{x^2}{2}}\,dx$
- $\int \frac{1}{1+\sin x}\,dx$

# Integration by Parts

- If functions $u = u(x)$ and $v = v(x)$ are continuously differentiable, then we should have the following differentiation result:
- $\frac{d(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$
- $u\frac{dv}{dx} = \frac{d(uv)}{dx} - \frac{du}{dx}v$
- Integrate both sides gives:
- $\int \left(u\frac{dv}{dx}\right) dx = uv - \int \frac{du}{dx} v\, dx$
- A shorthand version of this equation is:
- $\int udv = uv - \int v\, du$

# Exercise

- Derive the following integration results:
- $\int x \cos x \, dx$
- $\int x \, e^x dx$
- $\int x^2 \, e^x dx$
- $\int x \ln x \, dx$
- $\int \arccos x \, dx$
- $\int x \arctan x \, dx$
- $\int e^x \sin x \, dx$
- $\int sec^3 \, x$
- $\int e^{\sqrt{x}} \, dx$

# Integrating Rational Functions

- A rational function can be expressed as the ratio between two polynomials: $f(x) = \frac{P(x)}{Q(x)}$. In general, when rational functions are to be integrated, it is always a good idea to disintegrate the original function into partial fractions, i.e.
- $f(x) = \frac{P(x)}{Q(x)} = \sum \frac{a_n}{h_n(x)}$
- Where $h_n(x)$ are polynomials that are factors of $Q(x)$.

# Exercise

- Derive the following integration results:
- $\int \frac{x+1}{x^2-5x+6}\,dx$
- $\int \frac{x+2}{(2x+1)(x^2+x+1)}dx$
- $\int \frac{x-3}{(x-1)(x^2-1)}dx$
- $\int \frac{1+\sin x}{\sin x(1+\cos x)}dx$
- $\int \frac{\sqrt{x-1}}{x}dx$
- $\int \frac{dx}{1+\sqrt[3]{x+2}}$
- $\int \frac{dx}{(1+\sqrt[3]{x})\sqrt{x}}$