Computing Galois groups of Fano problems

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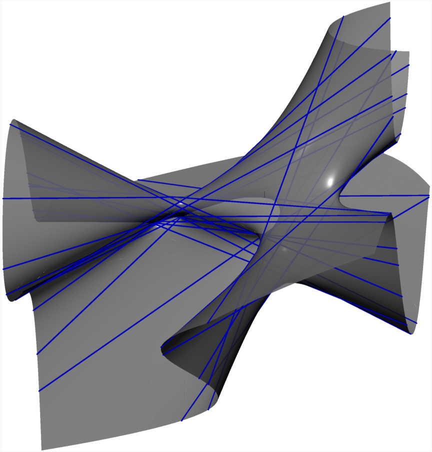
The problem of lines on a cubic surface

- Cayley and Salmon showed there are 27 distinct lines that lie on a smooth cubic surface.
- Schläfli determined these lines lie in a "remarkable configuration"
- A Fano problem is the problem of enumerating linear spaces of a fixed dimension on a variety.



Fano problems

- For fixed degrees $d_\bullet = (d_1, \ldots, d_s)$ choose homogeneous polynomials in $n+1$ variables, $F = (f_1, \ldots, f_s)$.
- Enumerate the r-planes that lie on the zero set $X_F = V(F) \subseteq \mathbb{P}^n$.

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- The combinatorial data $(r, n, d_\bullet)$ determines a Fano problem when $n - s - 2r \geq 0$ and

$$(r+1)(n-r) - \sum_{i=1}^{s} \binom{d_i + r}{r} = 0.$$

Examples

Debarre and Manivel explicitly determined the number of solutions to the Fano problem determined by the data $(r, n, d_\bullet)$ by intersection theoretic means. Write this number as $\deg(r, n, d_\bullet)$.

We list all Fano problems with less than 1000 solutions below.

r	n	$d_\bullet$	$\deg(r, n, d_\bullet)$	Galois group
1	4	$(2, 2)$	16	D_5
1	3	(3)	27	E_6
2	6	$(2, 2)$	64	D_7
3	8	$(2, 2)$	256	D_9
1	7	$(2, 2, 2, 2)$	512	S_{512}
1	6	$(2, 2, 3)$	720	S_{720}

Incidence correspondence

Write $\mathbb{C}^{(r,n,d_\bullet)}$ for the parameter space of homogeneous forms $F = (f_1, \ldots, f_s)$ in $n+1$ variables of degrees $d_\bullet = (d_1, \ldots, d_s)$.

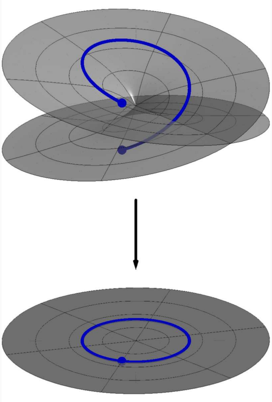
Fix $(r, n, d_\bullet)$. There is an incidence correspondence

$$\begin{tikzcd}
\Gamma_{(r,n,d_\bullet)} = \{(F,\ell) \in \mathbb{C}^{(r,n,d_\bullet)} \times \mathbb{G}(r,\mathbb{P}^n) : \ell \in V_r(X_F)\} \arrow[d, "\pi_{(r,n,d_\bullet)}"'] \arrow[dr] & \\
\mathbb{C}^{(r,n,d_\bullet)} & \mathbb{G}(r,\mathbb{P}^n)
\end{tikzcd}$$

- $\Gamma_{(r,n,d_\bullet)}$ is irreducible.
- $\deg \pi_{(r,n,d_\bullet)} = \deg(r, n, d_\bullet)$ and the fiber over $F \in \mathbb{C}^{(r,n,d_\bullet)}$ is the set of r-planes $V_r(X_F)$.
- $\pi_{(r,n,d_\bullet)}$ restricts to a covering space over a Zariski open set.

Galois groups of Fano problems

- The Galois group, $\mathcal{G}_{(r,n,d_\bullet)}$, of the Fano problem determined by $(r, n, d_\bullet)$ is the monodromy group of $\pi_{(r,n,d_\bullet)}$.
- Jordan/Harris: $\mathcal{G}_{(1,3,(3))} = E_6$
- Harris: $\mathcal{G}_{(1,n,(2n-3))}$ is symmetric for $n \geq 4$
- Hashimoto/Kadets: $\mathcal{G}_{(r,2r+2,(2,2))} = D_{2r+3}$
- Hashimoto/Kadets: If $d_\bullet \neq (3), (2,2)$ then $\mathcal{G}_{(r,n,d_\bullet)}$ contains the alternating group



Harris' method of proof

Lemma (Harris)

Let $\pi : Y \mapsto Z$ be a smooth map of degree k between irreducible varieties. If there exists a point $p \in Z$ such that the fiber $\pi^{-1}(p)$ consists of exactly $k-2$ simple points and one double point then the monodromy group of π contains a simple transposition.

We have a smooth map $\pi_{(r,n,d_\bullet)} : \Gamma_{(r,n,d_\bullet)} \to \mathbb{C}^{(r,n,d_\bullet)}$ of degree $\deg(r,n,d_\bullet)$ between irreducible varieties.

- Heuristically choose $F \in \mathbb{C}^{(r,n,d_\bullet)}$ so that $V_r(X_F)$ consists of $\deg(r,n,d_\bullet) - 2$ simple points and one double point.
- Solve a system $\overline{F}$ describing $V_r(X_F)$ in local coordinates.
- Verify claims with exact computation and numerical certification.

Choose $F \in \mathbb{C}^{(r,n,d_\bullet)}$, write $\overline{F}$, and solve

Choose local coordinates on $\mathbb{G}(r,\mathbb{P}^n)$ and for $\omega \in \mathbb{G}(r,\mathbb{P}^n)$ use local coordinates ω^* to parameterize ω.

- Write $\overline{F}$ as the system sending local coordinates of $\omega \in \mathbb{G}(r,\mathbb{P}^n)$ to the coefficients of $F|_\omega$ in local coordinates on ω.
- Choose $\ell \in \mathbb{G}(r,\mathbb{P}^n)$ and a nonzero vector $v \in \mathbb{C}^{(r+1)(n-r)}$.
- Randomly select $F \in \mathbb{C}^{(r,n,d_\bullet)}$ satisfying the linear conditions $\overline{F}(\ell^*) = 0$ and $D\overline{F}(\ell^*)v = 0$.
- Use your favorite solver to enumerate the solutions of $\overline{F}$.

<u>**Note:**</u> If ℓ and v have rational coordinates, F can be chosen to have rational coordinates as well.

Simple double roots and numerical certification

A simple double root of a system G is a root x satisfying $\ker DG(x) = \langle v \rangle$ and $D^2G(x)(v, v) \notin \operatorname{im} DG(x)$.

- By work of Shub, simple double roots are isolated solutions of multiplicity 2.
- If ℓ, v, and F have rational coordinates, these conditions can be checked exactly.

There are 2 methods of isolating the remaining solutions.

- α-theory
- Interval arithmetic

Smale's α-theory

Given a system $G : \mathbb{C}^m \to \mathbb{C}^m$ and $x \in \mathbb{C}^m$, there are associated quantities $\alpha(G, x)$, $\beta(G, x)$, and $\gamma(G, x)$.

Theorem (Smale et al.)
If G is a system and x is such that

$$\alpha(G, x) < \frac{13 - 3\sqrt{17}}{4},$$

then x converges (quadratically) under Newton's method to a solution y of G. Further, $||x - y|| \leq 2\beta(G, x)$.

- When G and x are given by exact coordinates, this provides an exact isolating ball for the solution y.
- `alphaCertified` will check the inequality above and certify balls isolating solutions are disjoint, with exact arithmetic.

Interval Arithmetic

Given a system $G : \mathbb{C}^m \to \mathbb{C}^m$, $x \in \mathbb{C}^m$, and $Y \in \mathrm{GL}_m(\mathbb{C})$, the Krawczyk operator $K_{x,Y}$ on the space of complex intervals generalizes the Newton operator.

Theorem

If G is a system, and x is a point, and I is a complex interval such that

$$K_{x,Y}(I) \subseteq I,$$

then I contains a zero of G.

- Certifies computations using floating point arithmetic.
- `HomotopyContinuation.jl` will attempt to find a complex interval around an approximate solution to isolate solutions.

Results and timings

Theorem (Y.)

The Fano problems with $d_\bullet \neq (3), (2,2)$ and less than 40,000 solutions have full symmetric Galois group.

Timing: (`NAG4M2`,`alphaCertified`,`HomotopyContinuation.jl`)

r	n	$d_\bullet$	$\deg(r,n,d_\bullet)$	`M2` (h)	`julia` (s)
1	7	$(2,2,2,2)$	512	2.66	.61
1	6	$(2,2,3)$	720	2.88	.87
2	8	$(2,2,2)$	1024	27.32	1.57
1	5	(3,3)	1053	2.69	.32
1	5	(2,4)	1280	6.09	.73
1	10	(2,2,2,2,2,2)	20480	-	15.44
1	9	(2,2,2,2,3)	27648	-	25.97
2	10	(2,2,2,2)	32768	-	36.67
1	8	(2,2,3,3)	37584	-	38.23

Moving forward

Data for these systems and code verifying the data is available at

`github.com/tjyahl/FanoGaloisGroups`

There is more to do!

- Generate systems with a single simple double root for larger Fano problems (in progress).
- Turn this into a proof for ALL Fano problems with $d_\bullet \neq (3), (2, 2)$.
- Explore using numerical certification for proving more about Galois groups.

References

Thank you all for your time!