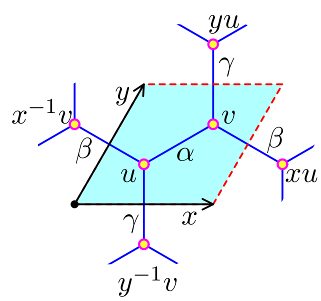
DISCRETE PERIODIC OPERATORS INTRO

MATTHEW FAUST

1. Graphene



A quick summary of background:

Notice the two vertices, u and v, these vertices together with the edges between them give us the *Fundamental domain* of the graphene.

Let L be the Laplace-beltrami operator defined by

f is just some function that takes vertices of the periodic graph to polynomials of finite support in the variables x and y. We also assume that f are also such that $f(x^m y^n u) = x^m y^n f(u)$.

Let

$$Lf(u) = \sum_{w \sim u} a_{(u,w)}(f(u) - f(w)).$$

Where $a_{(u,w)}$ is the weight of the edge between u and v. $w \sim u$ means we iterate over the neighbors of u.

So let us write out $Lf(u)$ one step at a time, first going over the summation.

$$Lf(u) = \alpha(f(u) - f(v)) + \beta(f(u) - f(x^{-1}v)) + \gamma(f(u) - f(y^{-1}v))$$

Then we use the property of f to get

$$Lf(u) = \alpha(f(u) - f(v)) + \beta(f(u) - x^{-1}f(v)) + \gamma(f(u) - f(v))$$

Then we just rewrite this in terms of $f(u)$ and $f(v)$:

$$Lf(u) = (\alpha + \beta + \gamma)f(u) - (\alpha + \beta x^{-1} + \gamma y^{-1})f(v)$$

Through a similar process we get:

$$Lf(v) = (\alpha + \beta + \gamma)f(v) - (\alpha + \beta x + \gamma y)f(u)$$

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$$L\begin{pmatrix} f(u) \\ f(v) \end{pmatrix} = \begin{pmatrix} Lf(u) \\ Lf(v) \end{pmatrix}$$

$$L = \begin{pmatrix} \alpha+\beta+\gamma & -\alpha-\beta x^{-1}-\gamma y^{-1} \\ -\alpha-\beta x-\gamma y & \alpha+\beta+\gamma \end{pmatrix}$$

We then let ϕ be the characteristic polynomial of L.

That is

$$\phi = det(L-\lambda I)$$

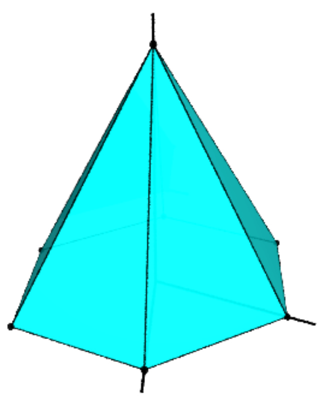
.

So we have

$$\phi = (\alpha+\beta+\gamma-\lambda)^2 - (\alpha+\beta x+\gamma y)(\alpha+\beta x^{-1}+\gamma y^{-1})$$

We then can assign values to α, β, γ and then look at the exponent vectors in terms of x, y, and λ.

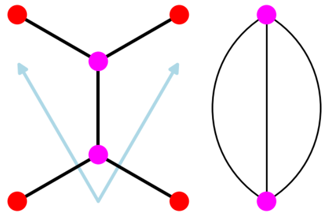
This should give us the following Newton polytope.



2. Brief idea of topological crystals

There are multiple fundamental domain for the same periodic graphs. However these fundamental domain correspond to different *quotient graphs*. A topolgical crystal (or periodic graph, as the graphene is) are *covering graphs* of quotient graphs. Any finite graph that is connected with no bridges, has a covering graph (no bridges means removing a single edge will not disconnect the graph).

For example here is another drawing of the fundamental domain of graphene (a fundamental domain has internal edges and external edges).



A finite graph can be equipped with a group (in particular which is a subgroup of something called the "first homology group" or fundamental group of the graph under some equivalence relation). This quotient graph subgroup combination corresponds uniquely pairs with a periodic graph (or topological crystal).

In the example of the two vertex fundamental domain of the graphene, its quotient graph equipped with its first homology group corresponds to it. If a fundamental domain corresponds to a quotient graph with its homology group, the periodic graph that fundamental domain produces is called a *Maximal abelian covering*.

There are unique conjectures pertaining to discrete periodic operators over such fundamental domains.

3. Examples

So we talked a little about what these objects are called. Without going into too much detail (I will provide this later if there is interest). Let us see how the graphene is identified by this quotient graph.

In the context of loops, what is called the first homology group of a bridgeless graph can be constructed as follows. We fix a vertex as the origin, lets say u, then the group of cycles that start and end at u give us the first homology group.

In constructing the periodic graph we identify vertices as paths (under an equivalence relation that is to be seen). We are going to cheat a little and assign our edges a direction, even though these are undirected graphs. This will allow us to view these edges as algebraic objects.

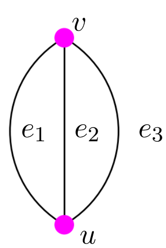
In the example below we have two vertices so we will just view e_i as an edge from u to v and its inverse $\bar{e_i}$ will take the same edge from v to u.

In this way we have paths u to v given by:

$[e_i]$, $[e_i\bar{e_j}e_k]$, $\ldots$ and so on.

We also have paths u to u given by:

$[]$, $[e_i\bar{e_j}]$, $\ldots$ and so on.



A universal cover of a graph is an infinite graph where each path is assigned a vertex, and two vertices $[\gamma]$ and $[\beta]$ have an edge between them if there is an edge e such that $[\gamma e] = [\beta]$.

A maximal abelian covering can be obtained from a universal cover by imposing an equivalence relation on these paths.

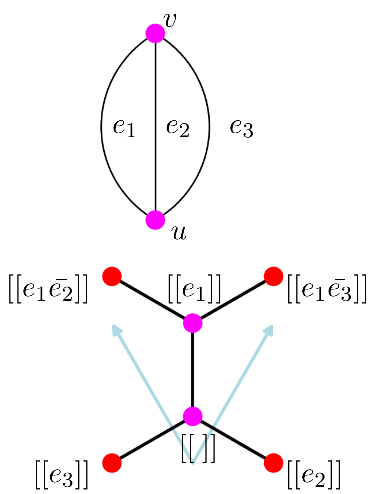
We say two paths $[\gamma]$ and $[\beta]$ are homologous, lets write this as $[[\gamma]] = [[\beta]]$ if one can be obtained by another by the following:

(1) Delete back tracking: that is if $[\gamma] = [\gamma_1 e_i \bar{e_i} \gamma_2]$ then $[[\gamma]] = [[\gamma_1\gamma_2]]$.
(2) Abelian loops that is if $[\gamma] = [\gamma_1\gamma_2\gamma_3\gamma_4]$ and γ_2 and γ_3 are loops (paths starting and ending at the same vertex) of the same vertex then $[[\gamma]] = [[\gamma_1\gamma_3\gamma_2\gamma_4]]$.

Under this relationship, the loops based at u form a group called the *fundamental group* with composition on paths as the operation, that is $[[\gamma]] \circ [[\beta]] = [[\gamma\beta]]$.

Fixing the universal covering under this relationship gives us a maximal abelian covering.

Let us look at how our 2 vertex fundamental domain of graphene relates to this.



We notice that our action x takes $[[\,]]$ to $[[e_1\bar{e_2}]]$ and y takes $[[\,]]$ to $[[e_1\bar{e_3}]]$. These are both loops based at u. This is no coincidence, as these two loops also generate all loops based at u **Can you prove this?** These in fact are a generating set of the fundamental group of the quotient graph (or seed graph).

You might be wondering why we chose to place our vertices in such a manner, this is a good question. This is merely one of many "geometric realizations" of the maximal abelian cover of this quotient graph.

We can then get other graphs by imposing more equivalences (i.e taking quotient groups). For example we could set $e_1\bar{e_2} \simeq e_1\bar{e_3}$ and we would just get a graph that we could represent as a line with vertices, but this would still be identified with a 2 vertex fundamental domain of that linear periodic graph.

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