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Computing Galois groups of Fano problems

Fano Problems

A Fano problem is the problem of enumerating linear spaces of a fixed dimension on a complete intersection.

- Let $X_F \subseteq \mathbb{P}^n$ be the complete intersection defined by homogeneous polynomials $F = (f_1, \dots, f_s)$ with $\deg f_i = d_i$. Write $d_\bullet = (d_1, \dots, d_s)$.
- The Fano scheme $V_r(X_F)$ is the subscheme of $\mathbb{G}(r, \mathbb{P}^n)$ of r-planes that lie on X_F.
- When $V_r(X_F)$ is finite, this is a Fano problem of type $(r, n, d_\bullet)$.

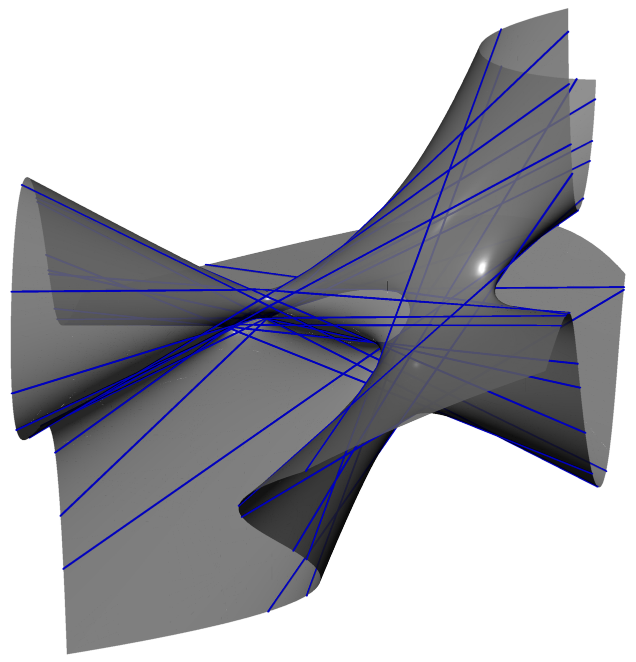


Figure: 27 lines on a cubic surface

Theorem (Debarre/Manivel)

The combinatorial data $(r, n, d_\bullet)$ determines a Fano problem when $n - s - 2r \geq 0$ and

$$(r+1)(n-r) - \sum_{i=1}^{s} \binom{d_i + r}{r} = 0.$$

r	n	$d_\bullet$	$N(r, n, d_\bullet)$
1	4	$(2, 2)$	16
1	3	(3)	27
2	6	$(2, 2)$	64
3	8	$(2, 2)$	256
1	7	$(2, 2, 2, 2)$	512
1	6	$(2, 2, 3)$	720

Figure: All Fano problems with ≤ 1000 solutions

Galois groups of Fano problems

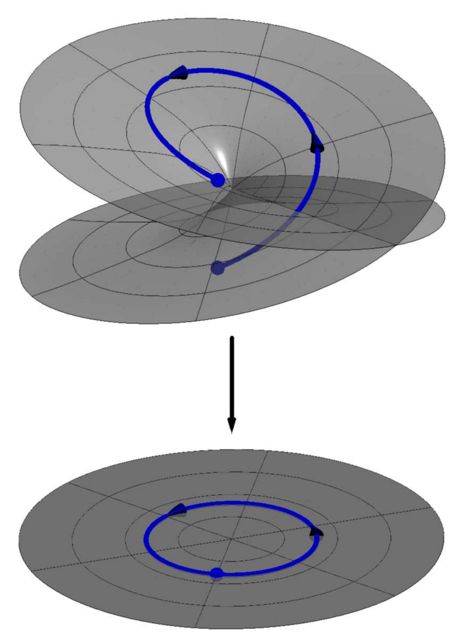
Write $\mathbb{C}^{(r,n,d_\bullet)}$ for the parameter space of homogeneous forms $F = (f_1, \dots, f_s)$ in $n+1$ variables of degrees $d_\bullet = (d_1, \dots, d_s)$. Consider the incidence correspondence.

$$\Gamma_{(r,n,d_\bullet)} = \{(F, \ell) \in \mathbb{C}^{(r,n,d_\bullet)} \times \mathbb{G}(r, \mathbb{P}^n) : \ell \in V_r(X_F)\}$$

$$\begin{tikzcd} \Gamma_{(r,n,d_\bullet)} \arrow[d, "\pi_{(r,n,d_\bullet)}"'] \arrow[dr] & \\ \mathbb{C}^{(r,n,d_\bullet)} & \mathbb{G}(r, \mathbb{P}^n) \end{tikzcd}$$

The Galois group, $\mathcal{G}_{(r,n,d_\bullet)}$, of the Fano problem determined by $(r, n, d_\bullet)$ is the monodromy group of $\pi_{(r,n,d_\bullet)}$.

- $\mathcal{G}_{(r,n,d_\bullet)}$ is transitive
- $\mathcal{G}_{(r,n,d_\bullet)}$ is a Galois group in the algebraic sense.



Theorem (Jordan, Harris, Hashimoto, Kadets)

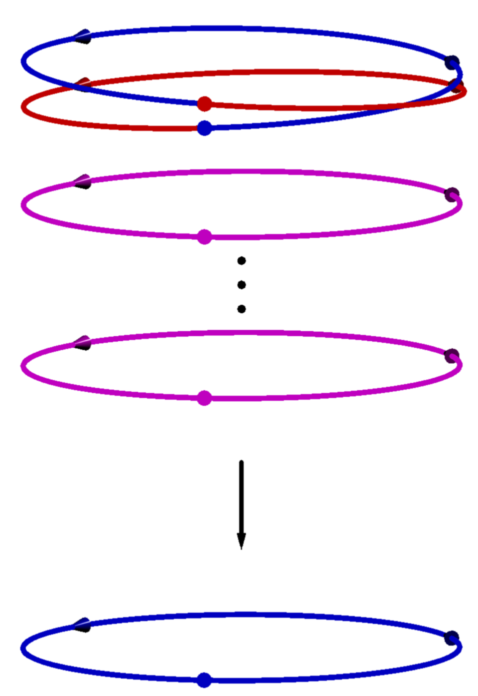
The following is known:

- [J],[Har] $\mathcal{G}_{(1,3,(3))} = E_6$.
- [Has,K] $\mathcal{G}_{(r,2r+2,(2,2))} = D_{2r+3}$.
- [Has,K] All other $\mathcal{G}_{(r,n,d_\bullet)}$ contain the alternating group.

Finding a simple transposition

Find $F \in \mathbb{C}^{(r,n,d_\bullet)}$ such that $V_r(X_F)$ is one point of multiplicity 2 and $N(r, n, d_\bullet) - 2$ smooth points.

- Heuristically choose $F \in \mathbb{C}^{(r,n,d_\bullet)}$.
- Show $V_r(X_F)$ contains a point of multiplicity 2 by exact computation.
- Show there are $N(r, n, d_\bullet) - 2$ smooth points by numerical certification.



Choose F by prescribing a subscheme of $V_r(X_F)$.

- Fix an r-plane $\ell \in \mathbb{G}(r, \mathbb{P}^n)$ and a tangent vector $v \in T_\ell \mathbb{G}(r, \mathbb{P}^n)$.
- Choose general F so that $\ell \in V_r(X_F)$ and $v \in T_\ell V_r(X_F)$.

When ℓ and v are chosen exactly, F can be chosen exactly as well.

- [Shub] simple double points are isolated.
- The remaining $N(r, n, d_\bullet) - 2$ solutions can be enumerated and isolated by certification.
  - α-theory
  - interval arithmetic

Results and timings

The reported timings for verifying the simple double point and certifying the remaining solutions is given below. (alphaCertified,HomotopyContinuation.jl)

r	n	$d_\bullet$	$N(r, n, d_\bullet)$	alCer (h)	HomCo (s)
1	7	$(2, 2, 2, 2)$	512	2.66	.61
1	6	$(2, 2, 3)$	720	2.88	.87
2	8	$(2, 2, 2)$	1024	27.32	1.57
1	5	(3,3)	1053	2.69	.32
1	5	(2,4)	1280	6.09	.73
1	10	(2,2,2,2,2,2)	20480	-	15.44
1	9	(2,2,2,2,3)	27648	-	25.97
2	10	(2,2,2,2)	32768	-	36.67
1	8	(2,2,3,3)	37584	-	38.23
1	8	(2,2,2,4)	47104	-	111.88

Theorem (Y.)

The Fano problems with $d_\bullet \neq (3), (2, 2)$ and less than 50,000 solutions have full symmetric Galois group.

Data for these systems and code verifying the data is available at

github.com/tjyahl/FanoGaloisGroups