Draft May 24, 2021

CRITICAL POINTS OF DISCRETE PERIODIC OPERATORS

M. FAUST AND F. SOTTILE

ABSTRACT. We study the spectrum of discrete periodic operators using methods from combinatorial algebraic geometry. Our main result is a bound on the number of complex critical points of the dispersion relation, together with an effective criterion for when this bound is attained. We show that this criterion holds in general for periodic graphs with sufficiently many edges. Our larger goal is to develop new methods to address the spectral edge conjecture—that for a general operator all real critical points are nondegenerate.

INTRODUCTION

The spectrum of a $\mathbb{Z}^n$-periodic self-adjoint operator consists of intervals (spectral bands) in $\mathbb{R}$. Floquet theory (Fourier expansion) reveals that the spectrum is the projection of a hypersurface in $(S^1)^n \times \mathbb{R}$, called the *dispersion relation*. An old and widely believed conjecture in mathematical physics concerns the structure of the dispersion relation near the edges of the spectrum. Namely, for L sufficiently general, the extrema of the dispersion relation have nondegenerate Hessians. This *spectral edges conjecture* is explicitly stated in [12, Conj. 5.25], but it appears in many other sources [4, 11, 13, 14]. Important notions, such as effective masses in solid state physics and the Liouville property, depend upon this conjecture.

This conjecture remains open for discrete periodic operators (operators on periodic graphs). For these, the dispersion relation becomes an algebraic variety (in appropriate coordinates) and tools from algebraic geometry become available. This perspective was used in [5] to establish a dichotomy: Given a family of discrete periodic operators depending algebraically on parameters, either almost all operators in the family satisfy the spectral edges conjecture or almost all do not. Furthermore, determining whether or not the conjecture holds generically may be determined by studying a single operator in the family. The utility of this dichotomy was illustrated on a family of Laplace-Beltrami difference operators on a $\mathbb{Z}^2$-periodic diatomic graph.

The first step was to establish an upper bound of 32 on the number (counted with multiplicity) of complex critical points of the dispersion relation for operators in that family. Next, for a single operator in the family, its dispersion relation was shown to have 32 nondegenerate critical points. Standard arguments from algebraic geometry then implied that generic members of this family satisfy the spectral edges conjecture.

2010 *Mathematics Subject Classification.* 81U30,14M25,81Q10.

Key words and phrases. Bloch Variety, Bernstein's Theorem.

Research supported in part by Simons Collaboration Grant for Mathematicians 636314.

We extend part of this argument to Laplace-Beltrami operators on a wide class of discrete periodic graphs. The (complexified) dispersion relation is a hypersurface in the product $(\mathbb{C}^\times)^n \times \mathbb{C}$ of a complex torus and the complex line defined by a Laurent polynomial $\Phi(z, \lambda)$. The critical point equations are a system of $n+1$ Laurent polynomials. By the Bernstein-Kushnirenko theorem [2, 10], the number of critical points is bounded my the mixed volume of their Newton polytopes, which turns out to be the volume of the Newton polytope of Φ. Bernstein[2] gives a criterion for when this bound is attained, which we show is equivalent to the dispersion relation being smooth 'at infinity'. Smoothness is quivalent to the nonvanishing of certain *facial discriminants* [9], which are universal integer polynomials in the coefficients of Φ. Consequently, the general member of a given family of operators has this expected number of critical points if the facial discriminants are not identically zero in that family.

Given a discrete periodic graph Γ, and a choice of fundamental domain for the $\mathbb{Z}^n$-action, let $W = (V, E)$ be the restriction of Γ to this fundamental domain. Each edge e incident to V either has both endpoints in V (so $e \in E$) or else it has one endpoint in V and the other in a translate $V + a$ of V ($a \in \mathbb{Z}^n$). Let $\mathcal{A}(\Gamma) \subset \mathbb{Z}^n$ be the collection of elements $a \in \mathbb{Z}^n$ such that Γ has an edge between V and $V + a$. Let P be its convex hull and $\mathrm{vol}(P)$ the Euclidean volume of P. As P is an integer polytope, $n!\mathrm{vol}(P) \in \mathbb{Z}$ is an integer.

Theorem. *Let L be Laplace-Beltrami operator on Γ. The number of isolated critical points on the dispersion relation for L is at most*

$$n!|V|^{n+1}\mathrm{vol}(P) .$$

When Γ is a dense periodic graph, this bound is attained for generic operators L on Γ.

We illustrate this on the example from [5]. Figure 1 shows part of a dense periodic graph Γ with $n = 2$, where the fundamental domain is the complete graph on two vertices, and the set $\mathcal{A}(\Gamma)$ is the columns of the matrix $\left(\begin{smallmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{smallmatrix}\right)$. It also displays the polytop P, which is a diamond. As $|V| = 2$ and $\mathrm{vol}(P) = 2$, we have $n!|V|^{n+1}\mathrm{vol}(P) =$

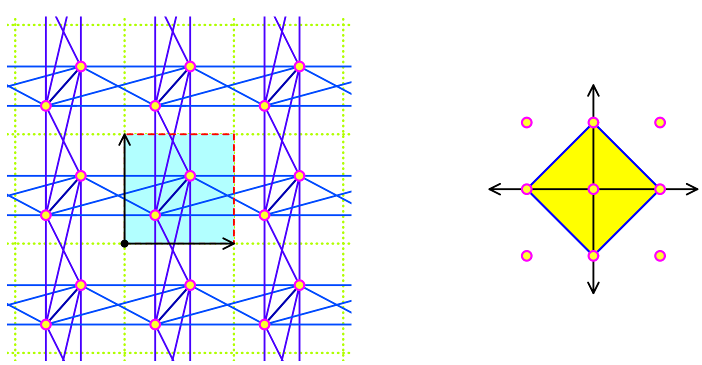


FIGURE 1. A dense periodic graph Γ with polytope P.

$2! \cdot 2^{2+1} \cdot 2 = 32$, which is the bound 32 from.... $\diamond$

Should mention toric varieties, and even that our next paper will introduce tools from arithmetic toric varieties in this study.

1. Background

Let n be a positive integer. We write $\mathbb{C}^\times := \mathbb{C} \smallsetminus \{0\}$ for the multiplicative group of nonzero complex numbers and $\mathbb{T} := \{z \in \mathbb{C}^\times \mid |z| = 1\}$ for its subgroup of unit complex numbers. Note that if $z \in \mathbb{T}$, then $\overline{z} = z^{-1}$. We write edges of a graph as pairs, (u, v) with u, v vertices, and understand that $(u, v) = (v, u)$.

1.1. **Discrete operators on periodic graphs.**

A *($\mathbb{Z}^n$-) periodic graph* is a locally finite simple (loopless, no multiple edges) graph Γ with a free cocompact action of $\mathbb{Z}^n$. That is, each vertex of Γ has finite degree and $\mathbb{Z}^n$ acts freely on the vertices, $V(\Gamma)$, and edges, $E(\Gamma)$, of Γ preserving incidences, and with finitely many orbits. Figure 2 shows two $\mathbb{Z}^2$-periodic graphs. One is the molecular structure of graphene and the other is an abelian cover of K_4.

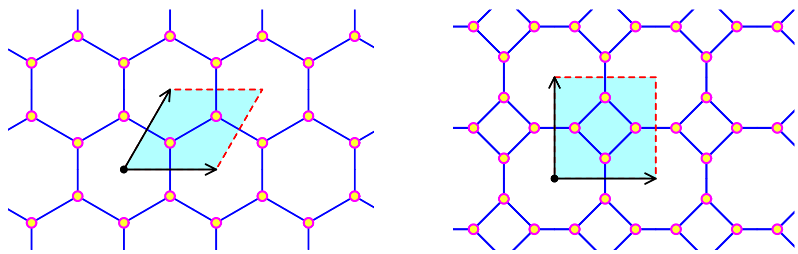


FIGURE 2. Two $\mathbb{Z}^2$-periodic graphs

It is useful to consider Γ to be immersed in $\mathbb{R}^n$ so that $\mathbb{Z}^n$ acts on Γ via translations. The graphs in Figure 2 are each immersed in $\mathbb{R}^2$, and each shows two independent vectors which generate the $\mathbb{Z}^2$-action.

Choose a fundamental domain for this $\mathbb{Z}^n$-action whose boundary does not contain a vertex of Γ. In Figure 2, we have shaded appropriate fundamental domains. Let $W = (V, E)$ be the restriction of Γ to the fundamental domain. Then V is a set of representatives of $\mathbb{Z}^n$-orbits of the vertices of Γ. Similarly, E consists of representatives of some orbits of the edges of Γ, but there are other orbits. An orbit that contains an edge of E is *internal*, and the remaining orbits are *external*. Each external orbit contains two edges that have one endpoint in V and another in a translate of V. These have the form $(v, u{+}a)$ and $(v{-}a, u)$ for some $u, v \in V$ and some $a \in \mathbb{Z}^n$. Let $\mathcal{A}(\Gamma) \subset \mathbb{Z}^n$ be the set of $a \in \mathbb{Z}^n$ such that $\{v, u + a\} \in E(\Gamma)$ for some $u, v \in V$. This finite set depends on the choice of fundamental domain and it is symmetric in that $\mathcal{A}(\Gamma) = -\mathcal{A}(\Gamma)$. For both graphs in Figure 2, this set forms the columns of $\left(\begin{smallmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{smallmatrix}\right)$.

An (edge) labeling of a periodic graph Γ is a function α from the edges of Γ to $\mathbb{R}$ or $\mathbb{C}$ that is $\mathbb{Z}^n$-invariant, i.e. constant on orbits of edges. Given a labeling α, we define a Laplace-Beltrami difference operator L_α that acts on functions φ on the vertices of Γ. It is defined by its value at $u \in V(\Gamma)$,

$$L_\alpha(\varphi)(u) \ := \sum_{(u,v)\in E(\Gamma)} \alpha_{(u,v)}(\varphi(u) - \varphi(v))\,.$$

We will call L_α a *discrete periodic operator* on Γ, and will often omit the subscript α. It is a bounded operator on the Hilbert space $\ell^2(\Gamma)$ of square-summable functions on $V(\Gamma)$. When the labels α are real, L_α is self-adjoint and has real spectrum, as it is represented by a real symmetric matrix.

1.2. **Floquet theory for discrete periodic operators.** As the action of $\mathbb{Z}^n$ on Γ commutes with the discrete periodic operator L_α, we may apply the Fourier transform, which reveals important structure of its spectrum. This is called *Floquet theory* and a standard reference is [1, Ch. 4] (see also [5, 11, 12]).

The complex torus $(\mathbb{C}^\times)^n$ is the group of characters of $\mathbb{Z}^n$. For $z \in (\mathbb{C}^\times)^n$ and $a \in \mathbb{Z}^n$, the corresponding character value is

$$z^a := z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} .$$

The character is *unitary* if $z \in \mathbb{T}^n$.

The Fourier transform $\hat{\varphi}$ of a function φ on $V(\Gamma)$ is a function on $(\mathbb{C}^\times)^n \times V(\Gamma)$ (or $\mathbb{T}^n \times V(\Gamma)$) such that for $z \in (\mathbb{C}^\times)^n$ (or in $\mathbb{T}^n$) and $u \in V(\Gamma)$,

$$\hat{\varphi}(z, u + a) = z^a \hat{\varphi}(z, u) \qquad \text{for } a \in \mathbb{Z}^n . \tag{1}$$

Thus $\hat{\varphi}$ is determined by its values at the vertices in a fundamental domain $W = (V, E)$. Then Fourier transform is an isomorphism $l^2(\Gamma) \xrightarrow{\sim} L^2(\mathbb{T}^n, \mathbb{C}^V)$, from $l^2(\Gamma)$ to square-integrable functions on $\mathbb{T}^n$ with values in the vector space $\mathbb{C}^V$.

Let $f \in L^2(\mathbb{T}^n, \mathbb{C}^V)$. Then for $u \in V$, $f(u)$ is a function of z. The action of the difference operator L_α under the Fourier transform gives the formula

$$L_\alpha(f)(u) = \sum_{\substack{v \in V \\ a \in \mathbb{Z}^n}} \alpha_{(u,v+a)} (f(u) - z^a f(v)) . \tag{2}$$

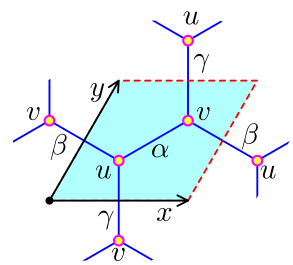
Recall that by (1), $f(v + a) = z^a f(v)$, and we set $\alpha_{\{u,w\}} = 0$ if $\{u, w\}$ is not an edge of Γ. This is a finite sum as the degree of u is finite, and we may further restrict a to lie in $\mathcal{A}(\Gamma)$.

Thus in the standard basis for $\mathbb{C}^V$, the difference operator L_α is multiplication by a square matrix whose rows and columns are indexed by elements of V. The matrix entry in position (u, v) is the function

$$\begin{aligned} - \sum_{a \in \mathcal{A}(\Gamma)} \alpha_{(u,v+a)} z^a \quad &\text{if} \quad u \neq v , \\ \sum_{(u,w) \in E(\Gamma)} \alpha_{(u,w)} - \sum_{a \in \mathcal{A}(\Gamma)} \alpha_{(u,u+a)} z^a \quad &\text{if} \quad u = v . \end{aligned}$$

Such a finite sum of monomials with exponents from $\mathcal{A}(\Gamma)$ is a Laurent polynomial with support $\mathcal{A}(\Gamma)$.

Example 1.1. Suppose that Γ is the graphene graph from Figure 2. We label the neighborhood of a fundamental domain of Γ.



Thus $V = \{u, v\}$ consists of two vertices and there are three (orbits of) edges, with labels α, β, γ. Let $(x, y) \in \mathbb{T}^2$ as indicated. The operator $L = L_{\alpha,\beta,\gamma}$ is

$$\begin{aligned} L(f)(u) &= \alpha(f(u) - f(v)) + \beta(f(u) - x^{-1}f(v)) + \gamma(f(u) - y^{-1}f(v)) \,, \\ L(f)(v) &= \alpha(f(v) - f(u)) + \beta(f(v) - xf(u)) \quad + \gamma(f(v) - yf(u)) \,. \end{aligned}$$

Collecting coefficients of $f(u), f(v)$, we may represent L by the 2×2-matrix,

$$L = \begin{pmatrix} \alpha+\beta+\gamma & -\alpha - \beta x^{-1} - \gamma y^{-1} \\ -\alpha - \beta x - \gamma y & \alpha+\beta+\gamma \end{pmatrix} \,,$$

whose entries are Laurent polynomials in x, y. When the labels α, β, γ are real and $x, y \in \mathbb{T}$, then $L^T = \overline{L}$, so that L is Hermitian and thus the operator L is self-adjoint. $\diamond$

What we saw for graphene is true in general. In the standard basis for $\mathbb{C}^V$, $L_\alpha(z)$ is multiplication by a $|V| \times |V|$-matrix whose entries are Laurent polynomials, and when the labels α are real and $z \in \mathbb{T}^n$, this matrix is Hermitian, so that $L_\alpha(z)$ is self-adjoint. Its spectrum is real and consists of its $|V|$ eigenvalues

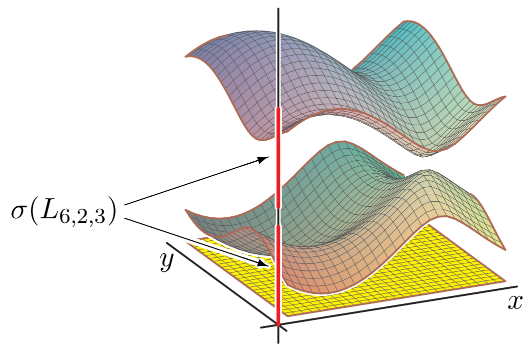
$$\lambda_1(z) \leq \lambda_2(z) \leq \cdots \leq \lambda_{|V|}(z) \,.$$

These eigenvalues vary continuously, and we call $\lambda_j(z)$ the jth *band function*. They are the roots of its characteristic polynomial

$$\Phi_\alpha(z, \lambda) := \det(L_\alpha(z) - \lambda Id) \,.$$

The characteristic polynomial defines a hypersurface in $\mathbb{T}^n \times \mathbb{R}$, called the *Bloch variety* or *dispersion relation* of the operator L_α. Its projection to $\mathbb{R}$ is the spectrum $\sigma(L_\alpha)$ of the operator L_α. The image is a union of closed intervals called *spectral bands*.

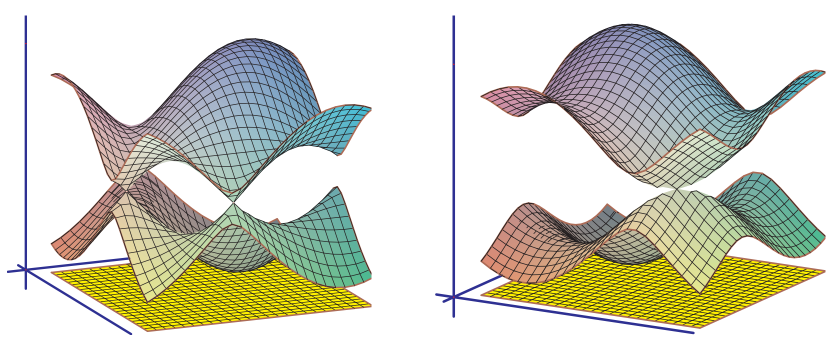
We show this for the graphene operator $L_{6,2,3}$—for this we unfurl $\mathbb{T}^2$, representing it by a fundamental domain $[-\frac{\pi}{2}, \frac{3\pi}{2}]^2$ in its universal cover. (This fundamental domain is called a *Brillouin zone* in physics.)

(3) 

An endpoint of a spectral band (*spectral edge*) is the image of an extremum of some band function $\lambda_j(z)$. For graphene at the parameters (6,2,3), each band function has two extrema, and these give the four spectral edges.

The *spectral edges conjecture* [12, Conj. 5.25] asserts that this holds in general. More precisely, the spectral edges conjecture for a discrete periodic graph Γ asserts that for generic values of the parameters α, each spectral edge is attained by a single band, and that the extrema are isolated and are nondegenerate critical points on the dispersion relation. This third condition is that the Hessian of $\lambda_j(z)$ at the extremum corresponding to a spectral edge is nondegenerate.

More is true for graphene at the parameters (6,2,3): Each band function is a perfect Morse function on $\mathbb{T}^2$—it has four nondegenerate critical points, a minimum, a maximum and two saddle points. The ranges of the two functions are also disjoint. This does not hold in general. Some calculations show that if α, β, γ are positive and form the sides of a triangle, then the gap between the bands disappears, and the band functions acquire one or two singularities. We show this for the Laplacian, which has parameters $(1, 1, 1)$ and with two singularities and for the parameters $(5, 2, 3)$ which has one singularity.



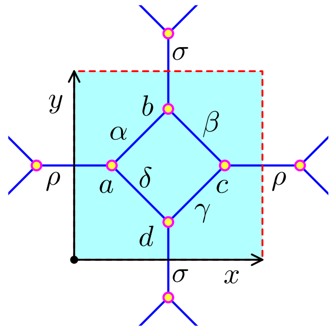
(4)

These three cases are completely determined by the geometry of the three parameters (when they are positive). If they do not form a triangle, then we have perfect Morse functions and a gap as in (3). If they form a degenerate triangle (one is the sum of the other two), there is a single singularity as on the right in (4), and if they form a triangle, there are two singularities, as on the left in (4).

Since in the singular cases, the singular value is $\lambda = \alpha + \beta + \gamma$, which is in the interior of the single spectral band, so the spectral gap conjecture holds for graphene.

1.3. **Sparse polynomial systems and combinatorial convexity.**

1.4. **Bernstein's Theorem.**



2. Dense Periodic Graphs

2.1. The case of dimension 2.

2.2. General Dense Periodic Graphs.

3. General result for Periodic operators

Theorem 3.1. *Suppose that $\phi \in \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}, \lambda]$ and $\Phi := \phi = z_1 \frac{\partial \phi}{\partial z_1} = \cdots = z_n \frac{\partial \phi}{\partial z_n} = 0$.*

We have that Φ has $(n+1)!\operatorname{Vol}(\mathrm{N}(\phi))$ isolated solutions counted with multiplicity in $(\mathbb{C}^*)^n \times \mathbb{C}$ if and only if ϕ_w is nonsingular for all $w \neq (0, \ldots, 0, a) \in \mathbb{R}^n$, $a > 0$.

Proof. Let $\phi_i = z_i \frac{\partial \phi}{\partial z_i}$. Consider the system $\Psi =: \phi = \psi_1 = \cdots = \psi_n = 0$ where $\psi_i = r_i \phi + \phi_i$, where r_i is a real number chosen such that $\operatorname{supp}(\phi) = \operatorname{supp}(\psi_i)$. Notice that if $b \in (\mathbb{C}^*)^n \times \mathbb{C}$ is a solution to Φ if and only if it is also a solution to Ψ. We also have that $\operatorname{supp}(\phi_i) \subset \operatorname{supp}(\phi)$. Finally remark that $\operatorname{MV}(\mathrm{N}(\phi), (\psi_1), \ldots, \mathrm{N}(\psi_n)) = (n+1)!\operatorname{Vol}(\mathrm{N}(\phi))$

- $\Leftarrow$ Suppose that ϕ_w is nonsingular. Then Φ_w has no solutions. If Ψ_w has a solution b then $\phi_w = 0$ and $\psi_i = 0$ for all i. Let ϕ_w be a polynomial in k variables, $z_1, \ldots, z_k$. Then $(\psi_j)_w = r_j \phi_w + (\phi_j)_w = (\phi_j)_w = 0$ for all $j \in (1, \ldots, k)$. Contradiction. Thus if ϕ_w is nonsingular for all $w \neq (0, \ldots, 0, a) \in \mathbb{R}^n$, $a > 0$ then Ψ_w has no solutions. By Bernstein's other theorem, Ψ has $\operatorname{MV}(\mathrm{N}(\phi), (\psi_1), \ldots, \mathrm{N}(\psi_n)) = (n+1)!\operatorname{Vol}(\mathrm{N}(\phi))$ solutions counted with multiplicity. Thus Φ has $(n+1)!\operatorname{Vol}(\mathrm{N}(\phi))$ solutions.
- $\Rightarrow$ Let Φ have $(n+1)!\operatorname{Vol}(\mathrm{N}(\phi))$ solutions. Then Ψ has $(n+1)!\operatorname{Vol}(\mathrm{N}(\phi))$ solutions. Suppose that ϕ_w is singular for some $w \neq (0, \ldots, 0, a) \in \mathbb{R}^n$. Let ϕ_w be a polynomial in k variables, $z_1, \ldots, z_k$. Then ϕ_w has a solution say b, and at b then $(\psi_i)_w = 0$ for $i \in (1, \ldots, k)$.

  Consider $j > k$. Then $\operatorname{supp}((\phi_j)_w) \cap \operatorname{supp}(\phi_w) = \emptyset$. This implies that $\operatorname{supp}((\psi_j)_w) = \operatorname{supp}(\phi_w)$, that is $(\psi_j)_w = r_j \phi_w = 0$.

  Then we must have that Ψ_w has solutions, but then by Bernstein's other theorem we have that Ψ has less than $\operatorname{MV}(N(\phi), (\psi_1), \ldots, \mathrm{N}(\psi_n)) = (n+1)!\operatorname{Vol}(\mathrm{N}(\phi))$ solutions, but then so too does Φ. This gives a contradiction.

  Thus we have that ϕ_w is nonsingular for all $w \neq (0, \ldots, 0, a) \in \mathbb{R}^n$, $a > 0$.

$\square$

Remark 3.2. Applying Bernstein's theorem to this proof, along with the fact that a polynomial system with generic coefficients has no solutions at infinity, allows us to talk about the structure of the underlying polytope. Namely, $\mathrm{MV}(\mathrm{N}(\phi), (\phi_1), \ldots, \mathrm{N}(\phi_n)) = (n+1)!\,\mathrm{Vol}(\mathrm{N}(\phi))$ if and only if for every $w \in \mathbb{R}^n$ when $\mathrm{N}(\phi_w)$ is a k-face then at least k of the distinct $\mathrm{N}((\phi_i)_w)$ have nontrivial intersection with $\mathrm{N}(\phi_w)$.

This can be viewed as a case of [15, Cor. 9] and a restatement of [3, Cor. 3.7]. In particular [3, Cor. 3.7].

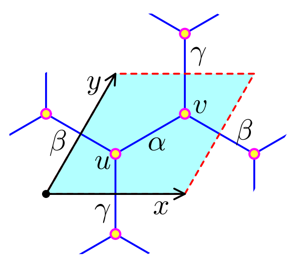
Corollary 3.3. *Suppose that* $\phi \in \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}, \lambda]$ *such that* $\phi = \det(A - I\lambda)$*, where* A *is a representation of a Laplace-Baltrami operator, and* $\Phi := \phi = \phi_1 = \cdots = \phi_n = 0$ *where* $\phi_i = z_i \frac{\partial \phi}{\partial z_i}$*. Then we have that* $\mathrm{MV}(\mathrm{N}(\phi), \mathrm{N}(\phi_1), \ldots, \mathrm{N}(\phi_n)) = (n+1)!\,\mathrm{Vol}(\mathrm{N}(\phi))$.

Proof. Let F be a k-face of $P = \mathrm{N}(\Phi)$ identified by the innernormal w. Then, because P is a cone with a symmetric base containing the origin, k distinct variables z_i must occur in ϕ_w, WLOG assume these are $z_1, \ldots, z_k$. Then we have that $\mathrm{N}(\phi_{iw})$ has nontrivial intersection with F for each $i \in \{1, \ldots, k\}$.

Thus by the previous remark, we have that $\mathrm{MV}(\mathrm{N}(\phi), \mathrm{N}(\phi_1), \ldots, \mathrm{N}(\phi_n)) = (n+1)!\,\mathrm{Vol}(\mathrm{N}(\phi))$. $\square$

4. General Result applied to relevant examples

4.1. **Graphene.** We begin our demonstration of examples by first looking at the $\mathbb{Z}^2$-periodic graphene.



Applying the difference operator we get that

$$(Af)(u) = L_\alpha(u) = \beta(f(u) - x^{-1}f(v)) + \gamma(f(u) - y^{-1}f(v)) + \alpha(f(u) - f(v))$$

$$(Af)(v) = L_\alpha(v) = \beta(f(v) - xf(u)) + \gamma(f(v) - yf(u)) + \alpha(f(v) - f(u))$$

We get the following matrix after applying the difference operator.

$$A = \begin{bmatrix} \beta+\gamma+\alpha & -(\beta x^{-1} + \gamma y^{-1} + \alpha) \\ -(\beta x + \gamma y + \alpha) & \beta+\gamma+\alpha \end{bmatrix}$$

Let

$$\psi = \det A - \lambda I.$$

Taking the Newton polytope of ψ and its partial derivatives with respect to x and y quickly shows us that we will not be able to use Rojas's theorem for solving the mixed volume. This is because the newton polytope of $\frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi}{\partial y}$ are only two dimensional polytope. We could go ahead and calculate mixed volume to find that the mixed volume is $12 = 3! * \mathrm{MV}(\mathrm{N}(\psi))$ as we would have hoped, but if we are able to satisfy theorem 8.6 for any choice of edge weights then we obtain the mixed volume of the original system is $3! * \mathrm{Vol}(\mathrm{N}(\psi))$, no calculations required.

This example works particularly well. You need only to look at the polyhedron below.

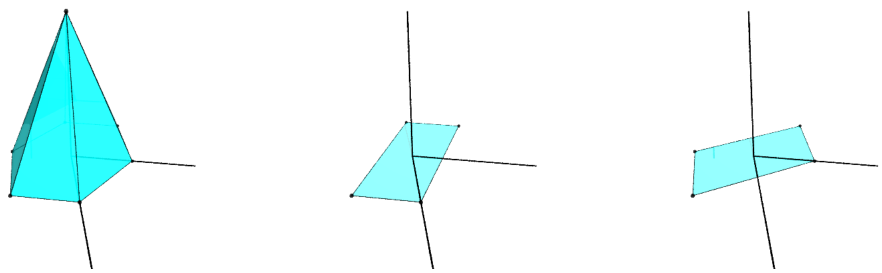


FIGURE 3. Graphene polytopes

By the theorem we need only consider inner normals $w \in \mathbb{Z}^n$ of $\mathrm{N}(\psi) = P$ for $w \neq (0, 0, 1)$. Examing the system of polytope, we can see for any facet F of $\mathrm{N}(\psi)$ where F is not the base, the intersection of F with at least one of $\mathrm{N}(x\frac{\partial\psi}{\partial x})$ and $\mathrm{N}(y\frac{\partial\psi}{\partial y})$ is a monomial. Due to this, no facial system has a solution, that is ψ_w is nonsingular for each w. By the theorem, for generic choices of edge weights, we have that that ψ has 12 critical points in $(\mathbb{C}^*)^2 \times \mathbb{C}$ counted to multiplicity.

Moreover, in this particular case ψ has 12 critical points, counted according to multiplicity, so long as all edge weights are nonzero.

4.2. **OcGr.** We move on to a $\mathbb{Z}^2$-periodic abelian cover of the $K4$ crystal.

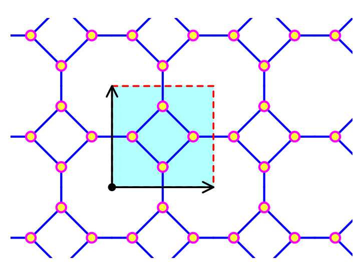


FIGURE 4. K4 labeled need to label and remove explanation

We get the following matrix after applying the difference operator.

$$A = \begin{bmatrix} \alpha_1 + \alpha_3 + \alpha_6 & -\alpha_3 & -\alpha_1 x^{-1} & -\alpha_6 \\ -\alpha_3 & \alpha_2 + \alpha_3 + \alpha_4 & -\alpha_4 & -\alpha_2 y \\ -\alpha_1 x & -\alpha_4 & \alpha_1 + \alpha_4 + \alpha_5 & -\alpha_5 \\ -\alpha_6 & -\alpha_2 y^{-1} & -\alpha_5 & \alpha_2 + \alpha_5 + \alpha_6 \end{bmatrix}$$

As before we let

$$\psi = \det A - \lambda I$$

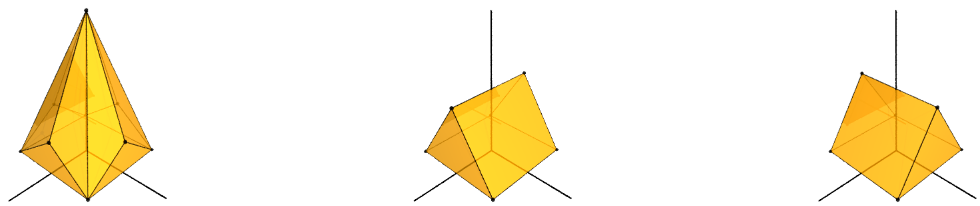


FIGURE 5. K4 polytopes

Unlike in the case of the graphene, this system does not fit our theorem. However, like the graphene we need only look at the polytope to understand why. Take the two-face identified by $w = (1, 0, 0, 0)$. Notice that $\psi_w = [x]\psi_w$, and so we can divide ψ_w by x, to get ψ'_w and $\psi'_{wx} = 0$. This yields system of equations with identical solutions and Ψ'_w is a a system of 2 equations in 2 variables, and so for generic edge weights there will be a solution.

4.3. **The Diamond crystal, and two vertex maximal abelian coverings.** The diamond cystal has a fundamental domain consisting of K_2 and 3 edges leaving the fundamental domain for each edge.

Applying the difference operator to this graph we get the following matrix:

$$A = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & -(\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z) \\ -(\alpha_1 + \alpha_2 x^{-1} + \alpha_3 y^{-1} + \alpha_4 z^{-1}) & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{bmatrix}$$

Let $\psi = A - \lambda I$

This looks errily similar to our graphene. This is to be expected, as the two are closely related. The diamond crystal is an abelian maximal covering of the same graph as the graphene, but with one additional edge.

Unlike the graphene however, the diamond crystal does have a facial system with solutions. In particular if w is an inner normal with a single 0 entry in the first three indices and the other two of the first three entries are either both positive or both negative, then ψ_w is singular. Take $w = (1, 0, 1, 0)$ or $w = (0, -1, -1, 1)$ for example.

4.4. **Maximal abelian coverings of two vertex selfloopless graphs.** The graphene and the diamond crystal are part of a larger family of topological crystals which are maximal abelian covers the graphs of 2 vertices with n parallel edges between them. The nth member of this family, would yield the following matrix for the difference operator.

$$A = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n & -(\alpha_1 + \alpha_2 z_1 + \alpha_3 z_2 + \cdots + \alpha_n z_{n-1}) \\ -(\alpha_1 + \alpha_2 z_1^{-1} + \alpha_3 z_2^{-1} + \cdots + \alpha_{n-1} z_{n-1}^{-1}) & \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{n-1} \end{bmatrix}$$

As with the diamond, we have when $n > 3$, these systems will have solutions on the facial system identified by the inner normal $w = (1, 1, 0, \dots, 0)$.

4.5. **The Graphene Dice Family.** Let D_i be a graph of $i + 1$ order vertices for $i \geq 1$ such that each vertex shares 3 parallel edges with its predecessor. Denote the vertices $v_1, \dots, v_i$.

Question to Frank: Should I introduce topological crystals fully or just refer the reader to Sunada's or Baez's paper? Or should we just save this for some other time / paper where it would make more sense to include these and the background?

Now consider the family of maximal abelian covers of D_i, denoted $\bar{X}_i$. To make our construction concise, let v_1 be our base point [1] and let $e_(l,j)$ be the jth parallel edge between v_l and v_{l+1}. Assume all edges are oriented from v_l to v_{l+1}. Since edges are oriented [2], a path that traverses $e_{l,j}$ from v_{l+1} to v_l, would actually traverse the algebraic inverse to $e_{l,j}$, written $\bar{e_{l,j}}$.

Our D_i have a total of $2i$ holes, and so the homology group has $2i$ elements, these are the equivalence classes of simple loops of v_1. The maximal abelian cover of a graph is obtained by quotienting the universal cover of that graph by the homology group. The universal cover is an infinite graph such that vertices are one to one with equivalence classes of paths starting at v_1, with edges between two vertices $[p]$ and $[q]$ if there is an edge e such that $[pe] = [q]$. Let c_l be the path $e_{1,1}e_{2,1}\dots e_{l,1}$ for $l \geq 1$, let c_0 be the trivial path. In the case of D_i the homology group H_i is generated by $[c_{l-1}e_{l,1}\bar{e_{l,2}}\bar{c_{l-1}}], [c_{l-1}e_{l,1}\bar{e_{l,3}}\bar{c_{l-1}}]$ for $l \in \{1, \dots, i)\}$. Thus $\bar{X}_i$ is given by the the universal cover of D_i quotiented by H_i.

We wish to look at a two dimensional covering graph of our D_i.

Consider the equivalence relation A_i on elements of H_i given by $[c_{s-1}e_{s,1}\bar{e_{s,2}}\bar{c_{s-1}}] \sim [c_{r-1}e_{r,1}\bar{e_{r,2}}\bar{c_{r-1}}]$ and $[c_{s-1}e_{s,1}\bar{e_{s,3}}\bar{c_{s-1}}] \sim [c_{r-1}e_{r,1}\bar{e_{r,3}}\bar{c_{r-1}}]$ for all $s, r \in (1, \dots, i)$.

Then let G_i be the family of topological crystals obtained from X_i/A_i. These G_i are two dimensional periodic graphs. In particular, G_1 is the graphene, and G_2 is the dice lattice.

I need to read more but I believe these are called the "standard realizations" of the D_i I gave.

Let W_i be a linear fundamental domain of G_i. For example we can take the vertices identified by the paths $[], [e_{1,1}], [e_{1,1}e_{2,1}], \dots, [e_{1,1}\dots e_{i,1}]$ and their edges. This subgraph is a fundamental domain.

---

[1] choice of base point doesn't matter except for making the construction explicit

[2] Oriented does not mean directed. These are NOT directed graphs.

Then we have that the Bloch variety over the difference operator has exactly $(i+1)*6$ critical points, counted according to multiplicity, in $(\mathbb{C}^*)^2 \times \mathbb{C}$ for a generic choice of edge weights.

The polytopes of the system obtained from the difference operator over G_i are as follows. (1) Take the convex hull of the base of the graphene with the point $(0,0,i)$. (2),(3) these polytope are the same as the polytope obtained from the graphene's partial derivatives. Thus the volume is obtained by dividing the graphene volume by it's height of 2 and multiplying by the new height i. These also satisfy our theorem for the same "eye balling" reason as graphene, that is, all facial systems except the base contain a monomial term.

4.6. **A more general m-Diamond Dice Family.** When I say smooth throughout this, I mean every facial system except the base has no solutions.

We might ask now, what if we consider $n+1$ vertices in order with m edges between each vertex and its predecessor. This appears to generalize well. Let us represent this as $(\underbrace{m, m, \dots, m}_{\text{n times}})$, with $G(m, \dots, m)$ being the topological crystals in question.

We will construct these $G(m, \dots, m)$ in basically the same manner, taking an equivalence relation over the elements of the homology group of $(m, \dots, m)$. In general we will have m relations of the form $[c_{s-1}e_{s,1}\bar{e_{s,i}}\bar{c_{s-1}}] \sim [c_{r-1}e_{r,1}\bar{e_{r,i}}\bar{c_{r-1}}]$ for all $s, r \in (1, \dots, i)$ and for each $i \in (1, \dots, m)$, call this set of equivalences A. Then $G(m, \dots, m)$ is the maximal abelian cover of $(m, \dots, m)$ quotiented by A.

these are also probably called "standard realizations", but again need to check.

In this way, $G(3)$ is the graphene, $G(3,3)$ is the dice lattice, $G(4)$ is the diamond crystal.

It may come as a surprise, as the diamond crystal system is not smooth, but other members of its 4-diamond dice family are. That is $G(4,4)$ generic facial systems are smooth, $G(4,4,4)$ generic facial systems are smooth, and so on.

We also have in general that $G(2m+1, \dots, 2m+1)$ has facial systems that are not smooth, for $m \geq 2$ (graphene dice family is the only exception due to monomials in every facial system except the base). In particular the system we obtain will always have solutions on the face identified by $w = (\underbrace{1, \dots, 1}_{\text{n times}}, 0, \dots, 0)$.

We also have $G(6,6)$ is not smooth. However, $G(6,6,6)$ is smooth.

Conjecture: In general we have $G(\underbrace{2m, \dots, 2m}_{\text{n times}})$, is smooth for $n \geq m$.

Furthermore these systems will have $(n+1) * \binom{2(2m-1)}{(2m-1)}$ solutions counted according to multiplicity.

As you can see, the ith member of the graphene dice family has $(i+1)*6 = i\binom{2(2)}{2}$ critical points counted with multiplicity, thus it agrees with this equation.

Might be interesting to look at mixed systems. For example $G(3,4,5,4,3)$.

We can also view the dense graphs as topological crystals which are quotients of a maximal abelian graph (in fact all periodic graphs should have such a realization).

5. Old Scratch Work

We start with a given fundamental domain $W = (V, E)$ of a periodic graph Γ [3] such that the fundamental domain has m vertices $(a_1, \ldots, a_m)$ and n basis actions $(z_1, \ldots, z_n)$.

We define a function f on the vertices such that if a vertex v is obtained from the action z_i on a vertex u then $f(v) = z_i f(u)$. Remark vertices not in the fundamental domain are induced by the edges of the domain.

We define the difference operator

$$L_\alpha(u) = \sum_{(u,v)\in E} \alpha_{u,v}(f(u) - f(v))$$

such that α is a set of edge weights with each $\alpha_i \in \mathbb{R}$.

A periodic graph Γ living in $m > 1$ dimensional ambient space, equipped with $G = Z^n$, is *dense* if there exist a fundamental domain W of Γ which is isomorphic to K_m for some $m > 1$.

We will bring our focus to the characteristic polynomial of the matrix A with entries $A_{i,j} = [f(a_j)]L_\alpha(a_i)$, where $\{a\}$ are the set of vertices of W. We will denote this characteristic polynomial as ψ.

6. Background

Since $G = Z^n$ our polynomials are real *Laurent polynomials* that is they exist in the ring $\mathbb{R}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$.

Given a Laurent polynomial ϕ the *Newton polytope* of ϕ, $\mathrm{N}(\phi)$, is the is the convex hull of $\mathrm{supp}(\phi) = \{s \mid [x^s]\phi \neq 0\}$.

Let P be the Newton polytope of the characteristic equation

$$\psi = det(A - \lambda I) \in \mathbb{R}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, \lambda],$$

and we will let P_i be the Newton polytope of the Laurent polynomial

$$\psi_i = z_i \frac{\partial \psi}{\partial z_i}.$$

It is not hard to see that P is a pyramid with an n dimensional cross-polytope (or n-fold bipyramid) as its base. [4].

A result we will make use of throughout the paper is the Bernstein-Kushnirenko theorem ([2, Thm. A]).

Theorem 6.1. *Bernstein-Kushnirenko Theorem*

A system

$$\Phi := \phi_1 = \phi_2 = \cdots = \phi_n$$

of n polynomials in l variables with supports $S_1, S_2, \ldots, S_n$ respectively has at most $\mathrm{MV}(S_1, \ldots, S_n)$ isolated solutions in $\mathbb{C}^n \backslash \{0\}$. This bound is saturated for a generic system.

---

[3] See [5] for details.

[4] Multiplication by z_i will have no effect on the mixed volume of the system since it merely shifts the entire polytope. Further no solutions are added since we will be considering our variables $z_1, \ldots, z_n$ on the complex torus.

We will also be utilizing a stronger result known as Bernstein's Theorem ([2, Thm. B])).

In order to state this theorem we must introduce the notion of facial systems.

Let $w \neq 0 \in Z^n$, S be a finite set in Z^n, $\langle w, s\rangle = w^T s$ for $s \in Z^n$, and $m_w = \min s \in S\langle w, s\rangle$.

When $S = \mathrm{supp}(\phi)$ then $S_w = \{c_s x^s \mid \langle w, s\rangle = m_w, s \in S, c_s x^s \in \phi, c_s \neq 0\}$.

Denote $\phi_w := \sum_{b \in S_w} c_b x^b$.

Then we have a facial system, denoted Φ_w, is given by

$$\phi_{w1} = \phi_{w2} = \cdots = \phi_{wn} = 0.$$

with corresponding supports $S_{w1}, \ldots, S_{wn}$.

Theorem 6.2. *Bernstein's Theorem*

Let $\Phi := \phi_1 = \phi_2 = \cdots = \phi_n = 0$ with supports $S_1, \ldots, S_n$ respectively, if Φ_w does not have any roots in $T = (\mathbb{C}^*)^n$ for every $w \neq 0$, then all roots of L are isolated and they are counted, according to multiplicity, exactly by $\mathrm{MV}(S_1, \ldots, S_n)$.

From the proof of Bernstein's theorem we also get the following corollary, which will prove convenient for us.

Corollary 6.3. *Suppose that Φ has a solution at Φ_w for $w \in \mathbb{Z}^n$ such that for some i we have $w_i > 0$ and $w_j = 0$ for $i \neq j$. Further unless ω is w times a positive scalar, we have Φ_ω has no solutions.*

Then we have that Φ has $\mathrm{MV}(S_1, \ldots, S_n)$ isolated solutions, counted according to multiplicity, in $(\mathbb{C}^*)^{i-1} \times \mathbb{C} \times (\mathbb{C}^*)^{n-i}$

Our goal is to utilize the Bernstein-Kushnirenko Theorem to obtain an upper bound on the number of solutions to the system of equations of $\Psi = (\psi, \psi_1, \ldots, \psi_n)$ by looking at the mixed volume of $(P, P_1, P_2, \ldots, P_n)$. Further we will utilize Bernstein's theorem to determined when this bound is saturated. We will go about solving for the mixed volume by employing a result of Roja's [15] utilized in [5].

Lemma 6.4. *The volume of P is*

$$\frac{2^n m^{n+1}}{(n+1)!}$$

for $n \geq 1$.

Proof. We notice that P is a pyramid with a m-dimensional cross polytope base, which is length $2m$ in each direction.

The volume of the base is given by, $(2m)^n$. Thus the volume of $P = \frac{m*(2m)^n}{(n+1)!} = \frac{2^n (m)^{n+1}}{(n+1)!}$. □

We begin with the goal of proving the following theorem.

Theorem 6.5. *When $m > 1$,*

$$\mathrm{MV}(P, P_1, \ldots, P_n) = 2^n m^{n+1}$$

for $n \geq 1$, where MV is the mixed volume, Before stating the proof we will develop some necessary preliminary results.

7. Preliminary Results on Mixed volume

Given a Laurent polynomial $\phi \in \mathbb{R}[z_1, z_2, \dots, z_n, z_1^{-1}, z_2^{-1}, \dots, z_n^{-1}, \lambda]$, let the Newton polytope of ϕ be Q, let Q_i be the Newton polytope of the Euler operator $z_i * (\frac{\partial f}{\partial z_i}) = \phi_i$. We also let H_i denote the (n)-plane where the ith coordinate is 0, $S = \operatorname{supp}(\phi)$ and $S_i = \operatorname{supp}(\phi_i)$

Fact 7.1. *We have that* $P_i = \operatorname{conv}(S_i \setminus H_i))$.

Proof. After each partial derivative say with respect to z_i terms of f that are constant when viewing f as a polynomial if z_i will vanish. These are exactly the terms corresponding to points on H_i. □

Let C be an $n+1$ dimensional cone with vertex $(0, \dots, 0, 1)$, and all other vertices of the form $(a_1, \dots, a_n, 0)$.

Lemma 7.2. *Let P be a polytope of the form nC, if a j-face F of P has a 0-face with a nonzero ith coordinate then P_i has a j-face that is a subset of F.*

Proof. Such a face j must either have the vertex that is a mutliple of n at the ith coordinate. In this case $F \cap P_i = F \backslash H_i = F_i \subset P_i$ since $m > 1$ F_i is still a j-face on P_i. Thus F_i is a face of P_i and a subset of F.

Notice that this is because for our polytope in question we have j edges leaving the vertex in question and since $m > 1$ [5] there will be integer points on these edges between this vertex and H_i and so subsets of these edges will not vanish and neither will the corresponding j face that their convex hulls define.
[6] □

With our lemmas established, we move onto proving a generalization of the proposed theorem 2.4.

Proof of a generalized Theorem 1.5

We will now prove theorem 1.5, or rather more generally:

Theorem 7.3. *If P is an $n+1$ dimensional polytope of the form $P = mC$ with C such that 0 lies internally on the base of the cone, then* $\mathrm{MV}(P, P_1, \dots, P_n) = (n+1)! Vol(P)$

In this statement and proof, by vertices we mean 0-dim faces of P.

[5] and in particular the end point of each edge is also of magnitude n in some direction, that is the magnitude of the two vertices the edges are connecting are not relatively prime and so there must lie integer points internally on the edge.

[6] Had longer very formal proof, but I think this is rather intuitive and obvious geometrically that it isn't really needed.

Proof. Notice $P = P \cup P_1 \cup \cdots \cup P_n$. We wish to use a result of Roja [15] that tells us, given a system $(Q, Q_1, \ldots, Q_s)$ such that $Q = Q \cup Q_1 \cup \cdots \cup Q_s$, if for each face k-face F of Q we have k of the Q_i have a k-face F_i that is a subset of on F.

If F lies on the base, then clearly there must be k vertices which are linearly independent as vectors from the origin, take then by the previous lemma there are k, P_i with k-faces F_i such that $F_i \subset F$,

In the case where F does not lie on the base, it must contain the apex of the pyramid. Viewing F as a k-dimensional polytope in the k-dimensional space determined by $AFF(F)$, we may take the apex to be the origin. Once again we must have k vertices excluding the origin which are linearly independent.

We now may apply Rojas's result with the known volume of P, obtaining that mixed volume is the normalized volume of P. In the case of theorem 1.2 where P is our original pyramid over the cross polytope, we have that

$$\mathrm{MV}(P, P_1, P_2, \ldots, P_n) = (n+1)!\,\mathrm{Vol}(P) = 2^n m^{n+1}$$

. □

Remark 7.4. Letting C be the $n+1$ dimensional pyramid with $v = (0, \ldots, 0, 1)$ with its base being the cross polytope we prove Theorem 1.2.

8. Saturating Bounds for facially smooth polynomials

Our system is a characteristic polynomial and its first n Euler operators and so genericity is not guaranteed. Hence, we must check the facial systems of our system of Newton polytope.

By Bernstein's theorem, we have if a system Φ_w has no solutions for each $w \in \mathbb{Z}^{n+1}$, then we will have shown that solutions to the system are isolated and they are counted by the mixed volume, up to multiplicity.

For our systems we are interested in, that is a characteristic polynomial and its n partial differentials, we must only check a single vector from each cone of the normal fan of $\mathrm{N}(\phi)$.

If any $\phi_{wi})$ is a monomial in $\mathbb{C}[z_1 \ldots, z_n, \lambda]$, then there are no solutions. Thus if w is the inner normal to a single 0-face for Q or any Q_i, then that facial system cannot have any solutions in the torus.

Lemma 8.1. *Φ_w has a solution only if ϕ_w is singular.* [7]

Proof. Given a face F identified by w, let us consider the following cases.

(1) Consider F such that F does not contain the apex of the pyramid.
Let $c_1, c_2, \ldots, c_p$ give the p the z_i such that z_i divides some term of ϕ_w. Then ϕ_{wc_j} is $z_{c_j}\frac{\partial \phi_w}{\partial z_{c_j}}$ [8], and further f_w is a function in $\mathbb{C}[z_{c_1}^{\pm}, \ldots, z_{c_p}^{\pm}]$.

---

[7]Because of the extra terms in the case $F_i \not\subset F$, where F is a face of $\mathrm{N}(\phi)$ we could have that ϕ_w is singular but there are still no solutions. Thus we do not necessarily have both directions.

[8]Unless $\frac{\partial \phi_w}{\partial z_{c_j}} = 0$ in which case we get a different polynomial for $\phi_{wi})$. For this proof, since we are assuming a solution exists, we may assume that this polynomial is zero. This observation is why it may

The solution restricted to the coordinates $c_1, \ldots, c_p$ must be a singular point of ϕ_w. That is, any solution to the facial system is a solution to ϕ_w of multiplicity greater than 1.

(2) Now consider the case when the apex of the pyramid is in F.

We wish to reduce Φ_w to a system of the same form as those described in case 1. If ϕ_w is a function in $\mathbb{C}[z_{c_1}^{\pm}, \ldots, z_{c_p}^{\pm}, \lambda]$ and $\mathrm{N}(\phi_w)$ is at most of p-face of P [9]. Thus we have that $\frac{\partial \phi_w}{\partial \lambda}$ can be written as a linear combination of ϕ_w and the $\frac{\partial \phi_w}{\partial z_{c_j}}$.

Thus if Φ_w has a solution then $\frac{\partial \phi_w}{\partial z_{c_j}} = \phi_w = 0$ and so $\frac{\partial \phi_w}{\partial \lambda} = 0$, and thus ϕ_w is singular.

□

Theorem 8.2. *The property, O, that a Laurent polynomial $\phi \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}, \ldots, z_n^{\pm}, \lambda]$ is such that ϕ_w has no singular points for any $w \in Z^{n+1}$ is generic.*

Proof. Let O be the set of polynomials having this property. Consider the set of polynomials that lie outside of O. Given some w, if we have that if ϕ_w has a singular point, then ϕ is in O. The property of having multiplicity one is generic, and so the property of having a singular point is given by an algebraic set, V_w. In particular we view V_w as variety of coefficient choices of the terms of ϕ_w, notice that coefficients of ϕ_w are also coefficients of ϕ. Since there are only finitely many faces, we only must consider $w_1, \ldots, w_l$ corresponding to each face. Then we have that ϕ lies outside O if and only if f has coefficients in $\bigcup V_{w_i}$ which is a finite union of algebraic sets and thus an algebraic set. Thus we have that the set of elements of $\mathbb{C}[z_1^{\pm}, z_2^{\pm}, \ldots, z_n^{\pm}, \lambda]$ which satisfy property O is the complement of an algebraic set and thus the property O is generic. □

If ϕ satisfies the property O presented in theorem 4.2, we will call ϕ *facially smooth*. Note that ϕ itself need not be smooth.

9. Back to the fundamentals

Our initial interest was in studying the dispersion relationship of discrete periodic operators, as such it would be nice to give a method of deciding when the characteristic polynomials presented in the introduction is facially smooth.

Our first family of investigation brings us back to the introduction. We call a fundamental domain *dense* if the atoms or vertices in the domain are saturated with edges, that is the subgraph formed by taking only the vertices in the fundamental domain is a complete graph, and if any atom not in the fundamental domain is saturated with edges to the fundamental domain. In particular we will considered a family of dense periodic graphs with n atoms and m dimensions, with membership identified by (m, n)

An example of a member of such a family can be found in [5], in particular this graph is $(2, 2)$.

be possible that Φ_w has no solutions, but ϕ_w is singular... should maybe look for counter example. Also this statement is a little off since if z_{c_i} appear in f_w then it cannot have partial derivative 0.

[9]Due to this there is a change of variables transforming f_w to just a polynomial of $\mathbb{C}[z_{c_1}^{\pm}, \ldots, z_{c_p}^{\pm}]$, and so λ can be seen as dependent of the other variables.

We assert that for $n, m > 1$ we have that given (m, n) with generic edge weights, the dispersion relationship of the operator Ψ_w has only non-degenerate extrema, given by the mixed volume of the polytope corresponding to the generators of the Bloch variety. Remark that this system particularly fits a system as the one mentioned up to **Remark 3.4**.

Thus we are left to show that we have that, for each $w \in \mathbb{Z}$, Ψ_w has no solutions.

Before we get ahead of ourselves, we should make sure to note that Bernstein's theorem is over Laurent Polynomials and so it gives us solutions on the algebraic torus. Unfortunately, in the case of the difference operator, we will often have a solution on the base.

Fact 9.1. *Solution at $\lambda = 0$ and $z_i = 1$*

Proof. Consider the original matrix from which we will obtain the characteristic matrix and set $\lambda = 0$.

$$M_{i,i} = (\sum_{j=1}^{n} 2a_{n*(i-1)+j} - a_{n*(i-1)+j}(z_j + z_j^{-1})) + \sum_{k=1}^{n} \sum_{j=1, j\neq i}^{m} (b_{i,j,k} + b_{j,i,k}) + \sum_{j=1, j\neq i}^{m} c_{i,j}$$

$$M_{i,j} = (\sum_{k=1}^{n} (b_{i,j,k} z_k + b_{j,i,k} z_k^{-1})) - c_{j,i}$$

where $a_{i,k}$ are edge weights of edges given by direct action out of the fundamental domain number such that each m consecutive a_i represent edges out of the domain in direction z_k. The $b_{i,j,k}$ are the edges remaining in the saturated graph not already counted by $\{a\}$ in the direction k from the fundamental domain. The c are such that $c_{i,j} = c_{j,i}$ and they represents the weight from the ith to jth atom in the complete subgraph of the fundamental domain.

It is easy to see that if $z_i = 1$ then the matrix becomes

$$M_{i,i} = \sum_{k=1}^{n} \sum_{j=1, j\neq i}^{m} (b_{i,j,k} + b_{j,i,k}) + \sum_{j=1, j\neq i}^{m} c_{i,j}$$

$$M_{i,j} = (\sum_{k=1}^{n} (b_{i,j,k} + b_{j,i,k})) - c_{j,i}$$

Then we can reduce this matrix to a matrix with 0's in the diagonal. Further since $c_{i,j} = c_{j,i}$ we are left with a symmetric matrix with all 0's in the diagonal, and so the determinate is 0.

We are left to show that each partial differential of the characteristic polynomial is 0 under this specialization. Without loss of generality, consider the partial differential with respect to z_1. We may start by specializing $z_i = 1$ for $i > 1$ since this will not affect our derivative. We get the following matrix.

$$M_{i,i} = 2a_{n*(i-1)+1} - a_{n*(i-1)+1}(z_1 + z_1^{-1}) + \sum_{k=1}^{n} \sum_{j=1, j\neq i}^{m} (b_{i,j,k} + b_{j,i,k}) + \sum_{j=1, j\neq i}^{m} c_{i,j}$$

$$M_{i,j} = -(b_{i,j,1}z_1 + b_{j,i,1}z_1^{-1})) + (\sum_{k=2}^{m} -(b_{i,j,k} + b_{j,i,k})) - c_{j,i}$$

When $z_1 = 1$ we have the following properties:

$$(1)\ \ \frac{\partial}{\partial z_1} M_{i,i} = 0$$

$$(2)\ \ \frac{\partial}{\partial z_1} M_{i,j} = -\frac{\partial}{\partial z_1} M_{j,i}$$

By Jacobi's formula we have that

$$\frac{d}{dz_1} \det(M)(z_1) = tr(adj(M)(z_1)\frac{dM}{dz_1})(z_1)$$

Evaluating at $z_1 = 1$, we focus on the adjoint matrix.

We have $\det(M) = 0$ and so $adj(M(1))M(1) = 0$. $M(1)$ was reduced to a matrix of rank $m - 1$, and so $adj(M(1))$ is of rank 1.

Thus $adj(M(1)) = hxy^T$ for some m dimensional complex vectors x, y and some polynomial $h \neq 0$ with edge weights as variables with $M(1)x = 0$ and $y^T M(1) = 0$.

By how $M(1)$ is defined we can let $x = (1, \ldots, 1)$ and $y = (1, \ldots, 1)$.

Thus we have that the $adj(M(1))$ is defined to be the matrix with every entry equal to h. Thus $\frac{d}{dz_1} \det(M)(z_1) = h \sum_{(i,j)\in([m]\times[m])} \frac{d}{dz_1} M_{i,j} = 0$. □

Fact 9.2. *Solution at $\lambda = 0$ and $z_i = 1$ is of degree 1 for a generic choice of edge weights.*

Proof. We notice that

$$\frac{\partial}{\partial z_1} M_{i,i} = a_{n*(i-1)+1} z_1^{-2} - a_{n*(i-1)+1}.$$

$$\frac{\partial}{\partial z_1} M_{i,j} = b_{j,i,1} z_1^{-2} - b_{i,j,1}$$

Clearly when $z_1 = 1$ we have the following properties:

$$(1)\ \ M_{i,j} = M_{j,i}$$

$$(2)\ \ \frac{\partial}{\partial z_1} M_{i,i} = 0$$

$$(3)\ \ \frac{\partial}{\partial z_1} M_{i,j} = -\frac{\partial}{\partial z_1} M_{j,i}$$

Need more work, but there is no reason for there to be a solution of degree 2 unless coefficients are particular.

We notice that

$$\frac{\partial^2}{\partial z_1^2} M_{i,i} = -2a_{n*(i-1)+1} z_1^{-3}.$$

$$\frac{\partial^2}{\partial z_1^2} M_{i,j} = -2b_{j,i,1} z_1^{-3}$$

By Jacobi's formula we have that

$$\frac{d}{dz_1}\det(M)(z_1) = tr(adj(M)(z_1)\frac{dM}{dz_1}(z_1))$$
$$\implies \frac{d^2}{dz_1^2}\det(M)(z_1) = tr((\frac{d}{dz_1}adj(M)(z_1))\frac{dM}{dz_1}(z_1)) + tr((adj(M)(z_1))\frac{d^2M}{dz_1^2}(z_1))$$

We also have that $\frac{d^2}{dz_1^2}\det(M)(z_1) = \frac{d^2}{dz_1^2}\sum_{\sigma\in S_n}\prod_{i=1}^m M_{i,\sigma(i)}$

Notice that either the same $M_(i,j)$ will have the the derivative taken twice or two different terms will, these refer to the right and left summands correspondingly.

Thus we have

$$tr((\frac{d}{dz_1}adj(M)(z_1))\frac{dM}{dz_1}(z_1))) = \sum_{j=1}^m\sum_{i=1}^m d(M_{i,j})d\left(\sum_{\sigma\in S_m|\sigma(j)=i}\prod_{k=1,k\neq j}^m M_{k,\sigma(k)}\right)$$

.

Notice because of properties (1),(2), and (3), this will reduce to

$$tr((\frac{d}{dz_1}adj(M)(z_1))\frac{dM}{dz_1}(z_1))) = 2\sum_{1\leq i<j\leq m}^m d(M_{i,j})d\left(\sum_{\sigma\in S_m|\sigma(j)=i}\prod_{k=1,k\neq j}^m M_{k,\sigma(k)}\right)$$

.

The right summand can be reduced to

$$tr((adj(M)(z_1))\frac{d^2M}{dz_1^2}(z_1)) = \sum_{j=1}^m\sum_{i=1}^m d^2(M_{i,j})\left(\sum_{\sigma\in S_m|\sigma(j)=i}\prod_{k=1,k\neq j}^m M_{k,\sigma(k)}\right)$$

.

Notice that the $a_{n*(i-1)+1}$ only can show up in the the second derivatives of the $M_{i,i}$ at $z_1 = 1$, thus as long as for some i we have that

$$\left(\sum_{\sigma\in S_m|\sigma(i)=i}\prod_{k=1,k\neq j}^m M_{k,\sigma(k)}\right)\neq 0$$

for generic choices of edge weights, we will have that for generic choices of edge weights we have a solution of degree 1 at $\lambda = 0, z_i = 0$.

Luckily for us, we already know exactly what these sums are. In particular, these correspond to the entry i,i in $adj(M)$ and as the proof of fact 4.1 mentioned, we have that each of entries are equal, particularly each equals a polynomial $h \in \mathbb{Z}[\{b\},\{c\}]$. We remark that this polynomial is not identitcally zero for generic edgeweights since $adj(M)$ is rank 1. Thus we have that the solution is of order greater than 1 when $p = (-2)a_kh + r = 0$ for some $r \in \mathbb{C}[\{a\}_{i\neq k},\{b\},\{c\}]$. It is clear that p is not identically 0 and thus $V(p)$ is a proper algebraic variety.

Thus if the edge weights are chosen from the generic set $\mathbb{C}^{|\{a\}|+|\{b\}|+|\{c\}|}\smallsetminus V(p)$, the solution at $\lambda = 0, z_i = 1$ will be of order 1.

$\square$

From here we consider the specialization where $b_{i,j} = 0$ unless

$$b_{i,j} = b_{1+(p \mod m),1+(1+p \mod m)},$$

$a_{n*(i-1)+1} = 0$ for each $i = 1, 2, \ldots, m$,and $c_{j,i} = 0$ for all pairs $(i, j) \in [m] \times [m]$. We notice that the resulting matrix is upper bidiagonal, and is such that $[z_1]M_{i,j} = 0$ unless $M_{i,j}$ is an index belonging to the upper diagonal, in which case $[z_1]M_{i,j} = -b_{i_j}$.

We get then that the coefficient of z_1^m of ϕ is independent of all other coefficients. Further we will specialize all a_i corresponding to z_1 to 0. We also specialize all the $c_{i,j}$ edges to 0 to avoid confusion in notation.

Lemma 9.3. *If P_w is a facial system of P identified by $w \neq (a, 0, \ldots, 0)$, then P_w contains an iterated circuit.*

Proof. Without loss of generality, let P_w be a k-facet of P containing the 0-dim facet $(m, 0, \ldots, 0) = v_1$. Notice that each edge of P_w containing $(m, 0, \ldots, 0)$ as an end point is linearly independent from the other edges. Further we have at least (or exactly) k edges containing v_1. Let $E_1, \ldots, E_k$ be these edges. Define B_i to be a set containing two distinct points of E_i that are not equal to v_1. Remark that since $m \geq 2$ and these edges contain at least $m + 1$ points including endpoints, thus the B_i are always constructible as desired. Under a parallel translation p we can consider $p(v_1)$ to be $\mathbf{0}$.

Let $B = p(v_1) \sqcup p(B_1) \sqcup \cdots \sqcup p(B_k)$. We wish to show B is an iterated circuit. Let L_i be the affine span of $p(v_1) \sqcup p(B_1) \sqcup \cdots \sqcup p(B_i)$. Since the B_j are linearly independent $L_{i+1} \setminus L_i$ is equal to the affine span of $p(v_1) \sqcup p(B_{i+1})$. Thus $L_{i+1} \to L_{i+1} \setminus L_i$ will map $\{0\} \sqcup p(B_{i+1})$ injectively onto 3 points in $L_{i+1} \setminus L_i$. We are left to remark that $L_{i+1} \setminus L_i \equiv \mathbb{R}$, and so these 3 distinct points give us a circuit in $L_{i+1} \setminus L_i$. Thus definition of iterated circuit given by Esterov in [6], we have that B is an interated circuit contained in P_w. □

Lemma 9.4. *If P_w is a facial system of $\mathrm{N}(\Psi)$ identified by w, that is not the base of the polytope, then for generic edge choices f_w is non-singular.*

Proof. Let P_w be a facet of dimension k. If $k = 0$ we are done since no monomial can have a solution on the torus. Thus we have that k is at least 1, and so P_w contains at least 2 0-dimensional faces. By making adequate rotation or change of variables, we can assume that $e = (m, 0, 0, \ldots, 0) \in P_w$, and so $e \in P_{wi} \subset P_w$ [10].

Let $\psi_w \in \mathbb{C}[z_1, z_2, \ldots, z_k, \lambda]$

By Lemma 5.3, P_w contains an iterated circuit and so P_w non defective [6] [11]. We have that the P_w-Discriminant, D_{P_w}, is an irreducible polynomial in $\mathbb{C}[P_w]$ [6] and $V(D_{P_w})$ is a hypersurface, where $V(D_{P_w})$ is such that given a polynomial γ with support F, then $\gamma \in V(D_{P_w})$ if γ has a singular point. We write a general polynomial γ of support P_w as $\gamma = \sum_{a \in F} c_a z^a$. By Lemma 3.9 of [7], we have that D_{P_w} is positive degree in c_a. By this we mean that, given a lattice point $e \in P_w$, D_{P_w} can be written as a polynomial of positive degree in variable c_e.

[10]This maybe important in the case we are currently in where we have not scaled by a monomial since there can be i such that P_{wi} has more vertices than P_w.

[11]This is absolutely not the way to put in citations!!!! Compare to the Introduction.

Suppose we fix all c_a for $a \in P_w$ where $a \neq e$. Then (D_{P_w}) becomes a single variable polynomial in $\mathbb{C}[e]$ of degree s, say $(D_{P_w})_e$. We wish to show that this polynomial is not identically 0 after the aforementioned specialization.

Under the specialization given notice that $\psi_w = (-1)^{m+1} z_1^m \prod_{p=1}^m b_{1+(p \mod m), 1+(1+p \mod m)} + g$, where $g \in \mathbb{C}[z_2, \ldots, z_k, \lambda]$, thus we have that $\psi_{1w} = m(-1)^{m+1} z_1^m \prod_{p=1}^m b_{1+(p \mod m), 1+(1+p \mod m)}$. In particular, this shows that for choices of $b_{i,j} \neq 0$ for (i,j) indices of the upper diagonal, we have that $\gamma \notin V(D_{P_w})$. Further we have that under this specialization we have that (D_{P_w}) as a polynomial of c_e must have terms of positive degree c_e that appear isolated from other c_a, since if all other coefficients are specialized to 0, we still have if c_e is nonzero, then $\gamma \notin V(D_{P_w})$. Thus since $\overline{V(D_{P_w})} \neq \emptyset$ we have that $V(D_{P_w})$ is a proper algebraic variety with respect to edge weights, and thus for a generic choice of edge weights we have that ψ_w is nonsingular.

$\square$

Through this we have proven that P_w has no solutions unless w identifies, is the inner normal, of the base.

We remark that by Bernstein's second theorem, the solutions at the base are exactly the solutions at the toric infinity $\lambda = 0$. Thus we indeed have all $2^n m^{n+1}$ isolated solutions in $(\mathbb{C}^*)^n \times \mathbb{C}$.

Theorem 9.5. *If W is a dense fundamental domain with m vertices in dimension n with generic edge weights α then the dispersion relation of the difference operator L_α has $2^n m^{n+1}$ degenerate isolated critical points.*

Wishing to append a slightly more general result.

Theorem 9.6. *Suppose $\phi \in \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}, \lambda]$ and $\Phi := \phi = z_1 \frac{\partial \phi}{\partial z_1} = \cdots = z_n \frac{\partial \phi}{\partial z_n} = 0$. Further suppose that except for the base, we have that ϕ_w is non singular.*

Then we have that Φ has $(n+1)! \operatorname{Vol}(\mathrm{N}(\phi))$ isolated solutions counted according to multiplicity in $(\mathbb{C}^*)^n \times \mathbb{C}$.

Proof. In order to do this we follow a method from [5]. We wish multiply ϕ by a monomial so that $[z_i^0]\phi = 0$, that is ϕ has no constant terms as a polynomial of z_i for each i. It suffices to pick $r \geq 1 + min_{i \in [m]}(deg_i(t) \mid t$ is a term of ϕ, and $deg_i(t)$ gives the degree of t with respect to i. Then we have $\phi' = (z_1^r \ldots z_n^r)\phi$ is a polynomial of degree greater than 1 with respect to each variable.

We now consider a system $\Phi' := \phi = \frac{z_1 \frac{\partial \phi'}{\partial z_1}}{z_1^r \ldots z_n^r} = \cdots = \frac{z_m \frac{\partial \phi'}{\partial z_n}}{z_1^r \ldots z_n^r} = 0$. Notice that $z_0 \in (\mathbb{C}^*)^n \times \mathbb{C}$ is a solution to Φ' if and only if z_0 is a solution to Φ.

In particular we have by Leibniz formula that $\frac{z_i \frac{\partial \phi'}{\partial z_i}}{z_1^r \ldots z_n^r} = \frac{r(z_1^r \ldots z_n^r)\phi + z_i \frac{\partial \phi}{\partial z_1}(z_i^r \ldots z_n^r)}{z_1^r \ldots z_n^r} = r\phi + z_i \frac{\partial \phi}{\partial z_1}$.

Further let $Q = \mathrm{N}(\phi)$, it is easy to see $\mathrm{N}(z_i \frac{\partial \phi'}{\partial z_i}) = Q$. Thus $\mathrm{MV}(Q, Q, \ldots, Q) = (n+1)! V(Q)$.

Since each ϕ_w is nonsingular Φ_w has no solutions, and thus Φ'_w has no solutions, for any w except possibly $w = (0, \ldots, 0, 1)$, then Φ'_w has $(n+1)! \operatorname{Vol}(Q)$ isolated solutions counted according to multiplicity in $(\mathbb{C}^*)^n \times \mathbb{C}$. Thus Φ_w has $(n+1)!\,\mathrm{Vol}(Q)$ isolated solutions, and further more

$$\mathrm{MV}(\mathrm{N}(\phi), \mathrm{N}(z_1\frac{\partial \phi}{\partial z_1}), \ldots, \mathrm{N}(z_n\frac{\partial \phi}{\partial z_n})) = (n+1)!\,\mathrm{Vol}(Q)$$

[12]. □

Question to Frank: Does this imply by theorem 4.2 that for generic Laurent polynomials ϕ in $n+1$ variables we almost always have that $\Phi =: \phi = z_1\frac{\partial \phi}{\partial z_1} = \cdots = z_n\frac{\partial f}{\partial z_n} = 0$ has $(n+1)!\,\mathrm{Vol}(\mathrm{N}(\phi))$ solutions and mixed volume. If so this seems like a decent result... I think my proof only works for ϕ that are characteristic polynomials, in which case this doesn't work for general ϕ.

REFERENCES

1. Gregory Berkolaiko and Peter Kuchment, *Introduction to quantum graphs*, Mathematical Surveys and Monographs, vol. 186, American Mathematical Society, Providence, RI, 2013.
2. David N. Bernstein, *The number of roots of a system of equations*, Functional Anal. Appl. **9** (1975), no. 3, 183–185.
3. Frédéric Bihan and Ivan Soprunov, *Criteria for strict monotonicity of the mixed volume of convex polytope*, Advances in Geometry **19** (2019), no. 4, 527–540.
4. Yves Colin de Verdière, *Sur les singularités de van Hove génériques*, Analyse globale et physique mathématique (Lyon, 1989), no. 46, Soc. Math. France, 1991, pp. 99–110.
5. Ngoc Do, Peter Kuchment, and Frank Sottile, *Generic properties of dispersion relations for discrete periodic operators*, J. Math. Phys. **61** (2020), no. 10, 103502, 19.
6. A. Esterov, *Newton polyhedra of discriminants of projections*, Discrete Comput. Geom. **44** (2010), no. 1, 96–148.
7. ______, *Galois theory for general systems of polynomial equations*, Compos. Math. **155** (2019), no. 2, 229–245.
8. G. Ewald, *Combinatorial convexity and algebraic geometry*, Graduate Texts in Mathematics, vol. 168, Springer-Verlag, New York, 1996.
9. I. M. Gel′ fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1994.
10. Anatoli G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), no. 1, 1–31.
11. Peter Kuchment, *Floquet theory for partial differential equations*, Operator Theory: Advances and Applications, vol. 60, Birkhäuser Verlag, Basel, 1993.
12. ______, *An overview of periodic elliptic operators*, Bull. Amer. Math. Soc. (N.S.) **53** (2016), no. 3, 343–414.
13. S. P. Novikov, *Bloch functions in the magnetic field and vector bundles. Typical dispersion relations and their quantum numbers*, Dokl. Akad. Nauk SSSR **257** (1981), no. 3, 538–543.
14. ______, *Two-dimensional Schrödinger operators in periodic fields*, Current problems in mathematics, Vol. 23, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983, pp. 3–32.
15. J. Maurice Rojas, *A convex geometric approach to counting the roots of a polynomial system*, Theoret. Comput. Sci. **133** (1994), no. 1, 105–140, Selected papers of the Workshop on Continuous Algorithms and Complexity (Barcelona, 1993).

[12] Follows from combining Bernstein's theorem with Theorem 4.12 of of chapter IV of Ewald [8]. Let $K, K', K_1, \ldots, K_{n-1}$ be convex bodies, and let $K \subset K'$, then $Vol(K, K_1, \ldots, K_{n-1}) \leq V(K', K_1, \ldots, K_{n-1})$ where V denotes the mixed volume (not normalized).

MATTHEW FAUST, DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843, USA
Email address: mfaust@math.tamu.edu
URL: http://www.math.tamu.edu/~mfaust

FRANK SOTTILE, DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843, USA
Email address: sottile@math.tamu.edu
URL: http://www.math.tamu.edu/~sottile