Numerical Integration

- Integration replaced by Summation
- Newton-Cotes
  - Equally spaced points
  - Trapezoidal Rule
  - Simpsons rule
- Gauss Quadrature
  - Unequally spaced points
- Romberg Integration
  - Newton-Cotes + Richardson Extrapolation

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} A_i\, f(x_i)$$

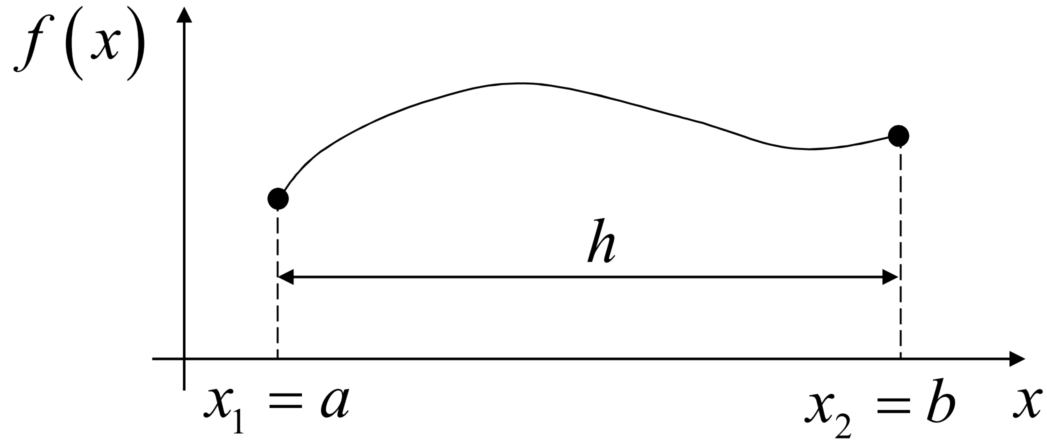
Sham Gurav

Numerical Integration: Trapezoidal Rule

Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} A_i\, f(x_i)$$

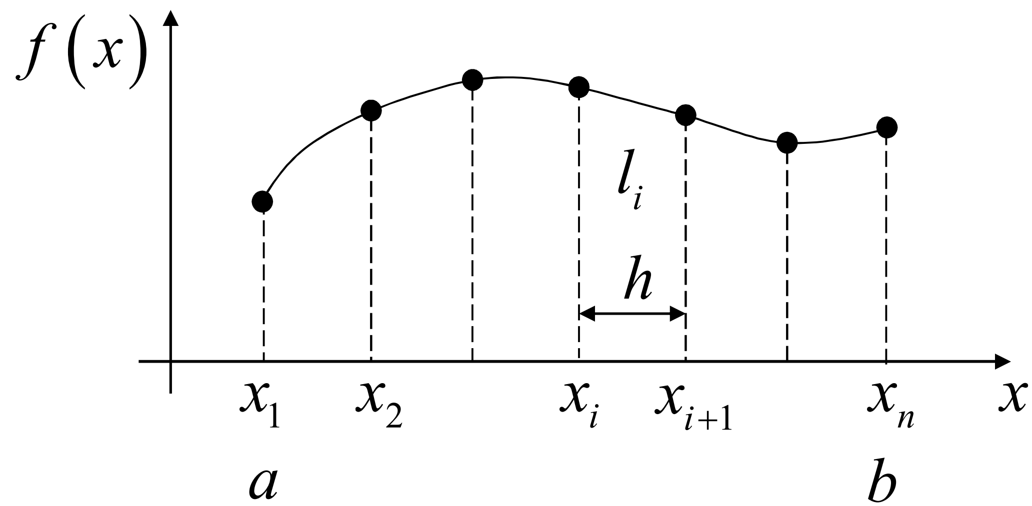
$$= \left[f(a) + f(b)\right]\frac{h}{2}$$



Composite Trapezoidal Rule:

$$\int_a^b f(x)\,dx$$

$$= \sum_{i=1}^{n-1}\left[f(x_i) + f(x_{i+1})\right]\frac{h}{2}$$



$$= \left[f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + +2f(x_{n-1}) + f(x_n)\right]\frac{h}{2}$$

Sham Gurav

Numerical Integration: Simpson's Rule

Simpson's (1/3) Rule:

$$\int_a^b f(x)\,dx = \sum_{i=1}^{3} A_i\, f(x_i) = \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]\frac{h}{3}$$

Simpson's (1/3) Composite Rule:

$$\int_a^b f(x)\,dx = \left[\begin{array}{l} f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + 2f(x_5) + \cdots \\ \qquad \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \end{array} \right]\frac{h}{3}$$

Simpson's (3/8) Rule:

$$\int_a^b f(x)\,dx = \sum_{i=1}^{4} A_i\, f(x_i) = \left[f(x_1) + 3f(x_2) + 3f(x_3) + f(x_4) \right]\frac{3h}{8}$$

Numerical Integration: Gauss Quadrature

- Unequally spaced points
- Higher Accuracy: Exact for polynomial of degree (2n-1)
- Integrals of the form $\int_a^b w(x) f(x)\, dx$
- Map Integration range from $\int_a^b$ to $\int_{-1}^1$ using transformation from x to ξ

$$x = \frac{b+a}{2} + \frac{b-a}{2}\xi$$

$$dx = \left(\frac{b-a}{2}\right) d\xi$$

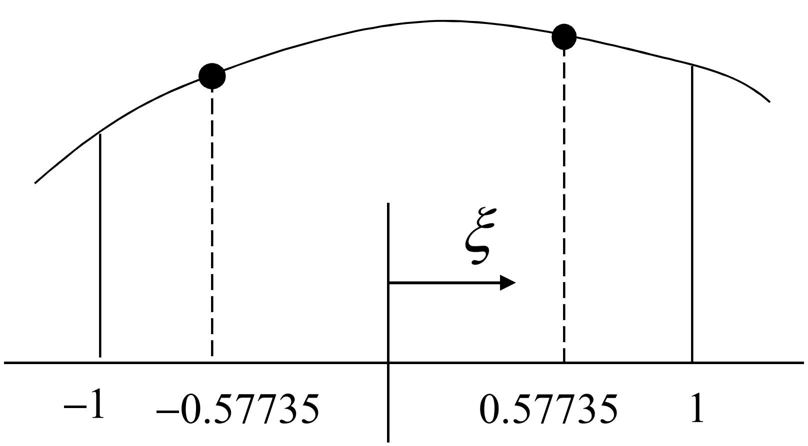
$$\int_a^b f(x)\, dx \approx \left(\frac{b-a}{2}\right) \int_{-1}^1 f(\xi)\, d\xi$$

$$\approx \left(\frac{b-a}{2}\right) \sum_{i=1}^{n} A_i\, f(\xi_i)$$

Numerical Integration: Gauss-Legendre Quadrature

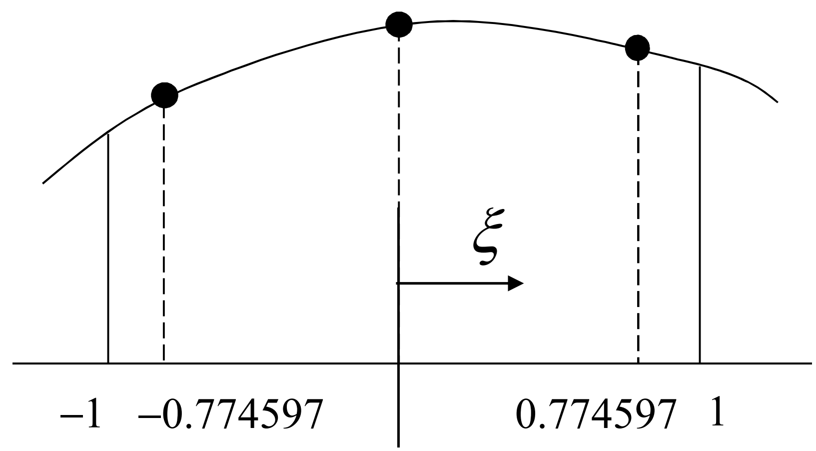
$$\int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{i=1}^{n} A_i\, f(\xi_i)$$

$$\int_{-1}^{1} f(\xi)\, d\xi \approx A_1\, f(\xi_1) + A_2\, f(\xi_2)$$



$n = 2$

$$\int_{-1}^{1} f(\xi)\, d\xi \approx A_1\, f(\xi_1) + A_2\, f(\xi_2) + A_3\, f(\xi_3)$$



$n = 3$

Numerical Integration: Gauss-Legendre Quadrature Example

Calculate $\int_{1.5}^{3} (-\cos x)\, dx$

For polynomial of order 5 :

$5 = 2n - 1$

$n = 3$

n	ξ_i	A_i
3	0.000000	0.888889
	±0.774597	0.555556

n	ξ_i	A_i
2	±0.577350	1.000000
3	0.000000	0.888889
	±0.774597	0.555556
4	±0.339981	0.652145
	±0.861136	0.347855
5	0.000000	0.568889
	±0.538469	0.478629
	±0.906180	0.236927
6	±0.238619	0.467914
	±0.661209	0.360762
	±0.932470	0.171324

Sham Gurav

Numerical Integration: Gauss-Legendre Quadrature Example

Calculate $\int_{1.5}^{3}(-\cos x)\,dx$

$$x = \left(\frac{3+1.5}{2}\right) + \left(\frac{3-1.5}{2}\right)\xi$$

$$x = 2.25 + 0.75\xi$$

$$\int_{1.5}^{3}(-\cos x)\,dx = \frac{(3-1.5)}{2}\int_{-1}^{1}\left(-\cos(2.25+0.75\xi)\right)d\xi = 0.75\sum_{i=1}^{3} A_i\, f(\xi_i)$$

$$A_1 = 0.888889 \qquad \xi_1 = 0.0$$

$$A_2 = 0.555556 \qquad \xi_2 = -0.774597$$

$$A_3 = 0.555556 \qquad \xi_3 = 0.774597$$

$$\int_{1.5}^{3}(-\cos x)\,dx$$

$$= 0.75\left[0.888889 \times 0.62816 + 0.555556 \times 0.098 + 0.555556 \times 0.95216\right]$$

$$= 0.85637$$

Sham Gurav