Computing Galois groups of Fano problems

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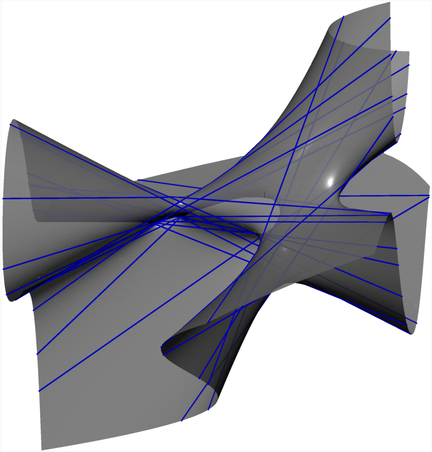
The problem of lines on a cubic surface

- Cayley and Salmon showed there are 27 distinct lines that lie on a smooth cubic surface.
- Schläfli determined these lines lie in a "remarkable configuration"
- A Fano problem is the problem of enumerating linear spaces of a fixed dimension on a variety.



Fano problems

- For fixed degrees $d_\bullet = (d_1, \ldots, d_s)$ choose homogeneous polynomials in $n+1$ variables, $F = (f_1, \ldots, f_s)$.
- Enumerate the r-planes that lie on the zero set $X = V(F) \subseteq \mathbb{P}^n$.

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- The combinatorial data $(r, n, d_\bullet)$ determines a Fano problem when $n - s - 2r \geq 0$ and

$$(r+1)(n-r) - \sum_{i=1}^{s} \binom{d_i + r}{r} = 0.$$

Examples

Debarre and Manivel determined the number of solutions to the Fano problem determined by the data $(r, n, d_\bullet)$ by intersection theoretic means. Write this number as $\deg(r, n, d_\bullet)$.

We list all Fano problems with less than 1000 solutions below.

r	n	$d_\bullet$	$\deg(r, n, d_\bullet)$	Galois group
1	4	$(2, 2)$	16	D_5
1	3	(3)	27	E_6
2	6	$(2, 2)$	64	D_7
3	8	$(2, 2)$	256	D_9
1	7	$(2, 2, 2, 2)$	512	S_{512}
1	6	$(2, 2, 3)$	720	S_{720}

Incidence correspondence

Write $\mathbb{C}^{(r,n,d_\bullet)}$ for the parameter space of homogeneous forms $(f_1, \ldots, f_s)$ in $n+1$ variables of degrees $d_\bullet = (d_1, \ldots, d_s)$.

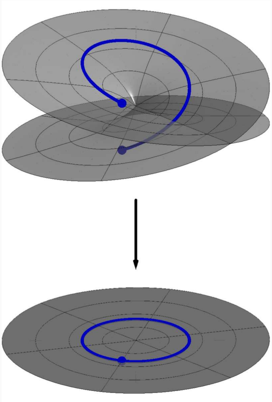
Fix $(r, n, d_\bullet)$. There is an incidence correspondence

$$\begin{array}{c} \Gamma_{(r,n,d_\bullet)} = \{(F, \ell) \in \mathbb{C}^{(r,n,d_\bullet)} \times \mathbb{G}(r,n) : F|_\ell = 0\} \\ \Big\downarrow \pi_{(r,n,d_\bullet)} \\ \mathbb{C}^{(r,n,d_\bullet)} \end{array}$$

- $\Gamma_{(r,n,d_\bullet)}$ is irreducible.
- $\deg \pi_{(r,n,d_\bullet)} = \deg(r, n, d_\bullet)$ and the fiber over $F \in \mathbb{C}^{(r,n,d_\bullet)}$ is the set of r-planes $V_r(F)$.
- π restricts to a covering space over a Zariski open set $U_{(r,n,d_\bullet)} \subseteq \mathbb{C}^{(r,n,d_\bullet)}$.

Galois groups of Fano problems

- The Galois group, $\mathcal{G}_{(r,n,d_\bullet)}$, of the Fano problem determined by $(r, n, d_\bullet)$ is the monodromy group of $\pi_{(r,n,d_\bullet)}$.
- Jordan/Harris: $\mathcal{G}_{(1,3,(3))} = E_6$
- Harris: $\mathcal{G}_{(1,n,(2n-3))}$ is symmetric for $n \geq 4$
- Hashimoto/Kadets: $\mathcal{G}_{(r,2r+2,(2,2))} = D_{2r+3}$
- Hashimoto/Kadets: All other $\mathcal{G}_{(r,n,d_\bullet)}$ contain the alternating group



Goal: use numerics prove remaining Galois groups contain a simple transposition and are therefore symmetric.

Harris' method of proof

Lemma (Harris)

Let $\pi : Y \mapsto X$ be a smooth map of degree n between irreducible varieties. If there exists a point $p \in X$ such that the fiber $\pi^{-1}(p)$ consists of exactly $n-2$ simple points and one double point then the monodromy group of π contains a simple transposition.

For remaining Fano problems of moderate size, we construct $p \in \mathbb{C}^{(r,n,d_\bullet)}$ satisfying the above.

Harris' method in context

We have a smooth map $\pi : \Gamma_{(r,n,d_\bullet)} \to \mathbb{C}^{(r,n,d_\bullet)}$ of degree $\deg(r, n, d_\bullet)$ between irreducible varieties. For $F \in \mathbb{C}^{(r,n,d_\bullet)}$, the fiber is the set of linear spaces $V_r(F)$.

- We construct $F \in \mathbb{C}^{(r,n,d_\bullet)}$ so that $V_r(F)$ consists of $\deg(r, n, d_\bullet) - 2$ simple points and one double point.
- To show our $F \in \mathbb{C}^{(r,n,d_\bullet)}$ has this property, we write our points as solutions to a system $\overline{F}$ with exact coefficients.
- We verify our system $\overline{F}$ has $\deg(r, n, d_\bullet) - 2$ simple solutions and one solution of multiplicity 2 by certification.

Constructing systems with a unique simple double root

- A simple double root of a system F is a root x satisfying $\ker DF(x) = \langle v \rangle$ and $D^2F(x)(v,v) \notin \operatorname{im} DF(x)$.
- By work of Shub, simple double roots are isolated solutions of multiplicity 2.

Construct a system with complex rational coefficients and a unique simple double root as follows.

1. Choose $x, v \in \mathbb{C}^n$ with complex rational coordinates.
2. Randomly choose $F \in \mathbb{C}^{(r,n,d_\bullet)}$ with complex rational coefficients such that $F(x) = 0$ and $DF(x)v = 0$.
3. Check that $\dim \ker DF(x) = 1$ and $D^2F(x)(v,v) \notin \operatorname{im} DF(x)$.

Smale's α-theory

Given a system $F : \mathbb{C}^n \to \mathbb{C}^n$ and $x \in \mathbb{C}^n$, there are associated quantities $\alpha(F, x)$, $\beta(F, x)$, and $\gamma(F, x)$.

Theorem (Smale et al.)
If $F : \mathbb{C}^n \mapsto \mathbb{C}^n$ is a system and $x \in \mathbb{C}^n$ with

$$\alpha(F, x) < \frac{13 - 3\sqrt{17}}{4},$$

then x converges (quadratically) under Newton's method to a solution x^* of F. Further, $||x - x^*|| \leq 2\beta(F, x)$.

- When F and x are given by complex rationals, this provides an exact isolating ball for the solution x^*.
- `alphaCertified` will check the inequality above and certify balls isolating solutions are disjoint, with exact arithmetic.

Results and timings

Theorem (Y.)
The Fano problems $(1, 7, (2, 2, 2, 2))$, $(1, 6, (2, 2, 3))$, $(2, 8, (2, 2, 2))$, $(1, 5, (3, 3))$, and $(1, 5, (2, 4))$ have symmetric Galois groups.

Timings (via my laptop) to check that the constructed systems have a simple double root and to certify the remaining solutions lie in disjoint balls.

r	n	$d_\bullet$	$\deg(r, n, d_\bullet)$	Computation time (h)
1	7	$(2, 2, 2, 2)$	512	2.66
1	6	$(2, 2, 3)$	720	2.88
2	8	$(2, 2, 2)$	1024	27.32
1	5	(3,3)	1053	2.69
1	5	(2,4)	1280	6.09

Moving forward

This is ongoing work, there are many avenues to explore!

- Generate systems with a single simple double root for larger Fano problems (in progress).
- Use interval arithmetic for faster certification (in progress).
- Show that such a system exists for all remaining Fano problems.
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References