Info retrieval sheet 4

Exercise 1

To show: for any prefix free encoding:
$\mathbf{E}\ L_x >= H(X) = \sum_i p_i * log(1/p_i)$
Given formula:
$\sum_i 2^{-L_i} <= 1$

We need to build the lagrange function:
$\rightarrow L(p_1, ..., p_k, L_1, ..., L_k, \lambda) = \sum_i p_i * L_1 + \lambda * (\sum_i 2^{-Li} - 1)$

$dL/dL_i = p_i - \lambda * ln(2) * 2^{-L_i} \overset{!}{=} 0$
$\leftrightarrow 2^{L_i} = p_i/(\lambda * ln(2))$
$\leftrightarrow -L_i = log_2(p_i) - log_2(\lambda * ln(2)$

Differentiate after lambda :
$dL/d\lambda = \sum_i 2^{-log_2(p_i/(\lambda * ln(2))} - 1 \overset{!}{=} 0$
$\leftrightarrow \sum_i p_i/(\lambda * ln(2)) = 1$
$\rightarrow (\lambda * ln(2)) = 1, becouse \sum_i p_i = 1$

$\rightarrow L_i = log_2(1/p_i) + log_2(1) = log_2(p_i)$
$\rightarrow \mathbf{E}L_x = \sum_i p_i * log_2(1/p_i) = H(X)$

Now we know that this is an extreme point, we only need to show that it is a minimum:
$d^2L/dL_i^2 = \lambda * ln(2)^2 * 2^{-L_i} > 0$
$\rightarrow$It is a minimum