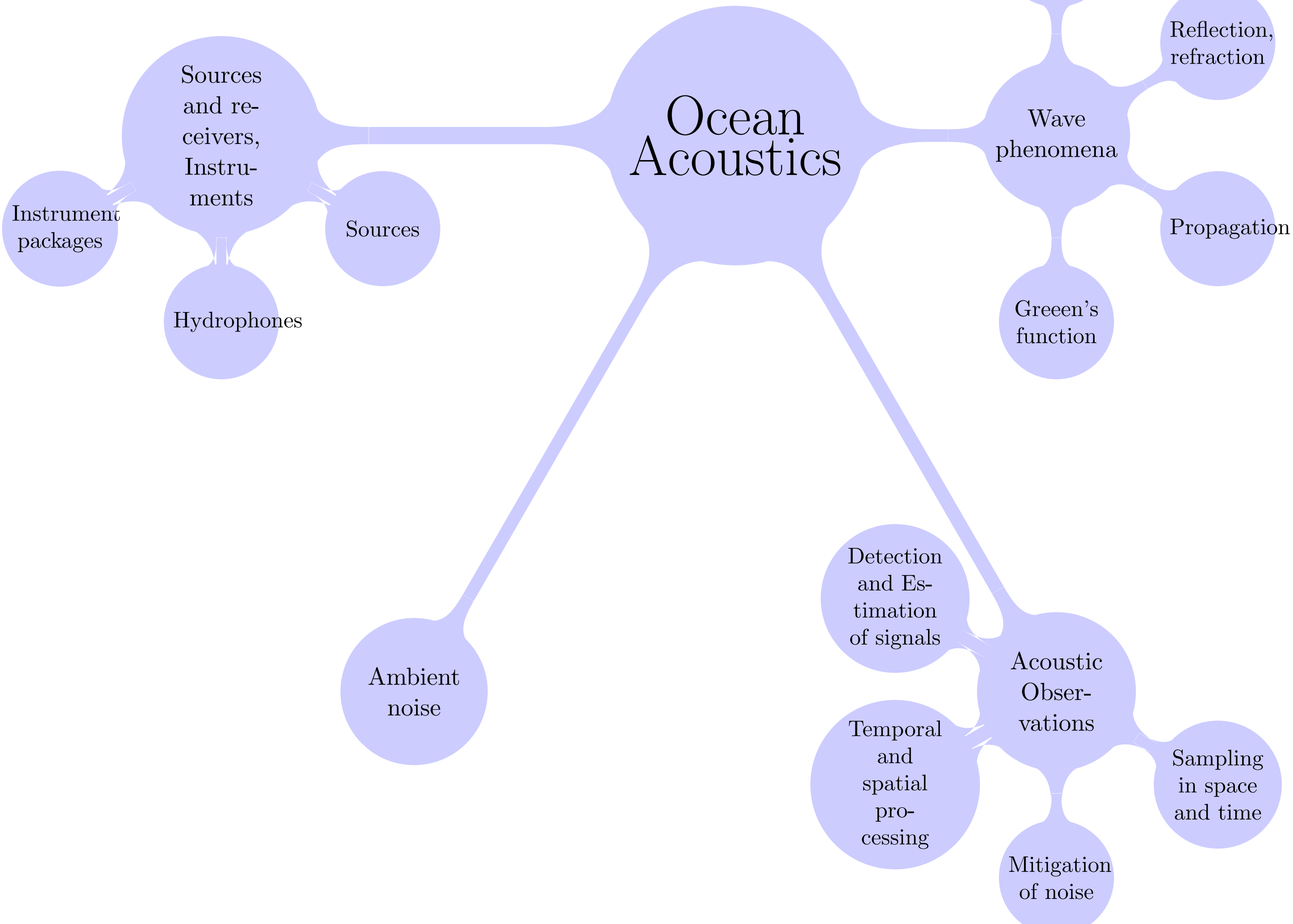
Ocean Acoustics
Sources and receivers, Instruments
Instrument packages
Hydrophones
Sources
Wave phenomena
Types of waves
Reflection, refraction
Propagation
Greeen's function
Ambient noise
Acoustic Observations
Detection and Estimation of signals
Temporal and spatial processing
Mitigation of noise
Sampling in space and time

Chapter 1

Introduction to the Course

The ocean is transparent to acoustics. This is especially true at low frequencies where sound travels to far distances with minimum attenuation. This makes sound an attractive tool for many applications such as communication, detection of targets, estimation of ocean depth, and studying marine life. Having said that, sound in the ocean has a few peculiar (or cool) features because of the medium. They are

- The bandwidth for sound is pretty low. The usable bandwidth is around 5 Hz to 500 KHz.
- The ocean contains temperature and salinity gradients by which sound is refracted several 10s of meters within short distances. This is called "anisotropy."
- The boundaries play an important role in the propagation. Sound interacts a lot with the ocean surface, and the ocean bottom.

Ocean acoustics thus deserves a separate subject of its own.

This course will teach the principles of acoustics, with an eye towards the applications. The list of subjects that this course will cover are

- Basics of waves
    1. Definitions of amplitude, and phase
    2. Wave propagation
    3. Interference phenomena

4. Refraction (Snell's law)
5. Attenuation
6. Fourier transforms to analyse waves, and vibrations

- Sound waves in the ocean

  1. Refraction and interference phenomena in the ocean
  2. Ray and mode theory
  3. Sound interaction with boundaries

- Observations

  1. Basics of hydrophones, and projectors
  2. Discrete time sampling, aliasing, and the Discrete Fourier Transform (DFT)
  3. Power spectral density calculation from observations
  4. Concepts such as bandwidth, and coherent gain
  5. Phased array processing, and spatial aliasing

Chapter 2

Oscillations of springs and vibration of beams

When sound propagates you have a disturbance along the way in the direction of propagation. This is called as longitudinal motion.

The direction along which the medium is compressed/expanded is the same along which the energy propagates. The vibrations are similar to a line of springs.

Now just think of one spring-mass system
The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{2.1}$$

Eq. 2.1 can be rewritten as

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{2.2}$$

Eq. 2.2 has harmonic solutions of the form $Acos(\omega_0 t)$, or $Acos(\omega_0 t + \phi)$. Let us consider one of the solutions

$$x = Acos(\omega_0 t + \phi) \tag{2.3}$$

In equation Eq. 2.3, A is referred to as the amplitude of vibration, ω as the angular frequency, and ϕ.

The spring-mass system in Fig. 2.2 does not yet consider the effect of a mechanical resistance. If there were friction say then the equation of motion becomes

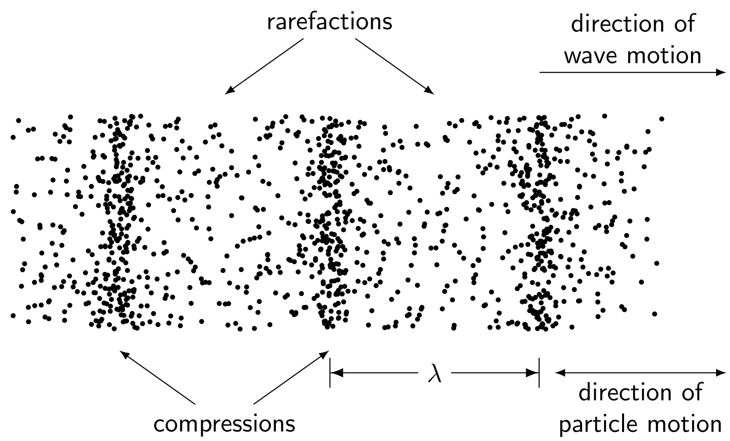

Figure 2.1: The propagation of a compressional wave

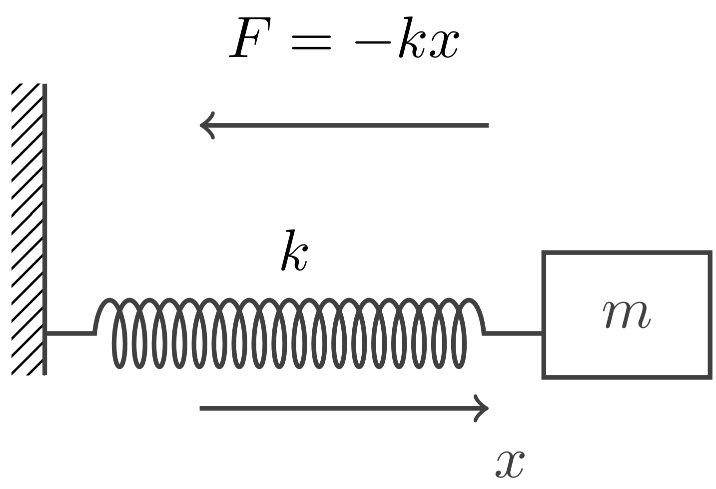

Figure 2.2: Spring mass system

$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + kx = 0 \tag{2.4}$$

Put in a solution as $x = Ae^{\gamma t}$ in Eq. 2.4. Then Eq. 2.4 becomes

$$\gamma^2 + \left(\frac{R_m}{m}\right)\gamma + \omega_0^2 = 0 \tag{2.5}$$

Equation. 2.5 has roots

$$\gamma = -\beta \pm \left(\beta^2 - \omega_0^2\right)^{1/2} \tag{2.6}$$

Say $\beta \ll \omega_0$ (a realistic assumption for acoustics), we have

$$\gamma = -\beta \pm j\omega_d \tag{2.7}$$

The solutions for x are then

$$x = e^{-\beta t}\left(A_1e^{j\omega_d t} + A_2e^{-j\omega_d t}\right) \tag{2.8}$$

For a certain choice of $A_1 = A_2 = A/2$ we get

$$x = Ae^{-\beta t}cos(\omega_d t + \phi) \tag{2.9}$$

So you have a cosinusoidal waveform (similar to Eq. 2.3), but that decays in time. That damping or decay is due to resistance.

We have not put in an explicit source term, but they can be added. These concepts or the wave amplitude, phase, and attenuation are all relevant to discussing acoustic waves.

Chapter 3

Fourier Transforms

Background Objectives

- Learn concepts such as Fourier series, Fourier transforms, and the Discrete Fourier Transform

3.1 Fourier Series

Let $f(t)$ be a periodic function with period T. Define the fundamental frequency $\omega_0 = 2\pi/T$. The function $f(t)$ can be represented by a Fourier series of sines and cosines:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \Big],$$

where the coefficients are given by

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos(n\omega_0 t)\, dt, \qquad b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin(n\omega_0 t)\, dt,$$

and

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t)\, dt.$$

These formulas arise from the orthogonality of $\{\cos(n\omega_0 t),\ \sin(n\omega_0 t)\}$ over one period. Equivalently, one can use the complex exponential form:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad c_k = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jk\omega_0 t}\, dt.$$

Here c_k are generally complex, and a_n, b_n can be recovered from c_k (e.g. $a_n = c_n + c_{-n}$, $b_n = j(c_n - c_{-n})$).

Dirichlet Conditions

Sufficient conditions for pointwise convergence of the Fourier series (Dirichlet conditions) include:

- $f(t)$ is absolutely integrable over one period.
- $f(t)$ has a finite number of maxima and minima in any finite interval.
- $f(t)$ has a finite number of discontinuities in any finite interval.
- At a discontinuity, the Fourier series converges to the midpoint of the jump (Gibbs phenomenon aside).

Example: Square Wave

Consider the 2π-periodic square wave defined by

$$f(t) = \begin{cases} 1, & 0 < t < \pi, \\ -1, & \pi < t < 2\pi, \end{cases} \qquad f(t + 2\pi) = f(t).$$

This function is odd about π, so $a_n = 0$ for all n. Computing b_n yields $b_n = \frac{4}{\pi n}$ for odd n, and $b_n = 0$ for even n. Thus its Fourier series is

$$f(t) = \frac{4}{\pi}\Big(\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\Big).$$

3.2 Fourier Transform

Let $x(t)$ be a non-periodic (non-decaying) signal. One obtains the continuous-time Fourier transform by letting the period $T \to \infty$ (so $\omega_0 \to 0$). Formally, the Fourier transform $X(\omega)$ and inverse transform are defined by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt, \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega.$$

These integrals require $x(t)$ to be absolutely integrable (or square integrable) and $X(\omega)$ to exist.

Properties

Important properties of the Fourier transform include:

- **Linearity:** If $y(t) = ax(t) + bz(t)$ then $Y(\omega) = aX(\omega) + bZ(\omega)$.
- **Time shift:** $x(t - t_0) \xrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$.
- **Frequency shift:** $e^{j\omega_0 t} x(t) \xrightarrow{\mathcal{F}} X(\omega - \omega_0)$.
- **Time scaling:** $x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$, for $a \neq 0$.

Example: Decaying Exponential

Consider $x(t) = e^{-at}u(t)$ with $a > 0$ (so $u(t)$ is the unit step function). Then the Fourier transform is

$$X(\omega) = \int_0^\infty e^{-at} e^{-j\omega t}\, dt = \int_0^\infty e^{-(a+j\omega)t}\, dt = \frac{1}{a + j\omega},$$

valid for $\Re\{a\} > 0$. The inverse transform recovers $x(t) = e^{-at}u(t)$.

3.3 Discrete Fourier Transform (DFT)

Let $x[n]$ be a finite-length sequence of length N (or one period of an N-periodic sequence). The discrete Fourier transform (DFT) is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}, \quad k = 0, 1, \ldots, N-1.$$

The inverse DFT is

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j(2\pi/N)kn}.$$

The DFT can be interpreted as the Fourier series coefficients of a periodic sequence or as sampling the discrete-time Fourier transform (DTFT) at N uniformly spaced frequencies.

In matrix form, define the DFT matrix W_N of size $N \times N$ with entries $W_N[k, n] = e^{-j(2\pi/N)kn}$. Then

$$\mathbf{X} = W_N\,\mathbf{x}, \quad \mathbf{x} = \frac{1}{N} W_N^H\,\mathbf{X},$$

where $\mathbf{x} = [x[0], \ldots, x[N-1]]^T$, $\mathbf{X} = [X[0], \ldots, X[N-1]]^T$, and $(\cdot)^H$ denotes the Hermitian transpose.

Example: Finite Sample of a Cosine

Consider a real cosine sequence $x[n] = \cos\!\left(\frac{2\pi m}{N} n\right)$ for $n = 0, 1, \ldots, N-1$, where m is an integer. Using Euler's formula,

$$x[n] = \frac{1}{2}\Big(e^{j\frac{2\pi m}{N}n} + e^{-j\frac{2\pi m}{N}n}\Big).$$

The DFT of $x[n]$ is then

$$X[k] = \frac{1}{2}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn}\Big(e^{j\frac{2\pi m}{N}n} + e^{-j\frac{2\pi m}{N}n}\Big).$$

Each term yields a Kronecker delta: $\sum_{n=0}^{N-1} e^{j2\pi(m-k)n/N} = N\,\delta_{k,m}$ (for integer m). Thus

$$X[k] = \frac{N}{2}\left[\delta_{k,m} + \delta_{k,N-m}\right],$$

showing two spectral lines at $k = m$ and $k = N - m$ (for $m \neq 0, N/2$). In other words, a sampled cosine yields two equal-magnitude peaks in its DFT.

Chapter 4

Speed of sound

Background Objectives

Learning Goals

- Try answer the question : How to calculate the speed of sound ?

Tools Required

- Concepts from thermodynamics of an adiabatic process, and isothermal process

Background Mathematics

- Calculus

$$\boxed{c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_0}} \tag{4.1}$$

Eq. 4.1 gives an expression for speed of sound c. The 0 in the subscript denotes steady state. We will call this velocity as c.

But how to calculate the value of c in Eq. 4.1 ?

Approach 1: Newton

$$\begin{aligned} p &\propto \frac{1}{V} \\ pV &= \text{constant} \end{aligned} \tag{4.2}$$

From Eq. 4.2

$$\begin{aligned} pdV + Vdp &= 0 \\ \implies \frac{dp}{p} + \frac{dV}{V} &= 0 \end{aligned} \tag{4.3}$$

But

$$\begin{aligned} V &= \frac{m}{\rho} \\ \implies \frac{dV}{V} &= -\frac{d\rho}{\rho} \end{aligned} \tag{4.4}$$

From Eq. 4.3, and 4.4

$$\frac{dp}{d\rho} = \frac{p}{\rho} \tag{4.5}$$

How does sound speed calculations work from Eq. 4.5 ? Let us verify it with what we know the speed of sound to be (330 m/s) in dry air. The pressure of air in Chennai is now

$$\begin{aligned} p &= \text{Density of } Hg \times \text{Height of mercury} \times g \\ &= 13600kg/m^3 \times 760 \times 10^{-3}m \times 9.8m/s^2 \end{aligned} \tag{4.6}$$

and

$$\rho = 1.299(kg/m^3) \tag{4.7}$$

Density of dry air

From Eq. 4.5, 4.6, and 4.7

$$\begin{aligned} c &= \sqrt{\frac{13600 \times 760 \times 10^{-3} \times 9.8}{1.299}} \\ &= 279.24m/s \\ &\neq 330m/s \end{aligned} \tag{4.8}$$

Newton assumed Boyle's law held for sound propagation. That is called the isothermal assumption. It did not work out. What we actually need is the adiabatic law, where no heat is exchanged with the surroundings. The adiabatic gas law is

$$\begin{aligned} pV &= nRT \\ \text{from which} \frac{dp}{p} + \frac{dV}{V} &= \frac{dT}{T} \end{aligned} \tag{4.9}$$

The work dQ done by the acoustic wave is

$$dQ = \left(\frac{\partial Q}{\partial V}\right) dV + \left(\frac{\partial Q}{\partial p}\right) dp \tag{4.10}$$

For conditions of constant pressure setting $dp = 0$ in Eqs. 4.9 and 4.10 we get

$$\begin{aligned} \left(\frac{\partial Q}{\partial V}\right) \frac{V}{T} &= \left(\frac{\partial Q}{\partial T}\right)_p \\ &= K_p \end{aligned} \tag{4.11}$$

The k_p in Eq. 4.11 is the specific heat of gas under constant pressure. Similarly for conditions of constant volume $dV = 0$ we get

$$\left(\frac{\partial Q}{\partial p}\right) \frac{p}{T} = K_v \tag{4.12}$$

Set $dQ = 0$ in Eq. 4.10, and use Eqs. 4.11 and 4.12 to get

$$\begin{aligned} \left(\frac{\partial Q}{\partial V}\right) dV + \left(\frac{\partial Q}{\partial P}\right) dP &= 0 \\ k_p \frac{dV}{V} + k_v \frac{dP}{P} &= 0 \end{aligned} \tag{4.13}$$

From Eq. 4.4 we had $\frac{dV}{V} = -\frac{d\rho}{\rho}$. We use it in Eq. 4.13 to get

$$\frac{dp}{d\rho} = \boxed{\frac{P}{\rho}} \left(\frac{k_v}{k_p}\right) \tag{4.14}$$

Newton's expression

Plugging in values for P and ρ from before and using $\frac{k_v}{k_p} = 1.4$, the specific heat for air you get

$$\begin{aligned} c &= \sqrt{\frac{dP}{d\rho}} \\ &= 279.24 \times \sqrt{1.4} \\ &= 330.5 m/s \end{aligned} \tag{4.15}$$

which is value that we come to expect. The speed of sound in air is less than the average (root-mean-square velocity) of the molecular velocity of air. The RMS velocity of air is around 500 m/s, which however includes all the random motions that air-molecules undergo. Sound (in air) propagates by the air molecules bumping into each other which takes place at a slower speed than the RMS velocity.

Speed of sound in water

For water it is not possible to derive analytical expressions like we did for air. The expression for speed of sound in water is

$$c^2 = \gamma \mathcal{B}_T / \rho_0 \tag{4.16}$$

where $\mathcal{B}_T$ is the isothermal bulk modulus, and γ is the ratio of specific heats. These quanities γ, $\mathcal{B}_T$, and ρ_0 depend on the ambient pressure, density, and temperature. You do not have analytical expressions for each of those cases. Thus you stick with expressions like

$$\begin{aligned} c(T,S,z) = {} & 1448.96 + 4.591\,T - 5.304 \times 10^{-2}\,T^2 + 2.374 \times 10^{-4}\,T^3 \\ & + 1.340\,(S-35) + 1.630 \times 10^{-2}\,z + 1.675 \times 10^{-7}\,z^2 \\ & - 1.025 \times 10^{-2}\,T\,(S-35) - 7.139 \times 10^{-13}\,T\,z^3 \end{aligned} \tag{4.17}$$

The Eq. 4.17 is an empirical speed equation [1]. Temperaure profile is similar to Fig. 4.1 ($T = T_0 exp(-z/500)$), and from it the sound speed prediction (Fig. 4.2).

You also have sound propagation through "fluid-like" sea bottoms such as red-clay, coarse silt, and sand with sound speeds of $1460m/s$, $1540m/s$, and $1730m/s$. The acoustic properties of these materials depend of properties such as grain size, how closely packed the sediment-particles are packed, and the amount of moisture.

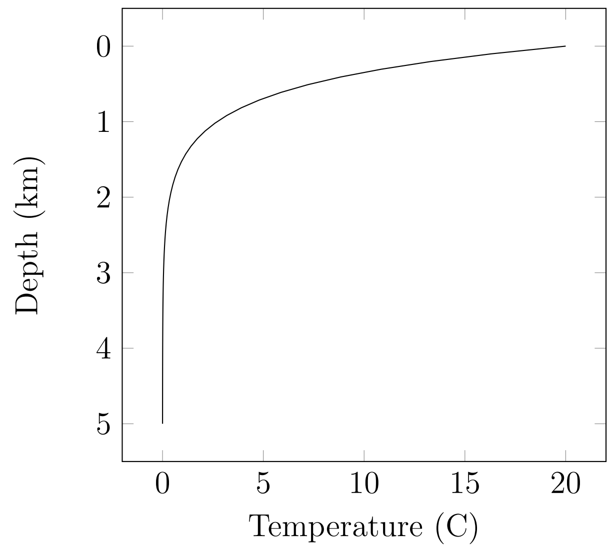

Figure 4.1: Temperature Profile

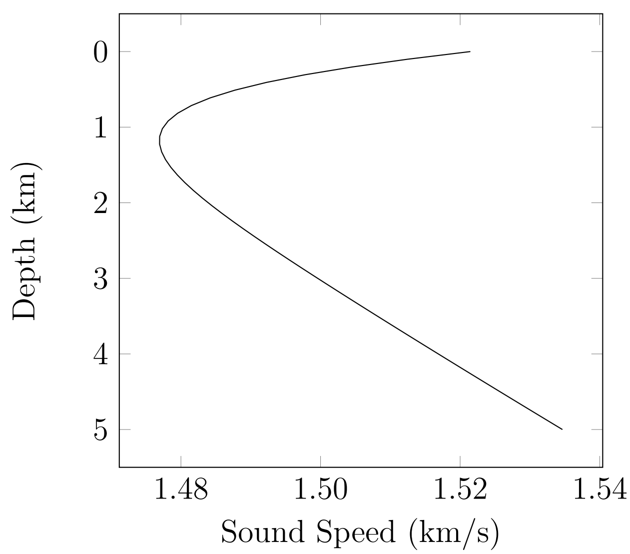

Figure 4.2: Sound Speed Profile

Sound propagation through solids
Ice has a sound speed of 2900-3200 m/s. Pyrex glass has a sound-speed of 5600 m/s.
How well does adiabtic approximation work ?
It however turns out that the adiabatic explanation for sound propagation is appropriate for frequencies as high as $10^9 Hz$ in air acoustics, and $2 \times 10^{12} Hz$ in water.

Chapter 5

Derivation of the acoustic wave equation

Background Objectives

Learning Goals

- Derive the linear acoustic wave equation

Tools Required

- Conservation of mass (mass is constant)
- Continuity of momentum ($F = ma$)
- Remember speed of sound is $c = \sqrt{\frac{dp}{d\rho}}$, and its calculation(s) from the previous chapter

Background Mathematics

- Calculus
- Calculus in 3D space, gradient, Laplacian

When a wave propagates through the parcel the following spatial and temporal changes happen

1. The pressure changes

$$p = p_0 + p' \tag{5.1}$$

The pressure p_0 is the ambient pressure which we assume to be a constant in space. Yet in Eq. 5.1 $p' \ll p_0$. Change in pressure is very small compared to the ambient pressure, which is a reasonable assumption to make.

2. The density of the parcel changes

$$\rho = \rho_0 + \rho' \tag{5.2}$$

Again $\rho' \ll \rho_0$.

3. The velocity of the fluid changes

$$\vec{u} = \vec{u}_0 + \vec{u}' \tag{5.3}$$

We assume $u_0 = 0$. The background velocity is 0.

The derivation for the acoustic wave equation relies on

1. The equation of conservation of mass
2. The equation of conservation of momentum
3. The expression for velocity of sound

5.1 Equation for conservation of mass

The rate of change of mass due to the change in density is

$$\frac{dm}{dt} = \frac{\partial \rho}{\partial t} dxdydz \tag{5.4}$$

For the dm in the LHS of Eq. 5.4 we use a concept called "mass-flux" which is the rate of flow of mass per unit area. The mass flux is ρu.

$$dm = (\rho u)_x dydzdt - (\rho u)_{x+dx} dydzdt \tag{5.5}$$

By approximating

$$(\rho u)_x - (\rho u)_{x+dx} = -\left(\frac{\partial(\rho u)}{\partial x}\right)dx \tag{5.6}$$

$$\frac{dm}{dt} = -\left(\frac{\partial(\rho u)}{\partial x}\right)dxdydz \tag{5.7}$$

$$\implies \frac{d\rho}{dt} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{5.8}$$

$$\tag{5.9}$$

5.2 Conservation of momentum

Let us take the second part which is the consservation of momentum

$$\underbrace{\boxed{\frac{D\int \rho\vec{u}dV}{Dt}}}_{\text{Convective derivative}} = \underbrace{\boxed{\vec{F}_s}}_{\text{Surface forces}} + \underbrace{\boxed{\vec{F}_b}}_{\text{Body forces (neglect except gravity waves at lowest frequencies)}} \tag{5.10}$$

The resultant surface forces that act on the total volume is

$$\vec{F}_s = \sum_{\Delta V \to 0} \vec{f}_s(\Delta V) \tag{5.11}$$

where

$$\vec{f}_s = (f_x \quad f_y \quad f_z) \tag{5.12}$$

Consider only the surface forces along the x-direction to get

$$f_x = -\left[p(x+dx, y, z)dydz - p(x, y, z)dydz\right] \tag{5.13}$$

The -ve sign is because the force acts in a direction that is opposite to the gradient of the pressure. Expanding Eq. 5.13 using only the first two terms of the Taylor's series

$$
\begin{aligned}
f_x &= -\left[p(x,y,z)dydz + \frac{\partial p}{\partial x}dxdydz - p(x,y,z)dydz\right] \\
&= -\left[\cancel{p(x,y,z)dydz} + \frac{\partial p}{\partial x}dxdydz - \cancel{p(x,y,z)dydz}\right] \\
&= -\left[\frac{\partial p}{\partial x}\right]dxdydz \\
&= -\frac{\partial p}{\partial x}dV
\end{aligned} \tag{5.14}
$$

Similar to the x direction, repeat Eq. 5.14 for y, and z also to get

$$
\begin{aligned}
\vec{f_s} &= -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)dV \\
&= -\nabla p dV.
\end{aligned} \tag{5.15}
$$

The total surface force by Eq. 5.11 and 5.15 is thus

$$
\vec{F_s} = -\int \nabla p dV \tag{5.16}
$$

Now we tackle the left hand side of Eq. 5.10.

$$
\begin{aligned}
\frac{D\int \rho\vec{u}dV}{Dt} &= \int \rho\frac{D\vec{u}}{Dt}dV \\
&= \int \rho\left(\frac{\partial \vec{u}}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial \vec{u}}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial \vec{u}}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial \vec{u}}{\partial z}\right)dV \\
&= \int \rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\cdot\vec{u}\right)dV
\end{aligned} \tag{5.17}
$$

Equate Eq. 5.17 with Eq. 5.16 (LHS to RHS) to get

$$
\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\cdot\vec{u}\right) = -\nabla p \tag{5.18}
$$

The 3 equations for us now are

1.

$$\begin{aligned} & p_0 + p' = p(\rho_0 + \rho')|_0 \\ \implies & p' = p(p') \\ & \quad = \left(\frac{\partial p}{\partial \rho}\right)_0 \rho' \\ \implies & p' = c^2 \rho' \end{aligned} \tag{5.19}$$

2. From conservation of mass we have

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_0 + \rho') &= -\nabla \cdot ((\rho_0 + \rho')u') \\ \implies \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot u' &= 0 \end{aligned} \tag{5.20}$$

   Remember only u', and no u_0 because $u_0 = 0$. Also $\rho' \times u'$ is a second order term that we neglect. Using Eq. 5.19 ($p' = c^2\rho'$) Eq. 5.20 becomes

$$\frac{1}{c^2}\frac{\partial p'}{\partial t} + \rho_0 \nabla \cdot \vec{u'} = 0 \tag{5.21}$$

3. From Euler we have

$$\begin{aligned} (\rho_0 + \rho')\frac{D\vec{u'}}{Dt} &= -\nabla(p_0 + p') \\ \implies \rho_0 \frac{\partial \vec{v'}}{\partial t} + \cancel{\left(\vec{u'} \cdot \nabla\right)} \vec{u'} &= -\nabla p' \\ \implies \rho_0 \frac{\partial \vec{u'}}{\partial t} &= -\vec{\nabla} p' \\ \implies \vec{\nabla} \cdot (\rho_0 \frac{\partial \vec{u'}}{\partial t}) &= \vec{\nabla} \cdot \left(-\vec{\nabla} p'\right) \\ \implies \rho_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{u'}\right) &= \nabla^2 p' \end{aligned} \tag{5.22}$$

Using Eq. 5.21 in Eq. 5.22 we get

$$\nabla^2 p' = \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} \tag{5.23}$$

Taking out the primes now to get

$$\boxed{\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}} \tag{5.24}$$

This is the acoustic wave equation.

Chapter 6

Solutions to the wave equation

What do the solutions to the wave equation (Eq. 5.24) look like ?

In the cartesian co-ordinate system we have for the Laplacian

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \tag{6.1}$$

Assume c^2 does not depend on the co-ordinates x, y, and z then Eq. 5.24 becomes

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{6.2}$$

One solution is

$$p = p_0 e^{i(k_x x + k_y y + k_z z + \omega t)} \tag{6.3}$$

Another solution is

$$p = p_0 sin(k_x x + k_y y + k_z z - \omega t) \tag{6.4}$$

This is a harmonic waveform in x, y, and z. The waveform repeats itself with spatial (angular) frequencies k_x, k_y, and k_z in the x, y, and z directions. The k_x, k_y and k_z are called wavenumbers. Put $k_x = 0$, $k_y = 0$, then the wave becomes $sin(k_z z - \omega t)$. The wavenumber is defined as

$$k_z = \frac{2\pi}{\lambda_z} \tag{6.5}$$

The λ_z is the wavelength, which is the spatial period. The phase of the wave is constant along x and y, but a function of z. If we drew the lines of constant phase for which $k_z z - \omega t$ were constant for different values of time t, then they would look like the lines in Fig. 6.1. The lines perpendicular to the direction of propagation are called wavefronts. They are also referred to as phasefronts.

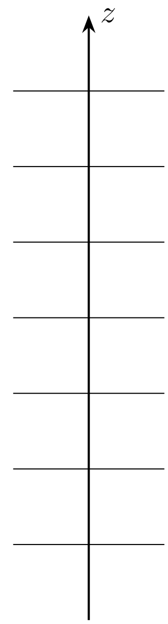

Figure 6.1: Wavefronts perpendicular to direction of propagation

The locus of constant phase at different points in time t and space z should satisfy

$$\begin{aligned} d(k_z z - \omega t) &= 0 \\ \implies \frac{dz}{dt} &= \frac{\omega}{k_z} \\ \text{or} \quad c_{ph} &= \frac{\omega}{k_z} \\ &= \frac{f}{\lambda_z} \end{aligned} \tag{6.6}$$

	Frequency	Wavenumber
Wave 1 p_1	$f_0 + \Delta f/2$	$k_0 + \Delta k/2$
Wave 2 p_2	$f_0 - \Delta f/2$	$k_0 - \Delta k/2$

Table 6.1: Frequencies and wavenumbers for two interfering waves

The c_{ph} in Eq. 6.6 is called the "phase velocity," which is the speed according to which the phasefronts propagate.

Complementary to the idea of phase velocity is the concept of group velocity. For that consider the resultant of a group of two waves at different frequncies.
The resultant of the two waves p_1 and p_2 in Table 6.1 is

$$\begin{aligned} y &= p_1(t) + p_2(t) \\ &= sin(k_1 z - \omega_1 t) + sin(k_2 z - \omega_2 t) \\ &= sin(k_0 z + \Delta k/2z - \omega_0 t - \Delta\omega/2t) + sin(k_0 z - \Delta k/2z - \omega_0 t + \Delta\omega/2t) \\ &= 2sin(k_0 z - \omega_0 t)cos(\Delta kz/2 - \Delta\omega t/2) \end{aligned} \tag{6.7}$$

The wave on the left is the comparatively high frequency wave, and the one on the right is the envelope which varies slow. The envelope travels at a speed c_g.

$$c_g = \frac{\Delta\omega}{\Delta k} \tag{6.8}$$

The c_g in Eq. 6.8 is called the "group velocity."

While we used Eq 6.1 for the Laplacian operator in the rectangular co-ordinate system, we could use the Laplacian in the cylindrical co-ordinate system as

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2} + \frac{\partial^2 p}{\partial z^2} \tag{6.9}$$

The solution for p from the wave equation in Eq. 5.24 is then

$$p = \boxed{\sqrt{\frac{2}{\pi kr}}} p_0 e^{i(kr - \omega t - \pi/4)} \tag{6.10}$$

Amplitude decreases unlike plane wave

The phasefronts follow the locus

$$d(kr - \omega t) = 0 \tag{6.11}$$

Similarly for a spherical wave

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial p}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 p}{\partial \phi^2} \tag{6.12}$$

The solution for the wave equation is then

$$p = \frac{1}{r}p_0 e^{i(kr - \omega t)} \tag{6.13}$$

Faster amplitude decreases faster than cylindrical

The phasefronts are spheres with radius r that are radiating from the source at origin.

Chapter 7

Intensity Calculations and the deciBel scale

When an acoustic wave proapgates there is an energy to the acoustic wave. There are two types of energies (kinetic because of the motion of the fluid), and potential because there is a compression of volume due to the pressure.

While measuring sound people usually resort to intensity. Intensity is the power per unit area. For these calculations assume that the pressure wave is

$$p(t,x) = P_0 sin(\omega t - kx) \tag{7.1}$$

The work done is

$$\begin{aligned} dW &= F dx \\ \text{Power} &= \frac{dW}{dt} \\ \implies \text{Power} &= Fu \\ &= pAu \end{aligned} \tag{7.2}$$

Note this power instantaneous, the power is a function of time. Now we have $u = \frac{p}{\rho c}$, and $F = p \times A$. The power is then

$$\begin{aligned} \text{Power} &= pAu \\ \implies &= \frac{p^2 A}{\rho c} \\ \implies \text{Intensity} &= \frac{p^2}{\rho c} \end{aligned} \tag{7.3}$$

Assume $p(t) = P_0 sin(\omega t - kx)$. The average intensity is then

$$\langle I \rangle_T = \frac{P_0^2}{2\rho c} \tag{7.4}$$

Or we could write

$$\langle I \rangle_T = \frac{p_{rms}^2}{\rho c} \tag{7.5}$$

where $p_{rms} = P_0/\sqrt{2}$.

Concept of the deciBel scale Sound is measured in dB because the log-scale compresses a wide range of values, and also that human hearing follows a log-scale. The deciBel was originally used to compare signal powers in telegraph circuits. The dB is defined as

$$SPL_{dB} = 10log_{10}\left(\frac{I}{I_{ref}}\right) \tag{7.6}$$

The reference intensity I_{ref} is

$$I_{ref} = \frac{P_{ref}^2}{\rho c} \tag{7.7}$$

The values for air are

$$\begin{aligned} P_{ref}^{air} &= 20\mu Pa \\ \rho &= 1kg/m^3 \\ c &= 330m/s \end{aligned}$$

The values for water are

$$\begin{aligned} P_{ref}^{water} &= 1\mu Pa \\ \rho &= 1000kg/m^3 \\ c &= 1500m/s \end{aligned}$$

Note the difference in the specific impedances

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