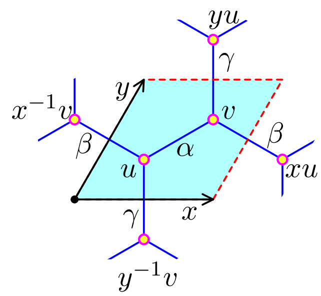
DISCRETE PERIODIC OPERATORS INTRO

MATTHEW FAUST

1. Graphene



A quick summary of background:

Notice the two vertices, u and v, these vertices together with the edges between them give us the *Fundamental domain* of the graphene.

Let L be the Laplace-beltrami operator defined by

f is just some function that takes vertices of the periodic graph to polynomials of finite support in the variables x and y. We also assume that f are also such that $f(x^m y^n u) = x^m y^n f(u)$.

Let

$$Lf(u) = \sum_{w \sim u} a_{(u,w)}(f(u) - f(w)).$$

Where $a_{(u,w)}$ is the weight of the edge between u and v. $w \sim u$ means we iterate over the neighbors of u.

So let us write out $Lf(u)$ one step at a time, first going over the summation.

$$Lf(u) = \alpha(f(u) - f(v)) + \beta(f(u) - f(x^{-1}v)) + \gamma(f(u) - f(y^{-1}v))$$

Then we use the property of f to get

$$Lf(u) = \alpha(f(u) - f(v)) + \beta(f(u) - x^{-1}f(v)) + \gamma(f(u) - f(v))$$

Then we just rewrite this in terms of $f(u)$ and $f(v)$:

$$Lf(u) = (\alpha + \beta + \gamma)f(u) - (\alpha + \beta x^{-1} + \gamma y^{-1})f(v)$$

Through a similar process we get:

$$Lf(v) = (\alpha + \beta + \gamma)f(v) - (\alpha + \beta x + \gamma y)f(u)$$

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$$L\begin{pmatrix} f(u) \\ f(v) \end{pmatrix} = \begin{pmatrix} Lf(u) \\ Lf(v) \end{pmatrix}$$

$$L = \begin{pmatrix} \alpha+\beta+\gamma & -\alpha-\beta x^{-1}-\gamma y^{-1} \\ -\alpha-\beta x-\gamma y & \alpha+\beta+\gamma \end{pmatrix}$$

We then let ϕ be the characteristic polynomial of L.

That is

$$\phi = det(L-\lambda I)$$

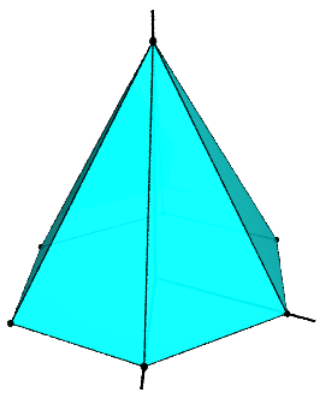
.

So we have

$$\phi = (\alpha+\beta+\gamma-\lambda)^2 - (\alpha+\beta x+\gamma y)(\alpha+\beta x^{-1}+\gamma y^{-1})$$

We then can assign values to α, β, γ and then look at the exponent vectors in terms of x, y, and λ.

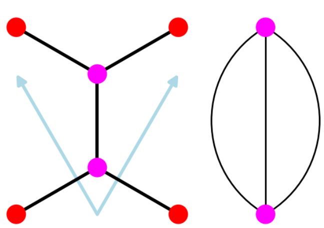
This should give us the following Newton polytope.



Some extra brief background:

There are multiple fundamental domain for the same periodic graphs. However these fundamental domain correspond to different *quotient graphs*. A topolgical crystal (or periodic graph, as the graphene is) are *covering graphs* of quotient graphs. Any finite graph that is connected with no bridges, has a covering graph (no bridges means removing a single edge will not disconnect the graph).

For example here is another drawing of the fundamental domain of graphene (a fundamental domain has internal edges and external edges).



A quotient graph can be equipped with a group (in particular which is a subgroup of something called the "fundamental group"of the graph). This quotient graph subgroup combination corresponds uniquely pairs with a periodic graph (or topological crystal).

In the example of the two vertex fundamental domain of the graphene, its quotient graph equipped with its fundamental group corresponds to it. If a fundamental domain corresponds to a quotient graph with its fundamental group, the periodic graph that fundamental domain produces is called a *Maximal abelian covering*.

There are unique conjectures pertaining to discrete periodic operators over such fundamental domains.

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