Computing Galois groups of Fano problems

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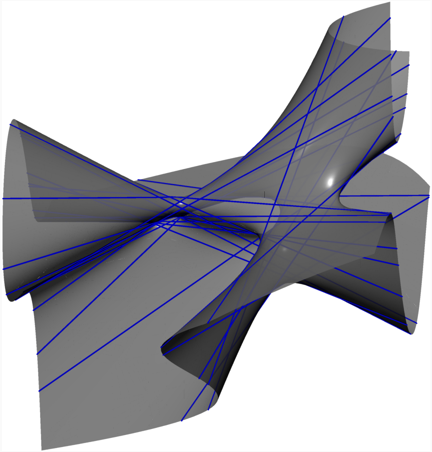
The problem of lines on a cubic surface

- Cayley and Salmon showed there are 27 distinct lines that lie on a smooth cubic surface.
- Schläfli determined these lines lie in a "remarkable configuration"
- This is an instance of a Fano problem.



Fano problems

- For fixed degrees $d_\bullet = (d_1, \ldots, d_s)$ choose homogeneous polynomials in $n+1$ variables, $F = (f_1, \ldots, f_s)$.
- Enumerate the r-planes that lie on the zero set $X = V(F) \subseteq \mathbb{P}^n$.

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- The combinatorial data $(r, n, d_\bullet)$ determines a Fano problem when $n - s - 2r \geq 0$ and

$$(r+1)(n-r) = \sum_{i=1}^{s} \binom{d_i + r}{r}.$$

Examples

Debarre and Manivel determined the number of solutions to the Fano problem determined by the data $(r, n, d_\bullet)$ by intersection theoretic means.

By exhaustion, we enumerate Fano problems of small size.

r	n	$d_\bullet$	# of solutions	Galois group
1	4	$(2,2)$	16	D_5
1	3	(3)	27	E_6
2	6	$(2,2)$	64	D_7
3	8	$(2,2)$	256	D_9
1	7	$(2,2,2,2)$	512	S_{512}
1	6	$(2,2,3)$	720	S_{720}

Incidence correspondence

Write $\mathbb{C}^{(r,n,d_\bullet)}$ for the parameter space of homogeneous forms $(f_1, \ldots, f_s)$ in $n+1$ variables of degrees $d_\bullet = (d_1, \ldots, d_s)$.

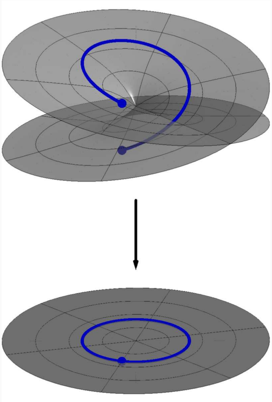
Fix $(r, n, d_\bullet)$. There is an incidence correspondence

$$\begin{tikzcd}
\Gamma_{(r,n,d_\bullet)} = \{(F,\ell) \in \mathbb{C}^{(r,n,d_\bullet)} \times \mathbb{G}(r,n) : F|_\ell = 0\} \arrow[d, "\pi_{(r,n,d_\bullet)}"'] \\
\mathbb{C}^{(r,n,d_\bullet)}
\end{tikzcd}$$

- $\Gamma_{(r,n,d_\bullet)}$ is irreducible.
- The fiber over $F \in \mathbb{C}^{(r,n,d_\bullet)}$ is the set of r-planes $V_r(F)$.
- π restricts to a covering space over a Zariski open set $U_{(r,n,d_\bullet)} \subseteq \mathbb{C}^{(r,n,d_\bullet)}$.

Galois groups of Fano problems

- The Galois group, $\mathcal{G}_{(r,n,d_\bullet)}$, of the Fano problem determined by $(r, n, d_\bullet)$ is the monodromy group of $\pi_{(r,n,d_\bullet)}$.
- Jordan showed $\mathcal{G}_{(1,3,(3))} \subseteq E_6$.
- Harris showed $\mathcal{G}_{(1,3,(3))} = E_6$ and $\mathcal{G}_{(1,n,(2n-3))}$ is symmetric.
- Hashimoto and Kadets showed $\mathcal{G}_{(r,2r+2,(2,2))} = D_{2r+3}$ and $\mathcal{G}_{(r,n,d_\bullet)}$ contains the alternating group for remaining Fano problems.



Goal: use numerics prove remaining Galois groups are symmetric.

Harris' method of proof

To show $\mathcal{G}_{(1,n,(2n-3))}$ is contains a simple transposition, Harris showed the existence of $p \in U_{(1,n,(2n-3))}$ satisfying the following.

Lemma (Harris)

Let $\pi : Y \mapsto X$ be a smooth map of degree n between irreducible varieties. If there exists a point $p \in X$ such that the fiber $\pi^{-1}(p)$ consists of exactly $n-2$ simple points and one double point then the monodromy group of π contains a simple transposition.

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- To prove the constructed systems satisfy the above, we make use of numerical certification.

Numerical certification

- Smale's α-theory
- Interval arithmetic

Example

Constructing systems with a unique simple double root

Results and timings

Moving forward

References