Computing Galois groups of Fano problems

Thomas Yahl
thomasjyahl@tamu.edu

Texas A&M University
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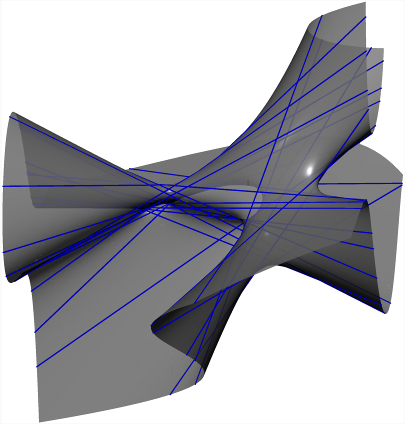
The problem of lines on a cubic surface

- Cayley and Salmon showed there are 27 distinct lines that lie on a smooth cubic surface.
- Schläfli determined these lines lie in a "remarkable configuration"
- A Fano problem is the problem of enumerating r-planes on a variety.



Fano problems

- Let $X_F \subseteq \mathbb{P}^n$ be the complete intersection defined by homogeneous polynomials $F = (f_1, \ldots, f_s)$ with $\deg f_i = d_i$.
- Enumerate the r-planes $\ell \in \mathbb{G}(r, \mathbb{P}^n)$ that lie on X_F.

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- [Debarre/Manivel] The combinatorial data $(r, n, d_\bullet)$ determines a Fano problem when $n - s - 2r \geq 0$ and
$$(r+1)(n-r) - \sum_{i=1}^{s} \binom{d_i + r}{r} = 0.$$

Examples

Debarre and Manivel determined the number of solutions to a general Fano problem of type $(r, n, d_\bullet)$. Write this number as $N(r, n, d_\bullet)$.

We list all Fano problems with less than 1000 solutions below.

r	n	$d_\bullet$	$N(r, n, d_\bullet)$
1	4	$(2, 2)$	16
1	3	(3)	27
2	6	$(2, 2)$	64
3	8	$(2, 2)$	256
1	7	$(2, 2, 2, 2)$	512
1	6	$(2, 2, 3)$	720

Incidence correspondence

Write $\mathbb{C}^{(r,n,d_\bullet)}$ for the parameter space of homogeneous forms $F = (f_1, \ldots, f_s)$ in $n+1$ variables of degrees $d_\bullet = (d_1, \ldots, d_s)$.

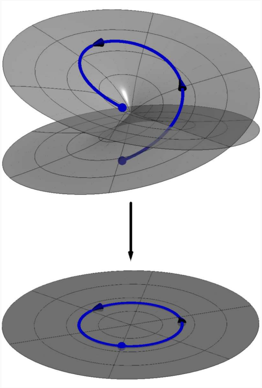
There is an incidence correspondence

$$\begin{tikzcd}
\Gamma_{(r,n,d_\bullet)} = \{(F,\ell) \in \mathbb{C}^{(r,n,d_\bullet)} \times \mathbb{G}(r,\mathbb{P}^n) : \ell \in V_r(X_F)\} \arrow[d, "\pi_{(r,n,d_\bullet)}"'] \arrow[dr] & \\
\mathbb{C}^{(r,n,d_\bullet)} & \mathbb{G}(r,\mathbb{P}^n)
\end{tikzcd}$$

- $\deg \pi_{(r,n,d_\bullet)} = N(r,n,d_\bullet)$.
- $\pi_{(r,n,d_\bullet)}$ restricts to a covering space over a Zariski open set.

Galois groups of Fano problems

- The Galois group, $\mathcal{G}_{(r,n,d_\bullet)}$, of the Fano problem determined by $(r,n,d_\bullet)$ is the monodromy group of $\pi_{(r,n,d_\bullet)}$.
- [Jordan],[Harris] $\mathcal{G}_{(1,3,(3))} = E_6$.
- [Harris] For $n \geq 4$, $\mathcal{G}_{(1,n,(2n-3))}$ is symmetric.
- [Hashimoto/Kadets] $\mathcal{G}_{(r,2r+2,(2,2))} = D_{2r+3}$.
- [Hashimoto/Kadets] If $d_\bullet \neq (3),(2,2)$ then $\mathcal{G}_{(r,n,d_\bullet)}$ contains the alternating group.

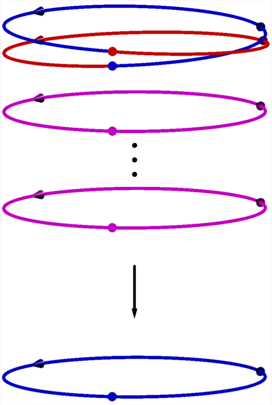


Harris' method of proof and our plan

<u>Goal:</u> Use computational methods to <u>prove</u> remaining Galois groups contain a simple transposition.

Find $F \in \mathbb{C}^{(r,n,d_\bullet)}$ such that $V_r(X_F)$ is one point of multiplicity 2 and $N(r,n,d_\bullet) - 2$ smooth points.

- Heuristically choose $F \in \mathbb{C}^{(r,n,d_\bullet)}$.
- Show $V_r(X_F)$ contains a point of multiplicity 2 by exact computation.
- Show there are $N(r,n,d_\bullet) - 2$ smooth points by numerical certification.



Choosing $F \in \mathbb{C}^{(r,n,d_\bullet)}$ and verifying conditions

Choose F by presecribing a subscheme of $V_r(X_F)$.

- Fix an r-plane $\ell \in \mathbb{G}(r, \mathbb{P}^n)$ and a tangent vector $v \in T_\ell \mathbb{G}(r, \mathbb{P}^n)$.
- Choose general F so that $\ell \in V_r(X_F)$ and $v \in T_\ell V_r(X_F)$.

When ℓ and v are chosen exactly, F can be chosen exactly as well.

- [Shub] simple double points are isolated.
- The remaining $N(r, n, d_\bullet) - 2$ solutions can be enumerated and isolated by certification.
  - α-theory
  - interval arithmetic

Smale's α-theory

Given a system $G : \mathbb{C}^m \to \mathbb{C}^m$ and $x \in \mathbb{C}^m$, there are associated quantities $\alpha(G, x)$, $\beta(G, x)$, and $\gamma(G, x)$.

Theorem (Smale et al.)
If G and x are such that

$$\alpha(G, x) < \frac{13 - 3\sqrt{17}}{4},$$

then x converges (quadratically) under Newton's method to a solution ξ of G. Further, $||x - \xi|| \leq 2\beta(G, x)$.

- Need G and x given by rational coordinates to certify a proof.
- `alphaCertified` will check the inequality above and certify the isolating balls are disjoint.

Interval Arithmetic

Given a system $G : \mathbb{C}^m \to \mathbb{C}^m$, $x \in \mathbb{C}^m$, and $Y \in \mathrm{GL}_m(\mathbb{C})$, the Krawczyk operator $K_{x,Y}$ acts on complex intervals.

Theorem (Krawczyk)
If G, x, Y, and I are such that

$$K_{x,Y}(I) \subseteq I,$$

then I contains a zero of G.

- Certifies computations using floating point arithmetic.
- `HomotopyContinuation.jl` will attempt to find a complex interval around an approximate solution to isolate solutions.

Results

Theorem (Y.)

The Fano problems with $d_\bullet \neq (3), (2,2)$ and less than 50,000 solutions have full symmetric Galois group.

Proof.
For those Fano problems, an F has been found such that $V_r(X_F)$ is a simple double point and $N(r, n, d_\bullet) - 2$ smooth points. $\square$

- This shows 10 Fano problems have full symmetric Galois group, which were previously unknown.

Data for these systems and code verifying the data is available at

`github.com/tjyahl/FanoGaloisGroups`

Timings

The reported timings for verifying the simple double point and certifying the remaining solutions is given below. (`alphaCertified`,`HomotopyContinuation.jl`)

r	n	$d_\bullet$	$N(r, n, d_\bullet)$	`alCer` (h)	`HomCo` (s)
1	7	$(2, 2, 2, 2)$	512	2.66	.61
1	6	$(2, 2, 3)$	720	2.88	.87
2	8	$(2, 2, 2)$	1024	27.32	1.57
1	5	(3,3)	1053	2.69	.32
1	5	(2,4)	1280	6.09	.73
1	10	(2,2,2,2,2,2)	20480	-	15.44
1	9	(2,2,2,2,3)	27648	-	25.97
2	10	(2,2,2,2)	32768	-	36.67
1	8	(2,2,3,3)	37584	-	38.23
1	8	(2,2,2,4)	47104	-	111.88

Moving forward & Bottlenecks

There is more to do!

- (In progress) Generate and verify data for larger Fano problems. Current bottleneck is memory!
- Turn this into a proof for ALL Fano problems with $d_\bullet \neq (3), (2, 2)$.
- Explore using numerical certification to prove more about Galois groups.

Thank you all for your time!

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[1, 2, 3, 4, 5, 6]