PROPERTIES NECESSARY FOR THE MINIMAL SUBGRAPHS

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1. Minimal subgraphs of a dense periodic graph

Let us consider a dense periodic graph on m vertices labeled $v_1, \dots, v_m$ living in an n dimensional ambient space with a fundamental domain W.

If a is a linear translation such that there is an edge of W to aW then we call aW an adjacent fundamental domain.

A fundamental domain of a periodic graph will always have finitely many edges leaving it. Because of this we will have finitely many adjacent fundamental domains. If U is an adjacent fundamental domain to W, then let $T(U) = b$ such that $bW = U$.

Let $P(W)$ be the collection of adjacent fundamental domains to W, then $Q(W) = \{T(U) | U \in P(W)\}$.

We say a fundamental domain $U \in P(W)$ where $T(U) = a \in Q(W)$ is maximally independent if ma cannot be expressed as a sum of m or less elements of $Q(w)$ except as ma.

Fact 1.1. *A minimal subgraph of a dense periodic graph must have at least m edges leaving W for each maximally independent adjacent fundamental domain. Further m of these edges must correspond to some member of S_m.*

Proof. The characteristic polynomial must contain each term x^{ma} for it's corresponding polytope to maintain the same volume. Thus the term x^a must appear in the matrix at least m times.

We have that x^{ma} is a term in the characteristic polynomial if it is a term in the determinant of the matrix representation of the Laplace-Beltrami operator say L.

We have that $det(L) = \sum_{\sigma \in S_m} sign(\sigma) \prod_{i=1}^{m} L_{i,\sigma(i)}$

Thus for some σ we must have x^a is a term in each $L_{i,\sigma(i)}$.

x^a is a term in $L_{i,\sigma(i)}$ exactly when there is an edge from v_i to $av_{\sigma(i)}$.

□

In an action adjacent dense periodic graph (the ones given by the function I gave), we have that each action is maximally independent and so we have that the graph must have $n * m$ edges leaving the fundamental domain.

In each direction the edges from F to aF contain a subset of edges corresponding to some permutation of $[m]$.

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