Maximum Likelihood on the Sphere

Statistics 98

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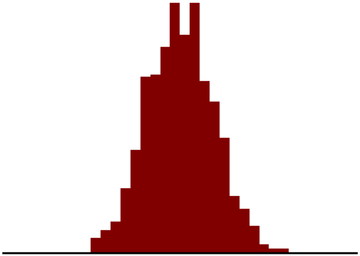
- We often have some data that we believe follows a known distribution.
- We want to estimate the parameters of this distribution using our data.



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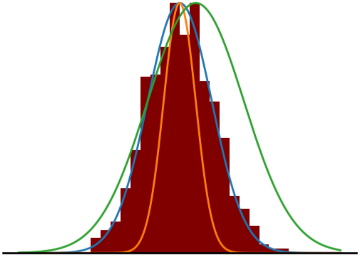
- Construct the **likelihood**—the joint probability of all of our data given a choice of parameters.
- The maximum of this function is called the **Maximum Likelihood Estimate**, or **MLE**.



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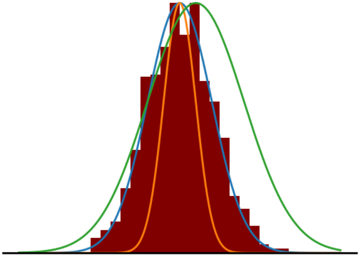

However, we have only worked with distributions whose support is Euclidean space. What if our data lies on a more complex manifold?

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  - Angles or orientations.
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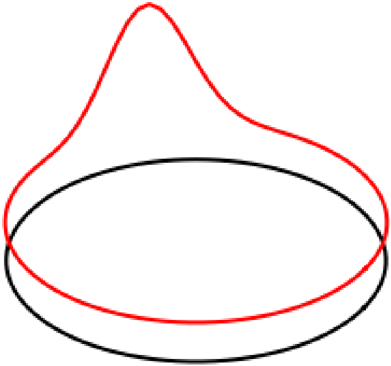

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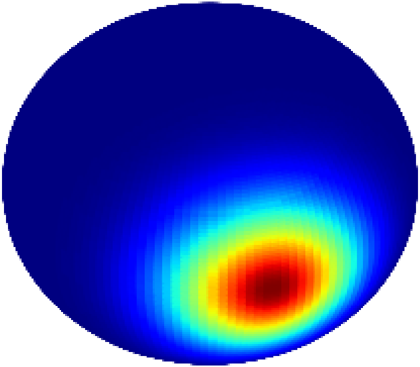
PDF given by

$$P(\mathbf{x}|\boldsymbol{\mu}) = N(\kappa)\exp\left(\kappa\boldsymbol{\mu}^T\mathbf{x}\right),$$

- Parameterized by *mean* unit vector μ and *concentration* $\kappa \geq 0$.
- Normalization constant $N(\kappa)$ found via numeric integration techniques.



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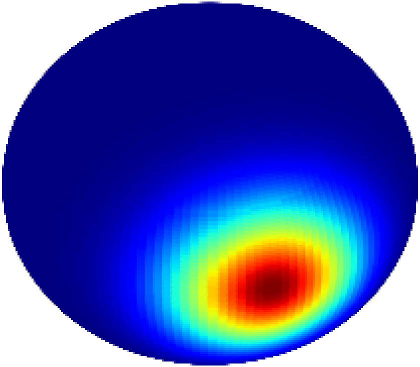

The log-likelihood for some $\{\mathbf{x_i}\}_{i=1}^n$, dropping additive constants, is given by

$$\ell(\boldsymbol{\mu}|\mathbf{x}) \approx \kappa\boldsymbol{\mu}^T\left(\sum_{i=1}^n \mathbf{x_i}\right) = n\kappa\boldsymbol{\mu}^T\overline{\mathbf{x}}.$$

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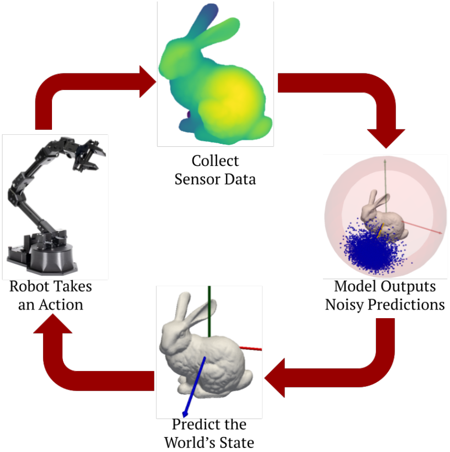

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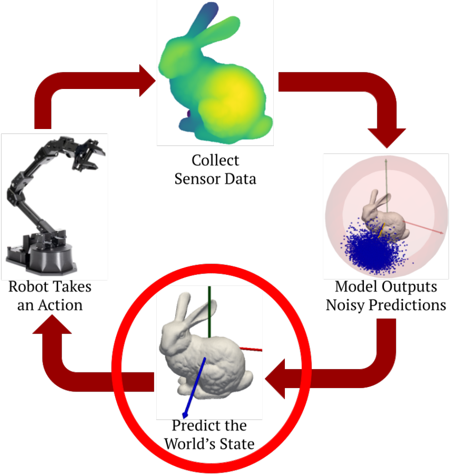

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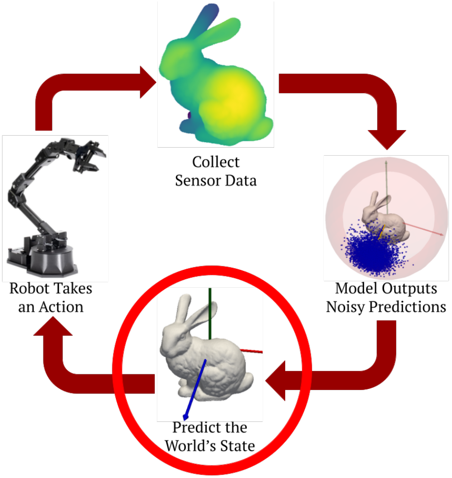
Let's think about this process in a bit more detail.



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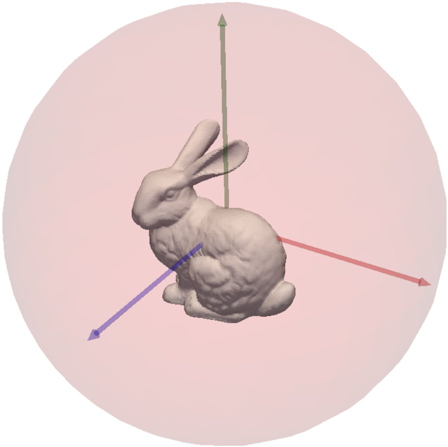

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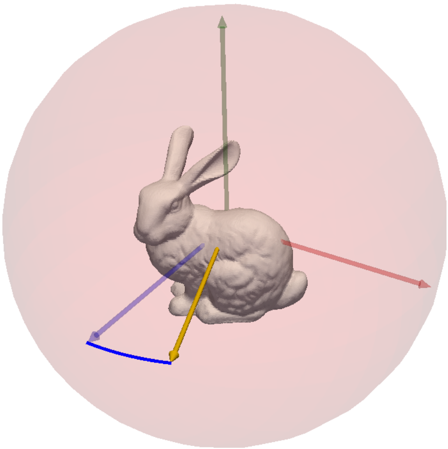

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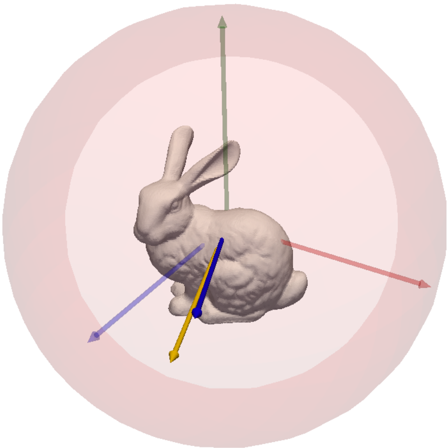

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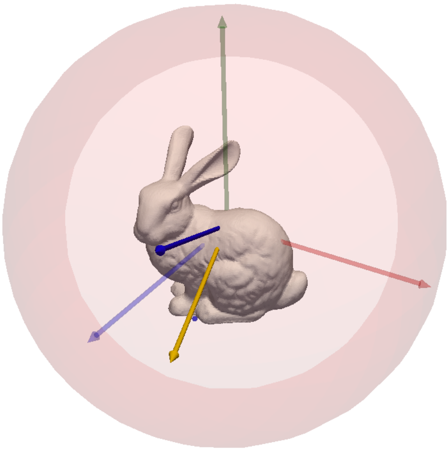

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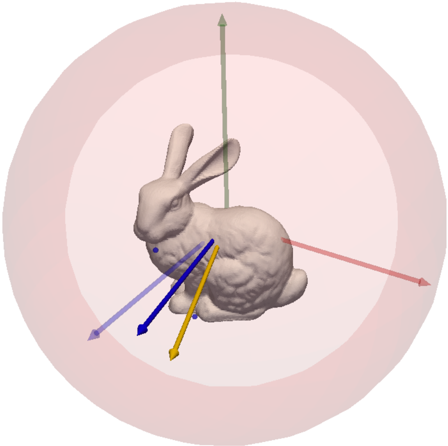

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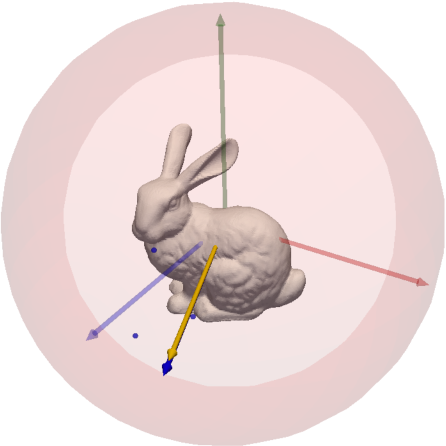

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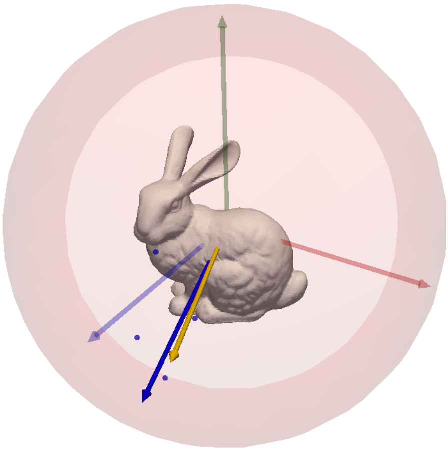

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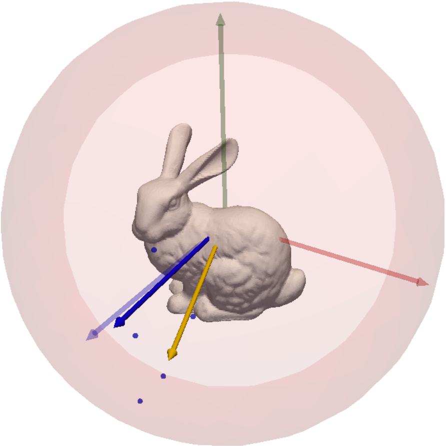

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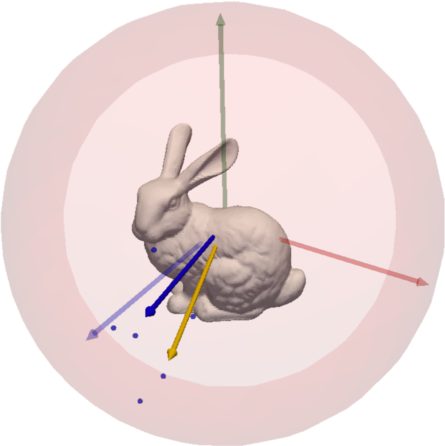

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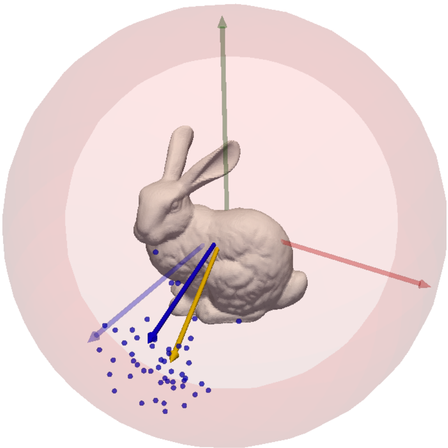

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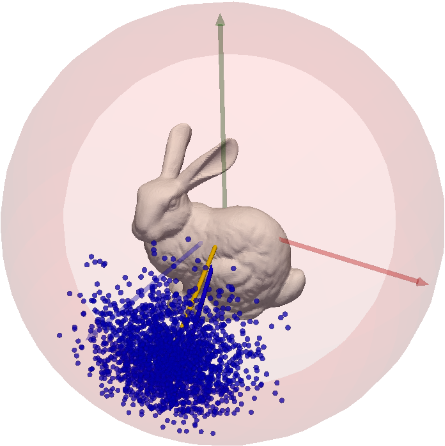

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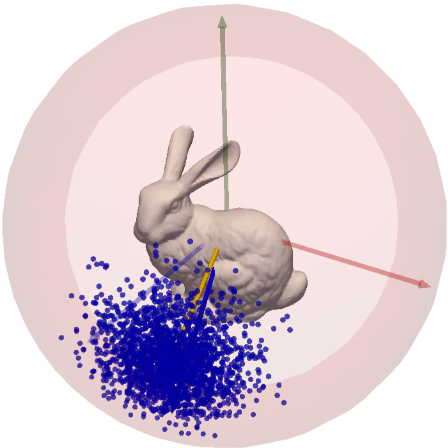

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- We find that the problem is now a **constrained optimization**—maximize $P(\boldsymbol{\mu}|\mathbf{x})$ subject to $||\boldsymbol{\mu}|| = 1$.

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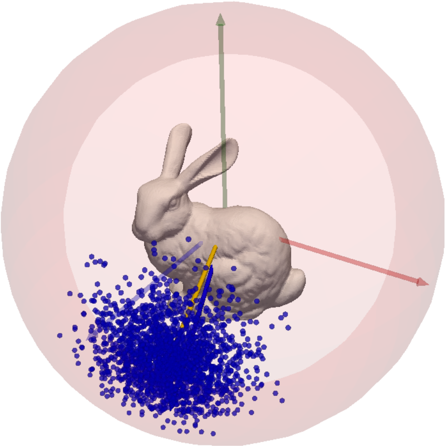

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