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# The Generalized Eigenvector Expansions of the Liouville Operator

**Wencai Liu · Zhenyou Huang**

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**Abstract** In this paper, we study the generalized eigenvector expansions of the Liouville operator, and construct the corresponding rigged Liouville space. As an example, we obtain the rigged Liouville space for the harmonic oscillator of one-dimensional.

**Keywords** Liouville operator · Rigged Liouville space · Generalized eigenvector

## 1 Introduction

### 1.1 Generalized Eigenvector Expansions of the Self-Adjoint Operator

In quantum mechanics, an observable is expressed as a self-adjoint operator $H$ on the Hilbert space $\mathscr{H}$ and a state is expressed as an element $\psi \in \mathscr{H}$; while in quantum statistical mechanics, the observable is expressed as the Liouville operator $L_H$ on the Liouville space $\mathscr{H} \otimes \mathscr{H}^{\times}$ and the state is expressed as the element in $\mathscr{H} \otimes \mathscr{H}^{\times}$, where $\mathscr{H}^{\times}$ is the conjugate dual space of $\mathscr{H}$ [10]. The Hamiltonian $H$ and its corresponding Liouvillian $L_H$ satisfy $L_H = \overline{H \otimes I - I \otimes H}$. If the self-adjoint operator $H$ has only point spectrum, the Hilbert space is enough for describing the quantum system [12], however, when $H$ has continuous spectrum, the rigged Hilbert space is needed [2]. In this paper, we discuss how to construct the rigged Liouville space for a Liouville operator.

Dirac creates the Dirac bracket system to describe quantum mechanics [5]. The following mathematical principles must be satisfied [3].

(1) Let $H$ be an observable, then $\forall \lambda \in \sigma(H)$, there exists an "eigenvector" $|\lambda\rangle$ such that

$$H|\lambda\rangle = \lambda|\lambda\rangle. \tag{1.1}$$

W. Liu · Z. Huang (✉)
Department of Mathematics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, People's Republic of China
e-mail: zyhuangh@mail.njust.edu.cn

W. Liu
e-mail: 12110180063@fudan.edu.cn

(2) Every wave function $\varphi$ can be expanded with the "eigenvectors"

$$\varphi = \int_{\sigma(H)} |\lambda\rangle\langle\lambda|\varphi\rangle d\lambda. \tag{1.2}$$

(3) "eigenvectors" are orthogonal, i.e.,

$$\langle\lambda|\lambda'\rangle = \delta(\lambda - \lambda'). \tag{1.3}$$

*Remark 1.1* The above description is only a special case, in general, differing with a weighted function $\mu(\lambda)$ is allowed, i.e.,

(2′) Every wave function $\varphi$ can be expanded with "eigenvectors"

$$\varphi = \int_{\sigma(H)} |\lambda\rangle\langle\lambda|\varphi\rangle d\mu(\lambda). \tag{1.4}$$

(3′) "eigenvectors" are orthogonal, i.e.,

$$\langle\lambda|\lambda'\rangle d\mu(\lambda') = \delta(\lambda - \lambda') d\lambda'. \tag{1.5}$$

The expanding form of a vector as (1.2) or (1.4) is called the generalized eigenvector expansions.

### 1.2 Gelfand Triples and the Gelfand-Maurin Theorem

**Definition 1.1** [8] Let $\mathscr{H}$ be a Hilbert space and $\Phi$ a subspace of $\mathscr{H}$, if there exist countably monotone inner products on $\Phi$, which are continuous with respect to the original inner product on $\mathscr{H}$, and $\Phi$ is complete with respect to the topology $\tau_\Phi$ decided by the countable inner products, then we call $\Phi$ a countable Hilbert space.

**Definition 1.2** [8] Suppose $\Phi \subset \mathscr{H}$ is a countable Hilbert space, the countably monotone inner products are

$$(\cdot,\cdot)_1, (\cdot,\cdot)_2, \dots, (\cdot,\cdot)_n, \dots.$$

Let $\Phi_n$ be the completion of $\Phi$ respected to the inner product $(\cdot,\cdot)_n$, if $\forall m$, there exists, an $n > m$ such that the embedded map $T^n_m$ from $\Phi_n$ to $\Phi_m$ is nuclear, i.e., $T^n_m$ have the form

$$T^n_m \varphi = \sum_{k=1}^{\infty} \lambda_k (\varphi_k, \varphi)_n \psi_k, \quad \forall \varphi \in \Phi_n,$$

where $\{\varphi_k\}$ and $\{\psi_k\}$ are the orthogonal bases for $\Phi_n$ and $\Phi_m$ respectively, $\lambda_k \geq 0$, and the series $\sum_{k=0}^{\infty} \lambda_k$ is convergent, then we call $\Phi$ a nuclear space. Furthermore, we call $\Phi \subset \mathscr{H} \subset \Phi^\times$ a Gelfand triple or rigged Hilbert space (RHS), where $\Phi^\times$ is the conjugate dual space of $\Phi$ with respect to the topology $\tau_\Phi$.

Let $H$ be an observable, i.e., a self-adjoint operator on the Hilbert space $\mathscr{H}$. If $H$ has no continuous spectrum, then $\forall \lambda \in \sigma(H) = \sigma_p(H)$, the "eigenvector" $|\lambda\rangle$ is just the corresponding eigenvector, i.e., $|\lambda\rangle \in \mathscr{H}$. Obviously, $|\lambda\rangle$ satisfies (1.1) and (1.3), and all the eigenvectors compose an orthogonal basis for the Hilbert space, i.e., (1.2) holds. Therefore, if an observable $H$ has only eigenvalues, then the Hilbert space is enough for studying the operator $H$.

However, when $H$ has continuous spectrum, $\forall\lambda \in \sigma_c(H)$, we need to know what kind of space that the corresponding "eigenvector" locate in.

**Theorem 1.2** (Gelfand-Maurin Theorem [8]) *Suppose $\Phi \subset \mathscr{H} \subset \Phi^\times$ is a RHS, and the realization of function space induced from $\mathscr{H}$ to space $L^2_\rho$ is $\varphi \to \varphi(\cdot)$, then $\forall\lambda \in \mathbb{R}$, we can construct a continuous conjugate linear functional $|\lambda\rangle$ on $\Phi$ such that $\forall\varphi \in \Phi$, equation*

$$\varphi(\lambda) = \langle\varphi|\lambda\rangle$$

*is correct for $\lambda \in \mathbb{R}$ (a.e. respected to the measure $\rho$).*

In the rigged Hilbert space $\Phi \subset \mathscr{H} \subset \Phi^\times$, $\Phi \subset \mathcal{D}(H)$, and $\forall\lambda \in \sigma(H)$, we can find a corresponding "eigenvector" $|\lambda\rangle \in \Phi^\times$, which could be a generalized eigenvector. The element $F \in \Phi^\times$ is called a generalized eigenvector for $H$ corresponding $\lambda \in \sigma(H)$, if $\forall\varphi \in \Phi$

$$F(H\varphi) = \lambda F(\varphi),$$

and $F$ is denoted as $|\lambda\rangle$.

**Theorem 1.3** (A special case of the Gelfand-Maurin Theorem [3, 8]) *Let $\Phi \subset \mathscr{H} \subset \Phi^\times$ be a RHS, $H$ be a self-adjoint operator on $\mathscr{H}$, $\Phi \subset \mathcal{D}(H)$, and $H$ has a cyclic vector, then $\forall\lambda \in \sigma(H)$, there exists a generalized eigenvector $F_\lambda \equiv |\lambda\rangle$ such that*

$$H^\times|\lambda\rangle = \lambda|\lambda\rangle,$$

*i.e.*,

$$\langle H\varphi|\lambda\rangle = \langle\varphi|H^\times|\lambda\rangle = \lambda\langle\varphi|\lambda\rangle, \quad \forall\varphi \in \Phi,$$

*moreover, there exists a unique positive measure $d\mu(\lambda)$ on $\sigma(H)$ such that*

$$(\psi, \varphi) = \int_{\sigma(A)} \langle\psi|\lambda\rangle\langle\lambda|\varphi\rangle d\mu(\lambda). \tag{1.6}$$

If $\Phi \subset \mathscr{H} \subset \Phi^\times$ is a RHS, where $\Phi \subset \mathcal{D}(H)$ and $H$ has a cyclic vector, then $\forall\lambda \in \sigma_c(H)$, $|\lambda\rangle \in \Phi^\times$ is not a usual eigenvector, but a generalized eigenvector. By Theorem 1.3, it is easy to prove that $|\lambda\rangle$ satisfies (1.1), (1.4) and (1.5).

*Remark 1.4* If $H$ doesn't have a cyclic vector, similarly, we can construct the generalized eigenvector expansions $|\lambda\rangle$ as (1.1), (1.4) and (1.5). But the technique of continuous direct sum [8] or direct integrals [7] of Hilbert spaces is needed.

*Remark 1.5* Even the self-adjoint operator $H$ has a cyclic vector, the corresponding Liouville operator $L_H$ usually doesn't have a cyclic vector, since "if the self-adjoint operator $H$ has a cyclic vector, then any eigen-subspace of $H$ must be one-dimensional." From [9], if $H$ contains more than two eigenvalues, then $0 \in \sigma_p(L_H)$ and $dimker(L_H) > 1$.

## 2 The Construction of Rigged Liouville Space

### 2.1 The Construction of Gelfand Triple in Liouville Space

Denote $\Phi \otimes_1 \Phi^\times = span\{|\varphi\rangle\langle\psi|\}_{\varphi,\psi\in\Phi}$, define the countable inner products on $\Phi \otimes_1 \Phi^\times$ by

$$\big(|\varphi_1\rangle\langle\psi_1|, |\varphi_2\rangle\langle\psi_2|\big)_n = (\varphi_1, \varphi_2)_n(\psi_2, \psi_1)_n, \quad n = 1, 2, \dots. \tag{2.1}$$

Obviously, the completion of $\Phi \otimes_1 \Phi^\times$ with respect to the $n$th inner product is just the Hilbert space $\Phi_n \otimes \Phi_n^\times$, moreover,

$$\Phi_1 \otimes \Phi_1^\times \supset \Phi_2 \otimes \Phi_2^\times \supset \cdots \supset \Phi_n \otimes \Phi_n^\times \supset \cdots \supset \Phi \otimes_1 \Phi^\times.$$

Denote by $\Phi \otimes \Phi^\times$ the completion of $\Phi \otimes_1 \Phi^\times$ according to the countable inner products in (2.1), i.e.,

$$\Phi \otimes \Phi^\times = \bigcap_{n=1}^{\infty} \Phi_n \otimes \Phi_n^\times.$$

Similarly, denote

$$\Phi^\times \otimes \Phi = \bigcup_{n=1}^{\infty} \Phi_n^\times \otimes \Phi_n,$$

then we can construct a countable Hilbert space $\Phi \otimes \Phi^\times$, for its inner products defined as (2.1), the completion of $\Phi \otimes \Phi^\times$ respected to the $n$th inner product is just $\Phi_n \otimes \Phi_n^\times$.

**Theorem 2.1** *The space $\Phi \otimes \Phi^\times \subset \mathscr{H} \otimes \mathscr{H}^\times \subset \Phi^\times \otimes \Phi$ constructed above is a RHS.*

*Proof* We only need to prove $\Phi \otimes \Phi^\times$ is nuclear.

Since $\Phi$ is a nuclear space, $\forall m$, $\exists n$ such that the embedded map $T_m^n$ from $\Phi_n$ to $\Phi_m$ is nuclear, i.e.,

$$T_m^n \varphi = \sum_{k=1}^{\infty} \lambda_k (\varphi_k, \varphi)_n \psi_k, \quad \varphi \in \Phi_n,$$

where $\{\varphi_k\}$ and $\{\psi_k\}$ are the orthogonal bases in $\Phi_n$ and $\Phi_m$ respectively, $\lambda_k \geq 0, k = 1, 2, \dots$, and the series $\sum_{k=0}^{\infty} \lambda_k$ is convergent. Thus the embedded map $\mathbb{T}_m^n$ from $\Phi_n \otimes \Phi_n^\times$ to $\Phi_m \otimes \Phi_m^\times$ satisfies

$$\begin{aligned}
\mathbb{T}_m^n \big(|\varphi\rangle\langle\psi|\big) &= |T_m^n \varphi\rangle\langle T_m^n \psi| \\
&= \left|\sum_{k=1}^{\infty} \lambda_k (\varphi_k, \varphi)_n \psi_k\right\rangle\!\left\langle\sum_{j=1}^{\infty} \lambda_j (\varphi_j, \psi)_n \psi_j\right| \\
&= \sum_{k,j} \lambda_k \lambda_j (\varphi_k, \varphi)_n (\psi, \varphi_j)_n |\psi_k\rangle\langle\psi_j| \\
&= \sum_{k,j} \lambda_k \lambda_j \big(|\varphi_k\rangle\langle\varphi_j|, |\varphi\rangle\langle\psi|\big)_n \big(|\psi_k\rangle\langle\psi_j|\big),
\end{aligned}$$

obviously, $|\varphi_k\rangle\langle\varphi_j|$ and $|\psi_k\rangle\langle\psi_j|$ are the orthogonal bases for $\Phi_n \otimes \Phi_n^\times$ and $\Phi_m \otimes \Phi_m^\times$ respectively, and

$$\sum_{k,j} \lambda_k \lambda_j = \left(\sum_{k=0}^{\infty} \lambda_k\right)\left(\sum_{j=0}^{\infty} \lambda_j\right) < \infty,$$

hence $\Phi \otimes \Phi^\times$ is nuclear.

$\Phi \otimes \Phi^\times \subset \mathscr{H} \otimes \mathscr{H}^\times \subset \Phi^\times \otimes \Phi$ is called the rigged Liouville space (RLS). □

### 2.2 The RLS of Liouville Operator in Quantum Statistical Mechanics

Although there exist many rigged Hilbert spaces $\Phi \subset \mathscr{H} \subset \Phi^\times$ satisfying $\Phi \subset \mathcal{D}(H)$, the smaller $\Phi$ is, the larger the space that the generalized eigenvectors located in, therefore, logically, the larger of $\Phi$ (the smaller of $\Phi^\times$), the better. In addition, the RHS $\Phi \subset \mathscr{H} \subset \Phi^\times$ should satisfy some physical conditions.

(1) If we only want to interpret $|\lambda\rangle$, then the RHS $\Phi \subset \mathscr{H} \subset \Phi^\times$ is only expected to satisfy

$$\Phi \subset \mathcal{D}(H). \tag{2.2}$$

(2) If we also want to consider the variance of the observable $H$

$$\Delta_\varphi H = \sqrt{\big(\varphi, H^2\varphi\big) - (\varphi, H\varphi)^2}, \tag{2.3}$$

then it is required that

$$\Phi \subset \mathcal{D}(H) \cap \mathcal{D}\big(H^2\big). \tag{2.4}$$

(3) In addition, we expect $\Phi$ is invariant under the action of $H$, so

$$\Phi \subset \bigcap_{n=1}^{\infty} \mathcal{D}\big(H^n\big). \tag{2.5}$$

(4) Furthermore, we need $\Phi$ is stable under the action of $H$, then

$$\Phi \subset \bigcap_{n=1}^{\infty} \mathcal{D}\big(H^n\big) \tag{2.6}$$

and $H$ is continuous with respect to the nuclear topology $\tau_\Phi$ on $\Phi$.

According to the different physical conditions, we can construct different RHS $\Phi \subset \mathscr{H} \subset \Phi^\times$, usually the above condition (4) is the most common.

**Theorem 2.2** *If $\Phi \subset \mathscr{H} \subset \Phi^\times$ is the RHS of a self-adjoint operator $H$ satisfying the above physical condition* (4), *then the RLS $\Phi \otimes \Phi^\times \subset \mathscr{H} \otimes \mathscr{H}^\times \subset \Phi^\times \otimes \Phi$ constructed as in Sect.* 2.1, *also satisfies physical condition* (4), *i.e., $L_H$ is continuous with respect to the nuclear topology $\tau_{\Phi\otimes\Phi^\times}$ and*

$$\Phi \otimes \Phi^\times \subset \bigcap_{n=1}^{\infty} \mathcal{D}\big(L_H^n\big). \tag{2.7}$$

**Lemma 2.3** [11] *Let $H$ be an operator on a nuclear space $\Phi$, invariant under $H$, then $H$ is continuous with respect to the topology $\tau_\Phi$ iff $\forall p$, $\exists m > p$, such that*

$$\|H\varphi\|_p \le M\|\varphi\|_m, \quad \forall\varphi \in \Phi, \tag{2.8}$$

*where $M = M(p, m)$.*

*Proof of Theorem 2.2*
(1) By the conditions and Lemma 2.3, $\forall p$, $\exists m > p$ such that

$$\|H\varphi\|_p \le M\|\varphi\|_m.$$

By the definition of Liouville operator $L_H$,

$$\begin{aligned}\big\|L_H\big(|\varphi\rangle\langle\psi|\big)\big\|_p &= \big\|\big(|H\varphi\rangle\langle\psi|\big) - \big(|\varphi\rangle\langle H\psi|\big)\big\|_p \\ &\le 2M\big\|\big(|\varphi\rangle\langle\psi|\big)\big\|_m.\end{aligned} \tag{2.9}$$

Since the span of $\{|\varphi\rangle\langle\psi|\}$ is dense in $\Phi \otimes \Phi^\times$,

$$\|L_H f\|_p \le 2M\|f\|_m, \quad \forall f \in \Phi \otimes \Phi^\times, \tag{2.10}$$

so, $L_H$ is continuous with respect to the topology $\tau_{\Phi\otimes\Phi^\times}$.
(2) We prove the nuclear space $\Phi \otimes \Phi^\times$ is invariant under $L_H$.
$\forall f \in \Phi \otimes \Phi^\times$, by the definition of $\Phi \otimes \Phi^\times$, $\exists\{f_n\} \subset \Phi \otimes_1 \Phi^\times$, such that

$$\lim_{n\to\infty} \|f_n - f\|_i = 0, \quad i = 1, 2, \dots. \tag{2.11}$$

By (2.10), $\{L_H f_n\}$ is a Cauchy sequence in $\Phi_i \otimes \Phi_i^\times$, $i = 1, 2, \dots$. Since $\Phi \otimes \Phi^\times$ is complete, $\exists g \in \Phi \otimes \Phi^\times$ such that

$$\lim_{n\to\infty} \|L_H f_n - g\|_i = 0, \quad i = 1, 2, \dots. \tag{2.12}$$

Since $L_H$ is closed,

$$L_H f = g \in \Phi \otimes \Phi^\times, \quad \forall f \in \Phi \otimes \Phi^\times,$$

$\Phi \otimes \Phi^\times$ is invariant under $L_H$. □

## 3 The RLS of Harmonic Oscillator

As an example, we obtain the RLS for Liouvillian $L_H$ of the harmonic oscillator.
For the harmonic oscillator, the Hamiltonian is

$$H = \frac{P^2}{2\mu} + \frac{\mu\omega^2 Q^2}{2},$$

where $P$ and $Q$ satisfy the Heisenberg equation

$$[P, Q] = -i\hbar I,$$

with $\hbar$, $\mu$ and $\omega$ being the Planck constant, inherent quality and frequency respectively. The realization of one-dimensional harmonic oscillator is as following [3]:

$$P \longleftrightarrow -i\hbar\frac{d}{dx}, \tag{3.1}$$

$$Q \longleftrightarrow x, \tag{3.2}$$

$$H \longleftrightarrow -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + \frac{\mu\omega^2}{2}x^2. \tag{3.3}$$

The RHS $\Phi \subset \mathscr{H} \subset \Phi^\times$ for $H$ is realized to be

$$\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})^\times, \tag{3.4}$$

where $\mathscr{S}(\mathbb{R})$ is the Schwartz space of one-dimensional, and the $n$th inner product of $\mathscr{S}(\mathbb{R})$ is defined as

$$(u, v)_n = \sum_{\alpha \le n} \int_{\mathbb{R}} \left(1 + x^2\right)^n \overline{\frac{d^\alpha u}{dx^\alpha}} \frac{d^\alpha v}{dx^\alpha} dx.$$

**Theorem 3.1** *If the RHS $\Phi \subset \mathscr{H} \subset \Phi^\times$ is*

$$\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})^\times,$$

*then the RLS $\Phi \otimes \Phi^\times \subset \mathscr{H} \otimes \mathscr{H}^\times \subset \Phi^\times \otimes \Phi$ constructed as in Sect. 2.1 is*

$$\mathscr{S}\left(\mathbb{R}^2\right) \subset L^2\left(\mathbb{R}^2\right) \subset \mathscr{S}\left(\mathbb{R}^2\right)^\times, \tag{3.5}$$

*where $\mathscr{S}(\mathbb{R}^2)$ is the Schwartz space of two-dimensional.*

*Proof* By the definition of the RLS, the $n$th inner product of $\Phi \otimes \Phi^\times$

$$\begin{aligned}
&\big(\varphi_1(x)\psi_1(y), \varphi_2(x)\psi_2(y)\big)_n \\
&\quad = \big(\varphi_1(x), \varphi_2(x)\big)_n \big(\psi_1(y), \psi_2(y)\big)_n \\
&\quad = \left(\sum_{\alpha_1 \le n} \int_{\mathbb{R}} \left(1 + x^2\right)^n \overline{\frac{d^{\alpha_1}\varphi_1(x)}{dx^{\alpha_1}}} \frac{d^{\alpha_1}\varphi_2(x)}{dx^{\alpha_1}} dx\right) \\
&\qquad \times \left(\sum_{\alpha_2 \le n} \int_{\mathbb{R}} \left(1 + y^2\right)^n \overline{\frac{d^{\alpha_2}\psi_1(y)}{dy^{\alpha_2}}} \frac{d^{\alpha_2}\psi_2(y)}{dy^{\alpha_2}} dy\right) \\
&\quad = \sum_{\alpha_1 \le n, \alpha_2 \le n} \iint_{\mathbb{R}^2} \left(1 + x^2\right)^n \left(1 + y^2\right)^n \overline{\frac{\partial^{\alpha_1+\alpha_2}\varphi_1(x)\psi_1(y)}{\partial x^{\alpha_1}\partial y^{\alpha_2}}} \frac{\partial^{\alpha_1+\alpha_2}\varphi_2(x)\psi_2(y)}{\partial x^{\alpha_1}\partial y^{\alpha_2}} dx dy.
\end{aligned} \tag{3.6}$$

By the basic knowledge of generalized function, $\Phi \otimes \Phi^\times$ is realized to $\mathscr{S}(\mathbb{R}^2)$, i.e.,

$$\Phi \otimes \Phi^\times \subset \mathscr{H} \otimes \mathscr{H}^\times \subset \Phi^\times \otimes \Phi$$

is realized to be

$$\mathscr{S}\left(\mathbb{R}^2\right) \subset L^2\left(\mathbb{R}^2\right) \subset \mathscr{S}\left(\mathbb{R}^2\right)^\times. \tag{3.7}$$

For harmonic oscillator, the corresponding Liouville operators on the Liouville space are realized as following,

$$L_P \longleftrightarrow -i\hbar\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right), \tag{3.8}$$

$$L_Q \longleftrightarrow x-y, \tag{3.9}$$

$$L_H \longleftrightarrow -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)+\frac{\mu\omega^2}{2}\left(x^2-y^2\right). \tag{3.10}$$

Since

$$\Phi=\bigcap_{n=1}^{\infty}\mathcal{D}\left(H^n\right),$$

the nuclear space $\mathscr{S}(\mathbb{R})$ is optimal for the harmonic oscillator, i.e., $\Phi$ the is maximum nuclear space satisfying

$$\Phi\subset\bigcap_{n=1}^{\infty}\mathcal{D}\left(H^n\right) \tag{3.11}$$

and $H$ is continuous with respect to the nuclear topology $\tau_\Phi$ on $\Phi$. However, we will see that the RLS $\mathscr{S}(\mathbb{R}^2)$ isn't optimal for the corresponding Liouville operator. □

**Lemma 3.2** [3] *Let*

$$N=\frac{1}{\omega\hbar}H-\frac{1}{2}I,$$

*then the self-adjoint operator $N$ has only point spectrum $\sigma(N)=\sigma_p(N)=\mathbb{N}$, and all eigenvalues are simple.*

Let $\{\phi_n\}$ be the normal eigenvectors of $N$, i.e,

$$N\phi_n=n\phi_n,\quad n=0,1,2,\ldots, \tag{3.12}$$

then

$$H\phi_n=\hbar\omega(n+1/2)\phi_n,\quad n=0,1,2,\ldots. \tag{3.13}$$

Obviously, $\{\phi_n\}$ is an orthogonal basis for $L^2(\mathbb{R})$. Define a unitary operator $U_1: L^2(\mathbb{R})\to l^2$,

$$U_1 f=\tilde{f},\quad \forall f\in L^2(\mathbb{R}), \tag{3.14}$$

where $\tilde{f}$ is the Fourier coefficient sequence of $f$, i.e.,

$$\tilde{f}=\left\{(f,\phi_n)\right\}_{n=0}^{\infty}.$$

Similarly, define the unitary operator $U_2: L^2(\mathbb{R})\otimes L^2(\mathbb{R})^\times\to l^2\otimes l^{2\times}$,

$$U_2|g\rangle\langle h|=|\tilde{g}\rangle\langle\tilde{h}|,\quad \forall g,h\in L^2(\mathbb{R}). \tag{3.15}$$

**Lemma 3.3** *Let $\tilde{H}=U_1HU_1^{-1}$, $\tilde{L}_H=U_2L_HU_2^{-1}$, then $\tilde{L}_H=L_{\tilde{H}}$.*

Let $\tilde{\mathscr{S}}(\mathbb{R}) = U_1\mathscr{S}(\mathbb{R})$, by unitary equivalence relations, we define a nuclear topology $\tau_{\tilde{\mathscr{S}}(\mathbb{R})}$ on $\tilde{\mathscr{S}}(\mathbb{R})$, i.e., the countable inner products on $\tilde{\mathscr{S}}(\mathbb{R})$ are

$$(\tilde{g}, \tilde{h})_p = (g, h)_p, \quad \forall g, h \in \mathscr{S}(\mathbb{R}), p = 1, 2, \ldots.$$

Similarly, define the nuclear topology $\tau_{\tilde{\mathscr{S}}(\mathbb{R}^2)}$ on $\tilde{\mathscr{S}}(\mathbb{R}^2)$.

**Theorem 3.4** [3] *We have the following representation*

$$\tilde{\mathscr{S}}(\mathbb{R}) = \left\{ \{\alpha_n\}_{n=0}^{\infty} \middle| \sum_{n=0}^{\infty} (n+1)^p |\alpha_n|^2 < \infty, p = 0, 1, 2, \ldots \right\}. \tag{3.16}$$

*If we define another countable inner products on* $\tilde{\mathscr{S}}(\mathbb{R})$,

$$\langle \alpha, \beta \rangle_p = \sum_{n=0}^{\infty} (n+1)^p \overline{\alpha_n} \beta_n, \quad p = 1, 2, \ldots, \tag{3.17}$$

*then* $\tilde{\mathscr{S}}(\mathbb{R})$ *is also a nuclear space with respect to these inner products, and the topology is equivalent to* $\tau_{\tilde{\mathscr{S}}(\mathbb{R})}$.

**Lemma 3.5** *The operator* $\tilde{H}$ *is self-adjoint on* $l^2$, *where*

$$\begin{aligned} \mathcal{D}(\tilde{H}) &= \left\{ \{h_n\}_{n=0}^{\infty} \middle| \sum_{n=0}^{\infty} |h_n|^2 < \infty, \sum_{n=0}^{\infty} (n+1)^2 |h_n|^2 < \infty \right\}, \\ \tilde{H}\{h_n\} &= \left\{ \hbar\omega(n+1/2)h_n \right\}, \quad \forall \{h_n\} \in \mathcal{D}(\tilde{H}). \end{aligned} \tag{3.18}$$

*Proof* In $L^2(\mathbb{R})$, the domain of $H$ is $\mathcal{D}(H) = \{\varphi \in L^2(\mathbb{R}) \mid H\varphi \in L^2(\mathbb{R})\}$. The conclusion follows by (3.13). □

**Theorem 3.6**

$$l^2 \otimes l^{2\times} = \left\{ \{h_{ij}\}_{i,j=0}^{\infty} \middle| \sum_{i,j} |h_{ij}|^2 < \infty \right\}, \tag{3.19}$$

*and*

$$\begin{aligned} \mathcal{D}(\tilde{L}_H) &= \left\{ \{h_{ij}\} \middle| \sum_{i,j} |h_{ij}|^2 < \infty, \sum_{i,j} \left|(i-j)h_{ij}\right|^2 < \infty \right\}, \\ \tilde{L}_H\{h_{ij}\} &= \left\{ \hbar\omega(i-j)h_{ij} \right\}, \quad \forall \{h_{ij}\} \in \mathcal{D}(\tilde{L}_H). \end{aligned} \tag{3.20}$$

**Theorem 3.7**

$$\tilde{\mathscr{S}}\left(\mathbb{R}^2\right) = \left\{ \{h_{ij}\} \middle| \sum_{i,j}^{\infty} (i+1)^p (j+1)^p |h_{ij}|^2 < \infty, p = 0, 1, 2, \ldots \right\}, \tag{3.21}$$

*and*

$$\bigcap_{n=1}^{\infty} \mathcal{D}\left(\tilde{L}_H^n\right) = \left\{ \{h_{ij}\} \middle| \sum_{i,j} |h_{ij}|^2 < \infty, \sum_{i,j} (i-j)^p |h_{ij}|^2 < \infty, p = 1, 2, \ldots \right\}. \tag{3.22}$$

From (3.21) and (3.22), $\tilde{\mathscr{S}}(\mathbb{R}^2)$ is a proper subset of $\bigcap_{n=1}^{\infty}\mathcal{D}(\tilde{L}_H^n)$, therefore, $\mathscr{S}(\mathbb{R}^2)$ is a proper subset of $\bigcap_{n=1}^{\infty}\mathcal{D}(L_H^n)$.

In the following, we discuss whether there exists a nuclear space $\tilde{\Psi}$ satisfying

$$\tilde{\mathscr{S}}\big(\mathbb{R}^2\big) \subsetneqq \tilde{\Psi} \subsetneqq \bigcap_{n=1}^{\infty}\mathcal{D}\big(\tilde{L}_H^n\big) \tag{3.23}$$

and $\tilde{L}_H$ is continuous with respect to the nuclear topology $\tau_{\tilde{\Psi}}$.

**Theorem 3.8** *There exists a nuclear space $\tilde{\Psi}$ satisfying*

$$\tilde{\mathscr{S}}\big(\mathbb{R}^2\big) \subsetneqq \tilde{\Psi} \subsetneqq \bigcap_{n=1}^{\infty}\mathcal{D}\big(\tilde{L}_H^n\big), \tag{3.24}$$

*with $\tau_{\tilde{\mathscr{S}}(\mathbb{R}^2)} \succ \tau_{\tilde{\Psi}}$ and $\tilde{L}_H$ is continuous with respect to nuclear topology $\tau_{\tilde{\Psi}}$ iff there exists a nuclear space $\tilde{\Phi}_1$ satisfying*

$$\tilde{\mathscr{S}}(\mathbb{R}) \subsetneqq \tilde{\Phi}_1 \subsetneqq l^2, \tag{3.25}$$

*with $\tau_{\tilde{\mathscr{S}}(\mathbb{R})} \succ \tau_{\tilde{\Phi}_1}$.*

*Proof* *Sufficiency*.

Denote by $\langle\cdot,\cdot\rangle_p$ the $p$th inner product of the nuclear space $\tilde{\Phi}_1$ which satisfies

$$\tilde{\mathscr{S}}(\mathbb{R}) \subsetneqq \tilde{\Phi}_1 \subsetneqq l^2.$$

We construct the space

$$\tilde{\Psi} = \left\{\{h_{ij}\}_{i,j=1}^{\infty} \,\middle|\, \sum_{i\neq j}^{\infty}(i+1)^p(j+1)^p|h_{ij}|^2 < \infty, \{h_{ii}\}_{i=1}^{\infty} \in \tilde{\Phi}_1, p = 0, 1, 2, \dots\right\}, \tag{3.26}$$

the $p$th inner product on $\tilde{\Psi}$ is defined as following

$$(\alpha,\beta)_p = \sum_{i\neq j}(i+1)^p(j+1)^p\overline{\alpha_{ij}}\beta_{ij} + \big\langle\{\alpha_{ii}\},\{\beta_{ii}\}\big\rangle_p. \tag{3.27}$$

It's easy to prove that $\tilde{\Psi}$ is nuclear. By (3.22), (3.25) and (3.26), we obtain

$$\tilde{\mathscr{S}}\big(\mathbb{R}^2\big) \subsetneqq \tilde{\Psi} \subsetneqq \bigcap_{n=1}^{\infty}\mathcal{D}\big(\tilde{L}_H^n\big),$$

with $\tau_{\tilde{\mathscr{S}}(\mathbb{R}^2)} \succ \tau_{\tilde{\Psi}}$ and $\tilde{L}_H$ is continuous with respect to the nuclear topology $\tau_{\tilde{\Psi}}$.

*Necessity*.

If the nuclear space $\tilde{\Psi}$ satisfies

$$\tilde{\mathscr{S}}\big(\mathbb{R}^2\big) \subsetneqq \tilde{\Psi} \subsetneqq \bigcap_{n=1}^{\infty}\mathcal{D}\big(\tilde{L}_H^n\big),$$

arrange the orthogonal basis $\{|e_i\rangle\langle e_j|\}$ for $l^2 \otimes l^{2\times}$ in the following order

$$\begin{array}{ccccccc}
|e_1\rangle\langle e_1| & \rightarrow & |e_1\rangle\langle e_2| & & |e_1\rangle\langle e_3| & \rightarrow & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & \\
|e_2\rangle\langle e_1| & & |e_2\rangle\langle e_2| & & |e_2\rangle\langle e_3| & & \cdots \\
\downarrow & \nearrow & & \swarrow & & & \\
|e_3\rangle\langle e_1| & & |e_3\rangle\langle e_2| & & |e_3\rangle\langle e_3| & & \\
 & \swarrow & & & & & \\
\cdots & & \cdots & & \cdots & &
\end{array}$$

then we can define a unitary operator $V : l^2 \otimes l^{2\times} \rightarrow l^2$ which makes a correspondence between $\{|e_i\rangle\langle e_j|\}$ and $\{e_k\}$ as the order above, i.e.,

$$\begin{aligned}
V|e_1\rangle\langle e_1| &= e_1, \\
V|e_1\rangle\langle e_2| &= e_2, \\
V|e_2\rangle\langle e_1| &= e_3, \\
V|e_3\rangle\langle e_1| &= e_4, \\
&\vdots
\end{aligned}$$

where $\{e_i\}$ is the natural orthogonal basis of $l^2$, i.e., the $i$th component of $e_i$ being 1, others being 0. Obviously, $V\tilde{\mathscr{S}}(\mathbb{R}^2) = \tilde{\mathscr{S}}(\mathbb{R})$, we only need consider $\tilde{\Phi}_1 = V\tilde{\Psi}$. □

**Lemma 3.9** *There exists a nuclear space $\tilde{\Phi}_1$ satisfying*

$$\tilde{\mathscr{S}}(\mathbb{R}) \subsetneqq \tilde{\Phi}_1 \subsetneqq l^2, \quad \tau_{\tilde{\mathscr{S}}(\mathbb{R})} \succ \tau_{\tilde{\Phi}_1}.$$

*Proof* Choose a vector $\varphi_0 \in l^2$, $\varphi_0 \notin \tilde{\mathscr{S}}(\mathbb{R})$, and let

$$\tilde{\Phi}_1 = span\{\tilde{\mathscr{S}}(\mathbb{R}) \cup \{\varphi_0\}\},$$

define the $p$th inner product on $\tilde{\Phi}_1$

$$\langle \varphi, \psi\rangle_p = (f, g)_p + \overline{\lambda_\varphi}\lambda_\psi, \quad \forall \varphi, \psi \in \tilde{\Phi}_1,$$

where

$$\varphi = f + \lambda_\varphi \varphi_0, \quad f \in \tilde{\mathscr{S}}(\mathbb{R})$$

and

$$\psi = g + \lambda_\psi \varphi_0, \quad g \in \tilde{\mathscr{S}}(\mathbb{R}),$$

where $(\cdot)_p$ is the $p$th inner product on $\tilde{\mathscr{S}}(\mathbb{R})$. It is easy to verify that $\tilde{\Phi}_1$ satisfies the lemma. □

**Theorem 3.10** *There exits a nuclear space $\tilde{\Psi}$ satisfies*

$$\tilde{\mathscr{S}}(\mathbb{R}^2) \subsetneqq \tilde{\Psi} \subsetneqq \bigcap_{n=1}^{\infty} \mathcal{D}(\tilde{L}_H^n),$$

*and $\tilde{L}_H$ is continuous respected to nuclear topology $\tau_{\tilde{\Psi}}$, i.e, $\tilde{\mathscr{S}}(\mathbb{R}^2)$ isn't the optimal nuclear space for $\tilde{L}_H$, and i.e., $\mathscr{S}(\mathbb{R}^2)$ isn't the optimal nuclear space for $L_H$.*

*Proof* By Theorem 3.8 and Lemma 3.9. □

*Remark 3.11* Theorem 3.10 suggests that even if $\Phi$ is optimal for $H$, however, the nuclear space $\Phi \otimes \Phi^\times$ constructed as Sect. 2.1 may not be optimal for the Liouville operator $L_H$. But it will be very difficult to improve our results. For the harmonic oscillator, the nuclear space $\Phi \otimes \Phi^\times$ constructed as Sect. 2.1 is optimal for the operator $T = \overline{H \otimes I + I \otimes H}$, since

$$\Phi \otimes \Phi^\times = \mathscr{S}\big(\mathbb{R}^2\big) = \bigcap_{n=1}^{\infty} \mathcal{D}\big(T^n\big),$$

$L_H$ is a hyperbolic operator, $T$ is a elliptic operator. Thanks to the complexion of hyperbolic operator, it's difficult to obtain a optimal nuclear space for the hyperbolic operator $L_H$.

## 4 Summary

The use of rigged Hilbert spaces (RHS) in quantum mechanics is necessary for a proper mathematical presentation of the Dirac formulation (see [4, 6]). In this paper, we establish a kind of generalized eigenfunction expansions for the Liouville operator on a rigged Liouville space, which is a direct generalization of the RHS (the idea appears in [1]).

It is very important to discuss the RHS for the Schrödinger operator $-\Delta + V$ with continuous spectrum, and it is natural and interesting to establish the RLS for the corresponding Liouville operator. By our construction, we obtain the RLS for harmonic oscillator, but which is not the best. In the future, we expect to obtain optimal RLS for harmonic oscillator, furthermore, for other Schrödinger operators with continuous spectrum.

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