Solving Semi-mixed Polynomial Systems with the Polyhedral Homotopy

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Semi-mixed Polynomial Systems

Let $F = (F_1, \ldots, F_n) = 0$ be a system of polynomial equations in $\mathbb{C}[x_1, \ldots, x_n]$.

Definition

A system of polynomial equations is called **semi-mixed** if some pair of polynomials has identical support.

Write $\mathcal{A} = (\mathcal{A}_1, k_1; \ldots ; \mathcal{A}_r, k_r)$ to denote the support of a semi-mixed system with distinct supports given by $\mathcal{A}_i$ and each $\mathcal{A}_i$ supporting k_i equations.

We may assume that the affine span of $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_r$ is all of $\mathbb{R}^n$. Otherwise, there are either no solutions or infinitely many solutions! Oh no!

Regular Subdivisions

Let $\mathcal{A}_i \subseteq \mathbb{R}^n$ be finite sets for $i = 1, \ldots, r$. A cell of $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_r)$ is a collection of subsets $C = (C_1, \ldots, C_r)$ with $C_i \subseteq \mathcal{A}_i$.

Definition

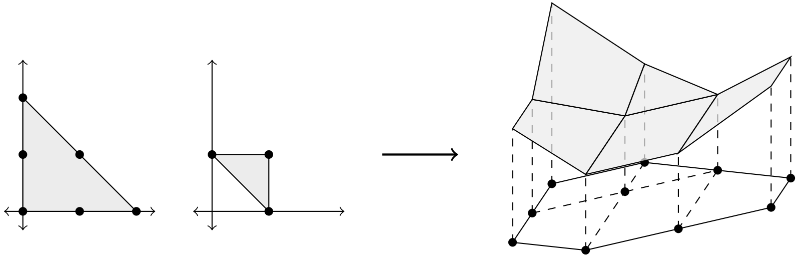
A **subdivision** of $\mathcal{A}$ is a collection of cells $C^{(1)}, \ldots, C^{(m)}$ such that

1. $\dim(\mathrm{conv}(C_1^{(i)} + \ldots + C_r^{(i)})) = n$
2. $\mathrm{conv}(C_1^{(i)} + \ldots + C_r^{(j)}) \cap \mathrm{conv}(C_1^{(j)} + \ldots + C_r^{(j)})$ is a face of both polytopes.
3. $\bigcup_{i=1}^{r} \mathrm{conv}(C_1^{(i)} + \ldots C_r^{(i)}) = \mathrm{conv}(\mathcal{A}_1 + \ldots + \mathcal{A}_r)$.

Nobody wants to check all this. That's why we like the class of **regular subdivisions!**

Regular Subdivisions

A regular subdivision of $\mathcal{A}$ can be described as follows. For each i, choose any function $\omega_i : \mathcal{A}_i \mapsto \mathbb{R}$ and let $\widehat{\mathcal{A}}_i$ be its graph. Each face of the lower hull of $\mathrm{conv}(\widehat{\mathcal{A}}_1 + \ldots + \widehat{\mathcal{A}}_r)$ corresponds uniquely to a cell of $\mathcal{A}$. This collection of cells is a subdivision!



For a semi-mixed system with support $\mathcal{A} = (\mathcal{A}_1, k_1; \ldots ; \mathcal{A}_r, k_r)$, a regular subdivision is defined to be a regular subdivision of the distinct supports $(\mathcal{A}_1, \ldots , \mathcal{A}_r)$.

Fine Mixed Subdivisions

Definition

A **mixed subdivision** of $\mathcal{A}$ is a subdivision $C^{(1)}, \ldots, C^{(m)}$ such that for each cell $C^{(j)}$,

$$\sum_{i=1}^{r} \dim(\mathrm{conv}(C_i^{(j)})) = n.$$

Definition

A **fine mixed subdivision** of $\mathcal{A}$ is a mixed subdivision $C^{(1)}, \ldots, C^{(m)}$ such that for each cell $C^{(j)}$,

$$\sum_{i=1}^{r} (\# C_i^{(j)} - 1) = n.$$

Fine Mixed Subdivisions

Let $\mathcal{A} = (\mathcal{A}_1, k_1; \ldots; \mathcal{A}_r, k_r)$ be the set of supports of a semi-mixed system. If $C = (C_1, \ldots, C_r)$ is a cell of $\mathcal{A}$, let

$$\operatorname{type}(C) = (\dim \operatorname{conv}(C_1), \ldots, \dim \operatorname{conv}(C_r)).$$

The BKK theorem tells us that with a generic choice of coefficients to this system, the number of roots is the mixed volume

$$\mathcal{M}(\mathcal{A}_1, k_1; \ldots; \mathcal{A}_r, k_r) = \mathcal{M}(\mathcal{A}_1, \ldots, \mathcal{A}_1, \ldots, \mathcal{A}_r, \ldots, \mathcal{A}_r).$$

Theorem

If $C^{(1)}, \ldots, C^{(m)}$ is a fine mixed subdivision of $\mathcal{A}$, then

$$\mathcal{M}(\mathcal{A}_1, k_1; \ldots; \mathcal{A}_r, k_r) = \sum_{type(C^{(j)})=(k_1,\ldots,k_r)} \left|\det V(C^{(j)})\right|.$$

The Polyhedral Homotopy

Let $F(x)$ be our semi-mixed system of polynomial equations with support $\mathcal{A} = (\mathcal{A}_1, k_1; \ldots; \mathcal{A}_r, k_r)$. Let $\omega = (\omega_1, \ldots, \omega_r)$ be lifting functions yielding a fine mixed subdivision $C^{(1)}, \ldots, C^{(m)}$. We may write our system as

$$f_{ij}(x) = \sum_{\alpha \in \mathcal{A}_i} c_{\alpha,j} x^\alpha,$$

for $1 \leq j \leq k_i$. let $F(x,t)$ be the one-parameter system defined by the polynomials

$$f_{ij}(x,t) = \sum_{\alpha \in \mathcal{A}_i} c_{\alpha,j} x^\alpha t^{\omega_i(\alpha)}.$$

For $t = 1$, we obtain our original system of equations.

The Polyhedral Homotopy

For each cell $C^{(p)}$, let $(\gamma, 1) \in \mathbb{R}^{n+1}$ be an inner normal of the face $\mathrm{conv}(\widehat{C}_1^{(p)} + \ldots + \widehat{C}_r^{(p)})$ corresponding to the cell $C^{(p)}$. make the substitution $x = \widetilde{x}t^{\gamma}$ in $F(x,t)$ to obtain polynomials

$$f_{ij}(\widetilde{x}, t) = \sum_{\alpha \in \mathcal{A}_i} c_{\alpha,j} \widetilde{x}^{\alpha} t^{\gamma \cdot \alpha + \omega_i(\alpha)} = \sum_{\alpha \in \mathcal{A}_i} c_{\alpha,j} \widetilde{x}^{\alpha} t^{(\gamma,1)\cdot(\alpha,\omega_i(\alpha))}.$$

$(\gamma, 1) \cdot (\alpha, \omega_i(\alpha))$ is minimized for $\alpha \in C_i^{(p)}$. Divide out by this minimal power of t. Plugging in $t = 1$ still gives our original system and plugging in $t = 0$ now gives

$$g_{ij}(\widetilde{x}) = \sum_{\alpha \in C_i^{(p)}} c_{\alpha,j} \widetilde{x}^{\alpha}.$$

Our start system corresponding to the cell $C^{(p)}$ is the collection of polynomials

$$g_{ij}(x) = \sum_{\alpha \in C_i^{(p)}} c_{\alpha,j} \widetilde{x}^{\alpha}$$

for each i, there are k_i polynomials g_{ij} with support $C_i^{(p)}$. Since $\#C_i^{(p)} = k_i + 1$ and conv$(C^{(p)})$ is a k_i-simplex, we can use Gaussian elimination to convert the system g_{ij} into a binomial system! Yaay!

Now we can numerically track the roots of the system g_{ij} to the roots of f_{ij} via the homotopy associated to the cell $C^{(p)}$.

That's all, folks!

HAPPY END OF THE SEMESTER!!