Wave Equation in Cylindrical Coordinates

Symmetric Case and Bessel Function Approximations

1 The Source-Free Wave Equation

We begin with the scalar wave equation in three dimensions:

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0. \tag{1}$$

This equation governs acoustics, electromagnetics (scalar components), and other linear wave phenomena in homogeneous media.

2 Cylindrical Coordinates

In cylindrical coordinates (r, ϕ, z), the Laplacian is

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2} + \frac{\partial^2 p}{\partial z^2}. \tag{2}$$

3 Symmetry Assumptions

We assume:

- azimuthal symmetry: $\partial_\phi p = 0$,
- axial (depth) symmetry: $\partial_z p = 0$.

Under these assumptions, the Laplacian reduces to

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right). \tag{3}$$

4 Wave Equation in Radial Form

Substituting into the wave equation yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0. \tag{4}$$

This is the wave equation for cylindrically symmetric fields.

5 Time-Harmonic Reduction

Assume harmonic time dependence:

$$p(r,t) = \psi(r)\, e^{-i\omega t}. \tag{5}$$

Substitution gives the radial Helmholtz equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) + k^2\psi = 0, \qquad k = \frac{\omega}{c}. \tag{6}$$

6 Radial Differential Equation

Expanding explicitly:

$$\frac{1}{r}\left(r\psi'' + \psi'\right) + k^2\psi = 0, \tag{7}$$

$$r\psi'' + \psi' + k^2 r\psi = 0. \tag{8}$$

Multiplying by r:

$$\boxed{r^2\psi'' + r\psi' + k^2r^2\psi = 0.} \tag{9}$$

7 Bessel Equation of Order Zero

Define the dimensionless variable

$$x = kr. \tag{10}$$

Then the equation becomes

$$\boxed{x^2\frac{d^2\psi}{dx^2} + x\frac{d\psi}{dx} + x^2\psi = 0.} \tag{11}$$

This is Bessel's equation of order zero.

8 General Solution

The general solution is

$$\boxed{\psi(r) = AJ_0(kr) + BY_0(kr),} \tag{12}$$

where

- J_0 is finite at $r = 0$,
- Y_0 is logarithmically singular at $r = 0$.

or in terms of Hankel functions

$$\boxed{\psi(r) = C(J_0(kr) \pm iY_0(kr)),} \tag{13}$$

Physical regularity at the axis requires $B = 0$ if $r = 0$ is included.

9 Approximate Forms of the Solution

9.1 Small-Argument Approximation ($kr \ll 1$)

Using the series expansion:

$$\boxed{J_0(x) \approx 1 - \frac{x^2}{4} + O(x^4).} \tag{14}$$

The second solution behaves as

$$\boxed{Y_0(x) \approx \frac{2}{\pi}\left(\ln\frac{x}{2} + \gamma\right),} \tag{15}$$

where γ is Euler's constant.

9.2 Large-Argument Approximation ($kr \gg 1$)

For large arguments, the solutions become oscillatory:

$$\boxed{J_0(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right),} \tag{16}$$

$$\boxed{Y_0(x) \approx \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4}\right).} \tag{17}$$

Using $x = kr$, and choosing the Hankel function representation with only the outgoing waves (with the - sign), the expression for the pressure field is

$$\boxed{p(r,t) = \sqrt{\frac{2}{\pi kr}}e^{ikr-\pi/4}} \tag{18}$$

The amplitude decays as $1/\sqrt{r}$ due to cylindrical spreading.

10 Summary

Cylindrical symmetry reduces the wave equation to a radial Bessel equation. Near the origin, the solution is approximately constant, while far from the origin it behaves like a decaying oscillatory wave.