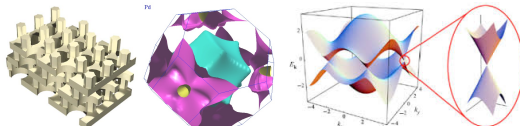


Dedication  
Instead of detailed references  
Preliminaries  
Dispersion relations and all that  
Spectra and solutions  
Some conferences  
Some references

# Introduction to periodic operators

Peter Kuchment, Texas A & M University

Periodic, Almost-Periodic, and Random Operators:  
Instructional School  
Isaac Newton Institute for Mathematical Sciences  
Cambridge, UK, January 5 – 16, 2015.



## Dedication

Instead of detailed references  
Preliminaries  
Dispersion relations and all that  
Spectra and solutions  
Some conferences  
Some references

# DEDICATION

**Dedicated to the memory of dear friends and colleagues  
M. Birman, L. Ehrenpreis, V. Geyler, S. Krein, M. Novitskii**

Dedication  
Instead of detailed references  
Preliminaries  
Dispersion relations and all that  
Spectra and solutions  
Some conferences  
Some references

## Some important names

Agmon, Aizenman, Allaire, Atiyah, Avellaneda, Avron, Bättig, Berkolaiko, Berry, Birman, Brooks, Brüning, Carlson, Colding, Colin de Verdière, Combes, Cox, Danilov, Derguzov, Dobson, Ehrenpreis, Eskin, Exner, Fefferman, Feldman, Figotin, Filonov, Firsova, Friedlander, Froese, Gelfand, Gérard, Geyler, Gieseke, Gordon, Helffer, Herbst, Hoffmann-Ostenhof, Jikov, Joannopoulos, Johnson, Kappeler, Karpeshina, Kirsch, Klein, Klopp, Knörrer, Korotyaev, Kotani, M&S Krein, Levendorskii, Li, Lin, Malgrange, Meshkov, Minicozzi, Molchanov, Morame, Moser, Nier, Novikov, Novitsky, Palamodov, G&V Papanicolaou, Parnowski, Pavlov, Pinchover, Post, Ralston, Shen, Shubin, Simon, Skriganov, Smyshlyaev, Sobolev, Struwe, Sunada, Suslina, Thomas, Trubowitz, Vainberg, Weinstein, Wilcox, Zaidenberg, Zelenko

# Table of Contents

- **Preliminaries:** 1D (brief sketch), lattices, periodic operators, Floquet transform, direct integral decomposition, band-gap structure
- **Dispersion relation an all that:** Bloch and Fermi varieties, analyticity, irreducibility, gap edge location and generic structure, Dirac cones, Bloch bundles, inverse problems
- **Spectra and solutions:** existence and number of gaps, AC-PP-SING, density of states, Floquet-Bloch solutions, Shnol'-Bloch theorem, Wannier functions, Liouville theorems
- **Miscellanea:** Homogenization,  $\Psi$ DOs, slow light, waveguides, photonic crystals, time-periodic evolution equations, non-abelian case.
- **References**

## A disclosure

### A Good Faith Disclosure:

There is no attempt made here to reach the highest generality level, e.g., in terms of weakest conditions on coefficients. Most issues arise already in the smooth case, so, unless directed otherwise, the attendees can think of  $C^\infty$ . For more detailed discussions see the books/papers in the reference list and the references therein.

The proofs are at most sketched.

See another good faith disclosure before the list of references.

Dedication  
Instead of detailed references  
**Preliminaries**  
Dispersion relations and all that  
Spectra and solutions  
Some conferences  
Some references

1D - just a brief sketch  
Lattices  
Periodic operators  
Floquet transform. Direct integral decomposition

## Preliminaries. Brief content

- Brief 1D story
- Lattices
- Various periodic operators
- Floquet transform
- Direct integral decomposition
- Band-gap structure of the spectrum

## “Standard” Floquet theory

$$\frac{dx}{dt} = A(t)x, t \in \mathbb{R}, x \in \mathbb{C}^n, \quad (*)$$

1-periodic  $n \times n$  matrix  $A(t)$ .  $X(t)$  - fundamental ( $n \times n$ ) solution.

**Floquet theorem**  $X(t) = P(t)e^{Ct}$ ,

where matrix-function  $P(t)$  is 1-periodic and  $C$  is a constant matrix.

Hint:  $X(1)$  is the **monodromy matrix**.

**Corollary** Any solution of  $(*)$  is a linear combination of

**Floquet solutions**  $x(t) = e^{ikt} \sum_j p_j(t) t^j$

with 1-periodic coefficients  $p_j(t)$ .

$z := e^{ik}$  - **Floquet multiplier**.  $k$  - **quasimomentum / crystal momentum**. Floquet multipliers  $z$  are eigenvalues of the monodromy matrix.

## Lyapunov reduction

**Lyapunov reduction theorem** *There exists a periodic invertible matrix-function  $B(t)$  such that the substitution  $x(t) = B(t)y(t)$  reduces the system  $(*)$  to the one with a constant matrix  $C$ :*

$$\frac{dy}{dt} = Cy(t).$$

**Lyapunov's theorem easily implies Floquet theorem.**

The analogs of Floquet and Lyapunov theorems MIGHT (but not always do) work for evolution PDEs. The situation is much worse

for non-evolution periodic PDEs.





## Hill's operator

$$H = -\frac{d^2 u}{dx^2} + V(x)$$

with a “nice” 1-periodic potential  $V$  on  $\mathbb{R}$ .

Spectral problem

$$Hu = \lambda u.$$

Fundamental system of two analytic in  $\lambda$  solutions  $\phi(x, \lambda), \psi(x, \lambda)$ :

$$\phi(0, \lambda) = \psi'_x(1, \lambda) = 1, \phi'_x(0, \lambda) = \psi(1, \lambda) = 0.$$

## Monodromy, discriminant=Lyapunov function

**Monodromy matrix**  $M(\lambda) := \begin{pmatrix} \phi(1, \lambda) & \psi(1, \lambda) \\ \phi'_x(1, \lambda) & \psi'_x(1, \lambda) \end{pmatrix}$ .

$\det M(\lambda) = 1$  (Wronskian)

$\Delta(\lambda) := \text{Tr } M(\lambda) = \phi(1, \lambda) + \psi'_x(1, \lambda)$  - **discriminant / Lyapunov function.**

The secular equation

$$z^2 - \Delta(\lambda)z + 1 = 0$$

provides Floquet multipliers and quasimomenta for a given  $\lambda$ :

$$e^{ik} = z = 0.5 \left( \Delta(\lambda) \pm \sqrt{\Delta(\lambda)^2 - 4} \right)$$

Dedication  
Instead of detailed references  
**Preliminaries**  
Dispersion relations and all that  
Spectra and solutions  
Some conferences  
Some references

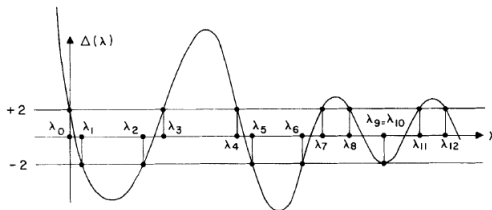
1D - just a brief sketch

Lattices

Periodic operators

Floquet transform. Direct integral decomposition

## Discriminant's properties. Real potential.



Entire function of exponential order  $1/2$ :

$$|\Delta(\lambda)| \leq C e^{C|\lambda|^{1/2}}.$$

Dedication  
Instead of detailed references  
**Preliminaries**  
Dispersion relations and all that  
Spectra and solutions  
Some conferences  
Some references

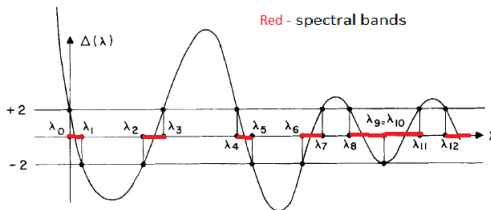
1D - just a brief sketch

Lattices

Periodic operators

Floquet transform. Direct integral decomposition

## Spectral structure



**Red** - **stability zones** = **spectral bands** (bounded solutions, real quasimomentum, zero Lyapunov exponent).

Between them - **instability zones** = **spectral gaps** (exponentially growing solutions, complex quasimomentum, positive Lyapunov exponent).

Dedication  
 Instead of detailed references  
**Preliminaries**  
 Dispersion relations and all that  
 Spectra and solutions  
 Some conferences  
 Some references

1D - just a brief sketch

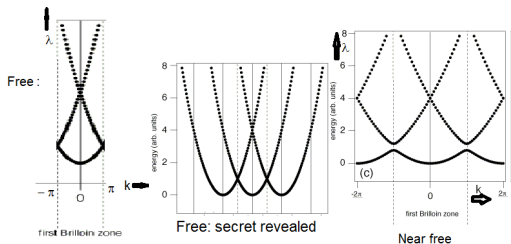
Lattices

Periodic operators

Floquet transform. Direct integral decomposition

## Spectral structure II

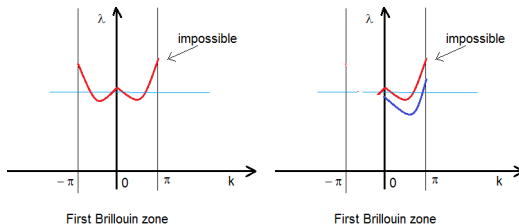
- Spectral bands **do not overlap**, but might touch
- Spectrum is absolutely continuous
- Dispersion relation ( $\lambda$  as a function of  $k$ )



- $k \mapsto -k$  symmetry

## Spectral structure III

- **Monotonicity!**



- Spectral **band edges** occur at  $k = 0$  or  $k = \pi$  only (periodic and anti-periodic spectra)

## Spectral structure IV

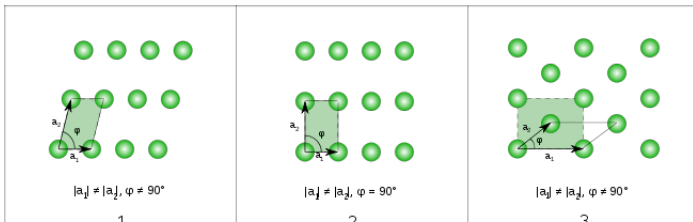
- **Generically, all gaps are present (open)**, so the bands do not touch (Simon)
- **Finite gap (finite zone) potentials** are very special and can be explicitly found
- **No gaps**  $\Leftrightarrow$  the potential is constant (Borg)
- **Gaps' sizes decay** determines smoothness of the potential
- There are **isospectral** potentials.
- The dispersion relation is **irreducible**, i.e. it can be obtained by analytic continuation of its small piece. (Kohn, Avron & Simon)
- The dispersion relation is generically **non-algebraic**

## (Bravais) Lattices

A (Bravais) **lattice**  $\Gamma$  in  $\mathbb{R}^n$  - integer linear combinations of  $n$  linearly independent vectors  $a_1, \dots, a_n$ :

$$\Gamma = \{\gamma \in \mathbb{R}^n \mid \gamma = \sum_{j=1}^n \gamma_j a_j, \gamma_j \in \mathbb{Z}\}.$$

Will be identified with the corresponding group of shifts of the space.





Dedication  
 Instead of detailed references  
**Preliminaries**  
 Dispersion relations and all that  
 Spectra and solutions  
 Some conferences  
 Some references

1D - just a brief sketch

**Lattices**

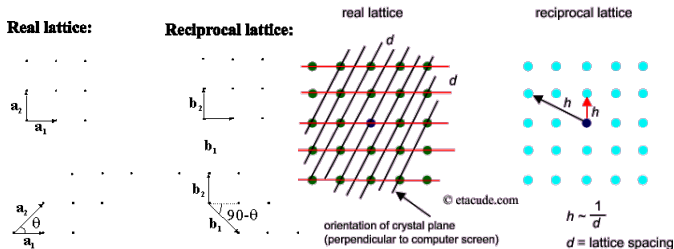
Periodic operators

Floquet transform. Direct integral decomposition

# Dual (reciprocal) lattice

The **dual (reciprocal) lattice**  $\Gamma^*$  in  $(\mathbb{R}^n)^*$

$$\Gamma^* = \{k \in (\mathbb{R}^n)^* \mid \langle k, \gamma \rangle \in 2\pi\mathbb{Z} \text{ for any } \gamma \in \Gamma\}.$$

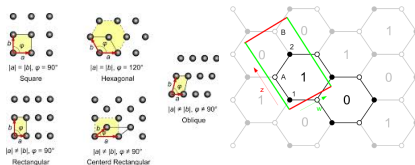


Dedication  
 Instead of detailed references  
**Preliminaries**  
 Dispersion relations and all that  
 Spectra and solutions  
 Some conferences  
 Some references

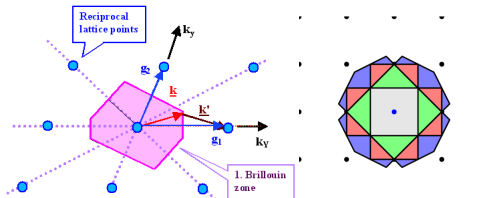
1D - just a brief sketch  
**Lattices**  
 Periodic operators  
 Floquet transform. Direct integral decomposition

## Fundamental domains

A fundamental domain  $W$  (**Wigner-Seitz cell**) of  $\Gamma$  in  $(\mathbb{R}^n)$ :



A fundamental domain  $\mathcal{B}$  (**Brillouin zone**) of  $\Gamma^*$  in  $(\mathbb{R}^n)^*$ :



# Tori

Two tori correspond to the two lattices :

$$\mathbb{T} := \mathbb{R}^n / \Gamma. \quad \mathbb{T}^* := (\mathbb{R}^n)^* / \Gamma^*.$$

Natural Haar measures on both. (We use the normalized ones.)

$\Gamma$  - ( $\Gamma^*$  -) periodic functions on  $\mathbb{R}^n$  (on  $(\mathbb{R}^n)^*$ )  
 = functions on  $\mathbb{T}$  ( $\mathbb{T}^*$ ).

Fourier series identify  $L^2(\mathbb{T})$  with the  $l^2$  space on  $\Gamma^*$ :

$$f(x) \mapsto \{f_k := \int_{\mathbb{T}} f(x) e^{-ik \cdot x} dx\}_{k \in \Gamma^*}$$

Analogously with  $L^2(\mathbb{T}^*)$  and  $l^2$  on  $\Gamma$ .

## Periodic operators

Our main “test” example: Schrödinger operator in  $\mathbb{R}^n$

$$L = -\Delta + V(x)$$

with a “sufficiently nice” potential  $V$ , periodic with respect to the group  $\Gamma = \mathbb{Z}^n$  (and thus  $\Gamma^* = 2\pi\mathbb{Z}^n$ ).

*Most techniques and many results apply to arbitrary lattices and much more general periodic elliptic operators or systems (e.g., Maxwell). Restriction to the 2nd order operators is often needed, as well as sometimes restrictions on the form of the operator. Boundary value problems in periodic domains (e.g., waveguides) can also be treated.*

## Periodic operators on abelian coverings

The natural mapping  $\mathbb{R}^n \mapsto \mathbb{T}$  is an **abelian covering** of the (compact) torus  $\mathbb{T}$ .

The techniques of these lectures apply to periodic operators on coverings

$$M \mapsto N,$$

subject to the following three conditions:

- The base  $N$  is compact
- The deck group  $\Gamma$  of the covering is abelian
- The operator on  $M$  is “elliptic” in the sense that being pushed down to  $N$ , it is a Fredholm operator in appropriate spaces.

In particular, periodic operators on abelian coverings of compact Riemannian manifolds, analytic manifolds, and even graphs, succumb gladly to the technique.

# Floquet transform

Fourier series transform functions on  $\Gamma$  into functions of  $z \in \mathbb{T}^*$ :

$$\{f_\gamma\}_{\gamma \in \Gamma} \mapsto \hat{f}(z) := \sum_{\gamma \in \Gamma} f_\gamma z^\gamma = \sum_{\gamma \in \Gamma} f_\gamma e^{lk \cdot \gamma}$$

Standard  $L^2$  isometry and Paley-Wiener type theorems hold.

Let now  $f(x)$  be a function on  $\mathbb{R}^n$ . We cover  $\mathbb{R}^n$  by the shifted copies of  $W$ :  $\mathbb{R}^n = \bigcup_{\gamma \in \Gamma} (W + \gamma)$  and define  $f_\gamma(x) = f|_{W+\gamma}(x + \gamma)$ .

**Floquet transform:**

$$f(x) \mapsto \mathcal{U}f(x, k) := \sum_{\gamma \in \Gamma} f_\gamma(x) e^{lk \cdot \gamma} = \sum_{\gamma \in \Gamma} f_\gamma(x) z^\gamma$$

Vector Fourier series, coefficients – pieces of  $f$  on  $W + \gamma$ .

$$\mathcal{U}f(x, k + k') = \mathcal{U}f(x, k), k' \in \Gamma^*$$

Dedication  
Instead of detailed references  
**Preliminaries**  
Dispersion relations and all that  
Spectra and solutions  
Some conferences  
Some references

1D - just a brief sketch  
Lattices  
Periodic operators  
**Floquet transform. Direct integral decomposition**

## Floquet transform inversion

Two inversion formulas:

$$f(x) = \int_{\mathbb{T}^*} \mathcal{U}f(x, k) dk, \quad x \in \mathbb{R}^n$$

or

$$f(x) = \int_{\mathbb{T}^*} \mathcal{U}f(x - \gamma, k) e^{ik \cdot \gamma} dk, \quad x \in W + \gamma$$

# Plancherel and Paley-Wiener type theorems

Vector valued analogs of Fourier series theorems give:

## Range Theorems

- $\mathcal{U}$  is isometry from  $L^2(\mathbb{R}^n)$  onto  $L^2(\mathbb{T}^*, L^2(W))$
- $\mathcal{U}$  is isometry from  $H^s(\mathbb{R}^n)$ ,  $s > 0$  onto a proper subspace of  $L^2(\mathbb{T}^*, H^s(W))$
- Exponential decay of  $f \iff$  analyticity of  $\mathcal{U}f$  in a complex neighborhood of the torus  $\mathbb{T}^*$ .
- Usual correspondences hold between power decay of  $f$  and smoothness of  $\mathcal{U}f$  on  $\mathbb{T}^*$ .



## What's “wrong” with $H^s$ ?

A simple explanation: In  $L^2$  there are no junction conditions between  $f$  on different copies  $W + \gamma$ . In  $H^s$  there are some!  
 A different look at Floquet transform:

$$f(x) \mapsto \mathcal{U}f(x, k) := \sum_{\gamma \in \Gamma} f(x - \gamma) e^{ik \cdot \gamma} = \sum_{\gamma \in \Gamma} f(x - \gamma) z^\gamma$$

No restriction  $x \in W$  imposed! Then calculate that for any  $x \in \mathbb{R}^n, \gamma \in \Gamma$

$$\mathcal{U}f(x + \gamma, k) = e^{ik \cdot \gamma} \mathcal{U}f(x, k). \quad (A_k)$$

I.e.,  $\mathcal{U}f(x, k)$  is  $\Gamma$ -**automorphic (cyclic, quasi-periodic)** w.r.t.  $x$  with the character  $e^{ik \cdot \gamma}$ .

$$H_k^s(W) := \{f|_W \mid f \in H_{loc}^s(\mathbb{R}^n) \text{ s.t. } f \text{ satisfies } (A_k)\} \quad (L_k^2 = L^2.)$$

## Linear bundles over $\mathbb{T}$ .

### $H^s$ Range Theorems

*The range theorems work in  $H^s$ , if  $L^2(\mathbb{T}^*, H^s(W))$  is replaced by the space  $L^2(\mathbb{T}^*, \mathcal{H}^s)$  of  $L^2$ -sections of the (analytic) vector bundle  $\mathcal{H}^s := \bigcup_{k \in \mathbb{T}^*} H_k^s(W)$  over  $\mathbb{T}^*$ .*

**Another interpretation** of  $\mathcal{U}f(x + \gamma, k) = e^{ik \cdot \gamma} \mathcal{U}f(x, k)$  (i.e., condition  $(A_k)$ ) is that rather than functions of  $x \in W$ , one deals with sections of a (depending on  $k$ ) linear bundle  $L_k$  over  $\mathbb{T}$ .

Its **advantage** is in dealing with elliptic equations on a closed smooth manifold  $W$  rather than elliptic BVP on non-closed and non-smooth  $W$ .

## Direct integral decomposition

Return to periodic operator  $H = -\Delta + V(x)$  in  $\mathbb{R}^n$ . Due to periodicity, it preserves the automorphicity condition  $(A_k)$ . Thus, it defines for each  $k$  the operator  $H(k)$  in  $L^2(W)$  with the domain  $H_k^2$ . Alternatively,  $H(k)$  is the restriction of  $H$  onto the sections of the linear bundle  $L_k$  over  $\mathbb{T}$ .

– **An elliptic operator in sections of a linear bundle! Elliptic theory applies.**

**Direct integral decompositions:**

$$L^2(\mathbb{R}^n) = \int_{\mathbb{T}^*}^{\oplus} L^2(W), \quad H^s(\mathbb{R}^n) = \int_{\mathbb{T}^*}^{\oplus} H_k^s, \quad H = \int_{\mathbb{T}^*}^{\oplus} H(k)$$

## Explicitly, what is $H(k)$ ?

- The operator  $H = -\Delta + V(x)$  on  $W$  with the domain  $H_k^2$ , i.e. with the cyclic boundary conditions  $(A_k)$ .
- Alternatively, the operator
$$-\Delta - 2ik \cdot \nabla + k^2 + V(x) \text{ on } \mathbb{T} \text{ (i.e., on } \Gamma\text{-periodic functions)}$$
- There are advantages and disadvantages of both representations. Use judiciously.

## Dispersion relations, etc. Brief content

- Dispersion relation = Bloch Variety
- Analyticity
- Irreducibility, absence of flat components
- Spectral edges - location and structure
- Dirac cones
- Bloch bundles
- Fermi surfaces
- Inverse problems

# Dispersion relation = Bloch variety

- $H(k)$  is bounded below and has a discrete (real) spectrum

$$\sigma(H(k)) = \{\lambda_j(k) | \lambda_1(k) \leq \lambda_2(k) \leq \dots \leq \lambda_d(k), \dots \mapsto \infty\}$$

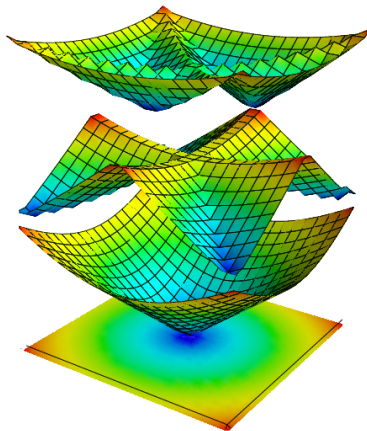
- $\lambda_j(k)$  – the  $j$ th **band function**. Continuous, p.-wise analytic.
- $k \mapsto \sigma(H(k))$  – **dispersion relation** / **Bloch variety**  $B_H$  of  $H$
- In other words,

$$B_H = \{(k, \lambda) \in \mathbb{C}^{n+1} \mid H(k)u = \lambda u \text{ has a non-zero solution}\} \subset \mathbb{C}^{n+1}$$

- The ability of considering the **complex** Bloch variety turns out to be very important.

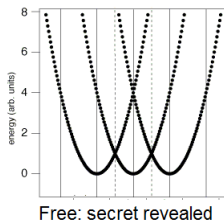
# Bloch variety

An example:



## Free case

For the free operator  $H = -\Delta$ ,  $B_H$  is the union of the paraboloid  $\lambda = k^2$  and its  $\Gamma^*$ -shifts:



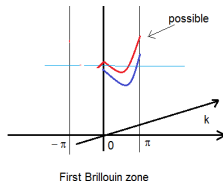


# Dispersion relation and the spectrum

- The **spectrum**:

$$\sigma(H) = \cup_k \sigma(H(k)) = \cup_j (\cup_k \{\lambda_j(k)\}) = \cup_j I_j$$

- $I_j$  - range of  $\lambda_j(k)$ ,  $j$ th **spectral band**. Bands **can** overlap!



- Choosing a **dense subset**  $S \subset B$ , one has often useful formula

$$\sigma(H) = \overline{\cup_{k \in S} \sigma(H(k))}$$

# Analyticity

We return to the (complex) Bloch variety  $B_H \subset \mathbb{C}^{n+1}$ .

## Theorem

*There exists an entire function  $f(k, \lambda)$  on  $\mathbb{C}^n$ , such that*

- 1.  $|f(k, \lambda)| \leq C_p e^{(|k|+|\lambda|)^p}$  for any  $p > n$ .*
- 2.  $B_H = \{(k, \lambda) \in \mathbb{C}^{n+1} \mid f(k, \lambda) = 0\}$*
- 3. Replacing  $p > n$  with  $p > n + 1$ , one can make sure that  $f$  is  $\Gamma^*$ -periodic w.r.t.  $k$ .*

- In particular,  $B_H$  is a  $\Gamma^*$ -periodic complex analytic sub-variety of co-dimension 1 in  $\mathbb{C}^{n+1}$
- The proof uses the theory of **regularized determinants** in Shatten-von Neumann classes.

## Floquet variety $B_H/\Gamma^*$

It is natural to reduce  $B_H$  w.r.t.  $\Gamma^*$ -periodicity (using Floquet multipliers instead of crystal momenta).

Introduce the notion (not commonly accepted):

**Floquet variety of  $H$**  is

$$B_H/\Gamma^* := \{(z, \lambda) \in (\mathbb{C}^*)^n \times \mathbb{C} \mid z = e^{ik} = (e^{ik_1}, \dots, e^{ik_n}), \quad (k, \lambda) \in B_H\},$$

where  $\mathbb{C}^*$  is the punctured complex plane  $\mathbb{C} \setminus \{0\}$ .

**In the free case**,  $B_H$  is the set of  $\Gamma^*$ -shifts of a single paraboloid, we see that

1. All irreducible components of  $B_H$  (shifted paraboloids) are algebraic.
2. The Floquet variety  $B_H/\Gamma^*$  is irreducible.

## Floquet variety $B_H/\Gamma^*$ , non-algebraicity

- In general, the irreducible components of Bloch variety are **not algebraic**.
- **Non-selfadjoint operators  $H$  with algebraic components of  $B_H$  do exist.**
- The **Floquet varieties** are **algebraic for discrete operators**.

# Irreducibility

What about irreducibility?

**Conjecture** *The Bloch variety of a self-adjoint  $H = -\Delta + V(x)$  is irreducible modulo  $\Gamma^*$ .*

**Theorem** *This is true in dimension 2. (Knörrer & Trubowitz)*

**Q:** Are the different  $\Gamma^*$ -shifts of a component, different components by themselves?

**Conjecture / Theorem?** *Unless an irreducible component  $C \subset B_H$  is algebraic, it is invariant with respect to some  $\Gamma^*$ -shifts.*

## Irreducibility II – no flat components

A simpler question (on its importance later):

**Q:** *Can  $B_H$  have any flat components  $\lambda = \text{const}$ ?*

Rephrasing: Can  $\lambda \in \mathbb{C}$  exist such that  $H(k)u = \lambda u$  has a non-zero solution for any  $k \in \mathbb{C}^n$ ?

**A: Yes**, it can for some s.-a. periodic elliptic operators, but

**Conjecture:** The answer is **NO** for elliptic **second order** operators with sufficiently “decent” coefficients.

Proven (see discussion later on) in many cases by many authors (Thomas, Simon, Danilov, Sobolev, Birman & Suslina, Shen, Morame, Friedlander, P.K. & Levendorskii, ... )

**In the periodic graph case, flat components can arise.**

## Where are spectral edges?

In  $1D$ , as we have seen, the extrema of the dispersion relation  $\lambda(k)$  occur only at  $k = 0$  and  $k = \pi$  in the first Brillouin zone.

This was due to the monotonicity.

$\Gamma^*$  is invariant with respect to the symmetry group generated by its shifts and “time reversion”  $k \mapsto -k$ . Points  $k = 0, \pi$  are the fixed by some of these symmetries.

In higher dimensions, the symmetry group can be larger.

**Popular belief (incorrect 🙄):** *For  $-\Delta + V(x)$  the extrema of dispersion relation must be attained at fixed points  $k \in \mathcal{B}$  of some symmetries (probably at the highest symmetry points).*

*Thus, computing the dispersion only around the symmetry points gives the correct spectrum as a set.*

Dedication  
Instead of detailed references  
Preliminaries  
**Dispersion relations and all that**  
Spectra and solutions  
Some conferences  
Some references

Dispersion relation = Bloch variety and its properties  
Fermi surfaces  
Inverse problems

## Where are spectral edges? II

No monotonicity (or any other) reason for this exists.

Some numerical evidence against this belief has been around, but not widely known/believed.

Discussed and disproved analytically (Harrison-P.K.-Sobolev-Winn, Exner-P.K.-Winn).



## Generic structure of spectral edges

Conjecture *Generically, the extrema of band functions*

- 1. are attained by a single band;*
- 2. are isolated;*
- 3. are non-degenerate, i.e. have non-degenerate Hessian.*

Why? Like “generic” family of s.a. matrices.

Counting dimensions.

**Theorem** (Klopp & Ralston) *1. holds.*

## Generic structure of spectral edges. II

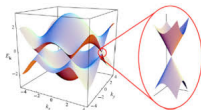
A bootstrap idea: a. prove for discrete periodic graphs  $\Rightarrow$  b. quantum graphs  $\Rightarrow$  c. full dimension.

The main difficulty lies in a.

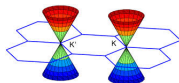
**Theorem** (Do & P.K. & Sottile) *a. holds in 2D for graphs with two atoms per unit cell.*

# Dirac cones

Dirac cone:



Some symmetries protect such a cone. Honeycomb, Graphene.



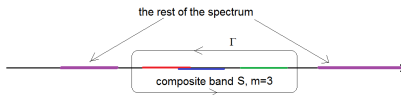
Analytically established for graph/quantum graphs (P.K.-Post) and Schrödinger operators (Grushin, Fefferman-Weinstein, Berkolaiko-Comech).

Other, non-honeycomb-symmetric cases (Do-P.K. and ref.

thorstein). No complete understanding at the moment.

## Bloch bundles

$S \subset \sigma(H)$  – subset consisting of  $m$  bands and surrounded by spectral gaps (a **composite band**):



Surround it with a contour  $\Gamma$  and introduce the  $m$ -dimensional spectral projector for  $H(k)$ :

$$P(k) := \frac{1}{2\pi i} \oint_{\Gamma} (\zeta - H(k))^{-1} d\zeta.$$

## Bloch bundles II

Projectors  $P(k)$  depend analytically on  $k$  in a complex neighborhood of  $\mathbb{T}^*$  in  $\mathbb{C}^n$ .

Its ranges form the **Bloch bundle** over this neighborhood corresponding to the composite band  $S$ .

Sections of this bundle form the invariant subspace for  $H$  that corresponds to  $S \subset \sigma(H)$ .

The bundle can be non-trivial (e.g., in presence of magnetic potential) (Thouless).

**Theorem** *Topological triviality of the Bloch bundle over  $\mathbb{T}^*$  is equivalent to its analytic triviality in a complex neighborhood.*

## Fermi surfaces

Consider the dispersion relation  $B_H$  as the graph of the multiple valued function

$$k \in \mathcal{B} \mapsto \sigma(H(k)).$$

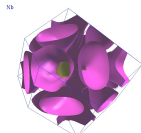
**Definition** The **Fermi surface**  $F_{\lambda,H}$  of  $H$  at a scalar value  $\lambda$  is the level  $\lambda$  set of the dispersion relation. I.e.,

$$\begin{aligned} F_{\lambda,H} &:= \{k \in \mathbb{C}^n \mid (k, \lambda) \in B_H\} \\ &= \{k \in \mathbb{C}^n \mid H(k)u = \lambda u \text{ has a non-zero solution}\}. \end{aligned}$$

**Theorem**  $F_{\lambda,H}$  is the zero set of an entire function  $f(k)$  of the same exponential order as for  $B_H$ .

## Fermi surfaces II

Just to scare you: an example of Fermi surface



**Conjecture** *The Fermi surface for  $H = -\Delta + V(x)$  is irreducible (modulo  $\Gamma^*$ ) for any  $\lambda \in \mathbb{R}$  (or at least except a discrete set of points  $\lambda$ ).*

Proven for discrete Schrödinger on  $\mathbb{Z}^2$ , for  $n = 2, 3$  and separable potential, and for  $n = 3$  and potential  $U(x_1, x_2) + V(x_3)$ , where the axis are oriented along the basis vectors of  $\Gamma$ .

## Analyticity in the space of parameters

One can add to the spectral parameter  $\lambda$  and quasimomenta  $k$  an infinite-dimensional space  $P$  of parameters of the periodic operator (i.e., an appropriate Banach space of periodic potentials)

It is very often useful to refer to

**Meta-theorem** *Bloch and Fermi surfaces are analytic subsets in the enlarged space  $\mathbb{C}^{n+1} \times P$  (respectively  $\mathbb{C}^n \times P$ ).*

Easy to prove in each particular case.



## Inverse problems. Borg's theorem

### **Borg Theorem in 1D TFAE:**

1. *the potential is constant.*
2. *There are no spectral gaps.*
3. *There exists an entire function, whose graph  $\lambda = f(k)$  belongs to the dispersion relation. (Avron & Simon)*

Equivalence 1.  $\Leftrightarrow$  2. fails miserably in dimensions  $n > 1!!$

### **Conjecture TFAE:**

1. *the potential is constant.*
2. *There exists an entire function, whose graph  $\lambda = f(k)$  belongs to the dispersion relation.*

Proven in 2D (Knörrer & Trubowitz).

## Inverse problems

**Remark:**  $V(\pm x_1 + a_1, \dots, \pm x_n + a_n)$  and  $V(x)$  with the lattice  $\mathbb{Z}^n$ .

**Theorem** (Gieseke-Knörrer-Trubowitz). *Let  $H_j = -\Delta + V_j$  be  $\Gamma$ -periodic discrete on  $\mathbb{Z}^2$ . There exists Zariski open set  $L \subset L^2(\mathbb{Z}^2/\Gamma)$  such that if  $V_1 \in L$ ,  $V_2 \in L^2(\mathbb{Z}^2/\Gamma)$  and  $B_{H_1} = B_{H_2}$ , then  $V_2(x, y) = V_1(\pm x + x_0, \pm y + y_0)$ .*

**Theorem** (Kappeler). *Analog holds for any  $n \geq 2$ .*

## Floquet rigidity

Analogous, but more restricted and less explicit results for periodic Schrödinger operators in  $\mathbb{R}^n$  by Eskin-Ralston-Trubowitz (ERT).

Various relations established between isospectrality of periodic problems (i.e. for  $k = 0$ ), equality of Bloch varieties, and isospectrality for  $1D$  “sub-potentials.”

Usually analyticity of potentials and some arithmetic restrictions on the lattice are required.

Microlocal analysis techniques heavily involved.

Largely untouched after ERT.

## Spectra and solutions. Brief content

- Spectral gaps - existence and number
- Eigenfunction expansion
- Types of spectra
- Density of states
- Floquet-Bloch solutions
- Shnol'-Bloch theorem
- Wannier functions
- Liouville theorems

## Band-gap structure: $1D$ vs higher dimensions

	$n = 1$	$n > 1$
overlaps	none	frequent
gaps	generically all open	open set of gapless potentials
no gaps	constant potential	all small potentials
free operator bands	touch to cover $\mathbb{R}^+$	overlap to cover $\mathbb{R}^+$

## Gaps - Bethe-Sommerfeld conjecture

**Bethe-Sommerfeld Conjecture (BSC):** *When  $n > 1$ , there can only be a finite number of gaps in the spectrum of any periodic Schrödinger operator.*

Has been proven in full generality, mostly due to the impressive works by Dahlberg & Trubowitz, L. Parnovsky, M. Skriganov, and A. Sobolev.

**A stronger BSC:-** *for any second order periodic elliptic operator, waveguide problems, etc.*

Proven for  $(-\Delta)^m +$  lower order periodic terms (e.g., magnetic periodic Schrödinger), mostly due to Parnovsky and Sobolev, as well as Barbatis, Karpeshina, Morame, and Veliev.

**BSC fails for periodic graph/quantum graph operators.**

## Abelian coverings

Consider the standard covering  $\mathbb{R}^n \mapsto \mathbb{T} (= \mathbb{R}^n / \mathbb{Z}^n)$ . Its deck group is  $\mathbb{Z}^n$ , which coincides with  $H_1(\mathbb{T}, \mathbb{Z})$ .

It can also be understood as the quotient of the universal cover of the torus by the commutator subgroup of the fundamental group  $\pi_1(\mathbb{T})$ .

**Definition:** *The maximal abelian covering of a compact manifold  $X$  is the covering  $Y \mapsto X$  with the deck group  $H_1(X, \mathbb{Z})$ , obtained as the quotient of the universal cover of  $X$  by the commutator subgroup of the fundamental group  $\pi_1(X)$ .*

**Theorem** (Sunada) *Let  $Y \mapsto X$  be a Riemannian covering with an amenable deck group. Then  $\sigma(-\Delta_X) \subset \sigma(-\Delta_Y)$ .*

## No gap conjecture

Thus, the spectrum of the Laplacian on the maximal abelian covering is the largest (and thus has the fewest gaps) among all abelian coverings of the same base.

**No gap conjecture:** (Sunada) *Let  $Y \mapsto X$  be the maximal abelian covering of a compact Riemannian manifold  $X$ . The spectrum of the Laplace-Beltrami operator  $-\Delta_Y$  on  $Y$  has no gaps.*

**Not proven.**

**Analogous conjecture for regular graphs** proven for all graphs of even degree and in various examples for graphs of odd degree.



## Gap creation

**Schrödinger operator**  $-\Delta + V(x)$ .

Spectral gaps can be created by periodically placing (at large distances) potential wells.

On **more general periodic coverings**, gaps can be created by making the geometry “high contrast” (Post, Hempel).

The potential well technique fails for **Maxwell** and some other operators, but (very) high contrast approach works (Figotin & P.K.).

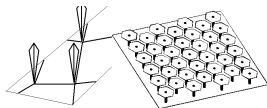
**Moral:** manipulating gaps in higher dimensions is hard!

## Resonant gap opening

A **different promising gap opening mechanism** : by spreading small identical resonators throughout the medium. No periodicity is needed. Allows gaps manipulation.

Suggested in continuous setting in 1987 (Pavlov). **Never made rigorous.**

Done for **graphs** (Aizenman & Schenker), quantum graphs (P.K.).



More complex “decorations” needed. Compare with expander constructions.

## Eigenfunction expansion

- eigenvalues  $\lambda_j(k)$  - continuous, piece-wise analytic
- Bloch eigenfunctions  $\psi_j(\cdot, k)$  - measurable , piece-wise analytic
- eigenfunction expansion  $f(\cdot, k) \mapsto \sum_j f_j(k) \psi_j(\cdot, k)$
- **The operator splits into direct sum of operators of multiplication by  $\lambda_j(k)$ !**

## Spectrum: singular, pure point, absolute continuous

The good model for understanding the structure of the spectrum is the operator of multiplication by a continuous, piece-wise analytic scalar function  $a(k)$ . The following is easy to establish:

**Proposition 1.** *There is no singular continuous spectrum. 2.*

$\lambda \in \sigma_{pp}$  if and only if there is a constant (flat) piece of the graph at the level  $\lambda$ , i.e. the level  $\lambda$  set of  $a(k)$  has positive measure.

Since the decomposition  $H = \int \oplus H(k)$  essentially reduces  $h$  to multiplication by the multivalued, continuous piecewise analytic function  $k \mapsto \{\lambda_j(k)\}_{j=1}^{\infty}$ , the argument works here as well, leading to

**Theorem 1.**  $\sigma_{sing}(H) = \emptyset$ . 2.  $\sigma_{pp}(H) = \emptyset$ , iff the Bloch variety  $B_H$  has no flat components.

## Absolute continuity

**Corollary** *The spectrum of  $H$  is absolutely continuous, iff  $B_H$  has no flat components.*

Since the famous work by L. Thomas in 70s, all proofs were based on proving absence of flat components.

Due to the often very technical work of many researchers (Thomas, Simon, Danilov, Sobolev, Birman & Suslina, Shen, Morame, Friedlander, P.K. & Levendorskii, Sunada, ... ), absolute continuity has been proven for many (**but still not all**) second order elliptic periodic operators, as well as periodic Pauli, Dirac, and in some cases Maxwell operators.

**Problem** *Prove absolute continuity for all second order periodic elliptic scalar operators with smooth coefficients.*

## Absolute continuity II

- The least technical approach was due to L. Friedlander. Regretfully, it requires some symmetry condition on the operator , which seems to be superfluous, but no one has succeeded in removing it.
- Absolute continuity **fails for some periodic elliptic operators of higher order**. The Plis' example of a 4th order operator which fails unique continuation property, i.e. has a compactly supported eigenfunction, can be massaged to produce a periodic counterexample, where  $\sigma_{pp} \neq \emptyset$ .

This relation to uniqueness of continuation theorems is not accidental.

## Absolute continuity III

- Filonov constructed an example of a 2nd order periodic elliptic operator with the leading coefficients just below Lipschitz condition with non-empty pure point spectrum.

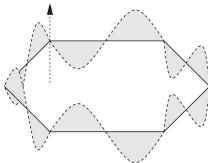
- Another evidence (and possible approach):

**Theorem** (P.K.) *Existence of an  $L^2$ -eigenfunction is equivalent to existence of a (different) super-exponentially decaying eigenfunction, i.e. such that  $|u(x)| \leq Ce^{-|x|^\gamma}$ , with some  $\gamma > 1$  (depending on the order of the operator and dimension).*

Existence of such a solution should violate a “unique continuity at infinity” result. Regretfully, the only such result known, due to Froese&Herbst& M.&T. Hoffman-Ostenhof and independently Meshkov, is not strong enough to lead to new absolute continuity theorems.

## Absolute continuity IV

- Equations on discrete or quantum graphs do not obey uniqueness of continuation. And sure enough, one can find compactly supported eigenfunctions for 2nd order elliptic periodic operators:





# Density of states

**Integrated density of states:**

$$\rho(\lambda) := \lim_{N \rightarrow \infty} \frac{1}{V_N} \# \{ \lambda_j < \lambda \} = \sum_j \mu \{ k \in \mathbb{T}^* \mid \lambda_j(k) < \lambda \}$$

Piecewise analytic! **Density of states**

$$g(\lambda) := \frac{d\rho}{d\lambda} = (2\pi)^{-n} \sum_j \int_{\lambda_j=\lambda} \frac{ds}{|\nabla_k \lambda_j|}$$

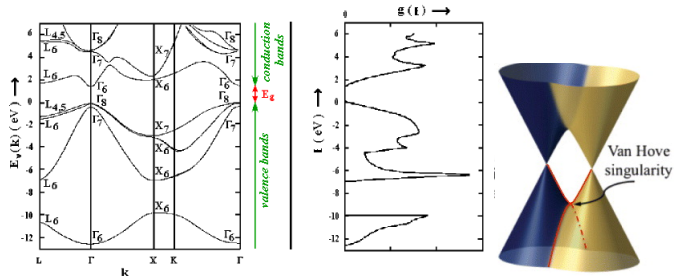
Integral of a holomorphic form over a real cycle on the Fermi surface  $F_\lambda$

Piece-wise analytic.

Dedication  
 Instead of detailed references  
 Preliminaries  
 Dispersion relations and all that  
**Spectra and solutions**  
 Some conferences  
 Some references

Spectral gaps  
 Spectra

## Density of states II. Van Hove singularities

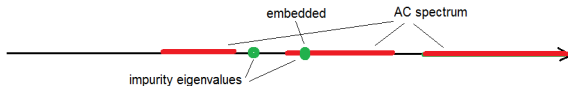


Van Hove singularities

## Embedded eigenvalues

Localized perturbation  $W(x)$  added to  $V(x)$  can only introduce new eigenvalues of finite multiplicity.

**Q:** Can embedded eigenvalues appear?



**A: Yes**, if  $V = 0$  and if the perturbation does not decay sufficiently fast (Wigner and von Neumann).

**A: No**, if  $V = 0$  and if the perturbation decays sufficiently fast.

**Q:** What happens in the periodic case?

**A:** In 1D the situation is similar: finiteness of the first moment implies absence of embedded eigenvalues (Roze-Beketov).

## Embedded eigenvalues, $n > 1$

The periodic situation in higher dimensions is much trickier.

**Theorem** (P.K. & Vainberg) *Let  $n \leq 4$  and the perturbation decay exponentially. If for a given  $\lambda \in \sigma(H)$  the Fermi surface at that level is irreducible, then  $\lambda \notin \sigma_{pp}(H + W)$ .*

*In particular, if all Fermi surfaces are irreducible, no embedded eigenvalues can arise.*

The irreducibility condition (or a weaker its version) seem to be significant: there are ODE (V. Papanicolaou) and quantum graph (Shipman) examples where irreducibility fails and embedded eigenvalues do arise.

**Theorem** (P.K. & Vainberg) *In the graph case, embedded eigenvalues might arise, but must be supported close to the support of the perturbation.*

## Threshold effects: Homogenization, Liouville theorems, etc.

- Homogenization
- Liouville theorems. (see [KP] and ref. therein and by the same authors Trans. AMS 359 (2007), no. 12, 5777-5815.)

**Theorem** (Classical Liouville theorem) *Any harmonic function in  $\mathbb{R}^n$  of polynomial growth of order  $N$  is a polynomial of degree  $N$ . The dimension of this space is*

$$\binom{n+N}{N} - \binom{n+N-2}{N-2}$$

. S.T. Yau's problem. Colding & Minicozzi. Periodic case - Avellaneda & Lin, Moser & Struwe, P.K. & Pinchover.

## Parabolic time-periodic equations

Fluid dynamics problems ask for analog of Floquet theory (e.g., completeness of and expansion into Floquet solutions) for parabolic time periodic problems in Banach/Hilbert spaces:

$$\frac{dx}{dt} = A(t)x, x(0) = x_0, A(t+1) = A(t) \quad (***).$$

E.g., the heat equation

$$\frac{du}{dt} = \Delta_x u + b(x, t) \quad (****)u$$

with time periodic  $b(x, t)$  in a cylinder, with say Dirichlet or Neumann BC on the boundary.

## Parabolic time-periodic equations II

There is a version of Floquet theory for hypoelliptic (e.g., parabolic) equations (P.K.). However, **the situation is embarrassing**. Even for the heat equation (\*\*\*\*), existence and completeness of Floquet solutions is proven only under very stringent conditions on  $b(x, t)$ . (Chow & Lu & Mallet-Paret, Miloslavskii, P.K.)

General operator theory methods fail, since there are perfect examples of abstract parabolic equations (\*\*\*\*), which do not have even a single Floquet solution. 🙄.

**The matter is up for grabs!**

Dedication  
Instead of detailed references  
Preliminaries  
Dispersion relations and all that  
**Spectra and solutions**  
Some conferences  
Some references

Spectral gaps  
**Spectra**

# Hyperbolic

Time periodic hyperbolic and Schrödinger type equations attract a lot of attention in physics.

However, most of the analytic theory described before fails here and other techniques are required.

The attendees are referred, for instance, to Yajima [Ya] and references therein.



## Skipped 🙄

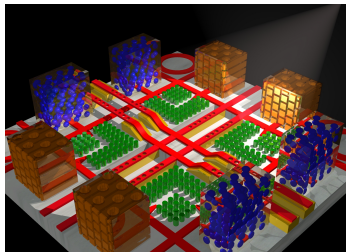
- Floquet-Bloch expansions. See [KF].
- Shnol' - Bloch theorem. See [KF].
- Wannier functions in presence of non-trivial Bloch bundles. See [KW].



- Positive solutions. See [KF] and Mem. AMS by V. Lin and Y. Pinchover.
- Inhomogeneous equations. See [KF]

## Skipped II

- Photonic crystals. See [KPBG]



- Slow light.



- Waveguides.

Dedication  
Instead of detailed references  
Preliminaries  
Dispersion relations and all that  
Spectra and solutions  
**Some conferences**  
Some references

## Conferences

- Mathematics of Novel Materials, 1 - 5 June, 2015, Mittag-Leffler Institute, Stockholm, Sweden.
- Mathematical Approach to Topological Phase in Spintronics, October 2015, Tohoku University and Advanced Institute for Material Research, Sendai, Japan
- Spectral Theory of Novel Materials, April 18 - 22 2016. Centre International de Rencontres Mathematiques, Marseille, France.

Dedication  
Instead of detailed references  
Preliminaries  
Dispersion relations and all that  
Spectra and solutions  
Some conferences  
Some references

## Ref. Disclosure

### Another Good Faith Disclosure:

The list of references below is FAR from being near to being partially complete. Most of the references given are not to the original works, but to books and surveys, where one can find the more detailed bibliography.

*Italics* indicates *book titles* (no publisher or year shown).

The titles of individual papers are not provided, only their authors, journal, volume, year, pages..

## Ref. 1D

- [De] Demidovich, *Leksii po matematicheskoi teorii ustoichivosti* (Russian, Polish)
- [Ea] Eastham, *The Spectral Theory of Periodic Differential Equations*
- [MW] Magnus & Winkler, *Hill's equation*
- [RS4] Reed & Simon, *Methods of modern math. physics*, v. 4
- , [YS] Yakubovich, Starzhinskii, *Linear differential equations with periodic coefficients*. v1– 2.

## Ref. Preliminaries

- [Ea,RS4]
- [AM] Ashcroft and Mermin, *Solid State Physics*
- [KF] Kuchment, *Floquet theory for partial differential equations*
- [KUs] Kuchment, Russian Math. Surveys, 37:4 (1982), 1–60
- [ParSob] Parnowski & Sobolev, Ann. H. Poincare, 2:3 (2001), 573–581.
- [Skr] Skriganov, *Geometric and arithmetic methods in the spectral theory of multidimensional periodic operators*
- [STC] Sunada, *Topological Crystallography*

## Ref. Dispersion relations, etc.

- [AM,KF,KUs,RS4,Skr]
- [AS] Avron & Simon, Ann. Phys. 110:1 (1978), 85–101.
- [BC] Berkolaiko & Comech, arXiv:1412.8096
- [Castro] Castro Neto, Guinea, Peres, Novoselov, Geim, Rev. Mod. Phys. 81 (2009), 109–162
- [CdV] Colin de Verdiere, Mém. Soc. Math. France (N.S.) 46 (1991), 99–110
- [ERT] Eskin, Ralston, Trubowitz, Com.Pure Appl.Math. 37 (1984), no.5, 647–676; no.6, 715–753.
- [FW] Fefferman and Weinstein, JAMS 25(2012), 1169–1220.

## Ref. Dispersion relations, etc. II

- [Ger] C. Gerard, *Resonance theory for periodic Schrödinger operators*
- [GKTb] Gieseke, Knörrer, Trubowitz, *The geometry of algebraic Fermi curves*
- [GKT] Gieseke, Knörrer, Trubowitz, Contemp. Math. v 116, AMS, 1991, 19-46.
- [YK] Karpeshina, *Perturbation theory for the Schrödinger operator with a periodic potential*
- [Graph] Katsnelson, *Graphene. Carbon in two dimensions*
- [KirSi] Kirsch & Simon, J. Funct. Anal. 75 (1987), 396410.



## Ref. Dispersion relations, etc. III

- [KIRal] Klopp & Ralston, Meth. Appl. Anal. **7** (2000), 459–463.
- [KT] Knörrer, H. and Trubowitz, E., Comment. Math. Helv., 65 (1990), no.1, 114–149.
- [KuPo] Kuchment & Post, CMP 275:3 (2007) 805–826.

## Ref. Spectra and solutions

- [Ea,KF,KUs,RS4,Skr,YK]
- [BirS] Birman, Suslina, St. Petersburg Math. J. 11 (2000), no. 2, 203232
- [Bush] Busch et al, J. Phys.: Condens.Matter **15** 2003, R1233–56
- [DT] Dahlberg & Trubowitz, Comm. Math. Helv. 57(1982), 130–134.
- [LFr] L. Friedlander, CMP, 229:1 (2002), 49–55.
- [KW] Kuchment, J. Phys. A. 42:2 (2009), 025203

## Ref. Spectra and solutions II

- [KLev] Kuchment & Levendorskii, Trans. AMS 354:2 (2002), 537–569.
- [KP] Kuchment & Pinchover, JFA 181:2 (2001), 402–446
- [KV] Kuchment & Vainberg, Comm. PDE 25(2000), 1809–1826
- [MarW] Marzari & Vanderbilt, Phys. Rev. B **56** (1997), no. 20, 12847–12865.
- [Par] Parnowski, Ann. H. Poincaré, 9 (2008), no. 3, 457–508
- [ParS] Parnowski & Sobolev, Invent. Math. 181 (2010), no.3, 467–540.
- [Sob] A. Sobolev, Invent. Math. 137 (1999), no.1, 85–112.

## Ref. Miscellanea

- [KF,KUs]
- [At] Atiyah, *Astérisque*, No. 32-33 (1976), 43–72.
- [TopIns] Bernevig, *Topological Insulators and Topological Semiconductors*
- [BSH] Birman & Suslina, *J. Math. Sci. (N. Y.)* 136 (2006), no. 2, 36823690.
- [LFRW] Friedlander, *Contemp. Math*, v. 339 (2003), 37–42
- [KPBG] Kuchment, in *Math. modeling in optical science*, SIAM, 2001, 207–272.
- [N] Novikov, Two-dimensional Schrödinger operators in periodic fields, *Current problems in mathematics*, Vol. 23, 1983, 3–32.

## Ref. Miscellanea II

- [SW] Sobolev, Walthoe, Proc. London Math. Soc. 85:3 (2002), 717–741.
- [Ya] Yajima, Time-periodic Schrödinger equations. in *Topics in the theory of Schrödinger operators*, 969, 2004

Dedication  
Instead of detailed references  
Preliminaries  
Dispersion relations and all that  
Spectra and solutions  
Some conferences  
Some references

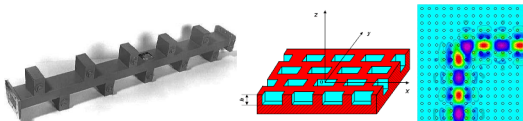
Thank you

Thank you for your patience!

Dedication  
Instead of detailed references  
Preliminaries  
Dispersion relations and all that  
Spectra and solutions  
Some conferences  
Some references

# Waveguides

**A periodic waveguide:** a tubular (or planar) periodic domain with a periodic elliptic boundary value problem inside.



One asks the same questions:

- **Is the spectrum waveguide AC? Proven for many, but not all situations** (Birman & Suslina, Derguzov, Friedlander, Hoang & Radosz, P.K., Shterenberg, Sobolev & Walthoe)
- A harder case – **“soft wall” guides** (Filonov, Frank, Hoang, Klopp, Shterenberg)
- **Does BSC hold for periodic waveguides? Not known.**