Critical Points of Discrete Periodic Operators

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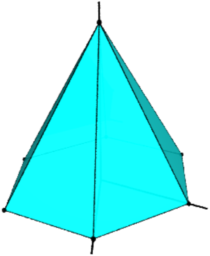
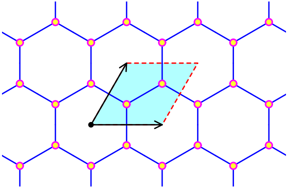
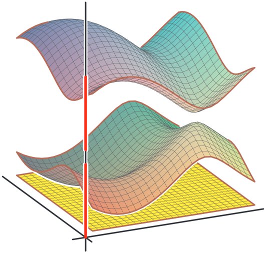


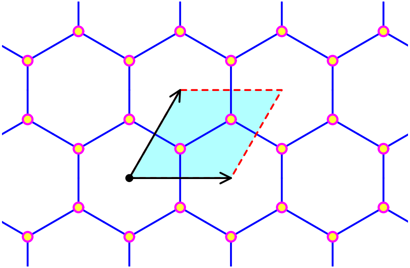




Joint work with Frank Sottile

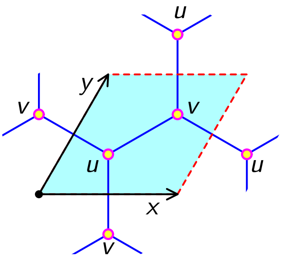
Definition

A $\mathbb{Z}^n$ **periodic graph** is a locally finite simple infinite graph Γ with finite orbits on both vertices and edges.



Graphene with a highlighted fundamental domain.

Discrete Laplace-Beltrami operator



a_e: weight of edge e.

g: function on vertices of Γ.

Laplace-Beltrami operator :

$$L_a g(v) = \sum_{w \sim v} a_{(w,v)}(g(v) - g(w))$$

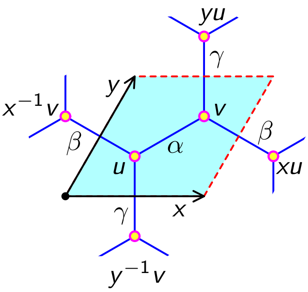
Consider quasi-periodic solutions to $L_a f = \lambda f$.

$f : V(\Gamma) \to \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}]$, such that $f(u + a) = z^a f(u)$ for $a \in \mathbb{Z}^n$.

$$z^a = z_1^{a_1} \cdots z_n^{a_n}$$

f are determined by values vertices of the fundamental domain.

Example: Graphene



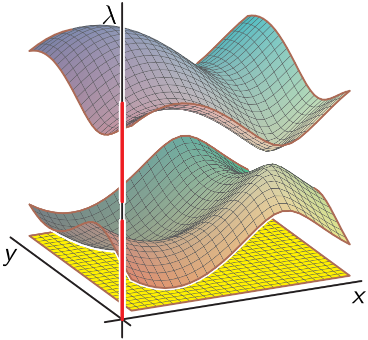
$$Lf(u) = (\alpha+\beta+\gamma)f(u)$$
$$-(\alpha+\beta x^{-1}+\gamma y^{-1})f(v)$$

$$Lf(v) = -(\alpha+\beta x+\gamma y)f(u)$$
$$+(\alpha+\beta+\gamma)f(v)$$

$$L\begin{pmatrix} f(u) \\ f(v) \end{pmatrix} = \begin{pmatrix} Lf(u) \\ Lf(v) \end{pmatrix}$$

$$L = \begin{pmatrix} \alpha+\beta+\gamma & -\alpha-\beta x^{-1}-\gamma y^{-1} \\ -\alpha-\beta x-\gamma y & \alpha+\beta+\gamma \end{pmatrix}$$

Bloch Variety and Spectral Gaps



Characteristic poly. : $\psi = \det(L - I_2\lambda)$

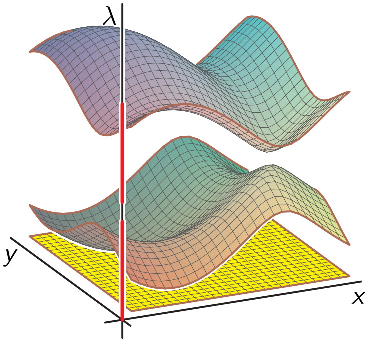
Bloch variety : $\psi = 0$

Spectral edge conjecture:

- An extremal value occurs in a single band.
- Extrema are isolated
- Extrema are non-degenerate.

Goal

We wish to work towards this conjecture by counting the critical points of the Bloch variety.



For a general Γ and L_a

BV: $\phi = \det(L_a - I_d\lambda) = 0$

$d = \#$ vertices in fundamental domain.

Critical point equations: $\Phi := \phi = z_1\dfrac{\partial\phi}{\partial z_1} = \cdots = z_n\dfrac{\partial\phi}{\partial z_n} = 0$

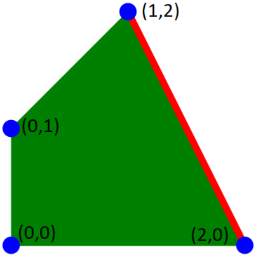
We can count solutions to Φ with Bernstein's theorem.

Newton Polytope Example

Definition

N(f) is the convex hull of the exponent vectors of f.

N(f):



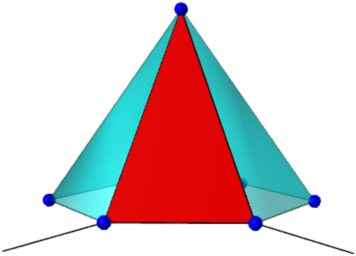
$$f = 3x^2 + 5xy^2 + 1 + 2y$$

f_F : restriction of f to a face.

Let F be the red face

$$f_F = 3x^2 + 5xy^2$$

Running Example



$$\Psi :=$$

$$\psi = (\alpha+\beta+\gamma-\lambda)^2 - (\alpha+\beta x+\gamma y)(\alpha+\beta x^{-1}+\gamma y^{-1})$$

$$(x\frac{\partial\phi}{\partial x}) = \alpha\beta(x-x^{-1}) + \beta\gamma(xy^{-1}-x^{-1}y)$$

$$(y\frac{\partial\phi}{\partial y}) = \alpha\gamma(y-y^{-1}) + \beta\gamma(x^{-1}y-xy^{-1})$$

$$\Psi_F :=$$

$$\psi_F = \lambda^2+\alpha\beta x+\alpha\gamma y$$

$$(x\frac{\partial\phi}{\partial x})_F = \alpha\beta x$$

$$(y\frac{\partial\phi}{\partial y})_F = \alpha\gamma y$$

Counting Critical Points

$$\phi = \det(L_a - I_d\lambda) \in \mathbb{C}[z_1^{\pm}, \dots, z_n^{\pm}, \lambda]$$

$$\Phi := \phi = z_1 \frac{\partial \phi}{\partial z_1} = \cdots = z_n \frac{\partial \phi}{\partial z_n} = 0$$

$$\mathrm{Vol}(\cdot) = \text{Euclidean volume.}$$

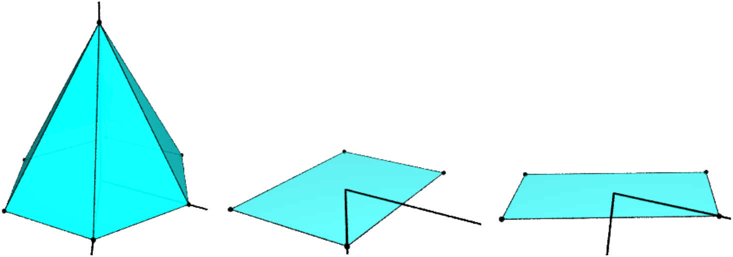
Theorem F. S.

Φ has at most $(n+1)! \cdot \mathrm{Vol}(\mathrm{N}(\phi))$ isolated solutions in $(\mathbb{C}^*)^n \times \mathbb{C}$.

Bound is exact if and only if Φ_F has no solutions for any face F.

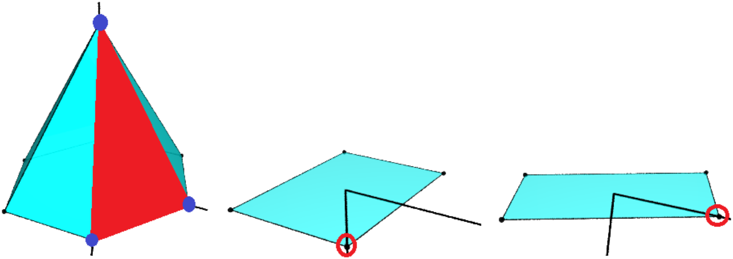
Remark: Solutions of Φ_F: singular points of BV at infinity.

Laplace-Beltrami operator over Graphene



Newton polytopes of ψ, $x\frac{\partial \phi}{\partial x}$ and $y\frac{\partial \phi}{\partial y}$

$3! \cdot Vol(N(\psi)) = 12$ critical points in $(\mathbb{C}^*)^2 \times \mathbb{C}$
(counted with multiplicity)



Newton polytopes of ψ, $x\frac{\partial \phi}{\partial x}$ and $y\frac{\partial \phi}{\partial y}$

$$\Psi_F := \lambda^2 + \alpha\beta x + \alpha\gamma y = \alpha\beta x = \alpha\gamma y = 0$$

$3! \cdot Vol(N(\psi)) = 12$ critical points in $(\mathbb{C}^*)^2 \times \mathbb{C}$
(counted with multiplicity)

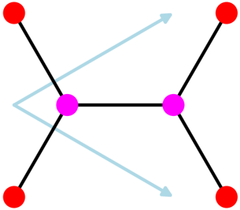
Dense Periodic Graphs

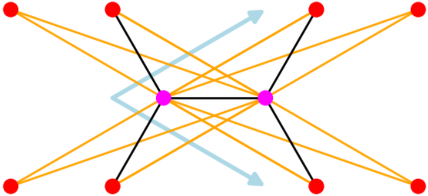
Definition

A $\mathbb{Z}^n$ **dense** periodic graph Γ has as many edges as possible:

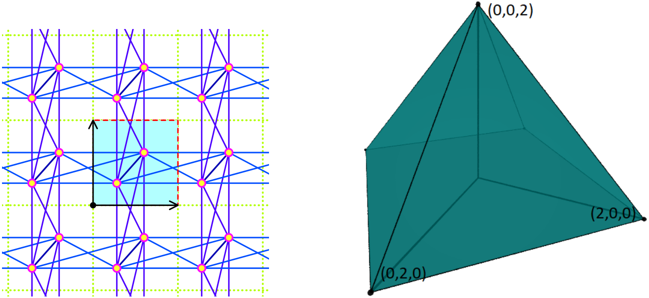
1. Fundamental domain, W, of Γ is a complete graph.
2. Edge W to $W+g \implies$ all edges possible W to $W+g$.

Every periodic graph is contained in a dense graph.





Dense periodic graph example



a dense periodic graph with polytope.

Do, Kuchment, and Sottile showed the corresponding Bloch variety has 32 critical points.

General Dense Periodic Graphs

Theorem F. S.

The Bloch variety of a dense periodic graph with generic edge weights has $(n+1)!\text{Vol}(\text{N}(\phi))$ critical points.

Idea behind proof:

Study the Discriminant of Φ_F.

Discriminant $= 0 \implies \phi_F$ is singular.

Structure of $\text{N}(\phi) \implies$ discriminant is a hypersurface.

Only vanishes for edge weights in a proper algebraic variety.

Thank you for listening.